TORIC VARIETIES WITH AMPLE TANGENT BUNDLE

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ABSTRACT. We give a simple combinatorial proof of the toric version of Mori's theorem that the only *n*dimensional smooth projective varieties with ample tangent bundle are the projective spaces \mathbb{P}^n .

1. INTRODUCTION

It is a well-known theorem that the only smooth projective varieties (over an algebraically closed field k) with ample tangent bundles are the projective spaces \mathbb{P}_k^n . This is first conjectured by Hartshorne [Har70, Problem 2.3] and later proved by Mori [Mor79] using the full force of his now-celebrated "bend and break" technique. Here we say that a vector bundle \mathcal{E} is ample (resp. nef) if the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on the projectivized bundle $\mathbb{P}\mathcal{E}$ is ample (resp. nef).

In this paper, we consider a toric version of this theorem and show that it admits a simple combinatorial proof.

Theorem 1.1. Let X be an n-dimensional smooth projective toric variety (over an algebraically closed field k) with ample tangent bundle \mathcal{T}_X . Then X is isomorphic to \mathbb{P}_k^n .

In the proof we consider the polytope $P \subseteq \mathbb{R}^n$ corresponding to X (together with any ample divisor D). The key observation we make is that the ampleness of \mathcal{T}_X implies that the sum of any pair of two adjacent angles on a 2-dimensional face of P is smaller than π . It follows that P has to be an n-simplex, and hence X is isomorphic to \mathbb{P}^n .

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2. Preliminaries

Here we list out some definitions and facts regarding toric varieties and toric vector bundles that we will use in this article. One may refer to [Ful93, CLS11] for more details about toric varieties, and [Pay08, DRJS18] for more details about toric vector bundles.

2.1. Toric varieties. We work throughout over an algebraically closed field k. By a toric variety, we mean an irreducible and normal algebraic variety X containing a torus $T \cong (k^*)^n$ as a Zariski open subset such that the action of T on itself (by multiplication) extends to an algebraic action of T on X.

Let M be the group of the characters of T, and N the group of the 1-parameter subgroups of T. Both Mand N are lattices of rank n (equal to the dimension of T), i.e. isomorphic to \mathbb{Z}^n . They are dual to each other in the sense that there is a natural pairing of M and N denoted by $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$.

Every toric variety X is associated to a fan Σ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} (\cong \mathbb{R}^n)$. A fan Σ is said to be *complete* if it supports on the whole $N_{\mathbb{R}}$, and is said to be *smooth* if every cone in Σ is generated by a subset of a \mathbb{Z} -basis of

N. A toric variety X is complete if and only if its associated fan Σ is complete, and X is smooth if and only if Σ is smooth.

There is an inclusion-reversing bijection between the cones $\sigma \in \Sigma$ and the *T*-orbits in *X*. Let $O_{\sigma} \subseteq X$ be the orbit corresponding to σ . The codimension of O_{σ} in *X* is equal to the dimension of σ . Each cone $\sigma \in \Sigma$ also corresponds to an open affine set $U_{\sigma} \in X$, which is equal to the union of all the orbits O_{τ} corresponding to cones τ contained in σ . Given a 1-dimensional cone $\rho \in \Sigma$, the closure of O_{ρ} is a Weil divisor, denoted by D_{ρ} . The class group of *X* is generated by the classes of the divisors D_{ρ} corresponding to the 1-dimensional cones in Σ .

2.2. Polytopes and toric varieties. Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. A lattice polytope P in $M_{\mathbb{R}}$ is the convex hull of finitely many points in M. The dimension of P is the dimension of the affine span of P. When dim $P = \dim M_{\mathbb{R}}$, we say that P is full dimensional.

Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope, and let $P_1, ..., P_m$ be the *facets* of P, i.e. codimension 1 faces of P. For each facet P_k , there exists a unique primitive lattice point $v_k \in N$ and a unique integer $c_k \in \mathbb{Z}$ such that

$$P_k = \{ u \in P \mid \langle u, v_k \rangle = -c_k \}$$

and $\langle u, v_k \rangle \geq -c_k$ for all $u \in P$.

Define Σ_P to be the complete fan whose 1-dimensional cones are exactly those generated by v_k . This fan Σ_P is called the *(inner) normal fan* of P. The toric variety X_{Σ_P} associated to Σ_P is called the toric variety of P, and denoted by X_P . Denote by D_k the divisor corresponding to the 1-dimensional cone generated by v_k . Then we may define a divisor on X_P by $D_P := \sum_{k=1}^m c_k D_k$. Such a divisor D_P is necessarily ample.

This process is reversible, and there is a 1-to-1 correspondence between full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ and a pair (X, D) of a complete toric variety X together with an ample T-invariant divisor D on X.

2.3. Toric vector bundles. A vector bundle $\pi : \mathcal{E} \to X$ over a toric variety $X = X_{\Sigma}$ is said to be toric (or equivariant) if there is a *T*-action on \mathcal{E} such that $t \circ \pi = \pi \circ t$ for all $t \in T$.

Given a cone $\sigma \in \Sigma$ and $u \in M$, define the line bundle $\mathcal{L}_u|_{U_{\sigma}}$ over U_{σ} to be the trivial line bundle $U_{\sigma} \times k$ equipped with the *T*-action given by $t.(x,z) := (t.x, \chi^u(t) \cdot z)$. If $u, u' \in M$ satisfy $u - u' \in \sigma^{\perp}$, then $\chi^{u-u'}$ is a non-vanishing regular function on U_{σ} which gives an isomorphism $\mathcal{L}_u|_{U_{\sigma}} \cong \mathcal{L}_{u'}|_{U_{\sigma}}$. In fact, the group of toric line bundles on U_{σ} is isomorphic to $M_{\sigma} := M/(M \cap \sigma^{\perp})$. Therefore, we also write $\mathcal{L}_{[u]}|_{U_{\sigma}}$, where $[u] \in M_{\sigma}$ is the class of u.

Let $\mathcal{E} \to X$ be a toric vector bundle of rank r. Its restriction to an invariant open affine set U_{σ} splits into a direct sum of toric line bundles with trivial underlying line bundles [Pay08, Proposition 2.2]; i.e. we have $\mathcal{E}|_{U_{\sigma}} \cong \bigoplus_{i=1}^{r} \mathcal{L}_{[u_i]}|_{U_{\sigma}}$ for some $[u_i] \in M_{\sigma}$. Define the associated characters of \mathcal{E} on σ to be the multiset $\mathbf{u}_{\mathcal{E}}(\sigma) \subset M_{\sigma}$ of size r that contains the $[u_i]$ showing up in the splitting.

Example 2.1 (Associated characters of tangent bundles). Let $X = X_{\Sigma}$ be an *n*-dimensional smooth projective toric variety, and consider its tangent bundle \mathcal{T}_X . Fix a maximal cone $\sigma \in \Sigma$. Since X is smooth, the dual cone $\check{\sigma}$ of σ is generated by some $u_1, ..., u_n \in M$ that form a Z-basis of M. Denote by $x_1, ..., x_n \in \Gamma(U_{\sigma}, \mathcal{O}_X)$ the coordinates on $U_{\sigma} \cong k^n$ corresponding to $u_1, ..., u_n$. Then $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is a local frame of \mathcal{T}_X on U_{σ} . Each non-vanishing section $\frac{\partial}{\partial x_i} \in \Gamma(U_{\sigma}, \mathcal{T}_X)$ naturally generates a toric line bundle on U_{σ} isomorphic to $\mathcal{L}_{u_i}|_{U_{\sigma}}$. Thus we have $\mathcal{T}_X|_{U_{\sigma}} \cong \bigoplus_{i=1}^n \mathcal{L}_{u_i}|_{U_{\sigma}}$, and hence the associated characters of \mathcal{T}_X on σ are $\mathbf{u}_{\mathcal{T}_X}(\sigma) = \{u_1, ..., u_n\}$. 2.4. Positivity of toric vector bundles. Let $X = X_{\Sigma}$ be a complete toric variety. By an *invariant curve* on X, we mean a complete irreducible 1-dimensional subvariety that is invariant under the T-action. Via the coneorbit correspondence, there is a one-to-one correspondence between the invariant curves and the codimension-1 cones; every invariant curve is the closure of an 1-dimensional orbit, which corresponds to a codimension-1 cone in Σ . For each codimension-1 cone $\tau \in \Sigma$, denote the corresponding invariant curve by C_{τ} .

The positivity of toric vector bundles can be checked on invariant curves according to the following result in [HMP10].

Theorem 2.2. [HMP10, Theorem 2.1] A toric vector bundle on a complete toric variety is ample (resp. nef) if and only if its restriction to every invariant curve is ample (resp. nef).

Note that every invariant curve is a \mathbb{P}^1 . By Birkhoff-Grothendieck theorem, every vector bundle on \mathbb{P}^1 splits into a direct sum of line bundles. Hence, the positivity of vector bundles on \mathbb{P}^1 is well understood, namely $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ is ample (resp. nef) if and only if every a_i is positive (resp. non-negative). It is common to call the *r*-tuple (or multiset) $(a_i)_{i=1}^r$ the splitting type of the vector bundle.

Fix a codimension-1 cone τ , and let σ, σ' be the two maximal cones containing τ . Given $u, u' \in M$ satisfying $u - u' \in \tau^{\perp}$, define a toric line bundle $\mathcal{L}_{u,u'}$ on $U_{\sigma} \cup U_{\sigma'}$ by glueing the toric line bundles $\mathcal{L}_{u}|_{U_{\sigma}}$ and $\mathcal{L}_{u'}|_{U_{\sigma'}}$ with the transition function $\chi^{u'-u}$. Since the invariant curve C_{τ} is contained in $U_{\sigma} \cup U_{\sigma'}$, we may restrict $\mathcal{L}_{u,u'}$ to get a toric line bundle $\mathcal{L}_{u,u'}|_{C_{\tau}}$ on C_{τ} .

Proposition 2.3. [HMP10, Corollary 5.5 and 5.10] Let X be a complete toric variety. Any toric vector bundle $\mathcal{E}|_{C_{\tau}}$ on the invariant curve C_{τ} splits equivariantly as a sum of line bundles

$$\mathcal{E}|_{C_{\tau}} = \bigoplus_{i=1}^{r} \mathcal{L}_{u_i, u_i'}|_{C_{\tau}}$$

The splitting is unique up to reordering.

Combining this with the following lemma that computes the underlying line bundle of $\mathcal{L}_{u,u'}|_{C_{\tau}}$, one gets the splitting type of $\mathcal{E}|_{C_{\tau}}$.

Lemma 2.4. [HMP10, Example 5.1] Let u_0 be the generator of $M \cap \tau^{\perp} \cong \mathbb{Z}$ that is positive on σ , and let m be the integer such that $u - u' = mu_0$. Then, the underlying line bundle of $\mathcal{L}_{u,u'}|_{C_{\tau}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(m)$.

3. Restricting \mathcal{T}_X to invariant curves

Let $X = X_{\Sigma}$ be a smooth complete toric variety of dimension n. In this section, we consider the restrictions of the tangent bundle \mathcal{T}_X to the invariant curves. The goal is to get the splitting types in terms of the combinatorial data of the fan Σ of X. This has in fact been done in [DRJS18, Example 5.1 and 5.2] and [Sch18, Theorem 2]. We repeat the calculation for the convenience of the readers.

Fix an (n-1)-dimensional cone $\tau \in \Sigma$. Let $\sigma, \sigma' \in \Sigma(n)$ be the two maximal cones containing τ . Let $v_1, ..., v_{n-1}, v_n, v'_n \in N$ be primitive vectors such that τ is generated by $\{v_1, ..., v_{n-1}\}$, σ is generated by $\{v_1, ..., v_{n-1}, v_n\}$, and σ' is generated by $\{v_1, ..., v_{n-1}, v'_n\}$. There are unique $u_i, u'_i \in M$ (i = 1, ..., n) such that $\langle u_i, v_i \rangle = \langle u'_i, v'_i \rangle = 1$ for all i and $\langle u_i, v_j \rangle = \langle u'_i, v'_j \rangle = 0$ for all $i \neq j$, where we define $v'_i = v_i$ for i = 1, ..., n-1. The dual cones $\check{\sigma}$ and $\check{\sigma}'$ are generated by $\{u_1, ..., u_n\}$ and $\{u'_1, ..., u'_n\}$, respectively.

By Example 2.1, the associated characters of \mathcal{T}_X on σ and σ' are given by

$$\mathbf{u}_{\mathcal{T}_X}(\sigma) = \{u_1, ..., u_n\}, \ \mathbf{u}_{\mathcal{T}_X}(\sigma') = \{u'_1, ..., u'_n\}$$

Following Section 2.4, let C_{τ} be the invariant curve corresponding to τ . The splitting of $\mathcal{T}_X|_{C_{\tau}}$ as in Proposition 2.3 is easy to get by the following fact.

Lemma 3.1. We have $u_i - u'_i \in \tau^{\perp}$ for all i = 1, ..., n, and $u_i - u'_j \notin \tau^{\perp}$ for all $i \neq j$.

Proof. The first part follows from $\langle u_i - u'_i, v_{i'} \rangle = 0$ for all i' = 1, ..., n - 1, and the second part follows from $\langle u_i - u'_j, v_i \rangle = -\langle u_i - u'_j, v_j \rangle = 1$, where at least one of i, j is not n.

Definition 3.2. Define $a_i \in \mathbb{Z}$ (i = 1, ..., n) to be the integers satisfying $u_i = u'_i + a_i u_n$. Such integers exist since u_n is a primitive generator of $\tau^{\perp} \cap M \cong \mathbb{Z}$. Note that $u'_n = -u_n$ so that $a_n = 2$.

Proposition 3.3. On the invariant curve C_{τ} , the restriction $\mathcal{T}_X|_{C_{\tau}}$ of the tangent bundle (as a toric vector bundle) splits into the following direct sum of toric line bundles

$$\mathcal{T}_X|_{C_{\tau}} \cong \bigoplus_{i=1}^n \mathcal{L}_{u_i,u_i'}|_{C_{\tau}}.$$

In particular, we have the following splitting of $\mathcal{T}_X|_{C_{\tau}}$ as a vector bundle

$$\mathcal{T}_X|_{C_{\tau}} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$$

Proof. By Proposition 2.3, $\mathcal{T}_X|_{C_{\tau}}$ splits into a direct sum of toric line bundles of the form $\mathcal{L}_{u,u'}|_{C_{\tau}}$. This gives a bijection $\iota : \mathbf{u}_{\mathcal{E}}(\sigma) \to \mathbf{u}_{\mathcal{E}}(\sigma')$ mapping u to u' whenever $\mathcal{L}_{u,u'}|_{C_{\tau}}$ shows up in the splitting. Note that $u_i - \iota(u_i) \in \tau^{\perp}$ by the definiton of $\mathcal{L}_{u,u'}$. Then Lemma 3.1 implies that we must have $\iota(u_i) = u'_i$ for all i, hence the splitting in the first part.

The second part follows directly from the first part together with Lemma 2.4.

Remark 3.4. The integers a_i are the same as the integers b_i that show up in the "wall relation"

$$b_1v_1 + \dots + b_{n-1}v_{n-1} + v_n + v'_n = 0,$$

mentioned in [Sch18] and [DRJS18]. Indeed we have $b_i = -\langle u_i, v'_n \rangle = a_i$ for all *i*.

Example 3.5. For each of the following toric surfaces X, we fix a 1-dimensional cone τ in its fan (as shown in Figure 3.6) and compute the splitting type of $\mathcal{T}_X|_{C_{\tau}}$.

- (1) $X = \mathbb{P}^2$. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \text{Cone}\{(-1,0), (-1,1)\}$ and $\check{\sigma} = \text{Cone}\{(0,-1), (1,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_{\tau}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. In fact, the restrictions of \mathcal{T}_X to the other two invariant curves have the same splitting type, so \mathcal{T}_X is ample by Proposition 2.2.
- (2) $X = \mathbb{P}^1 \times \mathbb{P}^1$. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \text{Cone}\{(-1,0), (0,1)\}$ and $\check{\sigma} = \text{Cone}\{(-1,0), (0,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_{\tau}} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. In fact, the restrictions of \mathcal{T}_X to the other three invariant curves have the same splitting type, so \mathcal{T}_X is nef (but not ample) by Proposition 2.2.
- (3) Let X be the Hirzebruch surface \mathbb{F}_1 , which is isomorphic to \mathbb{P}^2 blown up at one point. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \operatorname{Cone}\{(-1,0), (0,1)\}$ and $\check{\sigma} = \operatorname{Cone}\{(-1,1), (0,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_{\tau}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, and hence \mathcal{T}_X is not nef by Proposition 2.2.

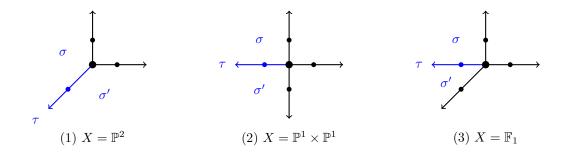


FIGURE 3.6. Fans of toric surfaces

4. Polytopes and ampleness of the tangent bundle

Let $X = X_{\Sigma}$, \mathcal{T}_X , τ , σ , σ' , u_i , u'_i , a_i be as in the previous section.

Fix an ample *T*-invariant divisor *D*, and let P = P(X, D) be the corresponding polytope. Note that *X* and Σ are simplicial as they are smooth; in particular, every maximal cone in Σ has exactly *n* faces of dimension (n-1), and every (n-1)-dimensional cone has exactly (n-1) faces of dimension (n-2). This implies that there are exactly *n* edges emanating from every vertex of *P* and that every edge of *P* is contained in exactly (n-1) faces of dimension 2.

Let $p_{\sigma} \in P$ be the vertex corresponding to the maximal cone σ . Denote by $P - p_{\sigma}$ the translation of P by $-p_{\sigma}$. The cone generated by $P - p_{\sigma}$ is given by $\{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq 0 \text{ for all } i = 1, ..., n\}$, which is exactly the dual cone $\check{\sigma}$ of σ . Thus, the n edges of P emanating from p_{σ} are parallel to $u_1, ..., u_n$. Similarly the n edges emanating from the vertex $p_{\sigma'}$ corresponding to σ' are parallel to $u'_1, ..., u'_n$.

Recall that the u_i and u'_i satisfy $u'_i = u_i - a_i u_n$ for all i = 1, ..., n - 1 and $u'_n = -u_n$. Since σ and σ' contain the (n-1)-dimensional cone τ as a common face, the convex hull of $\overline{p_{\sigma}, p_{\sigma'}}$ of p_{σ} and $p_{\sigma'}$ is an edge of P; it corresponds to τ and is parallel to u_n and u'_n . Fix a $j \in \{1, ..., n-1\}$. Consider the points $p_{\sigma} + u_j, p_{\sigma'} + u'_j \in M$. The point $p_{\sigma} + u_j$ is on an edge emanating from p_{σ} , and $p_{\sigma'} + u'_j$ is on an edge emanating from $p_{\sigma'}$. In addition, since $(p_{\sigma} + u_j) - (p_{\sigma'} + u'_j) = (p_{\sigma} - p_{\sigma'}) + m_j u_n, \overline{p_{\sigma} + u_j, p_{\sigma'} + u'_j}$ is parallel to $\overline{p_{\sigma}, p_{\sigma'}}$. Thus, the four points $p_{\sigma}, p_{\sigma'}, p_{\sigma} + u_i, p_{\sigma'} + u'_i$ are contained in a common 2-dimensional face $A_i \subseteq P$. In fact, A_i is the 2-dimensional face of P corresponding to the (n-2)-dimensional cone $\tau \cap (u_i)^{\perp} = \tau \cap (u'_i)^{\perp}$.

Denote the angles at p_{σ} and $p_{\sigma'}$ on A_j by $\theta(p_{\sigma}, A_j)$ and $\theta(p_{\sigma'}, A_j)$, respectively. Their sum is related to the integer a_j in the following way.

Proposition 4.1. The sum $\theta(p_{\sigma}, A_j) + \theta(p_{\sigma'}, A_j)$ is smaller than π if and only if $a_j > 0$, equal to π if and only if $a_j = 0$, and greater than π if and only if $a_j < 0$.

Proof. Suppose $a_j < 0$. Consider the quadrilateral with vertices $p_{\sigma}, p_{\sigma'}, p_{\sigma'} + u'_j, p_{\sigma} + u_j$. It is a trapezoid with the edges $\overline{p_{\sigma} + u_j, p_{\sigma'} + u'_j}$ and $\overline{p_{\sigma}, p_{\sigma'}}$ parallel to each other. Since

$$((p_{\sigma'} + u'_j) - (p_{\sigma} + u_j)) - (p_{\sigma'} - p_{\sigma}) = -a_j u_1,$$

the edge $\overline{p_{\sigma} + u_i, p_{\sigma'} + u'_i}$ is longer than $\overline{p_{\sigma}, p_{\sigma'}}$, implying $\theta(p_{\sigma}, A_j) + \theta(p_{\sigma'}, A_j) > \pi$.

The other two cases are similar.

Remark 4.2. Although the angles $\theta(p_{\sigma}, A_j), \theta(p_{\sigma'}, A_j)$ themselves are not invariant under a change of bases of M, whether their sum is smaller than, equal to, or greater than π is.

Example 4.3. In Figure 4.4 are polytopes $P(X, -K_X)$ corresponding to the toric surfaces X in Example 3.5 together with their anticanonical line bundles $-K_X$. The cones τ, σ, σ' are the same as in Example 3.5.

- (1) $X = \mathbb{P}^2$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 1 > 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) < \pi$.
- (2) $X = \mathbb{P}^1 \times \mathbb{P}^1$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) = \pi$.
- (3) $X = \mathbb{F}_1$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = -1 < 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) > \pi$.

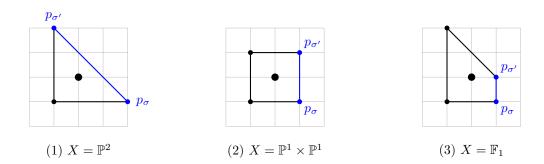


FIGURE 4.4. Polytopes $P(X, -K_X)$ of toric surfaces

5. Proof of Theorem 1.1

Proof of Theorem 1.1. As promised, we will show that the polytope P corresponding to X (together with any ample T-invariant divisor D) is an n-simplex.

Let A be a 2-dimensional face of P. Let m be the number of vertices of A, and let $p_1, ..., p_m$ be the vertices of A, ordered so that p_k is adjacent to p_{k+1} for all k = 1, ..., m (where $p_{m+1} := p_1$). Since \mathcal{T}_X is ample, its restriction to every invariant curve is ample. Then, by Proposition 3.3 and Proposition 4.1, $\theta(p_k, A) + \theta(p_{k+1}, A) < \pi$ for all k. This implies

$$m\pi > \sum_{k=1}^{m} (\theta(p_k, A) + \theta(p_{k+1}, A)) = 2 \sum_{k=1}^{m} \theta(p_k, A) = 2(m-2)\pi.$$

We get m < 4, implying A is a triangle. The same is true for all 2-dimensional faces of P.

Now, we start with a vertex q_0 of P. Recall that every vertex of P is adjacent to exactly n vertices since X is smooth and hence simplicial. Let $q_1, ..., q_n$ be the n points adjacent to q_0 . Given $1 < j \le n$, let A_j be the 2-dimensional face containing the edges $\overline{q_0q_1}$ and $\overline{q_0q_j}$. Since A_j is in fact a triangle, q_1 is also adjacent to q_j . Thus q_1 is adjacent to $q_0, q_2, ..., q_n$. Similarly, every p_j is adjacent to exactly $p_0, ..., \widehat{p_j}, ..., p_n$. Consequently, $p_0, p_1, ..., p_n$ are the only vertices of P, and hence P is the n-simplex with vertices $p_0, p_1, ..., p_n$.

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