

THE EXISTENCE OF POSITIVE SOLUTION FOR AN ELLIPTIC PROBLEM WITH CRITICAL GROWTH AND LOGARITHMIC PERTURBATION

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ABSTRACT. We consider the existence and nonexistence of positive solution for the following Brézis-Nirenberg problem with logarithmic perturbation:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u + \mu u \log u^2 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda, \mu \in \mathbb{R}$, $N \geq 3$ and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. The uncertainty of the sign of $s \log s^2$ in $(0, +\infty)$ has some interest in itself. We will show the existence of positive ground state solution which is of mountain pass type provided $\lambda \in \mathbb{R}$, $\mu > 0$ and $N \geq 4$. While the case of $\mu < 0$ is thornier. However, for $N = 3, 4$ $\lambda \in (-\infty, \lambda_1(\Omega))$, we can also establish the existence of positive solution under some further suitable assumptions. And a nonexistence result is also obtained for $\mu < 0$ and $-\frac{(N-2)\mu}{2} + \frac{(N-2)\mu}{2} \log(-\frac{(N-2)\mu}{2}) + \lambda - \lambda_1(\Omega) \geq 0$ if $N \geq 3$. Comparing with the results in Brézis, H. and Nirenberg, L. (Comm. Pure Appl. Math. 1983), some new interesting phenomenon occurs when the parameter μ on logarithmic perturbation is not zero.

Keywords: Brézis-Nirenberg Problem, Critical exponents, Positive solution, Logarithmic perturbation

1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the existence and nonexistence of positive solution for the following Brézis-Nirenberg problem with a logarithmic term:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u + \mu u \log u^2 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda, \mu \in \mathbb{R}$, $N \geq 3$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Here $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ equipped with the norm $\|u\| := (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$.

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Our motivation to consider (1.1) is that it resembles some variational problems in geometry and physics, which is lack of compactness. The most notorious example is Yamabe's problem: finding a function u satisfying

$$\begin{cases} -4\frac{N-1}{N-2}\Delta u = R'|u|^{2^*-2}u - R(x)u & \text{on } \mathcal{M}, \\ u > 0 & \text{on } \mathcal{M}, \end{cases}$$

where R' is a constant, \mathcal{M} is an N -dimensional Riemannian manifold, Δ denotes the Laplacian and $R(x)$ represents the scalar curvature. Some other examples we refer to [2, 8, 10, 13–15] and the references therein.

When $\lambda = \mu = 0$, Eq.(1.1) is reduced to

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \quad (1.2)$$

Pohozaev [11] asserts that Eq.(1.2) has no nontrivial solutions when Ω is starshaped. But, as Brézis and Nirenberg have shown in [3], a lower-order terms can reverse this circumstance. Indeed, they considered the following classical problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.3)$$

with $\lambda \in \mathbb{R}$, $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. They found out that the existence of a solution depends heavily on the values of λ and N . Precisely, they showed that:

- (i) when $N \geq 4$ and $\lambda \in (0, \lambda_1(\Omega))$, there exists a positive solution for Eq.(1.3);
- (ii) when $N = 3$ and Ω is a ball, Eq.(1.3) has a positive solution if and only if $\lambda \in (\frac{1}{4}\lambda_1(\Omega), \lambda_1(\Omega))$;
- (iii) Eq. (1.3) has no solutions when $\lambda < 0$ and Ω is starshaped;

where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary value. Furthermore, Brézis and Nirenberg [3] also considered the following general case:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + f(x, u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.4)$$

where $f(x, u)$ satisfies some of the following assumptions :

- (f₁) $f(x, u) = a(x)u + g(x, u)$, $a(x) \in L^\infty(\Omega)$;
- (f₂) $\lim_{u \rightarrow 0^+} \frac{g(x, u)}{u} = 0$, uniformly in $x \in \Omega$;
- (f₃) $\lim_{u \rightarrow +\infty} \frac{g(x, u)}{u^{2^*-1}} = 0$, uniformly in $x \in \Omega$;
- (f₄) $\exists \alpha > 0$ such that $\int (|\nabla v|^2 - a(x)v^2)dx \geq \alpha \int v^2 dx$ for all $v \in H_0^1(\Omega)$;
- (f₅) $f(x, u) \geq 0$ for a.e $x \in \omega_0$ and for all $u \geq 0$, where ω_0 is some nonempty open subset of Ω ;
- (f₆) $f(x, u) \geq \delta_0 > 0$ for a.e $x \in \omega_0$ and for all $u \in I$, where ω_0 is given in (f₅), $I \subset (0, +\infty)$ is some nonempty open interval and $\delta_0 > 0$ is some constant;

- (f₇) $f(x, u) \geq \delta_1 u$ for a.e $x \in \omega_1$ and for all $u \in [0, A]$, or, $f(x, u) \geq \delta_1 u$ for a.e $x \in \omega_1$ and for all $u \in [A, +\infty]$, where ω_1 is some nonempty open subset of Ω and δ_1, A are two positive constants;
- (f₈) $\lim_{u \rightarrow +\infty} \frac{f(x, u)}{u^3} = +\infty$ uniformly in $x \in \omega_2$, where ω_2 is some nonempty open subset of Ω .

They showed that if the assumptions (f₁) – (f₄) hold and there exists some $0 \leq u_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\sup_{t \geq 0} I(tu_0) < \frac{1}{N} S^{\frac{N}{2}}$, then Eq.(1.4) has a positive solution. More precisely, they proved that:

- (i) If $N \geq 5$, Eq.(1.4) has a positive solution provided (f₁) – (f₆);
- (ii) If $N = 4$, Eq.(1.4) has a positive solution provided (f₁) – (f₅) and (f₇);
- (iii) If $N = 3$, Eq.(1.4) has a positive solution provided (f₁) – (f₅) and (f₈).

Some similar results can be seen in [1, 5, 7]. Barrios et al. [1] proved the existence of positive solution for a fractional critical problem with a lower-order term, and Gao and Yang [5], Li and Ma [7] considered the existence of positive solution to a Choquard equation with critical exponent and lower-order term in a bounded domain Ω and in \mathbb{R}^N , respectively.

Remark 1.1. *Compared with $|u|^{2^*-2}u$, $u \log u^2$ is a lower-order term at infinity. However, we note that the situation we considered in present paper is not covered above. Indeed, in the Eq. (1.1), $f(x, u) = \lambda u + \mu u \log u^2$. So (f₁) fails due to the fact of $\lim_{u \rightarrow 0^+} \frac{u \log u^2}{u} = -\infty$. That is $\lambda u = o(\mu u \log u^2)$ for u close to 0. So it is natural to believe that $\mu u \log u^2$ has much more influence than λu on the existence of positive solutions to Eq.(1.1). Hence, our main goal in present paper is to make clear this guess.*

To find a positive solution to Eq.(1.1), we define a modified functional:

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2^*} \int |u_+|^{2^*} dx - \frac{\lambda}{2} \int u_+^2 dx - \frac{\mu}{2} \int u_+^2 (\log u_+^2 - 1) dx, \quad u \in H_0^1(\Omega), \quad (1.5)$$

which can be rewritten by

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2^*} \int |u_+|^{2^*} dx - \frac{\mu}{2} \int u_+^2 (\log u_+^2 + \frac{\lambda}{\mu} - 1) dx, \quad u \in H_0^1(\Omega), \quad (1.6)$$

where $u_+ = \max\{u, 0\}$, $u_- = -\max\{-u, 0\}$. It is easy to see that I is well-defined in $H_0^1(\Omega)$ and any nonnegative critical point of I corresponds to a solution of Eq.(1.1).

Before stating our results, we introduce some notations. Hereafter, we use \int to denote $\int_{\Omega} dx$, unless specifically stated, and let S and $\lambda_1(\Omega)$ be the best Sobolev constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and the first eigenvalue of $-\Delta$ with zero Dirichlet boundary value respectively, i.e,

$$S := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}$$

and

$$\lambda_1(\Omega) := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} |u|^2 \, dx}.$$

We also set

$$\|v\|^2 := \int |\nabla v|^2, \quad v \in H_0^1(\Omega),$$

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} \mid g(u) = 0\},$$

and

$$c_g := \inf_{u \in \mathcal{N}} I(u), \quad c_M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad (1.7)$$

where

$$g(u) := \int |\nabla u|^2 - \int |u_+|^{2^*} - \lambda \int u_+^2 - \mu \int u_+^2 \log u_+^2,$$

and

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Let

$$A_0 := \{(\lambda, \mu) \mid \lambda \in \mathbb{R}, \mu > 0\},$$

$$B_0 := \left\{ (\lambda, \mu) \mid \lambda \in [0, \lambda_1(\Omega)), \mu < 0, \frac{1}{N} \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right)^{\frac{N}{2}} S^{\frac{N}{2}} + \frac{\mu}{2} |\Omega| > 0 \right\},$$

$$C_0 := \left\{ (\lambda, \mu) \mid \lambda \in \mathbb{R}, \mu < 0, \frac{1}{N} S^{\frac{N}{2}} + \frac{\mu}{2} e^{-\frac{\lambda}{\mu}} |\Omega| > 0 \right\}.$$

Here comes our main results.

Theorem 1.2. *If $(\lambda, \mu) \in A_0$ and $N \geq 4$, then problem (1.1) has a positive Mountain pass solution, which is also a ground state solution.*

Denote $f(s) := |s|^{2^*-2}s + \lambda s + \mu s \log s^2$ which is of odd. It is easy to see that $\mathcal{N} \neq \emptyset$ and $c_M \geq c_g$ if problem (1.1) has a positive mountain pass solution. On the other hand, when $\lambda \in \mathbb{R}$ and $\mu > 0$, $\frac{f(s)}{s}$ is strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$, which enable one to show that $c_M \leq c_g$ (See [18, Theorem 4.2]). Therefore, the ground state energy c_g equals to the Mountain pass level energy c_M , which implies that the mountain pass solution must be a ground state solution. So, in Theorem 1.2, we only need to show that problem (1.1) has a positive mountain pass solution.

The case of $\mu < 0$ is thorny. Indeed for $(\lambda, \mu) \in B_0 \cup C_0$, $I(u)$ still has the mountain pass geometry (See Lemma 2.1). However, in such a case, it holds that $c_g < c_M$. Since we can not check the $(PS)_{c_M}$ condition for $I(u)$, we apply the mountain pass theorem without $(PS)_{c_M}$ condition to gain a positive solution for Eq.(1.1) when $(\lambda, \mu) \in B_0 \cup C_0$. However, we don't know whether this solution is of mountain pass type or not.

Theorem 1.3. *Problem (1.1) possesses a positive solution provided one of the following condition holds:*

- (i) $N = 3$, $(\lambda, \mu) \in B_0 \cup C_0$;
- (ii) $N = 4$, $(\lambda, \mu) \in B_0 \cup C_0$ with $\frac{32e^{\frac{\lambda}{\mu}}}{\rho_{max}^2} < 1$, where $\rho_{max} := \sup\{r > 0 : \exists x \in \Omega \text{ s.t. } B(x, r) \subset \Omega\}$.

For the nonexistence of positive solutions for problem (1.1), we have the following partial result.

Theorem 1.4. *Assume that $N \geq 3$. If $\mu < 0$ and $-\frac{(N-2)\mu}{2} + \frac{(N-2)\mu}{2} \log(-\frac{(N-2)\mu}{2}) + \lambda - \lambda_1(\Omega) \geq 0$, then problem (1.1) has no positive solutions.*

The existence and nonexistence results given by Theorem 1.2 - Theorem 1.4 can be described on the (λ, μ) plane by Figure 1. The pink regions stand for the existence of positive solution, while the blue regions correspond the non-existence of positive solution. Here τ_1 , η_1 , η_2 and η_3 are curves given by

$$\begin{aligned} \tau_1 : & -\frac{(N-2)\mu}{2} + \frac{(N-2)\mu}{2} \log(-\frac{(N-2)\mu}{2}) + \lambda - \lambda_1(\Omega) = 0, \\ \eta_1 : & \frac{1}{N} \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right)^{\frac{N}{2}} S^{\frac{N}{2}} + \frac{\mu}{2} |\Omega| = 0, \\ \eta_2 : & \frac{1}{N} S^{\frac{N}{2}} + \frac{\mu}{2} e^{-\frac{\lambda}{\mu}} |\Omega| = 0, \\ \eta_3 : & 32e^{\frac{\lambda}{\mu}} = \rho_{max}^2, \quad \rho_{max} := \sup\{r \in (0, +\infty) : \exists x \in \Omega \text{ s.t. } B(x, r) \subset \Omega\}. \end{aligned}$$

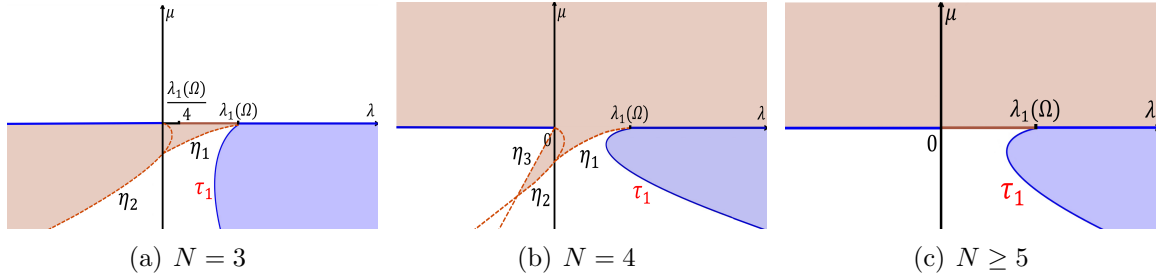


FIGURE 1. existence and nonexistence

Remark 1.5. *Comparing the results of [3] and FIGURE 1 above, we find that for the case of $N \geq 4$: Eq.(1.1) possesses a positive solution only for $\lambda \in (0, \lambda_1(\Omega))$ if $\mu = 0$. while it has a positive solution for all $\lambda \in \mathbb{R}$ if $\mu > 0$. So we see that $\mu u \log u^2$ ($\mu > 0$) really plays a leading role (compared with λu) in the effect on the existence of positive solution to Eq.(1.1). A similar phenomenon occurs for $N = 3$: Eq.(1.1) has a positive solution only for $\lambda \in (\lambda^*, \lambda_1(\Omega)) \subset (0, \lambda_1(\Omega))$ if $\mu = 0$, while it has a positive solution for all $\lambda \in (-\infty, \lambda_1(\Omega))$ if $\mu < 0$.*

Before closing the introduction, we give the outline of our paper. In Section 2, we will check the mountain pass geometry structure for $I(u)$, under different specific situations. We also give some other preliminaries. In Section 3, we are devoted to estimate the mountain pass level c_M for different parameters λ, μ and N . The proofs of our main Theorems 1.2, 1.3 and 1.4 are given in Section 4.

2. PRELIMINARIES

In this section, firstly we verify the mountain pass geometry structure for $I(u)$ when $(\lambda, \mu) \in A_0 \cup B_0 \cup C_0$. Secondly we show that $I(u)$ satisfies $(PS)_d$ condition provided $d < \frac{1}{N}S^{\frac{N}{2}}$. Finally, we deduce a existence result for Eq.(1.1) when $c_M \in (-\infty, 0) \cup (0, \frac{1}{N}S^{\frac{N}{2}})$ and $(\lambda, \mu) \in B_0 \cup C_0$.

Lemma 2.1. *Assume that $N \geq 3$ and $(\lambda, \mu) \in A_0 \cup B_0 \cup C_0$. Then the functional $I(u)$ satisfies the mountain pass geometry structure:*

- (i) *there exist $\alpha, \rho > 0$ such that $I(v) \geq \alpha$ for all $\|v\| = \rho$;*
- (ii) *there exists $\omega \in H_0^1(\Omega)$ such that $\|\omega\| \geq \rho$ and $I(\omega) < 0$.*

Proof. We divide the proof into three cases.

Case 1: $(\lambda, \mu) \in A_0$.

Since $\mu > 0$, it follows from the fact $s^2 \log s^2 \leq Cs^{2^*}$ for all $s \in [1, +\infty)$ that

$$\begin{aligned} & \mu \int u_+^2 (\log u_+^2 + \frac{\lambda}{\mu} - 1) \leq \mu \int u_+^2 (\log u_+^2 + \frac{\lambda}{\mu}) = \mu \int u_+^2 \log(e^{\frac{\lambda}{\mu}} u_+^2) \\ & = \mu \int_{\{e^{\frac{\lambda}{\mu}} u_+^2 \geq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}} u_+^2) + \mu \int_{\{e^{\frac{\lambda}{\mu}} u_+^2 \leq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}} u_+^2) \\ & \leq \mu \int_{\{e^{\frac{\lambda}{\mu}} u_+^2 \geq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}} u_+^2) \leq C\mu \int_{\{e^{\frac{\lambda}{\mu}} u_+^2 \geq 1\}} e^{\frac{(2^*-2)\lambda}{2\mu}} |u_+|^{2^*} \\ & \leq Ce^{\frac{(2^*-2)\lambda}{2\mu}} \mu \int |u|^{2^*} \leq Ce^{\frac{(2^*-2)\lambda}{2\mu}} \mu \|u\|^{2^*}. \end{aligned}$$

So

$$I(u) \geq \frac{1}{2} \|u\|^2 - C_1 \|u\|^{2^*} - C_2 \|u\|^{2^*} \quad \text{for some } C_1, C_2 > 0,$$

which implies that there exist $\alpha > 0$ and $\rho > 0$ such that $I(v) \geq \alpha > 0$ for all $\|v\| = \rho$.

Let $0 \leq \varphi \in H_0^1(\Omega) \setminus \{0\}$ be a fixed function, then

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2} \int |\nabla \varphi|^2 - \frac{|t|^{2^*}}{2^*} \int |\varphi|^{2^*} - \frac{\mu}{2} t^2 \int \varphi^2 (\log(t^2 \varphi^2) + \frac{\lambda}{\mu} - 1) \\ &= \frac{t^2}{2} \int |\nabla \varphi|^2 - \frac{|t|^{2^*}}{2^*} \int |\varphi|^{2^*} - \frac{\mu}{2} t^2 \int \varphi^2 (\log t^2 + \log \varphi^2 + \frac{\lambda}{\mu} - 1) \\ &= \frac{t^2}{2} \int |\nabla \varphi|^2 - \frac{|t|^{2^*}}{2^*} \int |\varphi|^{2^*} - \frac{\mu}{2} t^2 \log t^2 \int \varphi^2 - \frac{\mu}{2} t^2 \int \varphi^2 (\log \varphi^2 + \frac{\lambda}{\mu} - 1) \\ &\rightarrow -\infty \quad \text{as } t \rightarrow +\infty, \end{aligned}$$

since $\lim_{t \rightarrow +\infty} \frac{t^{2^*}}{t^2 \log t^2} = +\infty$. Therefore, we can choose $t_0 \in \mathbb{R}^+$ large enough such that

$$I(t_0\varphi) < 0 \quad \text{and} \quad \|t_0\varphi\| > \rho.$$

Case 2: $(\lambda, \mu) \in B_0$.

Since $\mu < 0$, we have

$$\begin{aligned}
& -\frac{\mu}{2} \int u_+^2 (\log u_+^2 - 1) = -\frac{\mu}{2} \int u_+^2 \log(e^{-1}u_+^2) \\
& = -\frac{\mu}{2} \int_{\{e^{-1}u_+^2 \geq 1\}} u_+^2 \log(e^{-1}u_+^2) - \frac{\mu}{2} \int_{\{e^{-1}u_+^2 \leq 1\}} u_+^2 \log(e^{-1}u_+^2) \\
& \geq -\frac{\mu}{2} \int_{\{e^{-1}u_+^2 \leq 1\}} u_+^2 \log(e^{-1}u_+^2) \\
& \geq -\frac{\mu}{2} e \int_{\{e^{-1}u_+^2 \leq 1\}} -e^{-1} dx \geq \frac{\mu}{2} |\Omega|.
\end{aligned}$$

It follows that

$$I(u) \geq \frac{1}{2} \frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \int |\nabla u|^2 - \frac{1}{2^*} S^{-\frac{2^*}{2}} \left(\int |\nabla u|^2 \right)^{\frac{2^*}{2}} + \frac{\mu}{2} |\Omega|. \quad (2.1)$$

Put $\alpha := \frac{1}{N} \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right)^{\frac{N}{2}} S^{\frac{N}{2}} + \frac{\mu}{2} |\Omega|$ and $\rho := \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right)^{\frac{N-2}{4}} S^{\frac{N}{4}}$, then $\alpha > 0$ and $\rho > 0$ due to the fact $(\lambda, \mu) \in B_0$. By (2.1),

$$I(v) \geq \frac{1}{2} \frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \rho^2 - \frac{1}{2^*} S^{-\frac{2^*}{2}} \rho^{2^*} + \frac{\mu}{2} |\Omega| = \frac{1}{N} \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right)^{\frac{N}{2}} S^{\frac{N}{2}} + \frac{\mu}{2} |\Omega| = \alpha > 0$$

for any $\|v\| = \rho$. Applying a similar argument as Case 1 above, we can find a function $w \in H_0^1(\Omega)$ such that $\|w\| \geq \rho$ and $I(w) < 0$.

Case 3: $(\lambda, \mu) \in C_0$.

Since $\mu < 0$, we have

$$\begin{aligned}
& -\frac{\lambda}{2} \int u_+^2 - \frac{\mu}{2} \int u_+^2 (\log u_+^2 - 1) = -\frac{\mu}{2} \int u_+^2 \left(\log u_+^2 + \frac{\lambda}{\mu} - 1 \right) \\
& = -\frac{\mu}{2} \int u_+^2 \log(e^{\frac{\lambda}{\mu}-1} u_+^2) \\
& = -\frac{\mu}{2} \int_{\{e^{\frac{\lambda}{\mu}-1} u_+^2 \geq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}-1} u_+^2) - \frac{\mu}{2} \int_{\{e^{\frac{\lambda}{\mu}-1} u_+^2 \leq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}-1} u_+^2) \\
& \geq -\frac{\mu}{2} \int_{\{e^{\frac{\lambda}{\mu}-1} u_+^2 \leq 1\}} u_+^2 \log(e^{\frac{\lambda}{\mu}-1} u_+^2) \\
& \geq -\frac{\mu}{2} e^{1-\frac{\lambda}{\mu}} \int_{\{e^{\frac{\lambda}{\mu}-1} u_+^2 \leq 1\}} -e^{-1} dx \geq \frac{\mu}{2} e^{-\frac{\lambda}{\mu}} |\Omega|.
\end{aligned}$$

It follows that

$$I(u) \geq \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2^*} S^{-\frac{2^*}{2}} \left(\int |\nabla u|^2 \right)^{\frac{2^*}{2}} + \frac{\mu}{2} e^{-\frac{\lambda}{\mu}} |\Omega|. \quad (2.2)$$

Put $\alpha := \frac{1}{N}S^{\frac{N}{2}} + \frac{\mu}{2}e^{-\frac{\lambda}{\mu}}|\Omega|$ and $\rho := S^{\frac{N}{4}}$, then $\alpha > 0$ due to that $(\lambda, \mu) \in C_0$. By (2.2),

$$I(v) \geq \frac{1}{2}\rho^2 - \frac{1}{2^*}S^{-\frac{2^*}{2}}\rho^{2^*} + \frac{\mu}{2}e^{-\frac{\lambda}{\mu}}|\Omega| = \frac{1}{N}S^{\frac{N}{2}} + \frac{\mu}{2}e^{-\frac{\lambda}{\mu}}|\Omega| = \alpha > 0$$

for any $\|v\| = \rho$. Similarly, it is not hard to find a function $w \in H_0^1(\Omega)$ such that $\|w\| \geq \rho$ and $I(w) < 0$. \square

Lemma 2.2. *Assume that $N \geq 3$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{0\}$. Then any $(PS)_d$ sequence $\{u_n\}$ of I must be bounded in $H_0^1(\Omega)$ for all $d \in \mathbb{R}$.*

Proof. By the definition of the $(PS)_d$ sequence, we have that, as $n \rightarrow +\infty$,

$$I(u_n) \rightarrow d \text{ and } I'(u_n) \rightarrow 0 \text{ in } H^{-1}(\Omega).$$

That is,

$$\frac{1}{2} \int |\nabla u_n|^2 - \frac{1}{2^*} \int |(u_n)_+|^{2^*} - \frac{\mu}{2} \int (u_n)_+^2 \log(u_n)_+^2 + \frac{\mu - \lambda}{2} \int (u_n)_+^2 = d + o_n(1), \quad (2.3)$$

and

$$\int |\nabla u_n|^2 - \int |(u_n)_+|^{2^*} - \lambda \int (u_n)_+^2 - \mu \int (u_n)_+^2 \log(u_n)_+^2 = o_n(1) \|u_n\| \quad (2.4)$$

as $n \rightarrow +\infty$. Now we divide the proof into two cases.

Case 1: $\mu > 0$.

It follows from (2.3) and (2.4) that

$$\begin{aligned} d + o_n(1) + o_n(1) \|u_n\| &= I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle \\ &= \frac{1}{N} \int |(u_n)_+|^{2^*} + \frac{\mu}{2} \int (u_n)_+^2 \geq \frac{\mu}{2} \int (u_n)_+^2, \end{aligned}$$

thus $|(u_n)_+|_2^2 \leq C + C \|u_n\|$. Using (2.3) and (2.4) again, we have, for n large enough,

$$\begin{aligned} 2d + \|u_n\| &\geq I(u_n) - \frac{1}{2^*} \langle I'(u_n), u_n \rangle \\ &= \frac{1}{N} \|u_n\|^2 - \frac{\lambda}{N} \int (u_n)_+^2 + \frac{\mu}{2} \int (u_n)_+^2 - \frac{1}{N} \mu \int (u_n)_+^2 \log(u_n)_+^2. \end{aligned}$$

Recalling the following inequality (see [12] or see [9, Theorem 8.14])

$$\int u^2 \log u^2 \leq \frac{a}{\pi} \|u\|^2 + (\log |u|_2^2 - N(1 + \log a)) |u|_2^2 \text{ for } u \in H_0^1(\Omega) \text{ and } a > 0,$$

we have that

$$\begin{aligned}
\frac{1}{N} \|u_n\|^2 &\leq 2d + \|u_n\| + C \int (u_n)_+^2 + \frac{1}{N} \mu \int (u_n)_+^2 \log(u_n)_+^2 \\
&\leq C + C \|u_n\| + \frac{1}{N} \mu \left[\frac{a}{\pi} \|u_n\|^2 + (\log |(u_n)_+|_2^2 - N(1 + \log a)) |(u_n)_+|_2^2 \right] \\
&\leq C + C \|u_n\| + \frac{1}{2N} \|u_n\|^2 + |(u_n)_+|_2^2 \log |(u_n)_+|_2^2 + C |(u_n)_+|_2^2 \\
&\leq C + C \|u_n\| + \frac{1}{2N} \|u_n\|^2 + C |(u_n)_+|_2^{2-\delta} + C |(u_n)_+|_2^{2+\delta} + C |(u_n)_+|_2^2 \\
&\leq C + C \|u_n\| + \frac{1}{2N} \|u_n\|^2 + C (C + C \|u_n\|)^{\frac{2-\delta}{2}} + C (C + C \|u_n\|)^{\frac{2+\delta}{2}},
\end{aligned}$$

where $a > 0$ with $\frac{a}{\pi} \mu < \frac{1}{2}$ and $\delta \in (0, 1)$. So there exists $C > 0$ such that $\|u_n\| < C$.

Case 2: $\mu < 0$.

For n large enough, we have

$$\begin{aligned}
2d + \|u_n\| &\geq \frac{1}{N} \|u_n\|^2 - \frac{\lambda}{N} \int (u_n)_+^2 + \frac{\mu}{2} \int (u_n)_+^2 - \frac{1}{N} \mu \int (u_n)_+^2 \log(u_n)_+^2 \\
&= \frac{1}{N} \|u_n\|^2 - \frac{1}{N} \mu \int (u_n)_+^2 \log(e^{\frac{\lambda}{\mu} - \frac{N}{2}} (u_n)_+^2) \\
&\geq \frac{1}{N} \|u_n\|^2 - \frac{1}{N} \mu \int_{\{e^{\frac{\lambda}{\mu} - \frac{N}{2}} (u_n)_+^2 \leq 1\}} (u_n)_+^2 \log(e^{\frac{\lambda}{\mu} - \frac{N}{2}} (u_n)_+^2) \\
&\geq \frac{1}{N} \|u_n\|^2 - \frac{1}{N} \mu \int_{\{e^{\frac{\lambda}{\mu} - \frac{N}{2}} (u_n)_+^2 \leq 1\}} -e^{\frac{N}{2} - \frac{\lambda}{\mu} - 1} dx \\
&\geq \frac{1}{N} \|u_n\|^2 + \frac{\mu}{N} e^{\frac{N}{2} - \frac{\lambda}{\mu} - 1} |\Omega|,
\end{aligned} \tag{2.5}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. \square

Lemma 2.3. *Let $\{u_n\}$ be a bounded sequence in $H_0^1(\Omega)$ such that $u_n \rightarrow u$ a.e in Ω as $n \rightarrow \infty$, then*

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n^2 \log u_n^2 dx = \int_{\Omega} u^2 \log u^2 dx, \tag{2.6}$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} (u_n)_+^2 \log (u_n)_+^2 dx = \int_{\Omega} u_+^2 \log u_+^2 dx. \tag{2.7}$$

Proof. We only prove (2.6). And (2.7) can be proved similarly.

Under the conditions, there exists some $C > 0$ such that

$$\left| \int_{\Omega} u_n^2 \log u_n^2 \right| \leq C \text{ and } \left| \int_{\Omega} u^2 \log u^2 \right| \leq C$$

By [12, Lemma 3.1], we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n^2 \log u_n^2 - |u_n - u|^2 \log |u_n - u|^2 = \int_{\Omega} u^2 \log u^2.$$

Since $|s^2 \log s^2| \leq Cs^{2-\delta} + Cs^{2+\delta}$ and the embedding of $H_0^1(\Omega) \hookrightarrow L^p(1 \leq p < 2^*)$ is compact, we obtain that

$$\left| \int_{\Omega} |u_n - u|^2 \log |u_n - u|^2 dx \right| \leq C \int_{\Omega} |u_n - u|^{2-\delta} + C \int_{\Omega} |u_n - u|^{2+\delta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n^2 \log u_n^2 = \int_{\Omega} u^2 \log u^2.$$

□

Lemma 2.4. *If $N \geq 3, \lambda \in \mathbb{R}, \mu > 0$ and $d < \frac{1}{N}S^{\frac{N}{2}}$, then $I(u)$ satisfies the $(PS)_d$ condition.*

Proof. Let $\{u_n\}$ be a $(PS)_d$ sequence of I . By Lemma 2.2, we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^q(\Omega), \quad 1 \leq q < 2^*, \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega. \end{aligned}$$

Since $\langle I'(u_n), \varphi \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any $\varphi \in C_0^\infty(\Omega)$, u is a weak solution to

$$-\Delta u = |u_+|^{2^*-2} u_+ + \lambda u_+ + \mu u_+ \log u_+^2,$$

which implies that

$$\int |\nabla u|^2 = \int |u_+|^{2^*} + \lambda \int u_+^2 + \mu \int u_+^2 \log u_+^2$$

and

$$\begin{aligned} I(u) &= \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2^*} \int |u_+|^{2^*} - \frac{\lambda}{2} \int u_+^2 - \frac{\mu}{2} \int u_+^2 (\log u_+^2 - 1) \\ &= \frac{1}{N} \int |u_+|^{2^*} + \frac{\mu}{2} \int u_+^2 \geq 0. \end{aligned} \tag{2.8}$$

Following from the definition of $(PS)_d$ sequence, we have

$$\int |\nabla u_n|^2 - \int |(u_n)_+|^{2^*} - \lambda \int (u_n)_+^2 - \mu \int (u_n)_+^2 \log (u_n)_+^2 = o_n(1)$$

and

$$\frac{1}{2} \int |\nabla u_n|^2 - \frac{1}{2^*} \int |(u_n)_+|^{2^*} - \frac{\lambda}{2} \int (u_n)_+^2 - \frac{\mu}{2} \int (u_n)_+^2 (\log (u_n)_+^2 - 1) = d + o_n(1).$$

Set $v_n = u_n - u$. Then

$$\int |\nabla v_n|^2 - \int |(v_n)_+|^{2^*} = o_n(1)$$

and

$$I(u) + \frac{1}{2} \int |\nabla v_n|^2 - \frac{1}{2^*} \int |(v_n)_+|^{2^*} = d + o_n(1).$$

Let

$$\int |\nabla v_n|^2 \rightarrow k, \text{ as } n \rightarrow \infty.$$

So

$$\int |(v_n)_+|^{2^*} \rightarrow k, \text{ as } n \rightarrow \infty.$$

By the definition of S , we have

$$|\nabla u|_2^2 \geq S |u|_{2^*}^2, \quad \forall u \in H_0^1(\Omega)$$

and

$$k + o_n(1) = \int |\nabla v_n|^2 \geq S \left(\int |(v_n)_+|^{2^*} \right)^{\frac{2}{2^*}} = S k^{\frac{N-2}{N}} + o_n(1).$$

If $k > 0$, then $k \geq S^{\frac{N}{2}}$. By (2.8), we have

$$0 \leq I(u) \leq d - \left(\frac{1}{2} - \frac{1}{2^*}\right)k \leq d - \frac{1}{N}S^{\frac{N}{2}} < 0,$$

which is impossible. So $k = 0$ and thus

$$u_n \rightarrow u, \quad \text{in } H_0^1(\Omega)$$

□

Lemma 2.5. *Assume that $N \geq 3, \lambda \in \mathbb{R}, \mu < 0$ and $c \in (-\infty, 0) \cup (0, \frac{1}{N}S^{\frac{N}{2}})$. If $\{u_n\}$ is a $(PS)_c$ sequence of I , then there exists a $u \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u is a nonnegative weak solution of (1.1).*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of I . By Lemma 2.2, we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^q(\Omega), \quad 1 \leq q < 2^*, \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega. \end{aligned}$$

Since $\langle I'(u_n), \varphi \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any $\varphi \in C_0^\infty(\Omega)$, u is a weak solution to

$$-\Delta u = |u_+|^{2^*-2} u_+ + \lambda u_+ + \mu u_+ \log u_+^2. \quad (2.9)$$

Assume that $u = 0$ and set $v_n := u_n - u$. Following from the definition of $(PS)_c$ sequence and Brezis-Lieb Lemma, we have

$$\int |\nabla v_n|^2 - \int |(v_n)_+|^{2^*} = o_n(1)$$

and

$$\frac{1}{2} \int |\nabla v_n|^2 - \frac{1}{2^*} \int |(v_n)_+|^{2^*} = c + o_n(1). \quad (2.10)$$

Let

$$\int |\nabla v_n|^2 \rightarrow k, \text{ as } n \rightarrow \infty.$$

Then

$$\int |(v_n)_+|^{2^*} \rightarrow k, \text{ as } n \rightarrow \infty.$$

It is easy to see that $k > 0$. In fact, if $k = 0$, then $\int |\nabla u_n|^2 = \int |\nabla v_n|^2 \rightarrow 0$, which implies that $I(u_n) \rightarrow 0$, contradicting to $c \neq 0$. Going on as Lemma 2.4, we can obtain that $k \geq S^{\frac{N}{2}}$. So, by (2.10), we have

$$\frac{1}{N}S^{\frac{N}{2}} \leq \frac{1}{N}k = \left(\frac{1}{2} - \frac{1}{2^*}\right)k = c < \frac{1}{N}S^{\frac{N}{2}},$$

a contradiction. Hence, $u \neq 0$.

By the density of C_0^∞ in $H_0^1(\Omega)$ and (2.9), we have that

$$\int |\nabla u_-|^2 = 0,$$

which implies that $u \geq 0$. Therefore, we can see that $u \in H_0^1(\Omega) \setminus \{0\}$ and u is a nonnegative weak solution of (1.1). \square

3. ESTIMATIONS ON c_M

In this section, we are going to give an estimation that $c_M < \frac{1}{N}S^{\frac{N}{2}}$, under different assumptions on parameters λ , μ and dimension N . Inspired by Brézis-Nirenberg[3], it is sufficient to find some suitable $U_\epsilon \in H_0^1(\Omega)$ such that $\sup_{t \geq 0} I(tU_\epsilon) < \frac{1}{N}S^{\frac{N}{2}}$. Without loss of generality, we may assume that $0 \in \Omega$, in particular, we suppose that 0 is the geometric center of Ω , i.e., $\rho_{\max} = \text{dist}(0, \partial\Omega)$.

It is well-known (see [4, 6, 16]) that the following problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ u > 0, \\ u(0) = \max_{x \in \mathbb{R}^N} u(x), \end{cases}$$

has a unique solution $\tilde{u}(x)$

$$\tilde{u}(x) = [N(N-2)]^{\frac{N-2}{4}} \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

And correspondingly, up to a dilations,

$$u_\epsilon(x) = [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{N-2}{2}}$$

is a minimizer for S .

We let $\varphi(x) \in C_0^\infty(\Omega)$ be such that $\varphi(x) \equiv 1$ for x in some neighborhood $B_\rho(0)$ of 0, and define

$$U_\epsilon(x) = \varphi(x)u_\epsilon(x). \quad (3.1)$$

Lemma 3.1. *If $N \geq 4$, then we have, as $\epsilon \rightarrow 0^+$,*

$$\int_\Omega |\nabla U_\epsilon|^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \quad (3.2)$$

$$\int_\Omega |U_\epsilon|^{2^*} = S^{\frac{N}{2}} + O(\epsilon^N), \quad (3.3)$$

and

$$\int_{\Omega} |U_{\epsilon}|^2 = \begin{cases} d\epsilon^2 |\ln \epsilon| + O(\epsilon^2), & \text{if } N = 4, \\ d\epsilon^2 + O(\epsilon^{N-2}), & \text{if } N \geq 5, \end{cases}$$

where d is a positive constant.

Proof. The proof can be found in [18]. □

Lemma 3.2. *If $N \geq 5$, then we have, as $\epsilon \rightarrow 0^+$,*

$$\int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 = C_0 \epsilon^2 \log \frac{1}{\epsilon} + O(\epsilon^2),$$

where C_0 is a positive constant.

Proof.

$$\begin{aligned} \int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 &= \int_{\Omega} \varphi^2 u_{\epsilon}^2 \log \varphi^2 + \int_{\Omega} \varphi^2 u_{\epsilon}^2 \log u_{\epsilon}^2 \\ &\triangleq I + II \end{aligned}$$

Since $|s^2 \log s^2| \leq C$ for $0 \leq s \leq 1$, we have

$$|I| \leq C \int_{\Omega} u_{\epsilon}^2 = O(\epsilon^2).$$

$$II = \int_{B_{\rho}(0)} u_{\epsilon}^2 \log u_{\epsilon}^2 + \int_{\Omega \setminus B_{\rho}(0)} \varphi^2 u_{\epsilon}^2 \log u_{\epsilon}^2 \triangleq II_1 + II_2.$$

Since $|s \log s| \leq C_1 s^{1-\delta} + C_2 s^{1+\delta}$ for all $s > 0$, where $0 < C_1 < C_2$ and $0 < \delta < \frac{1}{3}$ such that $(N-2)(1-\delta) \geq 2$,

$$\begin{aligned} |II_2| &\leq \int_{\Omega \setminus B_{\rho}(0)} |u_{\epsilon}^2 \log u_{\epsilon}^2| \\ &\leq C \int_{\Omega \setminus B_{\rho}(0)} (u_{\epsilon}^{2(1-\delta)} + u_{\epsilon}^{2(1+\delta)}) \\ &\leq C |\Omega| (\epsilon^{(N-2)(1-\delta)} + \epsilon^{(N-2)(1+\delta)}) \\ &= O(\epsilon^2), \end{aligned}$$

and

$$\begin{aligned} II_1 &= \int_{B_{(0,\rho)}} u_{\epsilon}^2 \log u_{\epsilon}^2 dx \\ &= C \epsilon^2 \int_{B_{\rho/\epsilon}(0)} \frac{1}{(1+|y|^2)^{N-2}} \log \left(C \epsilon^{-(N-2)} \frac{1}{(1+|y|^2)^{N-2}} \right) dy \\ &= C \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \int_{B_{\rho/\epsilon}(0)} \frac{1}{(1+|y|^2)^{N-2}} + C \epsilon^2 \int_{B_{\rho/\epsilon}(0)} \frac{1}{(1+|y|^2)^{N-2}} \log \frac{C}{(1+|y|^2)^{N-2}} \\ &= C \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} dy + O(\epsilon^2) + \epsilon^2 O \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2-\frac{1}{4}}} dy \right) \\ &= C \epsilon^2 \log \left(\frac{1}{\epsilon} \right) + O(\epsilon^2), \end{aligned}$$

where we have used the fact that

$$\int_{B_{\rho/\epsilon}^c(0)} \frac{1}{(1+|y|^2)^{N-2}} dy = O(\epsilon^{N-4}).$$

Thus

$$\int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 = C_0 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2).$$

We complete the proof. \square

Lemma 3.3. *If $N \geq 5$, $\lambda \in \mathbb{R}$ and $\mu > 0$, then $c_M < \frac{1}{N} S^{\frac{N}{2}}$.*

Proof. Let $g(t) \triangleq I(tU_{\epsilon})$. By Lemma 2.1, $g(0) = 0$ and $\lim_{t \rightarrow +\infty} g(t) = -\infty$, we can find $t_{\epsilon} \in (0, +\infty)$ such that

$$\sup_{t \geq 0} I(tU_{\epsilon}) = \sup_{t \geq 0} g(t) = g(t_{\epsilon}) = I(t_{\epsilon}U_{\epsilon}).$$

So

$$\int |\nabla U_{\epsilon}|^2 - t_{\epsilon}^{2^*-2} \int |U_{\epsilon}|^{2^*} - \lambda \int U_{\epsilon}^2 - \mu \int U_{\epsilon}^2 \log U_{\epsilon}^2 - \mu \log t_{\epsilon}^2 \int U_{\epsilon}^2 = 0,$$

which implies that, as $\epsilon \rightarrow 0^+$

$$\begin{aligned} 2S^{\frac{N}{2}} &\geq \int |\nabla U_{\epsilon}|^2 - \lambda \int U_{\epsilon}^2 - \mu \int U_{\epsilon}^2 \log U_{\epsilon}^2 \\ &= t_{\epsilon}^{2^*-2} \int |U_{\epsilon}|^{2^*} + \mu \log t_{\epsilon}^2 \int U_{\epsilon}^2 \\ &\geq t_{\epsilon}^{2^*-2} \left(\frac{1}{2} S^{\frac{N}{2}} \right) - c |\log t_{\epsilon}^2|. \end{aligned}$$

So there exists $c_1 > 0$ such that $t_{\epsilon} < c_1$.

On the other hand, as $\epsilon \rightarrow 0^+$,

$$\begin{aligned} \frac{1}{2} S^{\frac{N}{2}} &\leq \int |\nabla U_{\epsilon}|^2 - \lambda \int U_{\epsilon}^2 - \mu \int U_{\epsilon}^2 \log U_{\epsilon}^2 \\ &= t_{\epsilon}^{2^*-2} \int |U_{\epsilon}|^{2^*} + \mu \log t_{\epsilon}^2 \int U_{\epsilon}^2 \\ &\leq 2S^{\frac{N}{2}} t_{\epsilon}^{2^*-2} + C t_{\epsilon}^{2^*-2}, \end{aligned}$$

which implies that there exists $c_2 > 0$ such that $t_{\epsilon} > c_2$.

Therefore, combining with the definition of c_M , we have that, as $\epsilon \rightarrow 0^+$,

$$\begin{aligned}
c_M &\leq \sup_{t \geq 0} I(tu_\epsilon) \\
&= \frac{t_\epsilon^2}{2} \int |\nabla U_\epsilon|^2 - \frac{t_\epsilon^{2^*}}{2^*} \int |U_\epsilon|^{2^*} - \frac{\lambda}{2} t_\epsilon^2 \int U_\epsilon^2 - \frac{\mu}{2} \int t_\epsilon^2 U_\epsilon^2 (\log(t_\epsilon^2 U_\epsilon^2) - 1) \\
&\leq \left(\frac{t_\epsilon^2}{2} - \frac{t_\epsilon^{2^*}}{2^*}\right) S^{\frac{N}{2}} + O(\epsilon^2) + \frac{\mu}{2} t_\epsilon^2 (1 - \log t_\epsilon^2) \int U_\epsilon^2 - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 \\
&\leq \frac{1}{N} S^{\frac{N}{2}} - c\mu\epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) \\
&< \frac{1}{N} S^{\frac{N}{2}}.
\end{aligned}$$

We complete the proof. \square

The case for $N = 4$:

Let $\varphi(x) \in C_0^\infty(\Omega)$ be a radial function satisfying that $\varphi(x) = 1$ for $0 \leq |x| \leq \rho$, $0 \leq \varphi(x) \leq 1$ for $\rho \leq |x| \leq 2\rho$, $\varphi(x) = 0$ for $x \in \Omega \setminus B_{2\rho}(0)$, where $0 < \rho \leq 1$ with $\log\left(\frac{1}{8e^{3-\frac{\lambda}{\mu}}\rho^2}\right) > 1$.

Set

$$U_\epsilon = \varphi(x)u_\epsilon(x).$$

Lemma 3.4. *If $N = 4$, then, as $\epsilon \rightarrow 0^+$,*

$$\int_\Omega U_\epsilon^2 \log U_\epsilon^2 \geq 8 \log\left(\frac{8(\epsilon^2 + \rho^2)}{e(\epsilon^2 + 4\rho^2)^2}\right) \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2)$$

and

$$\int_\Omega U_\epsilon^2 \log U_\epsilon^2 \leq 8 \log\left(\frac{8e(\epsilon^2 + 4\rho^2)}{(\epsilon^2 + \rho^2)^2}\right) \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2).$$

Proof. Following from the definition of U_ϵ , we have, as $\epsilon \rightarrow 0^+$,

$$\begin{aligned}
\int_\Omega U_\epsilon^2 \log(U_\epsilon^2) &= 8 \int_\Omega \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2 \log \left[8\varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2\right] dx \\
&= 8 \int_\Omega \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2 \log \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2 dx \\
&\quad + 8 \log(8) \int_\Omega \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2 dx \\
&\quad + 8 \int_{\Omega \setminus B_\rho(0)} \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^2 \log \varphi^2 dx \\
&= I_1 + I_2 + O(\epsilon^2).
\end{aligned} \tag{3.4}$$

By direct computation, we obtain

$$\begin{aligned}
I_2 &= 8 \log 8 \int_{B_\rho(0)} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^2 dx + O(\epsilon^2) \\
&= 8 \log 8 \omega_4 \epsilon^2 \int_0^{\rho/\epsilon} \frac{1}{(1+r^2)^2} r^3 dr + O(\epsilon^2) \\
&= 4 \log 8 \omega_4 \epsilon^2 \left[\log(r^2 + 1) + \frac{1}{1+r^2} \right] \Big|_0^{\rho/\epsilon} + O(\epsilon^2) \\
&= 4 \log 8 \omega_4 \epsilon^2 \left[\log \left(\frac{\rho^2 + \epsilon^2}{\epsilon^2} \right) + \frac{1}{1 + \frac{\rho^2}{\epsilon^2}} - 1 \right] + O(\epsilon^2) \\
&= 8 \log 8 \omega_4 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) + O(\epsilon^2),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
I_1 &= 8 \int_{\Omega} \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^2 \log \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^2 dx \\
&= 8 \int_{B_{2\rho}(0)} \varphi^2 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^2 \log \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^2 dx \\
&= 8 \epsilon^2 \int_{B_{2\rho/\epsilon}(0)} \varphi^2(\epsilon x) \frac{1}{(1+|x|^2)^2} \log \left(\frac{1}{\epsilon^2} \frac{1}{(1+|x|^2)^2} \right) dx \\
&= 16 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \int_{B_{2\rho/\epsilon}(0)} \varphi^2(\epsilon x) \frac{1}{(1+|x|^2)^2} dx \\
&\quad + 8 \epsilon^2 \int_{B_{2\rho/\epsilon}(0)} \varphi^2(\epsilon x) \frac{1}{(1+|x|^2)^2} \log \frac{1}{(1+|x|^2)^2} dx \\
&\triangleq I_{11} + I_{12},
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
I_{11} &\geq 16 \omega_4 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \int_0^{\rho/\epsilon} \frac{1}{(1+r^2)^2} r^3 dr \\
&= 8 \omega_4 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \left[\log \left(\frac{1}{\epsilon^2} \right) + \log(\rho^2 + \epsilon^2) + \frac{\epsilon^2}{\rho^2 + \epsilon^2} - 1 \right] \\
&= 16 \omega_4 \epsilon^2 \left(\log \left(\frac{1}{\epsilon} \right) \right)^2 + 8 \omega_4 \log \left(\frac{\rho^2 + \epsilon^2}{\epsilon} \right) \epsilon^2 \log \left(\frac{1}{\epsilon} \right) + O(\epsilon^4 \log \left(\frac{1}{\epsilon} \right)),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
I_{11} &\leq 16 \omega_4 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \int_0^{2\rho/\epsilon} \frac{1}{(1+r^2)^2} r^3 dr \\
&= 8 \omega_4 \epsilon^2 \log \left(\frac{1}{\epsilon} \right) \left[\log \left(\frac{1}{\epsilon^2} \right) + \log(4\rho^2 + \epsilon^2) + \frac{\epsilon^2}{4\rho^2 + \epsilon^2} - 1 \right]
\end{aligned}$$

$$= 16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right)\right)^2 + 8\omega_4 \log\left(\frac{4\rho^2 + \epsilon^2}{e}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^4 \log\left(\frac{1}{\epsilon}\right)), \quad (3.8)$$

$$\begin{aligned} I_{12} &\geq -8\epsilon^2 \int_{B_{2\rho/\epsilon}(0)} \frac{1}{(1+|x|^2)^2} \log(1+|x|^2)^2 dx \\ &= -16\omega_4\epsilon^2 \int_0^{2\rho/\epsilon} \frac{1}{(1+r^2)^2} \log(1+r^2) r^3 dr \\ &= -8\omega_4\epsilon^2 \int_0^{2\rho/\epsilon} \frac{r^2+1-1}{(1+r^2)^2} \log(1+r^2) d(1+r^2) \\ &= -8\omega_4\epsilon^2 \int_0^{2\rho/\epsilon} \frac{1}{1+r^2} \log(1+r^2) d(1+r^2) \\ &\quad + 8\omega_4\epsilon^2 \int_0^{2\rho/\epsilon} \frac{1}{(1+r^2)^2} \log(1+r^2) d(1+r^2) \\ &\geq -4\omega_4\epsilon^2 (\log(1+r^2))^2 \Big|_0^{2\rho/\epsilon} \\ &= -4\omega_4\epsilon^2 \left(\log\left(1 + \frac{4\rho^2}{\epsilon^2}\right)\right)^2 \\ &= -4\omega_4\epsilon^2 \left[\log(\epsilon^2 + 4\rho^2) + 2\log\left(\frac{1}{\epsilon}\right)\right]^2 \\ &= -16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right)\right)^2 - 16\omega_4 \log(\epsilon^2 + 4\rho^2) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2). \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} I_{12} &\leq -8\epsilon^2 \int_{B_{\rho/\epsilon}(0)} \frac{1}{(1+|x|^2)^2} \log(1+|x|^2)^2 dx \\ &= -16\omega_4\epsilon^2 \int_0^{\rho/\epsilon} \frac{1}{(1+r^2)^2} \log(1+r^2) r^3 dr \\ &= -8\omega_4\epsilon^2 \int_0^{\rho/\epsilon} \frac{r^2+1-1}{(1+r^2)^2} \log(1+r^2) d(1+r^2) \\ &= -8\omega_4\epsilon^2 \int_0^{\rho/\epsilon} \frac{1}{1+r^2} \log(1+r^2) d(1+r^2) \\ &\quad + 8\omega_4\epsilon^2 \int_0^{\rho/\epsilon} \frac{1}{(1+r^2)^2} \log(1+r^2) d(1+r^2) \\ &\leq -4\omega_4\epsilon^2 (\log(1+r^2))^2 \Big|_0^{\rho/\epsilon} + 8\omega_4\epsilon^2 \int_0^{\rho/\epsilon} \frac{1}{(1+r^2)} d(1+r^2) \\ &= -4\omega_4\epsilon^2 \left[\log(\epsilon^2 + \rho^2) + 2\log\left(\frac{1}{\epsilon}\right)\right]^2 + 8\omega_4\epsilon^2 \left[\log(\epsilon^2 + \rho^2) + 2\log\left(\frac{1}{\epsilon}\right)\right] \end{aligned}$$

$$= -16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right) \right)^2 - 16\omega_4 \log\left(\frac{\epsilon^2 + \rho^2}{e}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2). \quad (3.10)$$

So, by (3.4)–(3.12), we have that

$$\begin{aligned} \int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 &\geq 8 \log 8\omega_4\epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + 16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right) \right)^2 + 8\omega_4 \log\left(\frac{\epsilon^2 + \rho^2}{e}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) \\ &\quad - 16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right) \right)^2 - 16\omega_4 \log(\epsilon^2 + 4\rho^2) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) \\ &= 8 \log \left(\frac{8(\epsilon^2 + \rho^2)}{e(\epsilon^2 + 4\rho^2)^2} \right) \omega_4\epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 &\leq 8 \log 8\omega_4\epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + 16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right) \right)^2 + 8\omega_4 \log\left(\frac{\epsilon^2 + 4\rho^2}{e}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) \\ &\quad - 16\omega_4\epsilon^2 \left(\log\left(\frac{1}{\epsilon}\right) \right)^2 - 16\omega_4 \log\left(\frac{\epsilon^2 + \rho^2}{e}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) \\ &= 8 \log \left(\frac{8e(\epsilon^2 + 4\rho^2)}{(\epsilon^2 + \rho^2)^2} \right) \omega_4\epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2). \end{aligned}$$

□

Lemma 3.5. *Assume that $N = 4$. If $(\lambda, \mu) \in B_0 \cup C_0$ and $\frac{32e^{\lambda}}{\rho_{max}^2} < 1$, or $\lambda \in \mathbb{R}$ and $\mu > 0$, then $c_M < \frac{1}{N} S^{\frac{N}{2}}$.*

Proof. Let $g(t) \triangleq I(tU_{\epsilon})$. Similar to the case of $N \geq 5$, we can find a $t_{\epsilon} \in (0, +\infty)$ such that

$$\sup_{t \geq 0} I(tU_{\epsilon}) = I(t_{\epsilon}U_{\epsilon})$$

and

$$\int |\nabla U_{\epsilon}|^2 - t_{\epsilon}^{2^*-2} \int |U_{\epsilon}|^{2^*} - \lambda \int U_{\epsilon}^2 - \mu \int U_{\epsilon}^2 \log U_{\epsilon}^2 - \mu \log t_{\epsilon}^2 \int U_{\epsilon}^2 = 0.$$

Similar to the case of $N \geq 5$ again, we can see that there exists $0 < C_2$ such that $t_{\epsilon} < C_2$ for any $\mu \in \mathbb{R} \setminus \{0\}$ and there exists $C_1 > 0$ such that $t_{\epsilon} > C_1$ for $\mu > 0$.

Case 1: $\mu > 0$

So

$$\mu \log t_{\epsilon}^2 \int U_{\epsilon}^2 = O(\epsilon^2 |\ln \epsilon|),$$

and

$$\begin{aligned} t_{\epsilon}^{2^*-2} &= \frac{\int |\nabla U_{\epsilon}|^2 - \lambda \int U_{\epsilon}^2 - \mu \int U_{\epsilon}^2 \log U_{\epsilon}^2 - \mu \log t_{\epsilon}^2 \int U_{\epsilon}^2}{\int |U_{\epsilon}|^{2^*}} \\ &= \frac{S^{\frac{N}{2}} + O(\epsilon^2 (\log(\frac{1}{\epsilon}))^2)}{S^{\frac{N}{2}} + O(\epsilon^N)} \longrightarrow 1 \text{ as } \epsilon \rightarrow 0^+, \end{aligned}$$

which implies that,

$$\mu \log t_\epsilon^2 \int U_\epsilon^2 = o(\epsilon^2 |\ln \epsilon|),$$

According to (3.5), we get that

$$\int_\Omega U_\epsilon^2 = 8\omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + O(\epsilon^2).$$

Therefore we have that, as $\epsilon \rightarrow 0^+$,

$$\begin{aligned} c_M &\leq I(t_\epsilon U_\epsilon) \\ &= \frac{t_\epsilon^2}{2} \int |\nabla U_\epsilon|^2 - \frac{t_\epsilon^{2^*}}{2^*} \int |U_\epsilon|^{2^*} + \frac{\mu - \lambda}{2} t_\epsilon^2 \int U_\epsilon^2 - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \left(\frac{t_\epsilon^2}{2} - \frac{t_\epsilon^{2^*}}{2^*}\right) S^{\frac{N}{2}} + O(\epsilon^2) + \frac{\mu - \lambda}{2} t_\epsilon^2 \int U_\epsilon^2 - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - \frac{t_\epsilon^2}{2} \int [\mu U_\epsilon^2 \log U_\epsilon^2 + (\lambda - \mu) U_\epsilon^2] + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - \frac{t_\epsilon^2}{2} \left(8\mu \log \left(\frac{8(\epsilon^2 + \rho^2)}{e(\epsilon^2 + 4\rho^2)^2} \right) \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + (\lambda - \mu) 8\omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) \right) + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - 4t_\epsilon^2 \log \left(\frac{8^\mu (\epsilon^2 + \rho^2)^\mu}{e^{2\mu - \lambda} (\epsilon^2 + 4\rho^2)^{2\mu}} \right) \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - 4t_\epsilon^2 \log \left(\frac{8^\mu}{25^\mu e^{2\mu - \lambda} \rho^{2\mu}} \right) \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - 4C_1^2 C \omega_4 \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\ &< \frac{1}{N} S^{\frac{N}{2}}, \end{aligned}$$

where we choose $\rho > 0$ small enough such that $\frac{8^\mu}{25^\mu e^{2\mu - \lambda} \rho^{2\mu}} > 1$.

Case 2: $\mu < 0$

When $(\lambda, \mu) \in B_0 \cup C_0$, we choose $\rho = \rho_{max}$. We can see that $t_\epsilon \not\rightarrow 0$. Otherwise, $0 < \alpha \leq c_M \leq I(t_\epsilon U_\epsilon) \rightarrow 0$, which is impossible. Similar to Case 1, we can see that $t_\epsilon \rightarrow 1$ and $t_\epsilon^2 \log t_\epsilon^2 = o(1)$. Therefore,

$$-\frac{\mu}{2} t_\epsilon^2 \log t_\epsilon^2 \int U_\epsilon^2 = o(\epsilon^2 |\ln \epsilon|). \quad (3.11)$$

Then we obtain that, as $\epsilon \rightarrow 0^+$,

$$\begin{aligned} c_M &\leq I(t_\epsilon U_\epsilon) \\ &= \frac{t_\epsilon^2}{2} \int |\nabla U_\epsilon|^2 - \frac{t_\epsilon^{2^*}}{2^*} \int |U_\epsilon|^{2^*} + \frac{\mu}{2} t_\epsilon^2 (1 - \log t_\epsilon^2) \int U_\epsilon^2 - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \left(\log U_\epsilon^2 + \frac{\lambda}{\mu} \right) \\ &\leq \left(\frac{t_\epsilon^2}{2} - \frac{t_\epsilon^{2^*}}{2^*}\right) S^{\frac{N}{2}} + \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 - \frac{\lambda}{2} t_\epsilon^2 \int U_\epsilon^2 + o(\epsilon^2 |\ln \epsilon|) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{N}S^{\frac{N}{2}} - \frac{\mu}{2}t_\epsilon^2 \left(\left(\frac{\lambda}{\mu} - 1\right)8\omega_4\epsilon^2 \log\left(\frac{1}{\epsilon}\right) + 8 \log\left(\frac{8e(\epsilon^2 + 4\rho^2)}{(\epsilon^2 + \rho^2)^2}\right) \omega_4\epsilon^2 \log\left(\frac{1}{\epsilon}\right) \right) + o(\epsilon^2 |\ln \epsilon|) \\
&\leq \frac{1}{N}S^{\frac{N}{2}} - 4\mu\omega_4t_\epsilon^2 \left(\left(\frac{\lambda}{\mu} - 1\right) + \log\left(\frac{8e(\epsilon^2 + 4\rho^2)}{(\epsilon^2 + \rho^2)^2}\right) \right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\
&\leq \frac{1}{N}S^{\frac{N}{2}} - 4\mu\omega_4t_\epsilon^2 \log\left(\frac{8e^{\frac{\lambda}{\mu}}(\epsilon^2 + 4\rho^2)}{(\epsilon^2 + \rho^2)^2}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\
&\leq \frac{1}{N}S^{\frac{N}{2}} - 4\mu\omega_4 \log\left(\frac{32e^{\frac{\lambda}{\mu}}}{\rho^2}\right) \epsilon^2 \log\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 |\ln \epsilon|) \\
&< \frac{1}{N}S^{\frac{N}{2}}
\end{aligned}$$

since the fact that $\frac{32e^{\frac{\lambda}{\mu}}}{\rho_{max}^2} < 1$.

We complete the proof. □

The case for $N = 3$:

Let $\varphi(x) \in C_0^1(\Omega)$ be a radial function satisfying that $\varphi(x) = 1$ for $0 \leq |x| \leq \rho$, $0 \leq \varphi(x) \leq 1$ for $\rho \leq |x| \leq 2\rho$, $\varphi(x) = 0$ for $x \in \Omega \setminus B_{2\rho}(0)$, where $0 < \rho$ is any fixed constant such that $B_{2\rho}(0) \subset \Omega$ and $4\rho^2 < 1$.

Set

$$U_\epsilon = \varphi(x)u_\epsilon(x).$$

Lemma 3.6. *If $N = 3$, then we have, as $\epsilon \rightarrow 0^+$,*

$$\int_{\Omega} |\nabla U_\epsilon|^2 dx = S^{\frac{3}{2}} + \sqrt{3}\omega_3 \int_{\rho}^{2\rho} |\varphi'(r)|^2 dr \epsilon + O(\epsilon^3), \quad (3.12)$$

$$\int_{\Omega} |U_\epsilon|^{2^*} dx = S^{\frac{3}{2}} + O(\epsilon^3), \quad (3.13)$$

$$\int_{\Omega} U_\epsilon^2 dx = \sqrt{3}\omega_3 \int_0^{2\rho} \varphi^2 dr \epsilon + O(\epsilon^2), \quad (3.14)$$

and

$$\int_{\Omega} U_\epsilon^2 \log U_\epsilon^2 dx = \sqrt{3}\omega_3 \int_0^{2\rho} \varphi^2 dr \epsilon \log \epsilon + O(\epsilon), \quad (3.15)$$

where ω_3 denotes the area of the unit sphere surface.

Proof. Following from the definition of U_ϵ , direct computations implies that

$$\begin{aligned}
& \int_{\Omega} |\nabla U_\epsilon|^2 dx \\
&= \int_{B_{2\rho}} \left(|\nabla \varphi|^2 \frac{\sqrt{3}\epsilon}{\epsilon^2 + |x|^2} - 2\nabla \varphi \cdot \varphi(r) \frac{\sqrt{3}\epsilon x}{(\epsilon^2 + |x|^2)^2} + \frac{\sqrt{3}\epsilon \varphi^2(r) x^2}{(\epsilon^2 + |x|^2)^3} \right) dx \\
&= \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} |\varphi'(r)|^2 \frac{r^2}{\epsilon^2 + r^2} dr - 2\sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \varphi'(r) r \varphi(r) \frac{r^2}{(\epsilon^2 + r^2)^2} dr \\
&+ \sqrt{3}\omega_3 \epsilon \int_0^{\rho} \frac{r^4}{(\epsilon^2 + r^2)^3} dr + \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \frac{\varphi^2(r) r^4}{(\epsilon^2 + r^2)^3} dr \\
&= \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} |\varphi'(r)|^2 dr + O(\epsilon^3) - 2\sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \varphi'(r) r \varphi(r) \frac{1}{\epsilon^2 + r^2} dr + O(\epsilon^3) \\
&+ \sqrt{3}\omega_3 \epsilon \int_0^{+\infty} \frac{r^4}{(\epsilon^2 + r^2)^3} dr - \sqrt{3}\omega_3 \epsilon \int_{\rho}^{+\infty} \frac{r^4}{(\epsilon^2 + r^2)^3} dr + \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \frac{\varphi^2(r) r^4}{(\epsilon^2 + r^2)^3} dr \\
&= \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} |\varphi'(r)|^2 dr - 2\sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \varphi'(r) \varphi(r) \frac{1}{r} dr + O(\epsilon^3) \\
&+ \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 - \sqrt{3}\omega_3 \epsilon \int_{\rho}^{+\infty} \frac{1}{\epsilon^2 + r^2} dr + \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} \varphi^2(r) \frac{1}{\epsilon^2 + r^2} dr + O(\epsilon^3) \\
&= S^{\frac{3}{2}} + O(\epsilon^3) \\
&+ \sqrt{3}\omega_3 \epsilon \left(\int_{\rho}^{2\rho} |\varphi'(r)|^2 dr - \int_{\rho}^{2\rho} 2\varphi'(r) \varphi(r) \frac{1}{r} dr - \int_{\rho}^{+\infty} \frac{1}{r^2} dr + \int_{\rho}^{2\rho} \varphi^2(r) \frac{1}{r^2} dr \right) \\
&= S^{\frac{3}{2}} + \sqrt{3}\omega_3 \epsilon \int_{\rho}^{2\rho} |\varphi'(r)|^2 dr + O(\epsilon^3)
\end{aligned}$$

$$\int_{\Omega} |U_\epsilon|^{2^*} dx = \int_{B_{\rho}(0)} |u_\epsilon|^{2^*} dx + O(\epsilon^3) = \int_{\mathbb{R}^3} |u_\epsilon|^{2^*} dx + O(\epsilon^3) = S^{\frac{3}{2}} + O(\epsilon^3),$$

and

$$\begin{aligned}
\int_{\Omega} U_\epsilon^2 dx &= \sqrt{3} \int_{B_{2\rho}(0)} \varphi^2 \frac{\epsilon}{\epsilon^2 + |x|^2} dx \\
&= \sqrt{3}\omega_3 \epsilon \int_0^{2\rho} \varphi^2 \frac{1}{\epsilon^2 + r^2} r^2 dr \\
&= \sqrt{3}\omega_3 \epsilon \int_0^{2\rho} \varphi^2 dr - \sqrt{3}\omega_3 \epsilon^3 \int_0^{2\rho} \varphi^2 \frac{1}{\epsilon^2 + r^2} dr \\
&= \sqrt{3}\omega_3 \epsilon \int_0^{2\rho} \varphi^2 dr + O(\epsilon^2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{\Omega} U_{\epsilon}^2 \log U_{\epsilon}^2 &= \sqrt{3} \int_{\Omega} \varphi^2 \frac{\epsilon}{\epsilon^2 + |x|^2} \log(\sqrt{3} \varphi^2 \frac{\epsilon}{\epsilon^2 + |x|^2}) dx \\
&= \sqrt{3} \int_{B_{2\rho}(0)} \varphi^2 \frac{\epsilon}{\epsilon^2 + |x|^2} \log(\sqrt{3} \varphi^2 \frac{\epsilon}{\epsilon^2 + |x|^2}) dx \\
&= \sqrt{3} \omega_3 \epsilon \int_0^{2\rho} \varphi^2(r) \frac{r^2}{\epsilon^2 + r^2} \left[\log \sqrt{3} + \log \epsilon + \log \varphi^2 + \log \frac{1}{\epsilon^2 + r^2} \right] dr \\
&= \sqrt{3} \log \sqrt{3} \omega_3 \epsilon \int_0^{2\rho} \varphi^2(r) \frac{r^2}{\epsilon^2 + r^2} dr \\
&\quad + \sqrt{3} \omega_3 \epsilon \log \epsilon \int_0^{2\rho} \varphi^2 \frac{r^2}{\epsilon^2 + r^2} dr \\
&\quad + \sqrt{3} \omega_3 \epsilon \int_0^{2\rho} \varphi^2 \log \varphi^2 \frac{r^2}{\epsilon^2 + r^2} dr \\
&\quad + \sqrt{3} \omega_3 \epsilon \int_0^{2\rho} \varphi^2(r) \frac{r^2}{\epsilon^2 + r^2} \log \frac{1}{\epsilon^2 + r^2} dr \\
&\triangleq I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.16}$$

By direct computation, we obtain that

$$I_1 = O(\epsilon), \quad I_3 = O(\epsilon), \tag{3.17}$$

$$\begin{aligned}
I_2 &= \sqrt{3} \omega_3 \epsilon \log \epsilon \int_0^{2\rho} \varphi^2 dr - \sqrt{3} \omega_3 \epsilon \log \epsilon \int_0^{2\rho} \varphi^2 \frac{\epsilon^2}{\epsilon^2 + r^2} dr \\
&= \sqrt{3} \omega_3 \epsilon \log \epsilon \int_0^{2\rho} \varphi^2 dr + O(\epsilon^2 \log \epsilon),
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
|I_4| &\leq -\sqrt{3} \omega_3 \epsilon \int_0^{\rho} \log(\epsilon^2 + r^2) dr + O(\epsilon) \\
&= -\sqrt{3} \omega_3 \epsilon r \log(\epsilon^2 + r^2) \Big|_0^{\rho} + \sqrt{3} \omega_3 \epsilon \int_0^{\rho} r \frac{1}{\epsilon^2 + r^2} 2r dr + O(\epsilon) \\
&= O(\epsilon).
\end{aligned} \tag{3.19}$$

It follows from (3.16)–(3.19) that

$$\int U_{\epsilon}^2 \log U_{\epsilon}^2 dx = \sqrt{3} \omega_3 \int_0^{2\rho} \varphi^2 dr \epsilon \log \epsilon + O(\epsilon).$$

We complete the proof. □

Lemma 3.7. *If $N = 3$ and $(\lambda, \mu) \in B_0 \cup C_0$, then $c_M < \frac{1}{N} S^{\frac{N}{2}}$.*

Proof. Assume that $g(t) := I(tU_\epsilon)$. Since $g(0) = 0$, $\lim_{t \rightarrow +\infty} g(t) = -\infty$ and Lemma 2.1, we can get a $t_\epsilon \in (0, +\infty)$ such that

$$\sup_{t \geq 0} I(tU_\epsilon) = I(t_\epsilon U_\epsilon).$$

Similar to the case of $N = 4$, we can see that there exist $0 < C_1 < C_2$ such that $t_\epsilon \in (C_1, C_2)$. Therefore, for ϵ small enough,

$$\begin{aligned} c_M \leq I(t_\epsilon U_\epsilon) &= \frac{t_\epsilon^2}{2} \int |\nabla U_\epsilon|^2 - \frac{t_\epsilon^{2^*}}{2^*} \int |U_\epsilon|^{2^*} - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 + O(\epsilon) \\ &= \left(\frac{t_\epsilon^2}{2} - \frac{t_\epsilon^{2^*}}{2^*} \right) S^{\frac{3}{2}} - \frac{\mu}{2} t_\epsilon^2 \int U_\epsilon^2 \log U_\epsilon^2 + O(\epsilon) \\ &\leq \frac{1}{N} S^{\frac{3}{2}} - \frac{\sqrt{3}\mu}{2} C_1^2 \omega_3 \int_0^{2\rho} \varphi^2 dr \epsilon \log(\epsilon) + O(\epsilon) \\ &< \frac{1}{N} S^{\frac{3}{2}}. \end{aligned}$$

We complete the proof. \square

4. THE PROOF OF MAIN THEOREMS

The proof of Theorem 1.2: Assume that $N \geq 4$ and $(\lambda, \mu) \in A_0$. By Lemma 2.1 and the mountain-pass theorem, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that, as $n \rightarrow \infty$,

$$\begin{aligned} I(u_n) &\rightarrow c_M, \\ I'(u_n) &\rightarrow 0, \quad \text{in } (H_0^1(\Omega))^{-1}, \end{aligned}$$

which, together with Lemma 2.2, implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Lemmas 2.4, 3.3 and 3.5, we can see that there exists $u \in H_0^1(\Omega)$ such that $u_n \rightarrow u$ in $H_0^1(\Omega)$, which implies that

$$I(u) = c_M \quad \text{and} \quad I'(u) = 0.$$

Thus,

$$0 = \langle I'(u), u_- \rangle = \int |\nabla u_-|^2,$$

which implies that $u \geq 0$.

Therefore, u is a nonnegative nontrivial weak solution of (1.1). By Moser's iteration, it is standard to prove that $u \in L^\infty(\Omega)$, then the Hölder estimate implies that $u \in C^{0,\gamma}(\Omega)$ ($0 < \gamma < 1$). Let $\beta : [0, +\infty) \mapsto \mathbb{R}$ be defined by

$$\beta(s) := \begin{cases} \frac{3|\mu|}{2} |s \log s^2|, & s > 0, \\ 0, & s = 0, \end{cases}$$

then for $a > 0$ small enough, one can see that

$$\Delta(u) = -u^{2^*-1} - \lambda u - \mu u \log u^2 \leq \beta(u) \quad \text{in } \{x \in \Omega : 0 < u(x) < a\}.$$

We may also assume that $a < \frac{1}{e}$, then $\beta'(s) = \frac{3|\mu|}{2}(-\log s^2 - 2) > |\mu|(-\log a - 1) > 0$ for $s \in (0, a)$. So we have that $\beta(0) = 0$ and $\beta(s)$ is nondecreasing in $(0, a)$. Furthermore,

$$\int_0^{\frac{a}{2}} (\beta(s)s)^{-\frac{1}{2}} ds = -\sqrt{\frac{2}{3|\mu|}} (-2\log s)^{\frac{1}{2}} \Big|_0^{\frac{a}{2}} = +\infty.$$

Hence, by [17, Theorem 1], we have that $u(x) > 0$ in Ω . In particular, for any compact $K \subset\subset \Omega$, there exists $c = c(K) > 0$ such that $u(x) \geq c, \forall x \in K$. Take $K \subset\subset K_1 \subset\subset \Omega$ and put $f(x) := -u(x)^{2^*-1} - \lambda u(x) - \mu u(x) \log u(x)^2$, then $\Delta u = f(x)$ in K_1 and f is of $C^{0,\gamma}$ in K_1 . So by the standard Schauder estimate, we see that $u \in C^{2,\gamma}(K)$. By the arbitrariness of K , we obtain that $u \in C^2(\Omega)$ and $u > 0$ in Ω . The proof is completed. \square

The proof of Theorem 1.3: By Lemma 2.1 and the mountain-pass theorem, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that, as $n \rightarrow \infty$,

$$\begin{aligned} I(u_n) &\rightarrow c_M, \\ I'(u_n) &\rightarrow 0, \quad \text{in } (H_0^1(\Omega))^{-1}, \end{aligned}$$

combining with Lemmas 2.5, 3.5 and 3.7, problem (1.1) has a nonnegative nontrivial weak solution u . Applying a similar argument as the proof of Theorem 1.2, we obtain that $u \in C^2(\Omega)$ and $u(x) > 0$ in Ω . \square

The proof of the Theorem 1.4: Assume that problem (1.1) has a positive solution u_0 and let $\varphi_1(x) > 0$ be the first eigenfunction corresponding to $\lambda_1(\Omega)$. Then

$$\begin{aligned} \int_{\Omega} (u_0^{2^*-1} + \lambda u_0 + \mu u_0 \log u_0^2) \varphi_1(x) &= \int_{\Omega} -\Delta u_0 \varphi_1(x) \\ &= \int_{\Omega} -\Delta \varphi_1(x) u_0 = \int_{\Omega} \lambda_1(\Omega) \varphi_1(x) u_0, \end{aligned}$$

which implies that

$$\int_{\Omega} (u_0^{2^*-2} + \lambda - \lambda_1(\Omega) + \mu \log u_0^2) u_0 \varphi_1(x) = 0. \quad (4.1)$$

Define

$$f(s) := s^{2^*-2} + \mu \log s^2 + \lambda - \lambda_1(\Omega), \quad s > 0,$$

then

$$f'(s) = (2^* - 2)s^{2^*-3} + 2\mu \frac{1}{s}.$$

By a direct computation, $f'(s) = 0$ has a unique root $s_0 = (-\frac{(N-2)\mu}{2})^{\frac{N-2}{4}}$. Furthermore, $f'(s) < 0$ in $(0, (-\frac{(N-2)\mu}{2})^{\frac{N-2}{4}})$ and $f'(s) > 0$ in $((-\frac{(N-2)\mu}{2})^{\frac{N-2}{4}}, +\infty)$. Hence,

$$f(s) \geq f((-\frac{(N-2)\mu}{2})^{\frac{N-2}{4}}) = -\frac{(N-2)\mu}{2} + \frac{(N-2)\mu}{2} \log(-\frac{(N-2)\mu}{2}) + \lambda - \lambda_1(\Omega) \geq 0. \quad (4.2)$$

Since $u_0 \in H_0^1(\Omega)$ and $u_0, \varphi_1 > 0$, we have $\int_{\Omega} f(u_0(x))u_0\varphi_1(x) > 0$. Otherwise, $f(u_0(x)) = 0$ a.e in Ω . That is, $u_0(x) = (-\frac{(N-2)\mu}{2})^{\frac{N-2}{4}}$ a.e in Ω , which contradicts to $u_0 \in H_0^1(\Omega)$. By (4.1) and (4.2), we have that

$$0 = \int_{\Omega} (u_0^{2^*-2} + \lambda - \lambda_1(\Omega) + \mu \log u_0^2)u_0\varphi_1(x) = \int_{\Omega} f(u_0(x))u_0\varphi_1(x) > 0,$$

a contradiction. Hence, problem (1.1) has no positive solutions. \square

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