ON GENERALIZATION OF BREUIL-SCHRAEN'S $\mathscr{L}$-INVARIANTS TO GL_n

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Abstract. Let p be prime number and K be a p-adic field. We systematically compute the higher Ext-groups between locally analytic generalized Steinberg representations (LAGS for short) of $\mathrm{GL}_n(K)$ via a new combinatorial treatment of some spectral sequences arising from the so-called Tits complex. Such spectral sequences degenerate at the second page and each Ext-group admits a canonical filtration whose graded pieces are terms in the second page of the corresponding spectral sequence. For each pair of LAGS, we are particularly interested their Ext-groups in the bottom two non-vanishing degrees. We write down an explicit basis for each graded piece (under the canonical filtration) of such an Ext-group, and then describe the cup product maps between such Ext-groups using these bases. As an application, we generalize Breuil's $\mathscr{L}$-invariants for $\mathrm{GL}_2(\mathbb{Q}_p)$ and Schraen's higher $\mathscr{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$ to $\mathrm{GL}_n(K)$. Along the way, we also establish a generalization of Bernstein–Zelevinsky geometric lemma to admissible locally analytic representations constructed by Orlik–Strauch, generalizing a result in Schraen's thesis for $\mathrm{GL}_3(\mathbb{Q}_p)$.

Contents

1. Introduction

Let p be a prime number and E be a sufficiently large finite extension of $\mathbb{Q}_p$. The theory of $\mathscr{L}$-invariant(s) has a long history, and was first introduced by Mazur-Tate-Teitelbaum [MTT86] to describe the derivative of the p-adic L-function (for a certain weight 2 modular form f) at its exceptional zero. If we write ρ_f for the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation attached to f via Langlands correspondence, then we are interested in the case when $\rho_{f,p}$ is semi-stable non-crystalline, and this derivative, written $\mathscr{L}(f)$, depends only on $\rho_{f,p}$ and can be read off explicitly from the filtered (φ, N)-module attached to $\rho_{f,p}$ via Fontaine's theory (see [Fon94]). We usually use the term *Fontaine-Mazur $\mathscr{L}$-invariant* for invariants (of semi-stable p-adic Galois representations) which are defined using an admissible Hodge filtration on a (φ, N)-module. Among many different definitions of the $\mathscr{L}$-invariant $\mathscr{L}(f)$ (which turn out to be equivalent, see the book [Ast] for a comprehensive study), Breuil constructs in [Bre04] an explicit finite length locally analytic representation $\Pi(\rho_{f,p})$ (written $\Sigma(2, \mathcal{L})$ in *loc.it.* with $\mathcal{L} = \mathscr{L}(f)$) of $\mathrm{GL}_2(\mathbb{Q}_p)$ whose isomorphism class recovers $\mathscr{L}(f)$, and shows in [Bre10] that $\Pi(\rho_{f,p})$ embeds into the f-isotypic component of completed cohomology of modular curves. (Such results are usually called p-adic local-global compatibility). Breuil's construction is actually one of the first instances of the p-adic Langlands correspondence, and the map $\rho_{f,p} \mapsto \Pi(\rho_{f,p})$ can be improved to be a bijection between the family of semi-stable non crystalline p-adic continuous representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(E)$ and an explicit family of locally analytic (E-)representations $\Pi(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. Motivated by the general philosophy of p-adic Langlands correspondence, it is natural to seek for generalizations of Breuil's $\mathscr{L}$-invariants to $\mathrm{GL}_n(K)$ with $[K : \mathbb{Q}_p] < \infty$, namely to recover all Fontaine-Mazur $\mathscr{L}$-invariants of a semi-stable $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(E)$ from a certain locally analytic representation $\Pi(\rho)$ of $\mathrm{GL}_n(K)$. When ρ is crystalline with generic φ-eigenvalues, it is expected that (for $\mathrm{GL}_n(K) \neq \mathrm{GL}_2(\mathbb{Q}_p)$) $\Pi(\rho)$ has to involve some locally analytic representations which do not appear in any parabolic induction. Hence, an explicit construction of $\Pi(\rho)$ (which recovers all Fontaine-Mazur $\mathscr{L}$-invariants) is not yet available beyond the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. However, according to Breuil's Ext^1 conjecture in [Bre19], we do expect the existence of some explicit $\Pi(\rho)$ that recovers *all* Fontaine-Mazur $\mathscr{L}$-invariants of ρ, when ρ is semi-stable with maximal rank monodromy (namely $N^{n-1} \neq 0$).

We assume from now that ρ is semi-stable with $N^{n-1} \neq 0$. We also assume for simplicity in this introduction that ρ is *ordinary*, in which case ρ is up to a twist of the following form

$$\begin{pmatrix} \varepsilon^{n-1} & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \tag{1.1}$$

with $\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to E^\times$ being the p-adic cyclotomic character. In [Ding19], Ding has constructed locally analytic representations of $\mathrm{GL}_n(K)$ that recover those Fontaine-Mazur $\mathscr{L}$-invariants of ρ coming from a two dimensional subquotient of ρ, which are morally the extension parameters at simple roots. The next objects of interest are therefore generalizations of Breuil's $\mathscr{L}$-invariants that conjecturally corresponds to Fontaine-Mazur $\mathscr{L}$-invariants of ρ *at non-simple roots*, the so-called *higher $\mathscr{L}$-invariants*. Much work has been done towards higher $\mathscr{L}$-invariants and there are essentially two different approaches which are closely related. Following Breuil's Ext^1 conjecture in [Bre19], Breuil-Ding construct explicit locally analytic representations $\Pi^{\mathrm{BD}}(\rho)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ (by which we mean either $\Pi(D)^-$ or $\Pi(D)$ in (3.111) and (3.112) of *loc.it.* with $\lambda = 0$), establish their (p-adic) local-global compatibility (crucially using p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$) in [BD20], and then study their image under certain functor towards (φ, Γ)-modules in [BD19]. In another direction, Schraen has done extensive computations in his thesis [Schr11] and find higher $\mathscr{L}$-invariants inside certain (higher) Ext-groups between locally analytic generalized Steinberg representations. Afterwards, part of the author's thesis has connected Breuil-Ding and Schraen's approaches in an explicit way, and in particular leading to a new family of locally analytic representations $\Pi^{\mathrm{Q}}(\rho)$ (see (1.1) of [Qian21]) which contains $\Pi^{\mathrm{BD}}(\rho)$. Breuil-Ding's approach has the advantage of constructing explicit true representations (instead of objects in certain derived category), but its generalization to $\mathrm{GL}_n(K)$ is à priori difficult due to complexity of $\Pi^{\mathrm{BD}}(\rho)$ and $\Pi^{\mathrm{Q}}(\rho)$. Schraen's approach is certainly not producing explicit representations, but has the obvious advantage of involving only locally analytic generalized Steinberg representations in its formulation. So it seems fair to first generalize Schraen's approach, which is the main focus of this paper.

1.1. **Statement of the main results.** We only state our main result for $\mathrm{GL}_n(\mathbb{Q}_p)$ for simplicity. Let B (resp. B^+, resp. T) be the lower triangular Borel subgroup (resp. upper triangular Borel subgroup, resp. diagonal maximal torus) of $\mathrm{GL}_{n/\mathbb{Z}}$ and Δ be the set of positive simple roots with respect to the pair B^+, T. Then for each subset $I \subseteq \Delta$, we can attach a parabolic subgroup $P_I \supseteq B$ and a standard Levi subgroup $L_I \subseteq \mathrm{GL}_n$. Note that $I' \subseteq I$ if and only if $P_{I'} \subseteq P_I$. For each $I \subseteq \Delta$, we define

$$i_I^{\mathrm{an}} \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}_{P_I(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1_{L_I(\mathbb{Q}_p)}\right)^{\mathrm{an}}$$

and then

$$v_I^{\mathrm{an}} \stackrel{\mathrm{def}}{=} i_I^{\mathrm{an}} / \sum_{I' \supsetneq I} i_{I'}^{\mathrm{an}},$$

which is the so-called *locally analytic generalized Steinberg representation* attached to I. Note that $\mathrm{St}_n^{\mathrm{an}} \stackrel{\mathrm{def}}{=} v_\emptyset^{\mathrm{an}}$ is called the *locally analytic Steinberg representation* and v_Δ^{an} is the trivial representation 1_n of $\mathrm{GL}_n(\mathbb{Q}_p)$. Similarly, for each $I \subseteq \Delta$, we have a *smooth generalized Steinberg representation* v_I^∞ (with $\mathrm{St}_n^\infty \stackrel{\mathrm{def}}{=} v_\emptyset^\infty$) which can be identified with the set of smooth vectors of v_I^{an}. Each v_I^{an} clearly factors as a locally analytic representation of $\mathrm{PGL}_n(\mathbb{Q}_p)$. We write $\mathrm{Ext}^\bullet_{\mathrm{PGL}_n(\mathbb{Q}_p)}(-,-)$ for locally analytic Ext-groups between admissible locally analytic representations of $\mathrm{PGL}_n(\mathbb{Q}_p)$(see Section 1.4 for further details). For each pair of subsets $I' \subseteq I \subseteq \Delta$, we write

$$\mathbf{E}_{I,I'} \stackrel{\mathrm{def}}{=} \mathrm{Ext}^{\#I-\#I'}_{\mathrm{PGL}_n(\mathbb{Q}_p)}(v_I^{\mathrm{an}}, v_{I'}^{\mathrm{an}}), \tag{1.2}$$

and note that each triple $I'' \subseteq I' \subseteq I$ induces a cup product map

$$\mathbf{E}_{I,I'} \otimes \mathbf{E}_{I',I''} \xrightarrow{\cup} \mathbf{E}_{I,I''}. \tag{1.3}$$

The following is our main result

Theorem 1.1 (Theorem 4.22, Corollary 5.1)**.** *The* Ext*-groups (1.2) and cup product maps (1.3) satisfy the following properties.*

(i) for each $\alpha \in \Delta$, we have a canonical isomorphism $\mathbf{E}_{\{\alpha\},\emptyset} \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$;
(ii) for each $I' \subseteq I$, we have a canonical isomorphism $\mathbf{E}_{I,I'} \cong \mathbf{E}_{I\setminus I',\emptyset}$;
(iii) the map (1.3) is injective, and is an isomorphism if $\alpha+\beta$ is not a root for any $\alpha \in I \setminus I'$ and $\beta \in I' \setminus I''$;
(iv) the map (1.3) together with item (ii) fits into the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{E}_{I,I'} & \otimes & \mathbf{E}_{I',I''} & \xrightarrow{\cup} & \mathbf{E}_{I,I''} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\varepsilon \\
\mathbf{E}_{I^*,I''} & \otimes & \mathbf{E}_{I,I^*} & \xrightarrow{\cup} & \mathbf{E}_{I,I''}
\end{array}$$

with $I^* = I'' \cup (I \setminus I')$ *and* $\varepsilon = (-1)^{(\#I\setminus I')(\#I'\setminus I'')}$*;*

(v) The map

$$\bigoplus_{\emptyset \neq I' \subsetneq I} \mathbf{E}_{I,I'} \otimes \mathbf{E}_{I',\emptyset} \to \mathbf{E}_{I,\emptyset}$$

has a codimension one image if $\sum_{\alpha \in I} \alpha$ *is a root, and is surjective otherwise.*

Remark 1.2. (1) We have

$$\mathrm{Ext}^k_{\mathrm{PGL}_n(\mathbb{Q}_p)}(v_I^{\mathrm{an}}, v_{I'}^{\mathrm{an}}) = 0$$

for each $k < \#I - \#I'$, and thus $\mathbf{E}_{I,I'}$ is the bottom degree non-vanishing Ext-group for each $I' \subseteq I \subseteq \Delta$.

(2) The space $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ admits a basis $\{\mathrm{val}, \log\}$ where $\mathrm{val} : \mathbb{Q}_p^\times \to \mathbb{Z} \hookrightarrow E$ is p-adic valuation and log is any branch of the p-adic logarithm. In particular, $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ contains a canonical line $E\mathrm{val}$ spanned by val.

(3) In Corollary 5.1, we also study $\mathbf{E}'_{I,I'} \overset{\mathrm{def}}{=} \mathrm{Ext}^{\#I-\#I'+1}_{\mathrm{PGL}_n(\mathbb{Q}_p)}(v_I^{\mathrm{an}}, v_{I'}^{\mathrm{an}})$ as well as cup product map of the form

$$\mathbf{E}_{I,I'} \otimes \mathbf{E}'_{I',I''} \xrightarrow{\cup} \mathbf{E}'_{I,I''},$$

and Theorem 1.1 has a natural variant in this setting.

(4) When $n = 2$, Theorem 1.1 goes back to [Bre04]. The $n = 3$ case is proven by Schraen in [Schr11] as a key ingredient in his definition of (higher) $\mathscr{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$.

When $n = 2$, we have a natural exact sequence

$$v_\Delta^{\mathrm{an}} \hookrightarrow i_\emptyset^{\mathrm{an}} \twoheadrightarrow v_\emptyset^{\mathrm{an}}$$

which induces canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E) \cong \mathrm{Ext}^1_{\mathbb{Q}_p^\times}(1, 1) \cong \mathrm{Ext}^1_{\mathrm{PGL}_2(\mathbb{Q}_p)}(i_\emptyset^{\mathrm{an}}, i_\emptyset^{\mathrm{an}}) \cong \mathrm{Ext}^1_{\mathrm{PGL}_2(\mathbb{Q}_p)}(v_\Delta^{\mathrm{an}}, v_\emptyset^{\mathrm{an}}). \tag{1.4}$$

Consequently, given a E-line $W \subseteq \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$, Breuil attaches a representation V_W that fits into

$$v_\emptyset^{\mathrm{an}} \hookrightarrow V_W \twoheadrightarrow v_\Delta^{\mathrm{an}}. \tag{1.5}$$

Note that $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ contains a canonical E-line $E\mathrm{val}$, and $W \neq E\mathrm{val}$ if and only if V_W is uniserial if and only if the set of smooth vectors $V_W^\infty \subseteq V_W$ is St_2^∞ if and only if the corresponding $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(E)$ is semi-stable non-crystalline (namely $N \neq 0$).

Now we consider the case $n = 3$ which is first studied in [Schr11]. Similar to (1.4), we have canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E) \cong \mathrm{Ext}^1_{\mathrm{PGL}_3(\mathbb{Q}_p)}(v_\Delta^{\mathrm{an}}, v_{\Delta\setminus\{\alpha\}}^{\mathrm{an}}) \cong \mathrm{Ext}^1_{\mathrm{PGL}_3(\mathbb{Q}_p)}(v_{\{\alpha\}}^{\mathrm{an}}, v_\emptyset^{\mathrm{an}}), \tag{1.6}$$

and we write val_α for the element corresponding to val under (1.6) in the second and third space, for each $\alpha \in \Delta$. Consequently, each E-line $W_\alpha \subseteq \mathrm{Ext}^1_{\mathrm{PGL}_3(\mathbb{Q}_p)}(v_{\{\alpha\}}^{\mathrm{an}}, v_\emptyset^{\mathrm{an}})$ determines a representation V_{W_α} that fits into

$$v_\emptyset^{\mathrm{an}} \hookrightarrow V_{W_\alpha} \twoheadrightarrow v_{\{\alpha\}}^{\mathrm{an}}.$$

We assume that

$$W_\alpha \neq E\mathrm{val}_\alpha \tag{1.7}$$

for each $\alpha \in \Delta$. Then Schraen computes that

$$\mathrm{Ext}^2_{\mathrm{PGL}_3(\mathbb{Q}_p)}(v_\Delta^{\mathrm{an}}, V_{W_{\alpha_1}, W_{\alpha_2}}) \cong \mathbf{E}_{\Delta,\emptyset} / \left(\sum_{i=1}^{2} \mathbf{E}_{\Delta,\{\alpha_i\}} \cup W_{\alpha_i}\right) \tag{1.8}$$

is two-dimensional, where $V_{W_{\alpha_1},W_{\alpha_2}}$ is the amalgamate sum of $V_{W_{\alpha_1}}$ and $V_{W_{\alpha_2}}$ over $v_\emptyset^{\mathrm{an}}$ with $\Delta = \{\alpha_1, \alpha_2\}$. (Note that (1.7) for each $\alpha \in \Delta$ if and only if the set of smooth vectors $V^\infty_{W_{\alpha_1},W_{\alpha_2}} \subseteq V_{W_{\alpha_1},W_{\alpha_2}}$ is St_3^∞.) We abuse $\mathrm{val}_{\alpha_1} \cup \mathrm{val}_{\alpha_2}$ for its image in (1.8). Hence, he can choose a third E-line W_{α_3} (with $\alpha_3 = \alpha_1 + \alpha_2$ being the unique non-simple positive root) inside (1.8) satisfying

$$W_{\alpha_3} \neq E\mathrm{val}_{\alpha_1} \cup \mathrm{val}_{\alpha_2} \tag{1.9}$$

and then define a derived object which (morally) fits into the following distinguished triangle (see (1.14) of [Schr11])

$$V_{W_{\alpha_1},W_{\alpha_2}} \to V_{W_{\alpha_1},W_{\alpha_2},W_{\alpha_3}} \to v_\Delta^{\mathrm{an}}[-1] \to . \tag{1.10}$$

Then he uses (1.10) to pin down a filtered (φ, N)-module from the de Rham complex of the Drinfeld upper half space. Note that the filtered (φ, N)-module in [Schr11] depends on some non-canonical choices of normalization of parameters, and it is à priori unclear which choice gives the correct ρ that "corresponds to" $V_{W_{\alpha_1},W_{\alpha_2},W_{\alpha_3}}$ under p-adic Langlands correspondence. We slightly reformulate Schraen's result by replacing the three E-lines W_{α_1}, W_{α_2} and W_{α_3} by a single hyperplane $W \subseteq \mathbf{E}_{\Delta,\emptyset}$ which is the preimage of W_{α_3} under

$$\mathbf{E}_{\Delta,\emptyset} \twoheadrightarrow \mathbf{E}_{\Delta,\emptyset}/\left(\sum_{i=1}^{2} \mathbf{E}_{\Delta,\{\alpha_i\}} \cup W_{\alpha_i}\right) \cong \mathrm{Ext}^2_{\mathrm{PGL}_3(\mathbb{Q}_p)}(v_\Delta^{\mathrm{an}}, V_{W_{\alpha_1},W_{\alpha_2}}). \tag{1.11}$$

It is clear that the hyperplane $W \subseteq \mathbf{E}_{\Delta,\emptyset}$ is determined by W_{α_1}, W_{α_2} and W_{α_3}. Conversely, W_{α_3} is the image of W under (1.11), and W_{α_i} is uniquely characterized by the equality

$$E\mathrm{val}_{\alpha_{3-i}} \cup W_{\alpha_i} = W \cap (E\mathrm{val}_{\alpha_{3-i}} \cup \mathbf{E}_{\{\alpha_i\},\emptyset}) \tag{1.12}$$

for each $i = 1, 2$. In other words, we simplify Schraen's definition of $\mathscr{L}$-invariants by gluing W_{α_1}, W_{α_2} and W_{α_3} together to a single hyperplane $W \subseteq \mathbf{E}_{\Delta,\emptyset}$.

In general, as $\mathbf{E}_{\{\alpha\},\emptyset} \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ contains a canonical E-line $E\mathrm{val}_\alpha$ for each $\alpha \in I$, we obtain a canonical E-line $\mathbf{E}^\infty_{I,I'} \cong \mathbf{E}^\infty_{I\setminus I',\emptyset}$ by cup product of $E\mathrm{val}_\alpha$ coming from each $\alpha \in I \setminus I'$. Hence, we can define $\widehat{\mathbf{E}}_{I,I'} \subseteq \mathbf{E}_{\Delta,\emptyset}$ as the image of

$$\mathbf{E}^\infty_{\Delta,I} \otimes \mathbf{E}_{I,I'} \otimes \mathbf{E}^\infty_{I',\emptyset} \xrightarrow{\cup} \mathbf{E}_{\Delta,\emptyset}. \tag{1.13}$$

We are therefore lead to the following simple definition of Breuil-Schraen $\mathscr{L}$-invariants for general GL_n, based on Theorem 1.1

Definition 1.3. A *Breuil-Schraen $\mathscr{L}$-invariant* for GL_n is a hyperplane $W \subseteq \mathbf{E}_{\Delta,\emptyset}$ such that

- $W \cap \widehat{\mathbf{E}}_{I,I'} \subsetneq W$ for each $I' \subseteq I \subseteq \Delta$ (and we write $W_{I,I'} \subseteq \mathbf{E}_{I,I'}$ for the unique hyperplane such that $\mathbf{E}^\infty_{\Delta,I} \otimes W_{I,I'} \otimes \mathbf{E}^\infty_{I',\emptyset}$ is the preimage of W under (1.13));
- (1.3) induces an isomorphism
$$\mathbf{E}_{I,I'}/W_{I,I'} \otimes \mathbf{E}_{I',I''}/W_{I',I''} \xrightarrow{\sim} \mathbf{E}_{I,I''}/W_{I,I''}$$
for each $I'' \subseteq I' \subseteq I \subseteq \Delta$.

As the first (resp. the second) condition is an open condition (resp. a closed condition), the moduli space of Breuil-Schraen $\mathscr{L}$-invariants is given by an explicit locally closed subscheme of the projective space $\mathbf{P}(\mathbf{E}_{\Delta,\emptyset})$. We prove in Theorem 5.9 that this moduli space is non-canonically isomorphic to U^+, the unipotent radical of B^+. Note that U^+ is naturally isomorphic (via Fontaine's $\mathbf{D}_{\mathrm{st}}$) to the moduli space of ρ of the form (1.1) that satisfies $N^{n-1} \neq 0$. Compared to the approach of [Schr11], our definition of $\mathscr{L}$-invariants has the advantage of being simple and independent of choices of bases of various $\mathbf{E}_{I,I'}$. When $n=3$, the first condition reduces to $W_{\alpha_1} \neq E\mathrm{val} \neq W_{\alpha_2}$ and (1.9), and the second condition reduces to (1.12).

We conjecture that, for each Breuil-Schraen $\mathscr{L}$-invariant $W \subseteq \mathbf{E}_{\Delta,\emptyset}$, one should be able to define

- a locally analytic representation $\mathcal{W}$ of $\mathrm{PGL}_n(\mathbb{Q}_p)$ with $\dim_E \mathrm{Hom}_{\mathrm{PGL}_n(\mathbb{Q}_p)}(\mathrm{St}_n^{\mathrm{an}}, \mathcal{W}) = 1$ and any embedding $\mathrm{St}_n^{\mathrm{an}} \hookrightarrow \mathcal{W}$ induces a surjection
$$\mathbf{E}_{\Delta,\emptyset} = \mathrm{Ext}^{n-1}_{\mathrm{PGL}_n(\mathbb{Q}_p)}(1_n, \mathrm{St}_n^{\mathrm{an}}) \twoheadrightarrow \mathrm{Ext}^{n-1}_{\mathrm{PGL}_n(\mathbb{Q}_p)}(1_n, \mathcal{W})$$
with kernel W;
- a unique semi-stable $\rho_W : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_n(E)$ such that $\mathcal{W}$ embeds into the conjectural representation $\Pi(\rho_W)$ given by p-adic Langlands correspondence.

Remark 1.4. The existence of such $\mathcal{W}$ and ρ_W is known for $n \leq 3$ thanks to the following work.

- When $n=2$, we can simply take $\mathcal{W}$ to be V_W as in (1.5), which was firstly introduced by Breuil in [Bre04] (see the paragraph in *loc.it.* before Conjecture 1.1.1) as $\Sigma(2,\mathcal{L})$ (with $\mathcal{L} \in E$ explicitly determined by W). Let f be a cusp new form of weight 2 whose attached Galois representation is ρ_f. Assume that $\rho_{f,p} \overset{\mathrm{def}}{=} \rho_f|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ is semi-stable non-crystalline. In Corollaire 1.1.6 of [Bre10], Breuil proves that (up to an unramified twist) $\Sigma(2,\mathcal{L})$ (or rather its completion $B(2,\mathcal{L})$) embeds into the f-isotypic Hecke eigenspace inside the completed cohomology of a tower of modular curves, if and only if the equality $\mathscr{L}(f) = \mathcal{L}$ holds with $\mathscr{L}(f)$ being the Fontaine-Mazur $\mathscr{L}$-invariant attached to $\rho_{f,p}$. In other words, $\mathscr{L}(f)$ is determined by W and we can take $\rho_W = \rho_{f,p}$. Note that we can also take $\mathcal{W}$ to be another uniserial locally analytic representation V_W^+ of length four that fits into the short exact seqnece
$$V_W \hookrightarrow V_W^+ \twoheadrightarrow \widetilde{I}(s \cdot 0)$$
where $\widetilde{I}(s\cdot 0)$ is another irreducible locally analytic principal series (cf. the paragraph before Lemma 3.13 of [BD20]). In terms of terminology of [BD20], our V_W (resp. V_W^+) is isomorphic to $\pi(0,\psi)^-$ (resp. $\pi(0,\psi)$) with $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ satisfying $W = E\psi$ (see (3.23) and (3.27) of *loc.it.*). According to [Col10] and Theorem 1.2.1 of [Eme11], the locally analytic representation $\Pi(\rho_W)$ attached to ρ_W via p-adic local Langlands correspondence is actually isomorphic to V_W^+. Therefore V_W and V_W^+ are the only two possible choices of $\mathcal{W}$ here.
- When $n=3$, following his Ext^1-conjecture in [Bre19], Breuil constructs an explicit family of locally analytic representations (Théorème 1.2 of *loc.it.*) and proves a weaker version of local-global compatibility (Théorème 1.3 of *loc.it.*) for them, compared with the strong version in Conjecture 1.4 of *loc.it.*. We write D for the rank three (φ,Γ)-module (over Robba ring) associated with an ordinary Galois representation ρ of the form (1.1) with $n=3$. In [BD20], Breuil-Ding attach to D two locally analytic representations $\Pi(D)^- \subseteq \Pi(D)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ (see (3.111) and (3.112) of *loc.it.*). Then they prove Conjecture 1.4 of [Bre19] (see Theorem 1.1 of [BD20]) crucially using the p-adic local Langlands correspondence of $\mathrm{GL}_2(\mathbb{Q}_p)$. We write $\widetilde{\Pi}(D)^-$ (resp. $\widetilde{\Pi}(D)$) for the amalgamate sum of $\Pi(D)^-$ (resp. $\Pi(D)$) and $\mathrm{St}_3^{\mathrm{an}}$ over the length five subrepresentation with Jordan–Hölder factors St_3^∞, $C_{1,1}$, $C_{2,1}$, $\widetilde{C}_{1,2}$, $\widetilde{C}_{2,2}$ (see (3.111) of *loc.it.* for the illustration of this subrepresentation with each factor defined in paragraphs before it). Then we can combine Proposition 6.8, Theorem 7.1 of [Qian21] with Theorem 1.1 of [BD20] and obtain

$$\mathbf{E}_{\Delta,\emptyset} = \mathrm{Ext}^2_{\mathrm{PGL}_3(\mathbb{Q}_p)}(1_3, \mathrm{St}_3^{\mathrm{an}}) \twoheadrightarrow \mathrm{Ext}^2_{\mathrm{PGL}_3(\mathbb{Q}_p)}(1_3, \widetilde{\Pi}^-(D)) \cong \mathrm{Ext}^2_{\mathrm{PGL}_3(\mathbb{Q}_p)}(1_3, \widetilde{\Pi}(D)) \tag{1.14}$$

with the last two terms being one dimensional. Consequently, if we write W for the kernel of (1.14), then we can let $\mathcal{W}$ be either $\widetilde{\Pi}(D)^-$ or $\widetilde{\Pi}(D)$, and ρ_W be the ρ that defines D. The minimal possible choice of $\mathcal{W}$ (under inclusion between representations) is actually

$$\ker(\widetilde{\Pi}(D)^- \twoheadrightarrow v_{\{\alpha\}}^{\infty})$$

for any $\alpha \in \Delta$, and in particular is not unique. On the other hand, under further restriction on $\mathcal{W}$ specified in Conjecture 5.11, it is possible that there exists a unique maximal choice of $\mathcal{W}$.

We refer the conjectural realization of ρ_W inside Drinfeld upper half space to Conjecture 5.18.

Remark 1.5. It is natural to ask if Theorem 1.1 admits natural generalization to other reductive groups (with v_I^{an} defined exactly the same way for each $I \subseteq \Delta$). Indeed, most combinatorial construction in this paper works for more general reductive groups and many Ext-groups between v_I^{an} can be computed. However, for general group, it seems unclear how to extend the definition of Breuil-Schraen $\mathscr{L}$-invariants using cup products involving only Ext-groups between v_I^{an}.

Remark 1.6. The space $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, E)$ contains a unique canonical E-line which is $E\mathrm{val}$, so $\mathbf{E}_{I,I'}$ contains a unique canonical E-line $E\mathrm{val}_\alpha$ whenever $I \setminus I' = \{\alpha\}$. For general $I' \subseteq I \subseteq \Delta$, there exists a minimal lattice of *canonical subspaces* of $\mathbf{E}_{I,I'}$ generated by the image of the cup product of various $\mathbf{E}_{I'',I'''}$ (for some choices of $I' \subseteq I''' \subseteq I'' \subseteq I$) with various $E\mathrm{val}_\alpha$ satisfying $\alpha \in I \setminus I'$. There exists an even more interesting filtration on $\mathbf{E}_{I,I'}$ induced from the layer structure of $v_{I'}^{\mathrm{an}}$ as a finite length admissible locally analytic representation. We will study this second filtration as well as its relative position with respect to the aforementioned lattice of canonical subspaces in a forthcoming work, which should lead to a proof of Conjecture 5.18 which generalizes Théorème 6.23 and Remarque 6.24 of [Schr11]. These computations also shed light on the construction (in the GL_n case) of locally analytic representations in Breuil's Ext^1 conjecture as well as those in Theorem 1.1 of [Qian21].

1.2. **Organization of the paper.** In Section 2, we systematically compute various locally analytic cohomology groups of locally analytic generalized Steinberg representations (LAGS for short). As each LAGS is derived equivalent to a complex of (direct sum of) parabolically induced principal series, we reduce our problem to the computation of a spectral sequence called $E_{\bullet,I_0,I_1}^{\bullet,\bullet}$ (see Section 2.2). The first page $E_{1,I_0,I_1}^{\bullet,\bullet}$ can be easily written down with an explicit basis using results from Section 2.1, so our main task is to explicitly compute the second page $E_{2,I_0,I_1}^{\bullet,\bullet}$ and show that the spectral sequence actually degenerates at its second page. To achieve this, we construct an explicit (and much smaller) combinatorial subcomplex $E_{1,I_0,I_1,\diamond}^{\bullet,k}$ of $E_{1,I_0,I_1}^{\bullet,k}$ for each $k \in \mathbb{Z}$, based on the notion of (I_0, I_1)-*atomic tuples* (see Definition 2.13). The key technical ingredient is to show that the $E_{1,I_0,I_1,\diamond}^{\bullet,k} \to E_{1,I_0,I_1}^{\bullet,k}$ is actually a quasi-isomorphism (see Proposition 2.23), which reduces our problem to the computation of cohomology of the complex $E_{1,I_0,I_1,\diamond}^{\bullet,k}$ for each $k \in \mathbb{Z}$. In Section 2.5, we complete the proof of Theorem 2.28 which is the main result of Section 2. In Section 2.6, we consider a variant of (I_0, I_1)-atomic tuples which will be crucially used in Section 4.3.

In Section 3, we prove a decomposition theorem for certain space of locally analytic distributions in Proposition 3.9, which is technically fundamental for computation of N-homology (with N being unipotent radical of some parabolic subgroup) of locally analytic representations using Bruhat decomposition. This generalizes results from Section 4.5 of [Schr11].

In Section 4, we compute various (higher) Ext-groups between LAGS. Using results from Section 3, we prove a generalization of Bernstein–Zelevinsky geometric lemma (see Section 2.12 of [BZ77]) to locally analytic setting, and in particular prove some vanishing results on N-(co)homology in Lemma 4.14 and Lemma 4.15. The main result of Section 4.1 is Proposition 4.16, which computes all (higher) Ext-groups between all i_I^{an}. Proposition 4.16 allows us to compute Ext-groups between LAGS by a spectral sequence, which turns out to be among the ones already treated in Section 2, so Theorem 4.22 easily follows. Section 4.3 is devoted to explicit computation of the cup product map using the commutative diagram (4.20) and (4.23), which ends up with Theorem 4.31.

In Section 5, we define and study what we call *Breuil-Schraen $\mathscr{L}$-invariants*. In particular, we propose Conjecture 5.11 that relate them to Breuil-Ding's approach of higher $\mathscr{L}$-invariants, and propose Conjecture 5.18 which generalizes [Schr11] on potential realization of $\mathscr{L}$-invariants inside the de Rham complex of Drinfeld upper half spaces. Some expectation on this realization as well as its relation to other objects are sketched in Remark 5.19.

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1.4. **Notation and preliminary.** Throughout this paper, p is prime number and K is a finite extension of $\mathbb{Q}_p$. We fix another p-adic field E as well as an embedding $E \hookrightarrow \overline{\mathbb{Q}_p}$ such that each embedding $\iota : K \hookrightarrow \overline{\mathbb{Q}_p}$ factors through it. Note that we write S for the set of all embeddings $\iota : K \hookrightarrow E$ and identify it with the set of all embeddings $K \hookrightarrow \overline{\mathbb{Q}_p}$ via the fixed embedding $E \hookrightarrow \overline{\mathbb{Q}_p}$.

We recall some standard facts on (admissible) locally analytic representations of p-adic analytic groups. Let G be a p-adic analytic group and $D(G) \overset{\mathrm{def}}{=} D(G,E)$ be the space of locally analytic distributions on G (see [ST03] for its definition). We write $\mathrm{Mod}_{D(G)}$ for the abelian category of (abstract) $D(G)$-modules, and $\mathcal{M}(G) \overset{\mathrm{def}}{=} D(\mathrm{Mod}_{D(G)})$ for its derived category. Given two bounded strict chain complexes of admissible ($\mathbb{Q}_p$-)locally analytic representations $\mathbf{C}_1$, $\mathbf{C}_2$ of G, we consider their strong dual $\mathbf{C}_1'$, $\mathbf{C}_2'$, and then define the locally analytic Ext-groups

$$\mathrm{Ext}_G^\bullet(\mathbf{C}_1,\mathbf{C}_2) \overset{\mathrm{def}}{=} \mathrm{Ext}^\bullet_{\mathcal{M}(G)}(\mathbf{C}_2',\mathbf{C}_1')$$

by Ext in the derived category $\mathcal{M}(G)$. In particular, we use the notation

$$H^\bullet(G,\Pi) \overset{\mathrm{def}}{=} \mathrm{Ext}_G^\bullet(1,\Pi)$$

for each admissible locally analytic representation Π of G.

Let $\mathbf{C}_1 = [\mathbf{C}_1^\ell]_{\ell\in\mathbb{Z}}, \mathbf{C}_2 = [\mathbf{C}_2^\ell]_{\ell\in\mathbb{Z}}$ be two bounded strict chain complexes of admissible locally analytic representations, and we abuse $\mathbf{C}_1,\mathbf{C}_2$ for the corresponding objects in $\mathcal{M}(G)$. For each $k,\ell \in \mathbb{Z}$, we can consider $\mathrm{Ext}_G^k(\mathbf{C}_1,\mathbf{C}_2^\ell)$ as well as $\mathrm{Ext}_G^k(\mathbf{C}_1^\ell,\mathbf{C}_2)$.

Lemma 1.7. *(i) There exists a spectral sequence converge to* $\mathrm{Ext}_G^\bullet(\mathbf{C}_1,\mathbf{C}_2)$ *whose first page has k-th row given by*

$$\cdots \to \mathrm{Ext}_G^k(\mathbf{C}_1,\mathbf{C}_2^\ell) \to \mathrm{Ext}_G^k(\mathbf{C}_1,\mathbf{C}_2^{\ell+1}) \to \cdots .$$

(ii) *There exists a spectral sequence converge to* $\mathrm{Ext}^\bullet_G(\mathbf{C}_1,\mathbf{C}_2)$ *whose first page has k-th row given by*

$$\cdots\to\mathrm{Ext}^k_G(\mathbf{C}_1^\ell,\mathbf{C}_2)\to\mathrm{Ext}^k_G(\mathbf{C}_1^{\ell-1},\mathbf{C}_2)\to\cdots.$$

Proof. This is standard inside the derived category $\mathcal{M}(G)$. □

Now assume that G is the set of $\mathbb{Q}_p$-points of a p-adic reductive group, and $P\subseteq G$ is a parabolic subgroup with unipotent radical N_P and Levi quotient L.

Lemma 1.8. *Let* $M\in\mathrm{Mod}_{D(G)}$ *and* $M_L\in\mathrm{Mod}_{D(L)}$ *with* M_L *satisfying the (FIN) condition in Section 6 of* [ST05]. *Then we have the following spectral sequence*

$$\mathrm{Ext}^{k_1}_{D(L)}(M_L,H^{k_2}(N_P,M))\Rightarrow\mathrm{Ext}^{k_1+k_2}_{D(G)}(D(G)\otimes_{D(P)}M_L,M).$$

Proof. This is equation (43) of [Bre19]. □

Let $n\geq 1$ be an integer. We set $G_n\overset{\mathrm{def}}{=}\mathrm{GL}_{n/K}$ and let B_n (resp. B_n^+, resp. T_n) be its lower triangular Borel subgroup (resp. its upper triangular Borel subgroup, resp. its diagonal maximal torus). We consider the set Δ_n of positive simple roots and fix a bijection $\Delta_n\cong\{1,2,\dots,n-1\}$ which sends $(i,i+1)$ to i for each $1\leq i\leq n-1$. There is a natural bijection between the set of subsets $I\subseteq\Delta_n$ and parabolic subgroups $P_{n,I}\subseteq G_n$ containing B_n (with $P_{n,\Delta_n}=G_n$ and $P_{n,\emptyset}=B_n$). We write $L_{n,I}\subseteq P_{n,I}$ for the standard Levi subgroup. Let Z_n be the center of G_n and $Z_{n,I}$ be the center of $L_{n,I}$. We set $\overline{G}_n\overset{\mathrm{def}}{=}G_n/Z_n$, $\overline{L}_{n,I}\overset{\mathrm{def}}{=}L_{n,I}/Z_n$ and $\overline{Z}_{n,I}\overset{\mathrm{def}}{=}Z_{n,I}/Z_n$ for each $I\subseteq\Delta_n$. We use the notation $\mathfrak{g}_n$ (resp. $\overline{\mathfrak{g}}_n$, resp. $\mathfrak{l}_{n,I}$, resp. $\overline{\mathfrak{l}}_{n,I}$) for the Lie algebra associated with G_n (resp. $\overline{G}_n$, resp. $L_{n,I}$, resp. $\overline{L}_{n,I}$).

Throughout this paper, we abuse the notation for group schemes (resp. Lie algebras) for their set of K-points, and therefore view them as p-adic Lie groups (resp. p-adic Lie algebras). For a Lie algebra $\mathfrak{g}$ over K, we set $\mathfrak{g}_{E,\iota}\overset{\mathrm{def}}{=}\mathfrak{g}\otimes_{K,\iota}E$ for each embedding $\iota:K\hookrightarrow E$, and note that $\mathfrak{g}_E\overset{\mathrm{def}}{=}\mathfrak{g}\otimes_{\mathbb{Q}_p}E\cong\prod_{\iota:K\hookrightarrow E}\mathfrak{g}_{E,\iota}$. We write $G_{n,E}$ for the base change of $\mathrm{Res}_{K/\mathbb{Q}_p}G_n$ to E, and similarly for other groups above.

For each p-adic reductive group G containing a parabolic subgroup P, we write $(\mathrm{Ind}_P^G\cdot)^{\mathrm{an}}$ (resp. $(\mathrm{Ind}_P^G\cdot)^\infty$) for the locally analytic (resp. smooth) parabolic induction functor. For each pair of subsets $I\subseteq I'\subseteq\Delta_n$, we set

$$i^{\mathrm{an}}_{n,I,I'}(\pi_I)\overset{\mathrm{def}}{=}\left(\mathrm{Ind}^{L_{n,I'}}_{P_{n,I}\cap L_{n,I'}}\pi_I\right)^{\mathrm{an}}$$

for each locally analytic representation π_I of $L_{n,I}$ and note that $i^{\mathrm{an}}_{n,I,I'}(\pi_I)$ is admissible if π_I is. If π_I is furthermore smooth, we define $i^\infty_{n,I,I'}(\pi_I)$ similarly.

Let $\lambda\in X(T_{n,E})$ be a weight which is dominant with respect to $B^+_{n,E}$. For each $I\subseteq\Delta_n$, we write $F_{n,I}(\lambda)$ for the algebraic representation of $L_{n,I,E}$ with highest weight λ (with respect to $B^+_{n,E}$), which induces a finite dimensional E-representation of the p-adic Lie group $L_{n,I}$. We define $i^\infty_{n,I}\overset{\mathrm{def}}{=}i^\infty_{n,I,\Delta_n}(1_{\overline{L}_{n,I}})$ and a *smooth generalized Steinberg representation* $V^\infty_{n,I}\overset{\mathrm{def}}{=}i^\infty_{n,I}/\sum_{I\subsetneq I'\subseteq\Delta_n}i^\infty_{n,I'}$ for each $I\subseteq\Delta_n$. We define $i^{\mathrm{alg}}_{n,I}(\lambda)\overset{\mathrm{def}}{=}F_{n,\Delta_n}(\lambda)\otimes_E i^\infty_{n,I}$ and the *locally algebraic generalized Steinberg representation* $V^{\mathrm{alg}}_{n,I}(\lambda)\overset{\mathrm{def}}{=}F_{n,\Delta_n}(\lambda)\otimes_E V^\infty_{n,I}$. For each $I\subseteq\Delta_n$, we know that $V^\infty_{n,I}$ and thus $V^{\mathrm{alg}}_{n,I}(\lambda)$ is irreducible. We write $\mathrm{Ext}^\bullet_{G_n,\lambda}$ for the Ext-groups fixing the central character which equals that of $F_{n,\Delta_n}(\lambda)$ (see Remarque 5.1.3 of [Bre19] for similar notation).

Similarly, we define

$$V^{\mathrm{an}}_{n,I,I'}(\lambda) \overset{\mathrm{def}}{=} i^{\mathrm{an}}_{n,I,I'}(F_{n,I}(\lambda))/\sum_{I\subsetneq I''\subseteq I'} i^{\mathrm{an}}_{n,I,I''}(F_{n,I''}(\lambda)) \tag{1.15}$$

for each $I \subseteq I' \subseteq \Delta_n$. In particular, we have $i^{\mathrm{an}}_{n,I}(\lambda) \overset{\mathrm{def}}{=} i^{\mathrm{an}}_{n,I,\Delta_n}(F_{n,I}(\lambda))$ and a *locally analytic generalized Steinberg representation* $V^{\mathrm{an}}_{n,I}(\lambda) \overset{\mathrm{def}}{=} V^{\mathrm{an}}_{n,I,\Delta_n}(\lambda)$ for each $I \subseteq \Delta_n$ with $\mathrm{St}^{\mathrm{an}}_n(\lambda) \overset{\mathrm{def}}{=} V^{\mathrm{an}}_{n,I}(\lambda)$. It is clear that $V^{\mathrm{an}}_{n,\Delta_n}(\lambda) \cong F_{n,\Delta_n}(\lambda)$.

We have a natural embedding $i^{\mathrm{alg}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I}(\lambda)$, which induces an embedding $V^{\mathrm{alg}}_{n,I}(\lambda) \hookrightarrow V^{\mathrm{an}}_{n,I}(\lambda)$ that identifies $V^{\mathrm{alg}}_{n,I}(\lambda)$ with the locally algebraic vectors of $V^{\mathrm{an}}_{n,I}(\lambda)$ (as we know the set of Jordan–Hölder factors (with multiplicity) of $V^{\mathrm{an}}_{n,I}(\lambda)$ thanks to [OS13] and [OS15]). We emphasize that, for each $I' \subseteq I \subseteq \Delta_n$, we have a distinguished embedding $\kappa^{\infty}_{I,I'} : i^{\infty}_{n,I} \hookrightarrow i^{\infty}_{n,I'}$, (resp. $\kappa^{\mathrm{alg}}_{I,I'}(\lambda) : i^{\mathrm{alg}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{alg}}_{n,I'}(\lambda)$, resp. $\kappa^{\mathrm{an}}_{I,I'}(\lambda) : i^{\mathrm{an}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I'}(\lambda)$) that sends various locally constant (resp. locally algebraic, resp. locally analytic) functions on $\overline{G}_n$ to themselves (concerning the definition of parabolic induction).

2. Cohomology of generalized Steinberg

In this section, we compute cohomologies of locally analytic generalized Steinberg representations via a careful combinatorial study of the first and second pages of a spectral sequence $E^{\bullet,\bullet}_{\bullet,I_0,I_1}$ (for some $I_0 \subseteq I_1 \subseteq \Delta_n$) induced from the locally analytic Tits complex $\mathbf{C}_{I_0,I_1}(\lambda)$ (see (2.10) for its definition and [OS13] for this terminology). The key combinatorial notion is that of (I_0, I_1)-atomic tuples (see Definition 2.13), and the main results of this section are Proposition 2.23 and Theorem 2.28.

2.1. **Standard bases of group cohomology.** Let $\mathfrak{g}$ be a (split) reductive lie algebra. It follows from Théorème 9.3 of [Kos50] that we have an isomorphism of graded algebra

$$H_*(\mathfrak{g}, 1) \cong (\wedge^*\mathfrak{g})^{\mathfrak{g}}.$$

Let $D(\mathfrak{g})$ be the subspace of $(\wedge^*\mathfrak{g})^{\mathfrak{g}}$ spanned by elements of the form $u \wedge v$ (namely decomposable). Note that we have a pairing $H_*(\mathfrak{g}, 1) \times H^*(\mathfrak{g}, 1)$ and let $P(\mathfrak{g})'$ be the subspace of $H^*(\mathfrak{g}, 1)$ given by the orthogonal complement of $D(\mathfrak{g})$. We set $P^k(\mathfrak{g}) \overset{\mathrm{def}}{=} P(\mathfrak{g})' \cap H^k(\mathfrak{g}, 1)$ for each $k \geq 0$.

Theorem 2.1 (Koszul, [Kos50]). *We have an isomorphism of graded algebra*

$$\wedge^* (P(\mathfrak{g})') \cong H^*(\mathfrak{g}, 1_{\mathfrak{g}}). \tag{2.1}$$

Moreover, (2.1) is functorial in the following sense: for each morphism of (split) reductive lie algebra $\varphi : \mathfrak{h} \to \mathfrak{g}$, we have an induced morphism $\varphi^* : P(\mathfrak{g})' \to P(\mathfrak{h})'$ which determines the morphism $\varphi^* : H^*(\mathfrak{g}, 1_{\mathfrak{g}}) \to H^*(\mathfrak{h}, 1_{\mathfrak{h}})$ completely, via (2.1).

The following classical theorem reduces various group cohomologies to Lie algebra cohomologies.

Theorem 2.2 (Casselman–Wigner, [CW74]). *The canonical morphism*

$$H^*(G, 1_G) \to H^*(\mathfrak{g}, 1_{\mathfrak{g}})$$

is an isomorphism of graded algebra if G is the set of $\mathbb{Q}_p$-points of a semisimple group over $\mathbb{Q}_p$.

Note that we do not require the semisimple group over $\mathbb{Q}_p$ in Theorem 2.2 to be split. Also note that similar results hold for compact p-adic analytic groups due to [La65].

We recall K, E, $S = \{\iota : K \hookrightarrow E\}$ and other notation from 1.4. For each $I \subseteq \Delta_n$, we set $r_I \stackrel{\text{def}}{=} n - \#I$ and there exists a partition $n_I = (n_d)_{1\leq d\leq r_I}$ of n such that we have a decomposition into Levi blocks

$$L_{n,I} \cong G_{n_1} \times G_{n_2} \times \cdots \times G_{n_{r_I}}$$

which induces

$$\overline{L}_{n,I} \cong \overline{G}_{n_1} \times \overline{G}_{n_2} \times \cdots \times \overline{G}_{n_{r_I}} \times \overline{Z}_{n,I}. \tag{2.2}$$

with $\overline{Z}_{n,I} \cong (K^\times)^{r_I-1}$.

For each $I \subseteq \Delta_n$ and each $1 \leq d \leq r_I$, we have an embedding

$$\overline{\mathfrak{g}}_{n_d} \hookrightarrow \overline{\mathfrak{l}}_{n,I} \hookrightarrow \overline{\mathfrak{g}}_n \tag{2.3}$$

which induces an embedding $\overline{\mathfrak{g}}_{n_d,E} \hookrightarrow \overline{\mathfrak{g}}_{n,E}$ and therefore a morphism

$$\mathrm{Res}^{k,d}_{n,I} :\ P^k(\overline{\mathfrak{g}}_{n,E}) \to P^k(\overline{\mathfrak{g}}_{n_d,E}). \tag{2.4}$$

The embedding $\overline{\mathfrak{g}}_{n_d,E} \hookrightarrow \overline{\mathfrak{g}}_{n,E}$ is the direct sum of $\overline{\mathfrak{g}}_{n_d,E,\iota} \hookrightarrow \overline{\mathfrak{g}}_{n,E,\iota}$ for each $\iota \in S$, and thus (2.4) decomposes into direct sum of

$$\mathrm{Res}^{k,d}_{n,I,\iota} :\ P^k(\overline{\mathfrak{g}}_{n,E,\iota}) \to P^k(\overline{\mathfrak{g}}_{n_d,E,\iota})$$

for all $\iota \in S$.

The following result on primitive classes for Lie algebra cohomology is well-known.

Theorem 2.3. *For each $n \geq 2$ and $\iota \in S$, we have $\dim_E P^k(\overline{\mathfrak{g}}_{n,E,\iota}) = 1$ if $k = 2m-1$ for some $2 \leq m \leq n$ and $P^k(\overline{\mathfrak{g}}_{n,E,\iota}) = 0$ otherwise. Moreover, the morphism $\mathrm{Res}^{k,d}_{n,I,\iota}$ is an isomorphism if $k = 2m-1$ for some $2 \leq m \leq n_d$ and is zero otherwise.*

Lemma 2.4. *The map $\mathrm{Res}^{k,d}_{n,I,\iota}$ depends only on n, k, n_d and ι.*

Proof. We fix n, $n' \leq n$, $3 \leq k \leq 2n'-1$, and $\iota \in S$ throughout the proof. Let f_1 be the embedding

$$\overline{\mathfrak{g}}_{n',E} \hookrightarrow \overline{\mathfrak{l}}_{n,I,E} \hookrightarrow \overline{\mathfrak{g}}_{n,E}$$

when $I = \{1, \dots, n'-1\}$ and $d = 1$. We consider the adjoint action of $\overline{G}_n$ on $\overline{\mathfrak{g}}_{n,E}$ and set $f_g \stackrel{\text{def}}{=} \mathrm{Ad}(g)(f_1) : \overline{\mathfrak{g}}_{n',E} \hookrightarrow \overline{\mathfrak{g}}_{n,E}$. For each $g \in \overline{G}_n$, f_g induces a map $f_g^* : P^k(\overline{\mathfrak{g}}_{n,E,\iota}) \to P^k(\overline{\mathfrak{g}}_{n',E,\iota})$ which is an isomorphism between one dimensional spaces thanks to Theorem 2.3. Consequently, there exists a unique morphism $\chi : \overline{G}_n \to \mathbb{G}_m$ such that $f_g^* = \chi(g) f_1^*$ for each $g \in \overline{G}_n$. As $\overline{G}_n$ is adjoint, χ must be constant and thus $f_g^* = f_1^*$ for each $g \in \overline{G}_n$. We finally notice that the map $\mathrm{Res}^{k,d}_{n,I,\iota}$ constructed from different choices of I (with the same n, k, $n_d = n'$ and ι) are just f_g^* for different choices of g, and therefore are all equal. $\square$

For each $n \geq 2$, $k \geq 3$ and $\iota \in S$ satisfying $P^k(\overline{\mathfrak{g}}_{n,E,\iota}) \neq 0$, we choose a basis $\{v^k_{n,\iota}\}$ of $P^k(\overline{\mathfrak{g}}_{n,E,\iota})$ such that they are compatible under all morphisms of the form $\mathrm{Res}^{k,d}_{n,I}$. For technical convenience, we set $v^0_n \stackrel{\text{def}}{=} 1 \in H^0(\mathfrak{g}_n, 1_{\mathfrak{g}_n})$ for each $n \geq 1$. For each $n \geq 2$ and $k \geq 3$, we define $\Sigma_{n,k}$ as the set of subsets

$$\Lambda = \{(m_1, \iota_1), \dots, (m_r, \iota_r)\} \subseteq \{3, 5, \dots, 2n-3, 2n-1\} \times S$$

for some $r \geq 1$ such that $\sum_{s=1}^r m_s = k$. We define P^Λ_n as the one dimensional subspace of $\wedge^* \left(P(\overline{\mathfrak{g}}_{n,E})'\right)$ given by

$$P^{m_1}(\overline{\mathfrak{g}}_{n,E,\iota_1}) \wedge \cdots \wedge P^{m_r}(\overline{\mathfrak{g}}_{n,E,\iota_r})$$

for each $\Lambda\in\Sigma_{n,k}$, which clearly does not depend on the choice of order on Λ. For each $\Lambda\in\Sigma_{n,k}$, we write $\widetilde{\Lambda}$ for an enhancement of Λ to an ordered set $\{(m_1,\iota_1),\cdots,(m_r,\iota_r)\}$, and then set $v_n^{\widetilde{\Lambda}}\stackrel{\mathrm{def}}{=}v_{n,\iota_1}^{m_1}\wedge\cdots\wedge v_{n,\iota_r}^{m_r}$. So different $v_n^{\widetilde{\Lambda}}$ for the same Λ differ by an explicit sign. For each $n\geq 1$, we also set $\Sigma_{n,0}\stackrel{\mathrm{def}}{=}\{\emptyset\}$ and $v_n^{\emptyset}\stackrel{\mathrm{def}}{=}v_n^0=1\in H^0(\overline{\mathfrak{g}}_{n,E},1_{\overline{\mathfrak{g}}_{n,E}})$ for later convenience.

Corollary 2.5. *We have canonical isomorphisms*

$$H^k(\overline{G}_n,1_{\overline{G}_n})\cong H^k(\overline{\mathfrak{g}}_n,1_{\overline{\mathfrak{g}}_n})\cong H^k(\overline{\mathfrak{g}}_{n,E},1_{\overline{\mathfrak{g}}_{n,E}})\cong\bigoplus_{\Lambda\in\Sigma_{n,k}}P_n^{\Lambda}\tag{2.5}$$

for each $k\geq 0$ (with $1_{\overline{G}_n}$ and $1_{\overline{\mathfrak{g}}_n}$ understood to be E-representation of $\overline{G}_n$ and $\overline{\mathfrak{g}}_n$ respectively). In particular, $H^k(\overline{G}_n,1_{\overline{G}_n})=H^k(\overline{\mathfrak{g}}_n,1_{\overline{\mathfrak{g}}_n})=0$ if $k>(n^2-1)[K:\mathbb{Q}_p]$.

Proof. The first isomorphism follows from Theorem 2.2. The second isomorphism follows from Theorem 2.1 and Theorem 2.3. The third isomorphism follows from the Chevalley-Eilenberg complexes that compute the source and the target. The vanishing is clear from $\dim\overline{G}_n=\dim\overline{\mathfrak{g}}_n=(n^2-1)[K:\mathbb{Q}_p]$, and can also be seen from the fact that

$$n^2-1=\sum_{s=2}^{n}2s-1$$

which implies that $\Sigma_{n,k}=\emptyset$ if $k>(n^2-1)[K:\mathbb{Q}_p]$. □

We will use without explanation the isomorphisms in Theorem 2.1 and Theorem 2.2 and view P_n^{Λ} as a subspace of $H^k(\overline{G}_n,1_{\overline{G}_n})$ from now on.

Lemma 2.6. *We have a canonical isomorphism*

$$H^k(\overline{Z}_{n,I},1_{\overline{Z}_{n,I}})\cong\wedge^k(\mathrm{Hom}(\overline{Z}_{n,I},E))$$

for each $n\geq 1$, $I\subseteq\Delta_n$ and $k\geq 0$.

Proof. This is standard as $\overline{Z}_{n,I}$ is abelian (cf. Corollaire 3.11 of [Schr11]). □

Note that we have an embedding $\overline{Z}_{n,\Delta_n\setminus\{i\}}\hookrightarrow\overline{Z}_{n,I}$ for each $i\in\Delta_n\setminus I$, which actually induces an isomorphism

$$\overline{Z}_{n,I}\cong\prod_{i\in\Delta_n\setminus I}\overline{Z}_{n,\Delta_n\setminus\{i\}}.\tag{2.6}$$

Let $\mathrm{val}:K^\times\to\mathbb{Z}\hookrightarrow E$ be the p-adic valuation function with $\mathrm{val}(p)=1$. We fix a choice of p-adic logarithm $\log:\overline{\mathbb{Q}_p}^\times\to E$ satisfying $\log(p)=0$ and write $\log_\iota\stackrel{\mathrm{def}}{=}\log\circ\iota$ for each $\iota\in S$, so that $\{\mathrm{val}\}\sqcup\{\log_\iota\mid\iota\in S\}$ forms a basis of $\mathrm{Hom}(K^\times,E)$. Using the standard isomorphism $\overline{Z}_{n,\Delta_n\setminus\{i\}}\cong K^\times$, we obtain a basis $\mathcal{B}_{n,\Delta_n\setminus\{i\}}\stackrel{\mathrm{def}}{=}\{\mathrm{val}_i\}\sqcup\{\log_{i,\iota}\mid\iota\in S\}$ of $\mathrm{Hom}(\overline{Z}_{n,\Delta_n\setminus\{i\}},E)$. Hence, we deduce from (2.6) that $\mathcal{B}_{n,I}\stackrel{\mathrm{def}}{=}\bigsqcup_{i\in\Delta_n\setminus I}\mathcal{B}_{n,\Delta_n\setminus\{i\}}$ forms a basis of $\mathrm{Hom}(\overline{Z}_{n,I},E)$. We fix a total order on $\mathcal{B}_{n,\emptyset}$ which induces a total order on $\mathcal{B}_{n,I}\subseteq\mathcal{B}_{n,\emptyset}$ for each $I\subseteq\Delta_n$. We write $v\subseteq\mathcal{B}_{n,I}$ for a subset, which together with the fixed total order determines a unique element of $\wedge^k\mathrm{Hom}(\overline{Z}_{n,I},E)$. Through this way, the set of subsets of $\mathcal{B}_{n,I}$ with cardinality k naturally corresponds to a basis of $\wedge^k\mathrm{Hom}(\overline{Z}_{n,I},E)$. Note that each $v\subseteq\mathcal{B}_{n,I}$ determines a unique maximal $I_v\subseteq\Delta_n$ containing I such that $v\subseteq\mathcal{B}_{n,I_v}$. From now on, we will abuse the notation v for both a subset of $\mathcal{B}_{n,I}$ and the corresponding element of $\wedge^{\#v}\mathrm{Hom}(\overline{Z}_{n,I},E)$ (using the fixed total order on $\mathcal{B}_{n,I}$).

Lemma 2.7. *The isomorphism (2.2) induces an isomorphism*

$$H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \cong \bigoplus_{k_0+k_1+\cdots+k_{r_I}=k} \wedge^{k_0}(\mathrm{Hom}(\overline{Z}_{n,I}, E)) \otimes_E \bigotimes_{d=1}^{r_I} H^{k_d}(\overline{G}_{n_d}, 1_{\overline{G}_{n_d}}).$$

Proof. This is simply Kunneth formula (cf. Théorème 3.10 of [Schr11]) combined with Lemma 2.6. □

For each $i \in \Delta_n$, the embedding $L_{n,\Delta\setminus\{i\}} \subseteq G_n$ induces a canonical morphism

$$\mathrm{Res}^k_{n,i}:\ H^k(\overline{G}_n, 1_{\overline{G}_n}) \to H^k(\overline{L}_{n,\Delta_n\setminus\{i\}}, 1_{\overline{L}_{n,\Delta_n\setminus\{i\}}}).$$

For each $\Lambda \in \Sigma_{n,k}$ and $i \in \Delta_n$, we write $\mathbf{D}_{n,\Lambda,i}$ for those subsets $\Lambda' \subseteq \Lambda$ satisfying $\max\{m \mid (m,\iota) \in \Lambda'\} \leq 2i-1$ and $\max\{m \mid (m,\iota) \in \Lambda \setminus \Lambda'\} \leq 2(n-i)-1$. Note that both Λ' and $\Lambda \setminus \Lambda'$ are allowed to be empty. In the following, we fix a total order on S, and then fix the choice of enhancement $\widetilde{\Lambda}$ to be the one with decreasing order on integers and satisfying $(m,\iota) < (m',\iota')$ whenever $\iota < \iota'$, and write v_n^{Λ} instead of $v_n^{\widetilde{\Lambda}}$.

Lemma 2.8. *Let $\Lambda \in \Sigma_{n,k}$ be a partition and $\widetilde{\Lambda}$ be an enhancement of Λ. Then $\mathrm{Res}^k_{n,i}(P_n^{\Lambda}) \neq 0$ if and only if $\mathbf{D}_{n,\Lambda,i} \neq \emptyset$. Moreover, we have*

$$\mathrm{Res}^k_{n,i}(v_n^{\Lambda}) = \sum_{\Lambda' \in \mathbf{D}_{n,\Lambda,i}} \varepsilon(\Lambda') v_i^{\Lambda'} \otimes_E v_{n-i}^{\Lambda\setminus\Lambda'}.$$

Proof. This follows from Theorem 2.1 and Theorem 2.3. □

More generally, for each $I' \subseteq I \subseteq \Delta_n$, the embedding $L_{n,I'} \subseteq L_{n,I}$ induces a canonical morphism

$$\mathrm{Res}^k_{n,I,I'}:\ H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \to H^k(\overline{L}_{n,I'}, 1_{\overline{L}_{n,I'}}). \tag{2.7}$$

Let $i \in I$ and $1 \leq d \leq r_I$ such that $1 + \sum_{d'=1}^{d-1} n_{d'} \leq i \leq \sum_{d'=1}^{d} n_{d'}$, then $\mathrm{Res}^k_{n,I,I\setminus\{i\}}$ can be clearly recovered from $\mathrm{Res}^{k_d}_{n_d,\overline{i}}$ (with $\overline{i} \overset{\mathrm{def}}{=} i - \sum_{d'=1}^{d-1} n_{d'}$) by tensoring with identity morphisms on cohomologies of other Levi blocks of $L_{n,I}$ and that of $\overline{Z}_{n,I}$, and then sum over all decompositions $k = \sum_{d=0}^{r_I} k_d$ as in Lemma 2.7.

Let $I \subseteq \Delta_n$ be a subset and $\underline{k} = \{k_d\}_{0\leq d\leq r_I}$ be a tuple of non-negative integers satisfying $|\underline{k}| \overset{\mathrm{def}}{=} \sum_{d=0}^{r_I} k_d = k$. We choose an element $\Lambda_d \in \Sigma_{n_d,k_d}$ for each $1 \leq d \leq r_I$ and then write $\underline{\Lambda} = \{\Lambda_d\}_{1\leq d\leq r_I}$ for the tuple. We set

$$\Sigma_{n,I,\underline{k}} \overset{\mathrm{def}}{=} \prod_{d=1}^{r_I} \Sigma_{n_d,k_d} \tag{2.8}$$

and

$$v_{n,I,\underline{k}}^{\underline{\Lambda}} \overset{\mathrm{def}}{=} \bigotimes_{d=1}^{r_I} v_{n_d}^{\Lambda_d} \in \bigotimes_{d=1}^{r_I} H^{k_d}(\overline{G}_{n_d}, 1_{\overline{G}_{n_d}})$$

for each $\underline{\Lambda} \in \Sigma_{n,I,\underline{k}}$. It is clear that $\{v_{n,I,\underline{k}}^{\underline{\Lambda}}\}_{\underline{\Lambda}\in\Sigma_{n,I,\underline{k}}}$ is a basis of $\bigotimes_{d=1}^{r_I} H^{k_d}(\overline{G}_{n_d}, 1_{\overline{G}_{n_d}})$. Moreover, Lemma 2.7 implies that

$$\{v \otimes_E v_{n,I,\underline{k}}^{\underline{\Lambda}} \mid v \subseteq \mathcal{B}_{n,I},\ \#v = k_0,\ \underline{\Lambda} \in \Sigma_{n,I,\underline{k}},\ |\underline{k}| = k\} \tag{2.9}$$

forms a basis of $H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$ for each $I \subseteq \Delta_n$ and $k \geq 0$.

2.2. **Standard spectral sequences.** For each pair of subsets $I_0 \subseteq I_1 \subseteq \Delta_n$, we consider the following complex

$$\mathbf{C}_{I_0,I_1}(\lambda): \ i^{\mathrm{an}}_{n,I_1}(\lambda) \to \cdots \to \bigoplus_{I_0\subseteq I'\subseteq I_1, \#I=\ell} i^{\mathrm{an}}_{n,I}(\lambda) \to \cdots \to i^{\mathrm{an}}_{n,I_0}(\lambda) \tag{2.10}$$

with $i^{\mathrm{an}}_{n,I_0}(\lambda)$ placed at degree $-\#I_0$. Now we specify each map of the complex (2.10). Recall that we have a distinguished embedding $\kappa^{\mathrm{an}}_{I,I'}(\lambda) : i^{\mathrm{an}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I'}(\lambda)$ for each $I' \subseteq I \subseteq \Delta_n$. For each $i \in I \subseteq \Delta_n$, we define $m(I,i)$ to be the number of $i' \in I$ satisfying $i' < i$. Then the map $i^{\mathrm{an}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I\setminus\{i\}}(\lambda)$ in (2.10) is given by $(-1)^{m(I,i)}\kappa^{\mathrm{an}}_{I,I'}(\lambda)$, for each $I_0 \subseteq I \subseteq I_1$ and $i \in I \setminus I_0$. One can easily check that such maps actually define a complex as illustrated by (2.10).

Remark 2.9. There is an easy way to understand the choice of the sign $(-1)^{m(I,i)}$ that appears in the definition of the complex. We assume $I_0 = \emptyset$, $I_1 = \Delta_n$ and $\lambda = 0$ for simplicity, then each $\kappa^{\mathrm{an}}_{I,I'}(0)$ restrict to the identity map on the space of constant functions on $\overline{G}_n$ (as a natural subspace of both $i^{\mathrm{an}}_{n,I}(0)$ and $i^{\mathrm{an}}_{n,I'}(0)$ by definition of parabolic induction). We consider the subcomplex $\mathbf{C}$ of $\mathbf{C}_{\emptyset,\Delta_n}(0)$ consisting of constant functions on $\overline{G}_n$, then our choice of sign $(-1)^{m(I,i)}$ above guarantees that $\mathbf{C}$ is exact. In fact, the exact sequence $\mathbf{C}$ is simply the $\#\Delta_n = n-1$-th tensor power of the exact sequence $0 \to E \xrightarrow{\mathrm{Id}} E \to 0$ that is supported in degree $[-1,0]$.

For later convenience, we also set $\mathbf{C}_{I_0,I_1}(\lambda) \overset{\mathrm{def}}{=} 0$ if $I_0 \not\subseteq I_1$. If we consider two extra sets I'_0, I'_1 satisfying $I_0 \subseteq I'_0 \subseteq I'_1 \subseteq I_1$, then we have a natural commutative diagram by considering truncation of complex

$$\begin{array}{ccccc} & & \mathbf{C}_{I_0,I_1}(\lambda) & & \\ & \nearrow & & \searrow & \\ \mathbf{C}_{I'_0,I_1}(\lambda) & & & & \mathbf{C}_{I_0,I'_1}(\lambda) \\ & \searrow & & \nearrow & \\ & & \mathbf{C}_{I'_0,I'_1}(\lambda) & & \end{array}$$

Note that $\mathbf{C}_{I_0,I_1}(\lambda) \cong i^{\mathrm{an}}_{n,I_1,\Delta_n}(V^{\mathrm{an}}_{n,I_0,I_1}(\lambda))[-\#I_0]$ (and in particular $\mathbf{C}_{I_0,\Delta_n}(\lambda) \cong V^{\mathrm{an}}_{n,I_0}(\lambda)[-\#I_0]$) in the derived sense.

It follows from Lemma 1.8 and Theorem 7.1 of [Koh11] that we have canonical isomorphisms

$$\begin{aligned}\mathrm{Ext}^k_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) &\cong \mathrm{Ext}^k_{L_{n,I},\lambda}(H_0(N_{n,I}, F_{n,\Delta_n}(\lambda)), F_{n,I}(\lambda)) \\ &\cong \mathrm{Ext}^k_{L_{n,I},\lambda}(F_{n,I}(\lambda), F_{n,I}(\lambda)) \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})\end{aligned}$$

for each $I \subseteq \Delta_n$. Moreover, for each $I' \subseteq I \subseteq \Delta_n$, the map

$$\mathrm{Ext}^\bullet_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \to \mathrm{Ext}^\bullet_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), i^{\mathrm{an}}_{n,I'}(\lambda))$$

induced from $\kappa^{\mathrm{an}}_{I,I'}(\lambda) : i^{\mathrm{an}}_{n,I}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I'}(\lambda)$ is simply the restriction map (2.7). It follows from item (i) of Lemma 1.7 that there exists a standard spectral sequence $\{E^{-\ell,k}_{r,I_0,I_1}\}_{r\geq 0, \#I_0\leq \ell\leq \#I_1, k\geq 0}$ which converges to

$$\mathrm{Ext}^\bullet_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), \mathbf{C}_{I_0,I_1}(\lambda)) \cong \mathrm{Ext}^{\bullet+\#I_0}_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), i^{\mathrm{an}}_{n,I_1,\Delta_n}(V^{\mathrm{an}}_{n,I_0,I_1}(\lambda))).$$

For each $k \geq 0$ and $\#I_0 \leq \ell \leq n-1$, we have

$$E_{1,I_0,I_1}^{-\ell,k} \cong \bigoplus_{I_0 \subseteq I \subseteq I_1, \#I=\ell} \mathrm{Ext}^k_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \cong \bigoplus_{I_0 \subseteq I \subseteq I_1, \#I=\ell} H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}),$$

and the differential $d_{1,I_0,I_1}^{-\ell,k} : E_{1,I_0,I_1}^{-\ell,k} \to E_{1,I_0,I_1}^{-\ell+1,k}$ has the form

$$\bigoplus_{i \in I \setminus I_0} (-1)^{m(I,i)} \mathrm{Res}^k_{n,I,I\setminus\{i\}}$$

after restricting to $H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$, for each $I_0 \subseteq I \subseteq I_1$ satisfying $\#I = \ell$.

Given two pair of subsets $I_0 \subseteq I_1 \subseteq \Delta_n$ and $I_0' \subseteq I_1' \subseteq \Delta_n$, if $I_0 \subseteq I_0'$ and $I_1 \subseteq I_1'$, then the natural map $\mathbf{C}_{I_0,I_1}(\lambda) \to \mathbf{C}_{I_0',I_1'}(\lambda)$ induces a map between spectral sequences $E_{\bullet,I_0,I_1}^{\bullet,\bullet} \to E_{\bullet,I_0',I_1'}^{\bullet,\bullet}$.

2.3. (I_0, I_1)**-standard and** (I_0, I_1)**-atomic tuples.** In this section, we introduce (I_0, I_1)-standard and (I_0, I_1)-atomic elements (see Definition 2.13) of the bases introduced in Section 2.1. These combinatorial notion will be crucial later for the proof of Theorem 2.28.

Definition 2.10. We say that a subset $I \subseteq \Delta_n$ is an *interval* if there exists $1 \leq i \leq j \leq n-1$ such that $I = \{i, i+1, \dots, j\}$. For each $I \subseteq \Delta_n$, there exists a unique decomposition $I = \bigsqcup_{d=1}^{r_I} I^d$ such that I^d corresponds to the set of positive simple roots in the d-th Levi block of $L_{n,I}$. Hence, I^d is either empty or a maximal subinterval of I with $n_d = \#I^d + 1$, and moreover $i < i'$ for each $i \in I^d$ and $i' \in I^{d'}$ satisfying $1 \leq d < d' \leq r_I$.

Let $I_0 \subseteq I \subseteq I_1$ be a subset satisfying $\#I = \ell$ and $\underline{k} = \{k_d\}_{1 \leq d \leq r_I}$ be a tuple satisfying $|\underline{k}| = k$. Let $v \subseteq \mathcal{B}_{n,I}$ with $\#v = k_0$ and $\underline{\Lambda} \in \Sigma_{n,I,\underline{k}}$ (see (2.8)), we use in the following the shortened notation $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ and write $x_\Theta \overset{\mathrm{def}}{=} v \otimes_E v^{\underline{\Lambda}}_{n,I,\underline{k}}$. So we always have $I_0 \subseteq I \subseteq I_v \cap I_1$.

The space $E_{1,I_0,I_1}^{-\ell,k}$ admits a basis of the form

$$\{x_\Theta \mid \Theta = (v, I, \underline{k}, \underline{\Lambda}),\ v \subseteq \mathcal{B}_{n,\emptyset},\ \underline{\Lambda} \in \Sigma_{n,I,\underline{k}},\ I_0 \subseteq I \subseteq I_v \cap I_1,\ |\underline{k}| = k,\ \#I = \ell,\ |v| = k_0\}.$$

Hence, we have a decomposition (with $E_{1,I_0,I_1,v}^{-\ell,k}$ the v-isotypic direct summand)

$$E_{1,I_0,I_1}^{-\ell,k} = \bigoplus_{v \subseteq \mathcal{B}_{n,\emptyset}} E_{1,I_0,I_1,v}^{-\ell,k}$$

which is compatible with the differential $d_{1,I_0,I_1}^{-\ell,k}$. For each $x \in E_{1,I_0,I_1,v}^{-\ell,k}$ and each $\Theta = (v, I, \underline{k}, \underline{\Lambda})$, we write $c_\Theta(x)$ for the coefficient of x attached to x_Θ.

We introduce two fundamental construction which (if defined) sends a given tuple $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ to a new tuple.

- For each $i \in \Delta_n \setminus I$, there exists a unique $1 \leq d \leq r_I$ such that $i = \sum_{d'=1}^{d} n_{d'}$, and we define $p_i^+(\Theta) \overset{\mathrm{def}}{=} (v, I \sqcup \{i\}, \underline{k}', \underline{\Lambda}')$ by the condition that $\Lambda'_{d'} = \Lambda_{d'}$ for each $1 \leq d' \leq d-1$, $\Lambda'_{d'} = \Lambda_{d'+1}$ for each $d+1 \leq d' \leq r_I - 1$ and $\Lambda'_d = \Lambda_d \sqcup \Lambda_{d+1}$. So $p_i^+(\Theta)$ is well defined if and only if $i \in \Delta_n \setminus I$ and $\Lambda_d \cap \Lambda_{d+1} = \emptyset$.
- For each $i \in I$, there exists a unique $1 \leq d \leq r_I$ such that $i \in I^d$, and we define $p_i^-(\Theta) \overset{\mathrm{def}}{=} (v, I \setminus \{i\}, \underline{k}', \underline{\Lambda}')$ by the condition that $\Lambda'_{d'} = \Lambda_{d'}$ for each $1 \leq d' \leq d-1$, $\Lambda'_{d'} = \Lambda_{d'-1}$ for each $d+2 \leq d' \leq r_I + 1$, $\Lambda'_d = \Lambda_d$ and $\Lambda'_{d+1} = \emptyset$. So $p_i^-(\Theta)$ is well defined if and only if $i \in I$ and $\max\{m \mid (m, \iota) \in \Lambda_d\} \leq 2(i - \sum_{d'=1}^{d-1} n_{d'}) - 1$.

Remark 2.11. For each $1 \leq d \leq r_I$, we call Λ_d the *d-th block of the tuple* Θ. Intuitively speaking, $p_i^+(\Theta)$ (when defined) is obtained from Θ by gluing two adjacent blocks separated by i, and $p_i^-(\Theta)$ (when defined) is obtained from Θ by splitting a block into two adjacent blocks separated by i, leaving the second block trivial.

Definition 2.12. Let $\Theta = (v, I, \underline{k}, \underline{\Lambda})$, $\Theta' = (v', I', \underline{k}', \underline{\Lambda}')$ be two tuple. We say that Θ' *is an improvement of* Θ *with level* d for some $1 \leq d \leq r_I$, if there exists $i \in I^d \setminus I_0$ and $i' = \sum_{d'=1}^{d} n_{d'} \in (I_v \cap I_1) \setminus I$ such that $\Theta' = p_{i'}^+(p_i^-(\Theta))$ (which forces $v = v'$, $\underline{k} = \underline{k}'$ and $\underline{\Lambda} = \underline{\Lambda}'$). We say that Θ *is smaller than* Θ' *with level* $\geq d$ for some $1 \leq d \leq r_I$, written $\Theta <_d \Theta'$, if there exists a sequence of tuples $\Theta = \Theta^0, \dots, \Theta^m = \Theta'$ such that $\Theta^{m'}$ is an improvement of $\Theta^{m'-1}$ with level $\geq d$ for each $1 \leq m' \leq m$. We say that Θ and Θ' are *equivalent* (with respect to (I_0, I_1)) if there exists Θ'' such that $\Theta <_1 \Theta''$ and $\Theta' <_1 \Theta''$.

For each $1 \leq d \leq r_I$ satisfying $I^d \cap I_0 \neq \emptyset$, there exists a unique maximal possible $1 \leq d' \leq r_{I_0}$ such that $\emptyset \neq I_0^{d'} \subseteq I^d$ and $I^d \setminus I_0^{d'} = I^{d,-} \sqcup I^{d,+}$ with $I^{d,-} \overset{\text{def}}{=} \{i \in I^d \mid i < i' \text{ for each } i' \in I_0^{d'}\}$ and $I^{d,+} \overset{\text{def}}{=} \{i \in I^d \mid i > i' \text{ for each } i' \in I_0^{d'}\}$. So we always have $I^{d,+} \cap I_0 = \emptyset$ but $I^{d,-} \cap I_0$ might be non-empty. If $I^{d,-} \neq \emptyset$, then we have $I^{d,-} \setminus I_0 \neq \emptyset$ and set $n_d^- \overset{\text{def}}{=} \min\{i \mid i \in I^{d,-} \setminus I_0\} - \sum_{d''=1}^{d-1} n_{d''} \leq \#I^{d,-}$.

Note that the sets $I \subseteq I_v \cap I_1$ determines a sequence

$$0 = r_{v,I_1,I}^0 < r_{v,I_1,I}^1 < \cdots < r_{v,I_1,I}^{r_{I_v \cap I_1}} = r_I$$

characterized by $\Delta_n \setminus (I_v \cap I_1) = \{\sum_{d=1}^{r_{v,I_1,I}^s} n_d \mid 1 \leq s \leq r_{I_v \cap I_1} - 1\}$. Given $1 \leq d \leq r_I$, note that $\sum_{d'=1}^{d-1} n_{d'} \in \{0\} \sqcup \Delta_n \setminus (I_v \cap I_1)$ if and only if there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r_{v,I_1,I}^{s-1} + 1$, and similarly $\sum_{d'=1}^{d} n_{d'} \in \{n\} \sqcup \Delta_n \setminus (I_v \cap I_1)$ if and only if there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r_{v,I_1,I}^s$.

Definition 2.13. For each tuple $\Theta = (v, I, \underline{k}, \underline{\Lambda})$, we define its *sign* as $\varepsilon(\Theta) \overset{\text{def}}{=} (-1)^{\sum_{i \in I} i}$. We say that Θ is (I_0, I_1)-*standard* if there exists $2 \leq d_{I_0,I_1,\Theta} \leq r_I$ such that

- $\sum_{d'=1}^{d_{I_0,I_1,\Theta}-1} n_{d'} \in (I_v \cap I_1) \setminus I$;
- $\Lambda^{d_{I_0,I_1,\Theta}} = \emptyset$ and $\Lambda^d \neq \emptyset$ for each $d_{I_0,I_1,\Theta} + 1 \leq d \leq r_I$;
- $I^{d_{I_0,I_1,\Theta}} \subseteq I_0$; and
- there does not exist Θ' such that $\Theta <_{d_{I_0,I_1,\Theta}} \Theta'$.

We say that Θ is *maximally* (I_0, I_1)-*atomic* if

- there does not exist Θ' such that $\Theta <_1 \Theta'$;
- if $k_d = 0$ (namely $\Lambda_d = \emptyset$) for some $1 \leq d \leq r_I$, then there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r_{v,I_1,I}^{s-1} + 1$;
- for each $1 \leq s \leq r_{I_v \cap I_1}$ satisfying $I^{r_{v,I_1,I}^s} \cap I_0 \neq \emptyset$, $\Lambda_{r_{v,I_1,I}^s} \neq \emptyset$ and $(\{2n_{r_{v,I_1,I}^s} - 1\} \times S) \cap \Lambda_{r_{v,I_1,I}^s} = \emptyset$, we have $I^{r_{v,I_1,I}^s,+} = \emptyset$ and there exists $n' \geq \#I^{r_{v,I_1,I}^s,-} + 1$ and $\iota \in S$ such that $(2n' - 1, \iota) \in \Lambda_{r_{v,I_1,I}^s}$; and
- for each $1 \leq s \leq r_{I_v \cap I_1}$ satisfying $I^{r_{v,I_1,I}^s} \cap I_0 = \emptyset$ and $\Lambda_{r_{v,I_1,I}^s} \neq \emptyset$, there exists $\iota \in S$ such that $(2n_{r_{v,I_1,I}^s} - 1, \iota) \in \Lambda_{r_{v,I_1,I}^s}$.

We say that Θ is (I_0, I_1)-*atomic* if there exists a maximally (I_0, I_1)-atomic Θ' such that $\Theta <_1 \Theta'$. For each equivalent class Ω of (I_0, I_1)-atomic tuples, we associate an (I_0, I_1)-atom

$$x_\Omega \stackrel{\mathrm{def}}{=} \sum_{\Theta=(v,I,\underline{k},\underline{\Lambda})\in\Omega} \varepsilon(\Theta) x_\Theta.$$

The *bidegree* of a (I_0, I_1)-atom x_Ω is defined to be $(-\ell_\Omega, k_\Omega) \stackrel{\mathrm{def}}{=} (-\#I, |\underline{k}|)$ for an arbitrary $\Theta = (v, I, \underline{k}, \underline{\Lambda}) \in \Omega$.

Lemma 2.14. *Let $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ be a tuple as before and $1 \leq d \leq r_I$ be an integer. If $I^d \cap I_0 = \emptyset$, then the following statements are equivalent*

(i) *there exists $i \in I^d$ such that $\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta) \neq 0$;*
(ii) *$n_d \geq 2$ and $\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta) \neq 0$ for $i \in \{-1 + \sum_{d'=1}^d n_{d'}, 1 + \sum_{d'=1}^{d-1} n_{d'}\}$;*
(iii) *$n_d \geq 2$ and $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$.*

If $I^d \cap I_0 \neq \emptyset$, then the following statements are equivalent

(iv) *there exists $i \in I^{d,+}$ such that $\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta) \neq 0$;*
(v) *$I^{d,+} \neq \emptyset$ and $\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta) \neq 0$ for $i = -1 + \sum_{d'=1}^d n_{d'}$;*
(vi) *$I^{d,+} \neq \emptyset$ and $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$.*

If $I^d \cap I_0 \neq \emptyset$ and $I^{d,+} = \emptyset$, then the following statements are equivalent

(vii) *there exists $i \in I^{d,-}$ such that $c_{p_i^-(\Theta)}(\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta)) \neq 0$;*
(viii) *$I^{d,-} \neq \emptyset$ and $c_{p_i^-(\Theta)}(\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta)) \neq 0$ for $i = \#I^{d,-} + \sum_{d'=1}^{d-1} n_{d'}$);*
(ix) *$I^{d,-} \neq \emptyset$ and $\max\{m \mid (m, \iota) \in \Lambda_d\} < 2\#I^{d,-} + 1$.*

Proof. We only treat the case $I^d \cap I_0 = \emptyset$ and the other two cases are similar. For each $i \in I^d$, it follows from Lemma 2.8 that $\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta) \neq 0$ if and only if $\mathbf{D}_{n_d,\Lambda_d,\bar{i}} \neq \emptyset$ where $\bar{i} \stackrel{\mathrm{def}}{=} i - \sum_{d'=1}^{d-1} n_{d'}$. Note that $\mathbf{D}_{n_d,\Lambda_d,\bar{i}} \neq \emptyset$ implies the existence of $\Lambda' \subseteq \Lambda$ such that $\max\{m \mid (m, \iota) \in \Lambda'\} \leq 2\bar{i} - 1$ and $\max\{m \mid (m, \iota) \in \Lambda_d \setminus \Lambda'\} \leq 2(n_d - \bar{i}) - 1$, which altogether implies that $\max\{m \mid (m, \iota) \in \Lambda_d\} \leq 2n_d - 3$ (which is equivalent to $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$). In other words, we have (i) implies (iii). As (iii) clearly implies (ii) (namely $\mathbf{D}_{n_d,\Lambda_d,n_d-1} \neq \emptyset \neq \mathbf{D}_{n_d,\Lambda_d,1}$) and (ii) clearly implies (i), we finish the proof. □

Lemma 2.15. *Let $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ be a maximally (I_0, I_1)-atomic tuple. Let $1 \leq d \leq r_I$ be an integer. Then we have*

- *If $I^d \cap I_0 = \emptyset$ and $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$, then $n_d = 1$, $\Lambda_d = \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$.*
- *If $I^d \cap I_0 \neq \emptyset$ and $I^{d,+} \neq \emptyset$, then there exists $\iota \in S$ such that $(2n_d - 1, \iota) \in \Lambda_d$;*
- *If $I^d \cap I_0 \neq \emptyset$, $I^{d,+} = \emptyset$ and $\max\{m \mid (m, \iota) \in \Lambda_d\} < 2\#I^{d,-} + 1$, then $I^{d,-} = \emptyset$, $\Lambda_d = \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$.*

Proof. We may assume throughout that $d \neq r^s_{v,I_1,I}$ for each $1 \leq s \leq r_{I_v \cap I_1}$ (see Definition 2.13). We treat the three claims separately

- Let d satisfy $I^d \cap I_0 = \emptyset$ and $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$. If $n_d \geq 2$, then $\Theta < p^+_{i_2} p^-_{i_1}(\Theta)$ for $i_1 = -1 + \sum_{d'=1}^d n_{d'}$ and $i_2 = \sum_{d'=1}^d n_{d'}$ (using Lemma 2.14), which contradicts the maximality of Θ. Hence, we have $n_d = 1$, $k_d = 0$ and $\Lambda_d = \emptyset$, which gives a $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$ by Definition 2.13.

- Let d satisfy $I^d \cap I_0 \neq \emptyset$ and $I^{d,+} \neq \emptyset$. If $(\{2n_d - 1\} \times S) \cap \Lambda_d = \emptyset$, then Lemma 2.14 implies that $\Theta < p^+_{i_4} p^-_{i_3}(\Theta)$ for $i_3 = -1 + \sum_{d'=1}^{d} n_{d'}$ and $i_4 = \sum_{d'=1}^{d} n_{d'}$, which contradicts the maximality of Θ.
- Let d satisfy $I^d \cap I_0 \neq \emptyset$, $I^{d,+} = \emptyset$ and $\max\{m \mid (m, \iota) \in \Lambda_d\} < 2\#I^{d,-} + 1$. If $I^{d,-} \neq \emptyset$, then Lemma 2.14 implies that $\Theta < p^+_{i_6} p^-_{i_5}(\Theta)$ for
$$i_5 = -1 - \#I^d \cap I_0 + \sum_{d'=1}^{d} n_{d'} = \#I^{d,-} + \sum_{d'=1}^{d-1} n_{d'}$$
and $i_6 = \sum_{d'=1}^{d} n_{d'}$, which contradicts the maximality of Θ. Hence, we have $I^{d,-} = \emptyset$, which together with $\max\{m \mid (m, \iota) \in \Lambda_d\} < 2\#I^{d,-} + 1$ forces $k_d = 0$ (namely $\Lambda_d = \emptyset$), and thus there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$ by Definition 2.13.

□

Now we classify (I_0, I_1)-atomic tuples when $k \in \{\ell + \#I_1 - 2\#I_0, \ell + \#I_1 - 2\#I_0 + 1\}$.

Lemma 2.16. *We have the following consequences of existence of (I_0, I_1)-atomic tuples.*

- *If $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ is (I_0, I_1)-atomic, then we have*
$$\ell - 2\#I_0 + \#I_1 \leq k \leq \ell - n + 1 + [K : \mathbb{Q}_p](n^2 - 1).$$
- *If $k = \ell + \#I_1 - 2\#I_0$, then $I_v \cup I_1 = \Delta_n$ and a maximally (I_0, I_1)-atomic Θ satisfies*
  - *there does not exist $i \in \Delta_n \setminus I_v$ such that $\#v \cap \mathcal{B}_{n,\Delta_n \setminus \{i\}} \geq 2$ (which implies that $k_0 = \#\Delta_n \setminus I_v$);*
  - *if $I^d \cap I_0 = \emptyset$, then $\Lambda_d \neq \{(2n_d - 1, \iota)\}$ for any $\iota \in S$ if and only if $n_d = 1$, $\Lambda_d = \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$;*
  - *if $I^d \cap I_0 \neq \emptyset$, then $I^d \cap I_0$ is an interval and $I^{d,+} = \emptyset$, and moreover $\Lambda_d \neq \{(2\#I^{d,+} + 1, \iota)\}$ for any $\iota \in S$ if and only if $I^{d,-} = \emptyset$, $\Lambda_d = \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$.*
- *If $k = \ell + \#I_1 - 2\#I_0 + 1$, then either $\#I_v \cup I_1 = n - 2$ and a maximally (I_0, I_1)-atomic Θ satisfies exactly the same three conditions as $k = \ell + \#I_1 - 2\#I_0$ case, or $I_v \cup I_1 = \Delta_n$ and a maximally (I_0, I_1)-atomic Θ satisfies the conditions in $k = \ell + \#I_1 - 2\#I_0$ case with exactly one of the following modifications*
  - *there exists exactly one $i \in \Delta_n \setminus I_v$ such that $\#v \cap \mathcal{B}_{n,\Delta_n \setminus \{i\}} \geq 2$ and the equality holds for that i;*
  - *for exactly one choice of $1 \leq s \leq r_{I_v \cap I_1}$ and $d = r^{s-1}_{v,I_1,I} + 1$, we have $I^{d,+} = \emptyset \neq I^{d,-}$ and there exists $\iota \in S$ such that $\Lambda_d = \{(2\#I^{d,-} + 1, \iota)\}$ if $I^d \cap I_0 \neq \emptyset$ and $\Lambda_d = \{(2n_d - 1, \iota)\}$ if $I^d \cap I_0 = \emptyset$.*

Proof. It is harmless to assume that our Θ is maximally (I_0, I_1)-atomic (see Definition 2.12 and Definition 2.13). We start with the trivial observation that $r_{I_v} - 1 \leq k_0 \leq ([K : \mathbb{Q}_p] + 1)(r_{I_v} - 1) \leq ([K : \mathbb{Q}_p] + 1)(r_I - 1)$, which implies that
$$\begin{aligned}\sum_{d=0}^{r_I} k_d &\leq k_0 + \sum_{d=1}^{r_I} [K : \mathbb{Q}_p](n_d^2 - 1) = k_0 + [K : \mathbb{Q}_p](-r_I + \sum_{d=1}^{r_I} n_d^2) \\ &\leq ([K : \mathbb{Q}_p] + 1)(r_I - 1) + [K : \mathbb{Q}_p](n^2 - 3r_I + 2) = (1 - 2[K : \mathbb{Q}_p])(r_I - 1) + [K : \mathbb{Q}_p](n^2 - 1) \\ &\leq \ell - n + 1 + [K : \mathbb{Q}_p](n^2 - 1).\end{aligned}$$

We have the following observations from Lemma 2.15.

- If $I^d \cap I_0 = \emptyset$ and there does not exist $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$, then $k_d \geq 2n_d - 1$ and the equality holds if and only if $\Lambda_d = \{(2n_d - 1, \iota)\}$ for some $\iota \in S$.
- If $I^d \cap I_0 = \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$, then either $n_d = 1$ (with $k_d = 0$) or $k_d \geq 2n_d - 1$. So we always have $k_d \geq 2n_d - 2$ and the equality holds if and only if $n_d = 1$.
- If $I^d \cap I_0 \neq \emptyset$ and there does not exist $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$, then $k_d \geq 2\#I^{d,-} + 1 \geq 2n_d - 2\#I^d \cap I_0 - 1$ and the equality holds if and only if $I^d \cap I_0$ is an interval, $I^{d,+} = \emptyset$ and $\Lambda_d = \{(2n_d - 2\#I^d \cap I_0 - 1, \iota)\}$ for some $\iota \in S$. (Note that $I^{d,+} \neq \emptyset$ would force $k_d \geq 2n_d - 1 > 2\#I^{d,-} + 1$.)
- If $I^d \cap I_0 \neq \emptyset$ and there exists $1 \leq s \leq r_{I_v \cap I_1}$ such that $d = r^{s-1}_{v,I_1,I} + 1$, then either $I^{d,+} = I^{d,-} = \emptyset$ (with $k_d \geq 0$) or $I^{d,+} = \emptyset \neq I^{d,-}$ (with $k_d \geq 2n_d - 2n_d^- + 1$) or $I^{d,+} \neq \emptyset$ (with $k_d \geq 2n_d - 1$). So we always have $k_d \geq 2n_d - 2\#I^d \cap I_0 - 2$ and the equality holds if and only if $I^{d,+} = I^{d,-} = \emptyset$.

Consequently, for each $1 \leq s \leq r_{I_v \cap I_1}$, we have

$$\sum_{d=r^{s-1}_{v,I_1,I}+1}^{r^s_{v,I_1,I}} k_d \geq -1 + \sum_{d=r^{s-1}_{v,I_1,I}+1}^{r^s_{v,I_1,I}} (2n_d - 2\#I^d \cap I_0 - 1).$$

Summing over $1 \leq s \leq r_{I_v \cap I_1}$, we have

$$\sum_{d=1}^{r_I} k_d \geq -r_{I_v \cap I_1} + 2n - r_I - 2\#I_0,$$

which together with $k_0 \geq r_{I_v} - 1$ implies that

$$k = \sum_{d=0}^{r_I} k_d \geq r_{I_v} - 1 - r_{I_v \cap I_1} + 2n - r_I - 2\#I_0 = (n - \#I_v) - 1 - (n - \#I_v \cap I_1) + 2n - (n - \ell) - 2\#I_0$$

$$= \ell + n - 1 + \#I_v \cap I_1 - \#I_v - 2\#I_0 = \ell + n - 1 + \#I_1 - \#I_v \cup I_1 - 2\#I_0 \geq \ell + \#I_1 - 2\#I_0.$$

The precise conditions on Θ for the equality $k = \ell + \#I_1 - 2\#I_0$ to hold are clear from the discussion above. The case $k = \ell + \#I_1 - 2\#I_0 + 1$ is similar and follows from the following observation: if Θ is a maximally (I_0, I_1)-atomic tuple such that $\max\{m \mid (m, \iota) \in \Lambda_d\} > 2\#I^{d,-} + 1$ for some $1 \leq s \leq r_{I_v \cap I_1}$ and $r^{s-1}_{v,I_1,I} + 1 \leq d \leq r^s_{v,I_1,I}$ satisfying $I^d \cap I_0 \neq \emptyset$, then we must have $k \geq \ell + \#I_1 - 2\#I_0 + 2$. □

2.4. **A combinatorial subcomplex of $E^{\bullet,k}_{1,I_0,I_1}$.** In this section, we construct a combinatorial subcomplex of $E^{\bullet,k}_{1,I_0,I_1}$ for each $k \geq 0$ using (I_0, I_1)-atomic tuples. The main result here is Proposition 2.23, where we prove that this combinatorial subcomplex is quasi-isomorphic to $E^{\bullet,k}_{1,I_0,I_1}$.

Definition 2.17. We consider all sets of the form $\mathbf{I}_{>d} \overset{\text{def}}{=} (I^{d''})_{d''>d}$ for some $I_0 \subseteq I \subseteq I_v \cap I_1$ with $\#I = \ell$ and $1 \leq d \leq r_I = n - \ell$. Then the set $\{\mathbf{I}_{>d}\}_{I,d}$ admits a natural partial order described as follows. Given $\mathbf{I}_{>d}$ and $\mathbf{I}'_{>d'}$ for some (I, d) and (I', d'), we say that $\mathbf{I}'_{>d'} < \mathbf{I}_{>d}$ if exactly one of the following holds

- $d' < d$ and $(I')^{d''} = I^{d''}$ for each $d'' > d$;
- there exists $d_\flat \geq \max\{d, d'\}$ such that $(I')^{d_\flat} \subsetneq I^{d_\flat}$ and $(I')^{d''} = I^{d''}$ for each $d'' > d_\flat$.

It is not difficult to check that this is a well defined partial order on $\{\mathbf{I}_{>d}\}_{I,d}$.

For each (I_0, I_1)-standard Θ, we set $i_{I_0,I_1,\Theta} \overset{\text{def}}{=} \sum_{d=1}^{d_{I_0,I_1,\Theta}-1} n_d \in (I_v \cap I_1) \setminus I$ and consider $\Theta^+ \overset{\text{def}}{=} p^+_{i_{I_0,I_1,\Theta}}(\Theta)$ which is clearly well defined. For each (I_0, I_1)-standard Θ, we associate the set $\mathbf{I}_\Theta \overset{\text{def}}{=} (I^d)_{d > d_{I_0,I_1,\Theta}}$.

Lemma 2.18. *For each $v \subseteq \mathcal{B}_{n,\emptyset}$ with $\#v = k_0$, the subset*

$$\{d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Theta^+}) \mid \Theta = (v, I, \underline{k}, \underline{\Lambda}) \textit{ is } (I_0, I_1)\textit{-standard }, |\underline{k}| = k, \#I = \ell\} \subseteq E^{-\ell,k}_{1,I_0,I_1,v}$$

is linearly independent.

Proof. Let $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ be a (I_0, I_1)-standard tuple satisfying $\Theta' \neq \Theta$ and

$$c_{\Theta'}\left(d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Theta^+})\right) \neq 0.$$

Then we have the following possibilities

- If $d_{I_0,I_1,\Theta'} < d_{I_0,I_1,\Theta} - 1$, then we have $\Theta' <_{d_{I_0,I_1,\Theta'}} p^+_{i'_{I_0,I_1,\Theta}}(p^-_{i_{I_0,I_1,\Theta}}(\Theta'))$ with $i'_{I_0,I_1,\Theta} \overset{\text{def}}{=} \sum_{d=1}^{d_{I_0,I_1,\Theta}} n_d \in (I_v \cap I_1) \setminus I$, which contradicts the fact that Θ' is (I_0, I_1)-standard.
- If $d_{I_0,I_1,\Theta'} = d_{I_0,I_1,\Theta} - 1$, then we have $\mathbf{I}_{\Theta'} = \mathbf{I}_\Theta \sqcup \{(I')^{d_{I_0,I_1,\Theta}}\}$.
- If $d_{I_0,I_1,\Theta'} \geq d_{I_0,I_1,\Theta}$, then $I^{d_{I_0,I_1,\Theta'}+1} \cap I_0 \neq \emptyset$, and $\Theta^+ = p^+_i(\Theta')$ with $i \in I^{d_{I_0,I_1,\Theta'}+1,-}$. Moreover, the tuple $\mathbf{I}_{\Theta'} = \{(I')^d \mid d > d_{I_0,I_1,\Theta'}\}$ satisfies $(I')^{d_{I_0,I_1,\Theta'}+1} \subsetneq I^{d_{I_0,I_1,\Theta'}+1}$ and $(I')^d = I^d$ for each $d > d_{I_0,I_1,\Theta'} + 1$.

Hence, we always have $\mathbf{I}_{\Theta'} < \mathbf{I}_\Theta$ for partial order introduced in Definition 2.17. The proof is thus finished by induction on this partial order. □

Lemma 2.19. *For each $v \subseteq \mathcal{B}_{n,\emptyset}$ and $0 \neq x \in E^{-\ell,k}_{1,I_0,I_1,v}$, there exists $x' \in E^{-\ell-1,k}_{1,I_0,I_1,v}$ such that*

$$c_\Theta(x - d^{-\ell-1,k}_{1,I_0,I_1}(x')) = 0$$

for each (I_0, I_1)-standard $\Theta = (v, I, \underline{k}, \underline{\Lambda})$.

Proof. This follows immediately from Lemma 2.18. In fact, we can construct x' as a linear combination of various x_{Θ^+} for (I_0, I_1)-standard tuples Θ, by an induction on the partial order introduced in Definition 2.17. □

For each tuple Θ, we define an integer

$$e_\Theta \overset{\text{def}}{=} \sum_{d=1}^{r_I} \max\{m \mid (m, \iota) \in \Lambda_d\}$$

which will be useful in our later induction argument. (Here we use the convention $\max\{m \mid (m, \iota) \in \emptyset\} = 0$.) It has the following simple property.

Lemma 2.20. *Let Θ be a tuple.*

- *If $i \in (I_v \cap I_1) \setminus I$ and $p^+_i(\Theta)$ is defined, then $e_{p^+_i(\Theta)} \leq e_\Theta$. Moreover, $e_{p^+_i(\Theta)} = e_\Theta$ if and only if either $\Lambda_d = \emptyset$ or $\Lambda_{d+1} = \emptyset$, where $1 \leq d \leq r_I$ satisfies $\sum_{d'=1}^{d} n_{d'} = i$.*
- *If $i \in I$ and $p^-_i(\Theta)$ is defined, then $e_{p^-_i(\Theta)} = e_\Theta$.*

Proof. This is immediate from the definition of $p^+_i(\Theta)$ (resp. $p^-_i(\Theta)$). We use the fact that, if $\Lambda_d \cap \Lambda_{d+1} = \emptyset$, then

$$\max\{m \mid (m, \iota) \in \Lambda_d \sqcup \Lambda_{d+1}\} \leq \max\{m \mid (m, \iota) \in \Lambda_d\} + \max\{m \mid (m, \iota) \in \Lambda_{d+1}\},$$

and the equality holds if and only if either $\Lambda_d = \emptyset$ or $\Lambda_{d+1} = \emptyset$. □

For each bidegree $(-\ell, k)$, we define $E^{-\ell,k}_{1,I_0,I_1,\diamond} \subseteq E^{-\ell,k}_{1,I_0,I_1}$ as the subspace spanned by x_Ω where Ω runs through equivalence classes of (I_0, I_1)-atomic tuples satisfying $(-\ell_\Omega, k_\Omega) = (-\ell, k)$. As different equivalence classes of (I_0, I_1)-atomic tuples do not intersect, it is clear that $\{x_\Omega \mid (-\ell_\Omega, k_\Omega) = (-\ell, k)\}$ is a basis of $E^{-\ell,k}_{1,I_0,I_1,\diamond}$, which induces a basis of $E^{-\ell,k}_{1,I_0,I_1,\diamond,v} \stackrel{\mathrm{def}}{=} E^{-\ell,k}_{1,I_0,I_1,\diamond} \cap E^{-\ell,k}_{1,I_0,I_1,v}$ for each $v \subseteq \mathcal{B}_{n,\emptyset}$.

Lemma 2.21. *We have $d^{-\ell,k}_{1,I_0,I_1}(E^{-\ell,k}_{1,I_0,I_1,\diamond,v}) \subseteq E^{-\ell+1,k}_{1,I_0,I_1,\diamond,v}$ for each $1 \leq \ell \leq n-1$, $k \geq 0$ and $v \subseteq \mathcal{B}_{n,\emptyset}$. In particular, $E^{\bullet,k}_{1,I_0,I_1,\diamond,v}$ is a subcomplex of $E^{\bullet,k}_{1,I_0,I_1,v}$ for each $k \geq 0$ and $v \subseteq \mathcal{B}_{n,\emptyset}$.*

Proof. It suffices to prove that $d^{-\ell,k}_{1,I_0,I_1}(x_\Omega) \in E^{-\ell+1,k}_{1,I_0,I_1,\diamond,v}$ for each equivalence class Ω of (I_0, I_1)-atomic tuples satisfying $(-\ell_\Omega, k_\Omega) = (-\ell, k)$. Let $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ be a tuple satisfying

$$c_{\Theta'}(\mathrm{Res}^k_{n,I,I\setminus\{i\}}(x_\Theta)) \neq 0$$

for some $\Theta = (v, I, \underline{k}, \underline{\Lambda}) \in \Omega$ and $i \in I$. Then we only have the following two possibilities

- The tuple $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ is not (I_0, I_1)-atomic, and there exists a unique $1 \leq s \leq r_{I_v \cap I_1}$ and $r^{s-1}_{v,I_1,I} + 2 \leq d \leq r^s_{v,I_1,I} - 1$ such that $k'_d = 0$ (namely $\Lambda'_d = \emptyset$). Moreover, if we write $i_1 \stackrel{\mathrm{def}}{=} \sum_{d'=1}^{d-1} n'_{d'}$ and $i_2 \stackrel{\mathrm{def}}{=} \sum_{d'=1}^{d} n'_{d'}$, then $\Theta \in \{p^+_{i_1}(\Theta'), p^+_{i_2}(\Theta')\} \subseteq \Omega$, and moreover any $\Theta'' \in \Omega$ that satisfies $c_{\Theta'}(\mathrm{Res}^k_{n,I'',I''\setminus\{i''\}}(x_{\Theta''})) \neq 0$ (with $\Theta'' = (v, I'', \underline{k}'', \underline{\Lambda}'') \in \Omega$ and $i'' \in I''$) must be either $p^+_{i_1}(\Theta')$ or $p^+_{i_2}(\Theta')$. Note that we have
$$\varepsilon(p^+_{i_2}(\Theta')) = (-1)^{i_2-i_1}\varepsilon(p^+_{i_1}(\Theta'))$$
and
$$c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_2}(\Theta')})) = (-1)^{m(I'\sqcup\{i_2\},i_2)-m(I'\sqcup\{i_1\},i_1)}c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_1}(\Theta')})) = (-1)^{i_2-i_1-1}c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_1}(\Theta')})).$$
Consequently, we have
$$\begin{aligned} c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_\Omega)) &= \varepsilon(p^+_{i_1}(\Theta'))c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_1}(\Theta')})) + \varepsilon(p^+_{i_2}(\Theta'))c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_2}(\Theta')})) \\ &= \varepsilon(p^+_{i_1}(\Theta'))c_{\Theta'}(d^{-\ell,k}_{1,I_0,I_1}(x_{p^+_{i_1}(\Theta')}))(1 + (-1)^{i_2-i_1}(-1)^{i_2-i_1-1}) = 0. \end{aligned}$$
- The tuple $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ is (I_0, I_1)-atomic and we write Ω' for its equivalence class. For each $\Theta'' = (v, I'', \underline{k}'', \underline{\Lambda}'') \in \Omega'$, there exists a unique $i_{\Theta''} \in (I_v \cap I_1) \setminus I''$ such that $\Theta''' \stackrel{\mathrm{def}}{=} p^+_{i_{\Theta''}}(\Theta'') = (v, I''', \underline{k}''', \underline{\Lambda}''') \in \Omega$ (by comparing $\underline{\Lambda}''' = \underline{\Lambda}$ and $\underline{\Lambda}'' = \underline{\Lambda}'$). This gives a natural map
$$\Omega' \to \Omega : \ \Theta'' \mapsto p^+_{i_{\Theta''}}(\Theta'').$$
Using the basis of $E^{-\ell+1,k}_{1,I_0,I_1}$ of the form $\{x_{\Theta''}\}$ for various tuples Θ'', we define $d^{-\ell,k}_{1,I_0,I_1,\Omega'}$ as the composition of $d^{-\ell,k}_{1,I_0,I_1}$ with the projection to the subspace spanned by $\{x_{\Theta''} \mid \Theta'' \in \Omega'\}$. We claim that there exists $\varepsilon(\Omega, \Omega') \in \{1, -1\}$ such that $d^{-\ell,k}_{1,I_0,I_1,\Omega'}(x_\Omega) = \varepsilon(\Omega, \Omega')x_{\Omega'}$. In fact, if we pick up a $\Theta'' \in \Omega'$ (which determines a $\Theta''' = p^+_{i_{\Theta''}}(\Theta'') \in \Omega$) as above, it suffices to show that $\varepsilon(\Theta'')\varepsilon(\Theta''')c_{\Theta''}(d^{-\ell,k}_{1,I_0,I_1}(x_{\Theta'''}))$ is independent of the choice of $\Theta'' \in \Omega'$. It is clear that $\varepsilon(\Theta'')\varepsilon(\Theta''') = (-1)^{i_{\Theta''}}$ and
$$c_{\Theta''}(d^{-\ell,k}_{1,I_0,I_1}(x_{\Theta'''})) = (-1)^{m(I''',i_{\Theta''})}c_{\Theta''}(\mathrm{Res}^k_{n,I''',I''}(x_{\Theta'''})).$$

Then we observe that both $i_{\Theta''}-m(I''',i_{\Theta''})$ and $c_{\Theta''}(\mathrm{Res}^k_{n,I''',I''\}}(x_{\Theta'''}))$ depend only on $\underline{\Lambda}''$ and $\underline{\Lambda}'''$, so does $\varepsilon(\Theta'')\varepsilon(\Theta''')c_{\Theta''}(d^{-\ell,k}_{1,I_0,I_1}(x_{\Theta'''}))$. As all different choices of $\Theta''\in\Omega'$ share the same $\underline{\Lambda}''$ and $\underline{\Lambda}'''$, $\varepsilon(\Omega,\Omega')\overset{\mathrm{def}}{=}\varepsilon(\Theta'')\varepsilon(\Theta''')c_{\Theta''}(d^{-\ell,k}_{1,I_0,I_1}(x_{\Theta'''}))$ is well defined.

As a summary, we deduce that

$$d^{-\ell,k}_{1,I_0,I_1}(x_\Omega)=\sum_{\Omega'}\varepsilon(\Omega,\Omega')x_{\Omega'}\in E^{-\ell+1,k}_{1,I_0,I_1,\diamond,v} \tag{2.11}$$

where Ω' runs through all equivalence classes of (I_0,I_1)-atomic tuples that appear in the second possibility above. The proof is thus finished. □

Lemma 2.22. *Let $v\subseteq\mathcal{B}_{n,\emptyset}$ and $x\in E^{-\ell,k}_{1,I_0,I_1,v}$ such that $d^{-\ell,k}_{1,I_0,I_1}(x)\in E^{-\ell+1,k}_{1,I_0,I_1,\diamond,v}$ and $c_\Theta(x)=0$ for each $\Theta=(v,I,\underline{k},\underline{\Lambda})$ which is either (I_0,I_1)-standard or maximally (I_0,I_1)-atomic. Then $x=0$.*

Proof. For each tuple $\Theta=(v,I,\underline{k},\underline{\Lambda})$, we define $d_{I_0,I_1,\Theta}$ as the maximal integer satisfying $k_{d_{I_0,I_1,\Theta}}=0$ and $r^{s-1}_{v,I_1,I}+2\leq d_{I_0,I_1,\Theta}\leq r^s_{v,I_1,I}$ for some $1\leq s\leq r_{I_v\cap I_1}$ if it exists, and as 1 otherwise. This clearly extends the definition of $d_{I_0,I_1,\Theta}$ when Θ is (I_0,I_1)-standard. We assume inductively that $c_{\Theta'}(x)=0$ for all Θ' satisfying either $\Theta<_{d_{I_0,I_1,\Theta}}\Theta'$ or $e_{\Theta'}<e_\Theta$.

It is harmless to assume that Θ is neither (I_0,I_1)-standard nor maximally (I_0,I_1)-atomic. Hence, there exists $s\leq s_1\leq r_{I_v\cap I_1}$ and $\max\{d_{I_0,I_1,\Theta},r^{s_1-1}_{v,I_1,I}+1\}\leq d_1\leq r^{s_1}_{v,I_1,I}$ and $i_1\in I^{d_1}$ such that $p^-_{i_1}(\Theta)$ is defined. We choose d_1 and i_1 to be maximal possible, and thus $p^-_{i_1}(\Theta)$ is (I_0,I_1)-standard. If $c_\Theta(x)\neq 0$, then there exists $i_2\in(I_v\cap I_1)\setminus I$ such that $c_{p^+_{i_2}p^-_{i_1}(\Theta)}(x)\neq 0$. If $i_2<i_1$, then $p^+_{i_2}p^-_{i_1}(\Theta)$ is (I_0,I_1)-standard and thus contradicts our assumption. If $i_2>\sum_{d'=1}^{d_1}n_{d'}$, then as $\Lambda_{d'}\neq\emptyset$ for each $d'>d_{I_0,I_1,\Theta}$, we deduce from Lemma 2.20 that $e_{p^+_{i_2}p^-_{i_1}(\Theta)}<e_\Theta$, which contradicts our inductive assumption. Hence, we must have $i_2=\sum_{d'=1}^{d_1}n_{d'}$ and thus $\Theta<_{d_{I_0,I_1,\Theta}}p^+_{i_2}p^-_{i_1}(\Theta)$, another contradiction. The proof is thus finished. □

Proposition 2.23. *For each $v\subseteq\mathcal{B}_{n,\emptyset}$ and $k\geq 0$, the subcomplex*

$$E^{\bullet,k}_{1,I_0,I_1,\diamond,v}\to E^{\bullet,k}_{1,I_0,I_1,v}$$

induces an quasi-isomorphism.

Proof. We fix an $\#I_0\leq\ell\leq\#I_1$ and $k\geq 0$.

We first show that the induced map on cohomology is surjective, namely for each $x\in E^{-\ell,k}_{1,I_0,I_1,v}$ satisfying $d^{-\ell,k}_{1,I_0,I_1,v}(x)=0$, there exists $x'\in E^{-\ell-1,k}_{1,I_0,I_1,v}$ and $x''\in E^{-\ell,k}_{1,I_0,I_1,\diamond,v}$ such that $x=x''+d^{-\ell-1,k}_{1,I_0,I_1}(x')$. In fact, given such x, we can choose x' as in Lemma 2.19, and then take

$$x''\overset{\mathrm{def}}{=}\sum_{\Omega}c_\Theta(x-d^{-\ell-1,k}_{1,I_0,I_1}(x'))\varepsilon(\Theta)x_\Omega\in E^{-\ell,k}_{1,I_0,I_1,\diamond,v}$$

where Ω runs through all equivalence classes of (I_0,I_1)-atomic tuples with the fixed bidegree $(-\ell,k)$ and v as above, and $\Theta\in\Omega$ is maximally (I_0,I_1)-atomic with sign $\varepsilon(\Theta)\in\{1,-1\}$ as in Definition 2.13. The equality $x=x''+d^{-\ell-1,k}_{1,I_0,I_1}(x')$ then follows from Lemma 2.22.

Now we show that the induced map on cohomology is injective, namely for each $y\in E^{-\ell,k}_{1,I_0,I_1,\diamond,v}$, and $y'\in E^{-\ell-1,k}_{1,I_0,I_1,v}$ satisfying $y=d^{-\ell-1,k}_{1,I_0,I_1}(y')$, there exists $y''\in E^{-\ell-1,k}_{1,I_0,I_1,\diamond,v}$ such that $y=d^{-\ell-1,k}_{1,I_0,I_1}(y'')$.

By Lemma 2.19 we may assume that $c_\Theta(y')=0$ for each (I_0,I_1)-standard Θ. Then we take

$$y''\stackrel{\mathrm{def}}{=}\sum_{\Omega}c_\Theta(y')\varepsilon(\Theta)x_\Omega\in E_{1,I_0,I_1,\diamond,v}^{-\ell-1,k}$$

where Ω runs through all equivalence classes of (I_0,I_1)-atomic tuples with the fixed bidegree $(-\ell-1,k)$ and v as above and $\Theta\in\Omega$ is maximally (I_0,I_1)-atomic with sign $\varepsilon(\Theta)\in\{1,-1\}$ as in Definition 2.13. Then we have

$$d_{1,I_0,I_1}^{-\ell-1,k}(y'-y'')=y-d_{1,I_0,I_1}^{-\ell-1,k}(y'')\in E_{1,I_0,I_1,\diamond,v}^{-\ell,k},$$

which together with Lemma 2.22 (with bidegree $(-\ell,k)$ there replaced with $(-\ell-1,k)$) implies that $y'-y''=0$ and thus $y=d_{1,I_0,I_1}^{-\ell-1,k}(y'')$. The proof is thus finished. □

Remark 2.24. If $I_0=I_0^1$ (namely $I_0=\{1,\dots,\#I_0\}$), then each (I_0,I_1)-atomic tuple is automatically maximal and thus each equivalence class of (I_0,I_1)-atomic tuples contains exactly one element. Moreover, one can check that $d_{1,I_0,I_1}^{-\ell,k}(E_{1,I_0,I_1,\diamond}^{-\ell,k})=0$ and thus $E_{1,I_0,I_1,\diamond}^{-\ell,k}\cong E_{2,I_0,I_1}^{-\ell,k}$ for each bidegree $(-\ell,k)$ in this case.

2.5. **Computations in bottom two degrees.** In this section, we finish the computation of $\mathrm{Ext}_{G_n,\lambda}^h(F_{n,\Delta_n}(\lambda),\mathbf{C}_{I_0,I_1}(\lambda))$ when $h\in\{\#I_1-2\#I_0,\#I_1-2\#I_0+1\}$.

Lemma 2.25. *Let Ω be an equivalence class of (I_0,I_1)-atomic tuples with $(-\ell_\Omega,k_\Omega)=(-\ell,k)$.*

- *(i) If $k=\ell+\#I_1-2\#I_0$, then $d_{1,I_0,I_1}^{-\ell,k}(x_\Omega)=0$.*
- *(ii) If $k=\ell+\#I_1-2\#I_0+1$, then $d_{1,I_0,I_1}^{-\ell,k}(x_\Omega)\neq0$ if and only if $I_v\cup I_1=\Delta_n$ and there exists a unique $1\leq s\leq r_{I_v\cap I_1}$ satisfying $\Lambda_{r_{v,I_1,I}^{s-1}+1}\neq\emptyset$ and $I^d\cap I_0\neq\emptyset$ for some $r_{v,I_1,I}^{s-1}+1\leq d\leq r_{v,I_1,I}^s$. Moreover, if $d_{1,I_0,I_1}^{-\ell,k}(x_\Omega)\neq0$, then there exists a unique equivalence class Ω' of (I_0,I_1)-atomic tuples such that $d_{1,I_0,I_1}^{-\ell,k}(x_\Omega)=\pm x_{\Omega'}$, and Ω' also determines Ω uniquely.*

Proof. Let Ω' be an arbitrary equivalence class of (I_0,I_1)-atomic tuples that appears in (2.11), then it is clear that $k_{\Omega'}=k_\Omega$ and $\ell_{\Omega'}=\ell_\Omega-1$. By the choice of Ω' (see the second possibility of Lemma 2.21), there exists $\Theta=(v,I,\underline{k},\underline{\Lambda})\in\Omega$, $\Theta'=(v,I',\underline{k}',\underline{\Lambda}')\in\Omega'$ and $i\in I\setminus I'$ such that $\Theta=p_i^+(\Theta')$. We note from Lemma 2.16 that $\#\Lambda_d\leq1$ for each $1\leq d\leq r_I$. If $k=\ell+\#I_1-2\#I_0$, then there must exist $1\leq s\leq r_{I_v\cap I_1}$ and $r_{v,I_1,I'}^{s-1}+2\leq d'\leq r_{v,I_1,I'}^s$ such that $\Lambda'_{d'}=\emptyset$, which contradicts the fact that Θ' is (I_0,I_1)-atomic. Item (ii) thus follows. Similar argument works for the case $k=\ell+\#I_1-2\#I_0+1$ and either $\#I_v\cup I_1=n-2$ or there exists a unique $i\in\Delta_n\setminus I_v$ such that $\#v\cap\mathcal{B}_{n,\Delta_n\setminus\{i\}}\geq2$. It suffices to treat the case $k=\ell+\#I_1-2\#I_0+1$, $I_v\cup I_1=\Delta_n$ and $k_0=\#\Delta_n\setminus I_v$, and there exists a unique $1\leq s\leq r_{I_v\cap I_1}$ such that $\Lambda_{r_{v,I_1,I}^{s-1}+1}\neq\emptyset$. If $I^d\cap I_0=\emptyset$ for all $r_{v,I_1,I}^{s-1}+1\leq d\leq r_{v,I_1,I}^s$, the same argument as in item (ii) proves that Ω' cannot exist. If $I^d\cap I_0\neq\emptyset$ for some $r_{v,I_1,I}^{s-1}+1\leq d\leq r_{v,I_1,I}^s$, then we have $I^{d,-}\neq\emptyset$ and $\Lambda_d=\{(2\#I^{d,-}+1,\iota)\}$ (for some $\iota\in S$) for each such d, and Ω' is characterized by the following conditions

- $r_{v,I_1,I'}^{s'}=r_{v,I_1,I}^{s'}$ for each $1\leq s'\leq s-1$ and $r_{v,I_1,I'}^{s'}=r_{v,I_1,I}^{s'}+1$ for each $s\leq s'\leq r_{I_v\cap I_1}$;
- $\Lambda'_{d'}=\Lambda_{d'}$ for each $1\leq d'\leq r_{v,I_1,I'}^{s-1}$, $\Lambda'_{r_{v,I_1,I'}^{s-1}+1}=\emptyset$ and $\Lambda'_{d'}=\Lambda_{d'-1}$ for each $r_{v,I_1,I'}^{s-1}+2\leq d'\leq r_{I'}$.

It is then clear that Ω and Ω' uniquely determine each other, and thus item (iii) follows. □

Let $\#I_0 \leq \ell \leq \#I_1$ be an integer. We write $\Psi_{I_0,I_1}^{-\ell,k}$ for the set of equivalence classes Ω of (I_0,I_1)-atomic tuples satisfying $(-\ell_\Omega,k_\Omega)=(-\ell,\ell+\#I_1-2\#I_0)$ if $k=\ell+\#I_1-2\#I_0$. We write $\Psi_{I_0,I_1}^{-\ell,k}$ for the set of equivalence classes Ω of (I_0,I_1)-atomic tuples satisfying $(-\ell_\Omega,k_\Omega)=(-\ell,\ell+\#I_1-2\#I_0+1)$ and $d_{1,I_0,I_1}^{-\ell,k}(x_\Omega)=0$ (see item (iii) of Lemma 2.25) if $k=\ell+\#I_1-2\#I_0+1$.

Recall that $E_{2,I_0,I_1}^{-\ell,k}=\ker(d_{1,I_0,I_1}^{-\ell,k})/\mathrm{im}(d_{1,I_0,I_1}^{-\ell-1,k})$ for each $\#I_0\leq\ell\leq\#I_1$ and $k\geq 0$.

Proposition 2.26.
For each $\#I_0\leq\ell\leq\#I_1$, the subset

$$\{x_\Omega\}_{\Omega\in\Psi_{I_0,I_1}^{-\ell,k}}\subseteq\ker(d_{1,I_0,I_1}^{-\ell,k}) \tag{2.12}$$

induces a basis of $E_{2,I_0,I_1}^{-\ell,k}$ if $k\in\{\ell+\#I_1-2\#I_0,\ell+\#I_1-2\#I_0+1\}$.

Proof. This follows clearly from Proposition 2.23 and Lemma 2.25. Note that $E_{1,I_0,I_1,\diamond}^{-\ell,\ell+\#I_1-2\#I_0-1}=0$ by Lemma 2.16. □

Lemma 2.27. *The spectral sequence $\{E_{r,I_0,I_1}^{-\ell,k}\}_{r\geq 1,\#I_0\leq\ell\leq\#I_1,k\geq 0}$ degenerates at the second page.*

Proof. We set $\mathcal{B}_{n,\emptyset}^{\infty}\overset{\mathrm{def}}{=}\{\mathrm{val}_i\mid i\in\Delta_n\}$ and there exists a natural map between power sets $2^{\mathcal{B}_{n,\emptyset}}\to 2^{\mathcal{B}_{n,\emptyset}^{\infty}}$ by mapping v to the intersection $v\cap\mathcal{B}_{n,\emptyset}^{\infty}$. It is clear that we have a decomposition

$$E_{\bullet,I_0,I_1}^{\bullet,\bullet}\cong\bigoplus_{v\subseteq\mathcal{B}_{n,\emptyset}}E_{\bullet,I_0,I_1,v}^{\bullet,\bullet}$$

into its v-isotypic components and we set

$${}_{v^\infty}E_{\bullet,I_0,I_1}^{\bullet,\bullet}\cong\bigoplus_{v\subseteq\mathcal{B}_{n,\emptyset},v\cap\mathcal{B}_{n,\emptyset}^{\infty}=v^\infty}E_{\bullet,I_0,I_1,v}^{\bullet,\bullet}$$

with differential ${}_{v^\infty}d_{\bullet,I_0,I_1}^{\bullet,\bullet}$. Hence, it suffices to show that ${}_{v^\infty}d_{2,I_0,I_1}^{\bullet,\bullet}=0$. The spectral sequence ${}_{v^\infty}E_{\bullet,I_0,I_1}^{\bullet,\bullet}$ actually arises from the double complex ${}_{v^\infty}E_{0,I_0,I_1}^{\bullet,\bullet}$ with

$${}_{v^\infty}E_{0,I_0,I_1}^{-\ell,k}\cong\bigoplus_{I_0\subseteq I\subseteq I_{v^\infty}\cap I_1,\#I=\ell}\wedge^k\bar{\mathfrak{l}}_{n,I}^*$$

with the column complexes being direct sum of Koszul complexes for various Levi, and differential of row complexes being direct sum of

$$(-1)^{m(I,i)}\mathrm{Res}_{n,I,I\setminus\{i\}}^k:\wedge^k\bar{\mathfrak{l}}_{n,I}^*\to\wedge^k\bar{\mathfrak{l}}_{n,I\setminus\{i\}}^*$$

with $I_0\subseteq I\subseteq I_{v^\infty}\cap I_1$ and $i\in I\setminus I_0$. Following Section 14 of [BT82] we write δ for the row differential and D for the column differential of the double complex ${}_{v^\infty}E_{0,I_0,I_1}^{\bullet,\bullet}$. Note that ${}_{v^\infty}E_{0,I_0,I_1}^{\bullet,\bullet}$ contains a sub double complex ${}_{v^\infty,\natural}E_{0,I_0,I_1}^{\bullet,\bullet}$ with

$${}_{v^\infty,\natural}E_{0,I_0,I_1}^{-\ell,k}\cong\bigoplus_{I_0\subseteq I\subseteq I_{v^\infty}\cap I_1,\#I=\ell}(\wedge^k\bar{\mathfrak{l}}_{n,I}^*)^{\bar{\mathfrak{l}}_{n,I}}.$$

Note that D restricts to zero on ${}_{v^\infty,\natural}E_{0,I_0,I_1}^{\bullet,\bullet}$ and in fact the embedding ${}_{v^\infty,\natural}E_{0,I_0,I_1}^{-\ell,\bullet}\hookrightarrow{}_{v^\infty}E_{0,I_0,I_1}^{-\ell,\bullet}$ is a quasi-isomorphism for each $\#I_0\leq\ell\leq\#I_1$. Then we observe that $x_\Omega\in{}_{v^\infty}E_{2,I_0,I_1}^{-\ell,k}$ (for each equivalence class Ω of (I_0,I_1)-atomic tuple that show up) can be lifted to an element of $\widetilde{x}_\Omega\in{}_{v^\infty,\natural}E_{0,I_0,I_1}^{-\ell,k}\subseteq{}_{v^\infty}E_{0,I_0,I_1}^{-\ell,k}$ satisfying $D(\widetilde{x}_\Omega)=0$ and $\delta(\widetilde{x}_\Omega)=0$, which implies that ${}_{v^\infty}d_{2,I_0,I_1}^{-\ell,k}(x_\Omega)=0$ (using the description of ${}_{v^\infty}d_{2,I_0,I_1}^{-\ell,k}$ on page 162 of [BT82]). The proof is thus finished. □

Theorem 2.28. *Let $I_0 \subseteq I_1 \subseteq \Delta_n$ be a pair of subsets.*

- *If $\mathrm{Ext}^h_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), \mathbf{C}_{I_0,I_1}(\lambda)) \neq 0$, then $\#I_1 - 2\#I_0 \leq h \leq n^2 - n$.*
- *For each $h \in \{\#I_1 - 2\#I_0, \#I_1 - 2\#I_0 + 1\}$, the space $M^h_{I_0,I_1} \overset{def}{=} \mathrm{Ext}^h_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), \mathbf{C}_{I_0,I_1}(\lambda))$ admits a canonical decreasing filtration*
$$0 = \mathrm{Fil}^{-\#I_0+1}(M^h_{I_0,I_1}) \subseteq \mathrm{Fil}^{-\#I_0}(M^h_{I_0,I_1}) \subseteq \cdots \subseteq \mathrm{Fil}^{-\#I_1}(M^h_{I_0,I_1}) = M^h_{I_0,I_1}$$
such that $\mathrm{Fil}^{-\ell}(M^h_{I_0,I_1})/\mathrm{Fil}^{-\ell+1}(M^h_{I_0,I_1}) \cong E^{-\ell,\ell+h}_{2,I_0,I_1}$ admits a basis indexed by $\Psi^{-\ell,\ell+h}_{I_0,I_1}$ for each $\#I_0 \leq \ell \leq \#I_1$.

Proof. This follows directly from Proposition 2.26 and Lemma 2.27. □

2.6. **Twisted (I_0, I_1)-atomic tuples.** In this section, we consider a variant of (I_0, I_1)-atomic tuple as in Definition 2.13. Let $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ be a maximally (I_0, I_1)-atomic tuple and Ω be its equivalence class. We consider an integer $1 \leq s_0 \leq r_{I_v \cap I_1}$ that satisfies

Condition 2.29.
- *$I^{r^{s_0-1}_{v,I_1,I}+1} \subseteq I_0$ and $\Lambda_{r^{s_0-1}_{v,I_1,I}+1} = \emptyset$;*
- *$\Lambda_d = \{(2n_d - 1, \iota_d)\}$ for some $\iota_d \in S$, for each $r^{s_0-1}_{v,I_1,I} + 2 \leq d \leq r^{s_0}_{v,I_1,I}$ satisfying $I^d \cap I_0 = \emptyset$;*
- *$I^{d,+} = \emptyset$, $I^{d,-} \neq \emptyset$ and $\Lambda_d = \{(2\#I^{d,-} + 1, \iota_d)\}$ for some $\iota_d \in S$, for each $r^{s_0-1}_{v,I_1,I} + 2 \leq d \leq r^{s_0}_{v,I_1,I}$ satisfying $I^d \cap I_0 \neq \emptyset$.*

Then we choose a $r^{s_0-1}_{v,I_1,I} + 1 \leq d_0 \leq r^{s_0}_{v,I_1,I}$ and write $i_d \overset{\mathrm{def}}{=} \sum_{d'=1}^{d} n_{d'}$ and $i'_d \overset{\mathrm{def}}{=} \max\{i \in I \setminus I_0 \mid i < i_d\}$ for each $r^{s_0-1}_{v,I_1,I} + 1 \leq d \leq d_0$. Then we set
$$\Theta^{s_0,d_0} = (v, I^{s_0,d_0}, \underline{k}^{s_0,d_0}, \underline{\Lambda}^{s_0,d_0}) \overset{\mathrm{def}}{=} p^-_{i'_{d_0}} p^+_{i_{d_0-1}} \cdots p^-_{i'_2} p^+_{i_1}(\Theta)$$
and call it the (s_0, d_0)-*twist of* Θ. Note that Θ^{s_0,d_0} is well-defined thanks to Condition 2.29 and satisfies $\Lambda^{s_0,d_0}_d = \Lambda_{d+1}$ for each $r^{s_0-1}_{v,I_1,I} + 1 \leq d \leq d_0 - 1$, $\Lambda^{s_0,d_0}_{d_0} = \emptyset$ and $\Lambda^{s_0,d_0}_d = \Lambda_d$ for each $d_0 + 1 \leq d \leq r^{s_0}_{v,I_1,I}$. In particular, Θ^{s_0,d_0} satisfies

Condition 2.30.
- *$(I^{s_0,d_0})^{d_0} \subseteq I_0$ and $\Lambda_{d_0} = \emptyset$;*
- *$\Lambda_d = \{(2n_d - 1, \iota_d)\}$ for some $\iota_d \in S$, for each $r^{s_0-1}_{v,I_1,I} + 1 \leq d \leq r^{s_0}_{v,I_1,I}$ satisfying $d \neq d_0$ and $I^d \cap I_0 = \emptyset$;*
- *$I^{d,+} = \emptyset$, $I^{d,-} \neq \emptyset$ and $\Lambda_d = \{(2\#I^{d,-} + 1, \iota_d)\}$ for some $\iota_d \in S$, for each $d_0 + 1 \leq d \leq r^{s_0}_{v,I_1,I}$ satisfying $I^d \cap I_0 \neq \emptyset$.*
- *$I^{d,-} = \emptyset$, $I^{d,+} \neq \emptyset$ and $\Lambda_d = \{(2\#I^{d,+} + 1, \iota_d)\}$ for some $\iota_d \in S$, for each $r^{s_0-1}_{v,I_1,I} + 1 \leq d \leq d_0 - 1$ satisfying $I^d \cap I_0 \neq \emptyset$.*

Given two tuples $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ and $\Theta'' = (v, I'', \underline{k}'', \underline{\Lambda}'')$, we say that Θ'' is a (s_0, d_0)-*twisted improvement* of Θ' of level d if one of the following holds

- Θ'' is an improvement of Θ' of level d with $d \leq r^{s_0-1}_{v,I_1,I'}$ or $d \geq r^{s_0}_{v,I_1,I'} + 1$ (see Definition 2.12);
- $d \geq d_0$ and there exists $i \in (I')^d \setminus I_0$ and $i' = \sum_{d'=1}^{d} n_{d'} \in (I_v \cap I_1) \setminus I'$ such that $i \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$ and $\Theta'' = p^+_{i'} p^-_i(\Theta')$;
- $d \leq d_0 - 1$ and there exists $i \in (I'')^d \setminus I_0$ and $i' = \sum_{d'=1}^{d} n_{d'} \in (I_v \cap I_1) \setminus I''$ such that $i' \leq \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}$ and $\Theta' = p^+_{i'} p^-_i(\Theta'')$.

Similar to Definition 2.12, we can use the notion (s_0, d_0)-twisted improvement to define (s_0, d_0)-*twisted smaller* and (s_0, d_0)-*twisted equivalent.* Then it is easy to see that Θ^{s_0,d_0} is (s_0, d_0)-twisted smaller than no other tuple, and thus is *maximally* (s_0, d_0)-*twisted* (I_0, I_1)-*atomic.* Then a tuple is (s_0, d_0)-*twisted* (I_0, I_1)-*atomic* if it is (s_0, d_0)-twisted smaller than some tuple of the form Θ^{s_0,d_0}. Given Ω above, we write Ω^{s_0,d_0} for the (s_0, d_0)-twisted equivalence class of Θ^{s_0,d_0}, and call it the (s_0, d_0)-twist of Ω. Finally, we set $x_{\Omega^{s_0,d_0}} \stackrel{\text{def}}{=} \sum_{\Theta\in\Omega^{s_0,d_0}} \varepsilon(\Theta)x_\Theta$.

Lemma 2.31. *Let $\Theta' = (v, I', \underline{k}', \underline{\Lambda}')$ which has the same bidegree $(-\ell, k)$ as Θ. Assume that Θ satisfies Condition 2.29 and Θ' satisfies*

- $r^s_{v,I_1,I'} = r^s_{v,I_1,I}$ *for each* $1 \leq s \leq r_{I_v\cap I_1}$*;*
- $(I')^d = I^d$ *for each* $1 \leq d \leq r_{I'}$ *satisfying either* $d \leq r^{s_0-1}_{v,I_1,I'}$ *or* $d \geq r^{s_0}_{v,I_1,I'} + 1$*.*

Then $\Theta' \in \Omega^{s_0,d_0}$ if and only if the following holds

(i) $\underline{\Lambda}' = \underline{\Lambda}^{s_0,d_0}$*;*
(ii) $n'_{r^{s_0-1}_{v,I_1,I'}+1} \leq n^{s_0,d_0}_{r^{s_0-1}_{v,I_1,I'}+1}$ *if* $d_0 \geq r^{s_0-1}_{v,I_1,I'} + 2$*, and* $n'_{r^{s_0}_{v,I_1,I'}} \leq n^{s_0,d_0}_{r^{s_0}_{v,I_1,I'}}$ *if* $d_0 \leq r^{s_0}_{v,I_1,I'} - 1$*;*
(iii) $\sum_{d'=1}^{d_0-1} n'_{d'} \leq \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}$ *and* $\sum_{d'=1}^{d_0} n'_{d'} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$*.*

Proof. The 'only if' part follows immediately from the definition of a (s_0, d_0)-twisted improvement, and it suffices to prove the 'if' part. It is harmless to replace Θ' with a (s_0, d_0)-twisted maximal tuple in its (s_0, d_0)-twisted equivalence class, and we need to show that $\Theta' = \Theta^{s_0,d_0}$. Item (iii) (which implies $(I')^{d_0} \supseteq (I^{s_0,d_0})^{d_0}$) together with Θ' being (s_0, d_0)-twisted maximal forces $\sum_{d'=1}^{d_0-1} n'_{d'} = \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}$, $\sum_{d'=1}^{d_0} n'_{d'} = \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$ and in particular $(I')^{d_0} = (I^{s_0,d_0})^{d_0} \subseteq I_0$. The fact that Θ' is (s_0, d_0)-twisted maximal also implies that $\max\{m \mid (m, \iota) \in \Lambda'_d\} \geq 2n'_d - 2\#(I')^d \cap I_0 - 1$ for each $r^{s_0-1}_{v,I_1,I'} + 2 \leq d \leq r^{s_0}_{v,I_1,I'} - 1$ satisfying $d \neq d_0$, using arguments in the proof Lemma 2.16. Using item (i) and Condition 2.30, we have

$$n'_d \leq \#(I')^d \cap I_0 + \frac{1}{2}(1 + \max\{m \mid (m, \iota) \in \Lambda'_d\}) = n^{s_0,d_0} - \#(I^{s_0,d_0})^d \cap I_0 + \#(I')^d \cap I_0 \tag{2.13}$$

for each $r^{s_0-1}_{v,I_1,I'} + 2 \leq d \leq r^{s_0}_{v,I_1,I'} - 1$ satisfying $d \neq d_0$. Item (ii) implies that (2.13) holds for $d \in \{r^{s_0-1}_{v,I_1,I'} + 1, r^{s_0}_{v,I_1,I'}\} \setminus \{d_0\}$ as well. Sum up all the inequalities for $r^{s_0-1}_{v,I_1,I'} + 1 \leq d \leq r^{s_0}_{v,I_1,I'}$ (with $n'_{d_0} = n^{s_0,d_0}_{d_0}$), we obtain $\sum_{d=r^{s_0-1}_{v,I_1,I'}+1}^{r^{s_0}_{v,I_1,I'}} n'_d \leq \sum_{d=r^{s_0-1}_{v,I_1,I'}+1}^{r^{s_0}_{v,I_1,I'}} n^{s_0,d_0}_d$ which has to be an equality. Hence, all inequalities are equalities and we have $I' = I^{s_0,d_0}$ and thus $\Theta' = \Theta^{s_0,d_0}$. □

Proposition 2.32. *Let $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ be a maximally (I_0, I_1)-atomic tuple and Ω its equivalence class. Assume that Condition 2.29 holds for some $1 \leq s_0 \leq r_{I_v\cap I_1}$ and thus Θ^{s_0,d_0} as well as Ω^{s_0,d_0} are defined for each $r^{s_0-1}_{v,I_1,I} + 1 \leq d_0 \leq r^{s_0}_{v,I_1,I}$. Then we have*

$$x_{\Omega^{s_0,d_0}} - x_\Omega \in d^{-\ell-1,k}_{1,I_0,I_1}(E^{-\ell-1,k}_{1,I_0,I_1})$$

for each $1 \leq s_0 \leq r_{I_v\cap I_1}$ and $r^{s_0-1}_{v,I_1,I} + 1 \leq d_0 \leq r^{s_0}_{v,I_1,I}$.

Proof. As $\Theta^{s_0,r^{s_0-1}_{v,I_1,I}+1} = \Theta$ by definition, it suffices to assume that $d_0 \geq r^{s_0-1}_{v,I_1,I} + 2$ and prove that

$$x_{\Omega^{s_0,d_0}} - x_{\Omega^{s_0,d_0-1}} \in d^{-\ell-1,k}_{1,I_0,I_1}(E^{-\ell-1,k}_{1,I_0,I_1}).$$

Let $\Theta' = (v, I', \underline{k}', \underline{\Lambda}') \in \Omega^{s_0,d_0}$ be a tuple, and then set $i_{\Theta'} \stackrel{\text{def}}{=} \sum_{d'=1}^{d_0-1} n'_{d'} \in (I_v \cap I_1) \setminus I'$. Then we write $\Omega^{s_0,d_0,+}$ for the set of tuples of the form $p^+_{i_{\Theta'}}(\Theta')$ for some $\Theta' \in \Omega^{s_0,d_0}$, and define $x_{\Omega^{s_0,d_0,+}} \stackrel{\text{def}}{=}$

$\sum_{\Theta''\in\Omega^{s_0,d_0,+}}\varepsilon(\Theta'')x_{\Theta''}$. We claim that

$$d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Omega^{s_0,d_0,+}})=\varepsilon(\Omega,s_0,d_0)(x_{\Omega^{s_0,d_0}}-x_{\Omega^{s_0,d_0-1}}) \tag{2.14}$$

for some $\varepsilon(\Omega,s_0,d_0)\in\{1,-1\}$, which is clearly sufficient for our purpose.

Let $\Theta''\in\Omega^{s_0,d_0,+}$ and $\Theta'=(v,I',\underline{k}',\underline{\Lambda}')$ be an arbitrary tuple satisfying

$$c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Theta''}))\neq 0. \tag{2.15}$$

There exists a unique $1\leq d\leq r_{I''}$ and $i\in(I'')^d\setminus I'$ such that $\Theta''=p^+_i(\Theta')$. Condition 2.29 on Ω and the definition of $\Omega^{s_0,d_0,+}$ imply that $\#\Lambda''_{d'}\leq 1$ for each $1\leq d'\leq r_{I''}$ and in fact $\#\Lambda''_{d'}=1$ for each $r^{s_0-1}_{v,I_1,I''}+1\leq d'\leq r^{s_0}_{v,I_1,I''}$. Hence, we have the following two possibilities

- $\Lambda'_d=\emptyset$, $\Lambda'_{d+1}=\Lambda''_d$ and we set $d_1\overset{\mathrm{def}}{=}d$;
- $\Lambda'_d=\Lambda''_d$, $\Lambda'_{d+1}=\emptyset$ and we set $d_1\overset{\mathrm{def}}{=}d+1$.

In particular, we always have $\Lambda'_{d_1}=\emptyset$ and thus d_1 is uniquely determined by Θ' (as the unique $r^{s_0-1}_{v,I_1,I'}+1\leq d'\leq r^{s_0}_{v,I_1,I'}$ satisfying $\Lambda'_{d'}=\emptyset$). Now we fix a tuple Θ' such that there exists $\Theta''\in\Omega^{s_0,d_0,+}$ that satisfies (2.15), and then write $i_{\Theta'}\overset{\mathrm{def}}{=}\sum_{d'=1}^{d_1}n'_{d'}$ and $i'_{\Theta'}\overset{\mathrm{def}}{=}\sum_{d'=1}^{d_1-1}n'_{d'}$ for short. We note that such Θ'' has to be either $p^+_{i_{\Theta'}}(\Theta')$ or $p^+_{i'_{\Theta'}}(\Theta')$. We clearly have $\varepsilon(p^+_{i_{\Theta'}}(\Theta'))=(-1)^{n'_{d_1}}\varepsilon(p^+_{i'_{\Theta'}}(\Theta'))$ and

$$c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i_{\Theta'}}(\Theta')}))=(-1)^{n'_{d_1}-1}c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i'_{\Theta'}}(\Theta')})).$$

We have the following observations for Θ'.

- Any $\Theta''\in\Omega^{s_0,d_0,+}$ satisfying (2.15) must be either $p^+_{i_{\Theta'}}(\Theta')$ or $p^+_{i'_{\Theta'}}(\Theta')$, and at least one of them is in $\Omega^{s_0,d_0,+}$ by our choice of Θ'.
- If $p^+_{i_{\Theta'}}(\Theta'),p^+_{i'_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$, then we have

$$c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Omega^{s_0,d_0,+}}))=\varepsilon(p^+_{i_{\Theta'}}(\Theta'))c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i_{\Theta'}}(\Theta')}))+\varepsilon(p^+_{i'_{\Theta'}}(\Theta'))c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i'_{\Theta'}}(\Theta')}))=0. \tag{2.16}$$

- If $p^+_{i_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$ and $p^+_{i'_{\Theta'}}(\Theta')\notin\Omega^{s_0,d_0,+}$, we have

$$c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Omega^{s_0,d_0,+}}))=\varepsilon(p^+_{i_{\Theta'}}(\Theta'))c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i_{\Theta'}}(\Theta')})). \tag{2.17}$$

- If $p^+_{i_{\Theta'}}(\Theta')\notin\Omega^{s_0,d_0,+}$ and $p^+_{i'_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$, we have

$$c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{\Omega^{s_0,d_0,+}}))=\varepsilon(p^+_{i'_{\Theta'}}(\Theta'))c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i'_{\Theta'}}(\Theta')})). \tag{2.18}$$

- If $d_1\notin\{d_0-1,d_0\}$, then we must have $p^+_{i_{\Theta'}}(\Theta'),p^+_{i'_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$.

We claim the following two facts

(i) If $d_1=d_0-1$, then we always have $p^+_{i_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$, and $p^+_{i'_{\Theta'}}(\Theta')\notin\Omega^{s_0,d_0,+}$ if and only if $\Theta'\in\Omega^{s_0,d_0-1}$.

(ii) If $d_1=d_0$, then we always have $p^+_{i'_{\Theta'}}(\Theta')\in\Omega^{s_0,d_0,+}$, and $p^+_{i_{\Theta'}}(\Theta')\notin\Omega^{s_0,d_0,+}$ if and only if $\Theta'\in\Omega^{s_0,d_0}$.

We first treat item (i), namely the case when $d_1 = d_0 - 1$. We note that $p^+_{i_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ (resp. $p^+_{i'_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$) forces $i'_{\Theta'} \leq \sum_{d'=1}^{d_0-2} n^{s_0,d_0}_{d'}$ (resp. $i_{\Theta'} \leq \sum_{d'=1}^{d_0-2} n^{s_0,d_0}_{d'}$), therefore we always have

$$\sum_{d'=1}^{d_0-2} n'_{d'} = i'_{\Theta'} \leq \sum_{d'=1}^{d_0-2} n^{s_0,d_0}_{d'} = \sum_{d'=1}^{d_0-2} n^{s_0,d_0-1}_{d'}.$$

So it follows from Lemma 2.31 (with d_0 replacing with d_0-1) that $\Theta' \in \Omega^{s_0,d_0-1}$ if and only if

$$\sum_{d'=1}^{d_0-1} n'_{d'} = i_{\Theta'} \geq \sum_{d'=1}^{d_0-1} n^{s_0,d_0-1}_{d'}. \tag{2.19}$$

We have the following possibilities

- If (2.19) holds, then we clearly deduce $p^+_{i'_{\Theta'}}(\Theta') \notin \Omega^{s_0,d_0,+}$ from the definition of $\Omega^{s_0,d_0,+}$ (namely any $\Theta'' \in \Omega^{s_0,d_0,+}$ should satisfy $\sum_{d'=1}^{d_0-2} n''_{d'} \leq \sum_{d'=1}^{d_0-2} n^{s_0,d_0}_{d'} = \sum_{d'=1}^{d_0-2} n^{s_0,d_0-1}_{d'}$).
- If (2.19) is false, then we deduce from $i_{\Theta'} \notin I_0$ that

$$\sum_{d'=1}^{d_0-1} n'_{d'} = i_{\Theta'} \leq \sum_{d'=1}^{d_0-2} n^{s_0,d_0-1}_{d'} = \sum_{d'=1}^{d_0-2} n^{s_0,d_0}_{d'}. \tag{2.20}$$

  Assume on the contrary that $p^+_{i'_{\Theta'}}(\Theta') \notin \Omega^{s_0,d_0,+}$, then we must have $p^+_{i_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ by the choice of Θ', and thus there exists $\Theta''' \in \Omega^{s_0,d_0}$ such that $p^+_{i_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''}}(\Theta''')$. But this together with (2.20) and Lemma 2.31 implies that $p^+_{i'_{\Theta'}}p^-_{i_{\Theta'}}(\Theta''') \in \Omega^{s_0,d_0}$ and

$$p^+_{i'_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''}}\left(p^+_{i'_{\Theta'}}p^-_{i_{\Theta'}}(\Theta''')\right) \in \Omega^{s_0,d_0,+},$$

  a contradiction. Hence we deduce that $p^+_{i'_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ and thus there exists $\Theta'''' \in \Omega^{s_0,d_0}$ such that $p^+_{i'_{\Theta'}}(\Theta') = p^+_{i'_{\Theta''''}}(\Theta'''')$ (with $i'_{\Theta''''} \leq \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}$ and $i_{\Theta''''} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$). We consider the unique tuple Θ''''' that satisfies $\Theta'''' = p^+_{i'_{\Theta'}}p^-_{i_{\Theta'}}(\Theta''''')$, which clearly satisfies $i'_{\Theta'''''} = i'_{\Theta''''} \leq \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}$ and $i_{\Theta'''''} = i_{\Theta''''} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$, and thus $\Theta''''' \in \Omega^{s_0,d_0}$ by Lemma 2.31. Consequently, we have $p^+_{i_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''''}}(\Theta''''') \in \Omega^{s_0,d_0,+}$.

Now we treat item (ii), namely the case when $d_1 = d_0$. We note that $p^+_{i'_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ (resp. $p^+_{i'_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$) forces $i_{\Theta'} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$ (resp. $i'_{\Theta'} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$), therefore we always have

$$\sum_{d'=1}^{d_0} n'_{d'} = i_{\Theta'} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}.$$

So it follows from Lemma 2.31 that $\Theta' \in \Omega^{s_0,d_0}$ if and only if

$$\sum_{d'=1}^{d_0-1} n'_{d'} = i'_{\Theta'} \leq \sum_{d'=1}^{d_0-1} n^{s_0,d_0}_{d'}. \tag{2.21}$$

We have the following possibilities

- If (2.21) holds, then we clearly deduce $p^+_{i_{\Theta'}}(\Theta') \notin \Omega^{s_0,d_0,+}$ from the definition of $\Omega^{s_0,d_0,+}$ (namely any $\Theta'' \in \Omega^{s_0,d_0,+}$ should satisfy $\sum_{d'=1}^{d_0-1} n''_{d'} \geq \sum_{d'=1}^{d_0} n^{s_0,d_0}_{d'}$).

- If (2.21) is false, then we deduce from $i'_{\Theta'} \notin I_0$ that

$$\sum_{d'=1}^{d_0-1} n'_{d'} = i'_{\Theta'} \geq \sum_{d'=1}^{d_0} n_{d'}^{s_0,d_0}. \tag{2.22}$$

Assume on the contrary that $p^+_{i_{\Theta'}}(\Theta') \notin \Omega^{s_0,d_0,+}$, then we must have $p^+_{i'_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ by the choice of Θ', and thus there exists $\Theta''' \in \Omega^{s_0,d_0}$ such that $p^+_{i'_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''}}(\Theta''')$ and $i'_{\Theta'''} \leq \sum_{d'=1}^{d_0} n_{d'}^{s_0,d_0}$. But this together with (2.22) and Lemma 2.31 implies that $p^+_{i_{\Theta'}} p^-_{i'_{\Theta'}}(\Theta''') \in \Omega^{s_0,d_0}$ and

$$p^+_{i_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''}}\left(p^+_{i_{\Theta'}} p^-_{i'_{\Theta'}}(\Theta''')\right) \in \Omega^{s_0,d_0,+},$$

a contradiction. Hence we deduce that $p^+_{i_{\Theta'}}(\Theta') \in \Omega^{s_0,d_0,+}$ and thus there exists $\Theta'''' \in \Omega^{s_0,d_0}$ such that $p^+_{i_{\Theta'}}(\Theta') = p^+_{i'_{\Theta''''}}(\Theta'''')$ (with $i'_{\Theta''''} \leq \sum_{d'=1}^{d_0-1} n_{d'}^{s_0,d_0}$ and $i_{\Theta''''} = i_{\Theta'} \geq \sum_{d'=1}^{d_0} n_{d'}^{s_0,d_0}$). We consider the unique tuple Θ''''' that satisfies $\Theta'''' = p^+_{i_{\Theta'}} p^-_{i'_{\Theta'}}(\Theta''''')$, which clearly satisfies $i'_{\Theta'''''} = i'_{\Theta''''} \leq \sum_{d'=1}^{d_0-1} n_{d'}^{s_0,d_0}$ and $i_{\Theta'''''} = i'_{\Theta'} \geq \sum_{d'=1}^{d_0} n_{d'}^{s_0,d_0}$, and thus $\Theta''''' \in \Omega^{s_0,d_0}$ by Lemma 2.31. Consequently, we have $p^+_{i'_{\Theta'}}(\Theta') = p^+_{i'_{\Theta'''''}}(\Theta''''') \in \Omega^{s_0,d_0,+}$.

Finally, we note that there exists a number $\varepsilon(\Omega, s_0, d_0) \in \{1, -1\}$ depending only on Ω, s_0 and d_0 such that, for each Θ' considered above, we have

$$\varepsilon(p^+_{i_{\Theta'}}(\Theta')) c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i_{\Theta'}}(\Theta')})) = -\varepsilon(\Omega, s_0, d_0)\varepsilon(\Theta')$$

if $d_1 = d_0 - 1$, and

$$\varepsilon(p^+_{i'_{\Theta'}}(\Theta')) c_{\Theta'}(d^{-\ell-1,k}_{1,I_0,I_1}(x_{p^+_{i'_{\Theta'}}(\Theta')})) = \varepsilon(\Omega, s_0, d_0)\varepsilon(\Theta')$$

if $d_1 = d_0$. This together with (2.17) and (2.18) finish the proof of (2.14). □

3. A result on decomposition of locally analytic distributions

In this section, we prove a technical result on (tensor) decomposition of certain space of locally analytic distributions in Proposition 3.9, which is essential for our computations of N-(co)homologies using Bruhat stratifications in Section 4.1.

Lemma 3.1. *Let I be a set and $(V_i)_{i\in I}$ be a projective system of locally convex E vector space and W another locally convex E vector space. Then we have a canonical topological isomorphism*

$$(\varprojlim_i V_i)\widehat{\otimes}_{E,\pi} W \cong \varprojlim_i (V_i \widehat{\otimes}_{E,\pi} W).$$

If there exists a locally convex E-vector space A with separately continuous algebra structure such that W is a separately continuous left A-module and V_i is a separately continuous right A-module for each $i \in I$, then we have a canonical isomorphism

$$(\prod_i V_i)\widehat{\otimes}_{A,\pi} W \cong \prod_i (V_i \widehat{\otimes}_{A,\pi} W).$$

Proof. This is standard (using for example Corollary 17.9 of [S02]). □

According to Proposition 17.6 of [S02], we can identify $V_1\widehat{\otimes}_{E,\iota}V_2$ and $V_1\widehat{\otimes}_{E,\pi}V_2$ for two Frechét spaces V_1, V_2 and sometimes write $V_1\widehat{\otimes}_E V_2$ for both of them. Note that a countable projective limit of Fréchet spaces is still Fréchet, but a countable inductive limit (in particular countable direct sum) of Fréchet spaces is in general not Fréchet. Consequently, the notation $V_1\widehat{\otimes}_{E,\iota}V_2$ would appear in the sequel typically when V_2 is not Fréchet (say a countable direct sum of Fréchet spaces).

For each p-adic manifold M (paracompact and finite dimensional), we can consider the space of E-valued locally analytic functions $C^{\mathrm{an}}(M,E)$, whose strong dual is the space of distributions $D(M,E)$. For each closed subset $C\subseteq M$, we consider the closed subspace $D(M,E)_C\subseteq D(M,E)$ consisting of distributions supported in C. If C is furthermore compact, we have an isomorphism $D(M,E)_C\cong(\varinjlim_U C^{\mathrm{an}}(U,E))'$ where U runs through open subset of M containing C. In particular, for each open subset $U\subseteq M$ containing C, we have a canonical isomorphism $D(U,E)_C\cong D(M,E)_C$.

Lemma 3.2. *Let M be a p-adic manifold and $C_1\subseteq C_2$ two compact subsets of M. Then we have a topological isomorphism*

$$D(M,E)_{C_2}/D(M,E)_{C_1}\cong\varprojlim_C D(M,E)_C$$

where C runs through compact open subsets of $C_2\setminus C_1$.

Proof. As C is compact open in C_2, $C_2\cong C\sqcup(C_2\setminus C)$ (with disjoint union topology) and thus $D(M,E)_{C_2}\cong D(M,E)_C\oplus D(M,E)_{C_2\setminus C}$ which induces a canonical projection $D(M,E)_{C_2}\twoheadrightarrow D(M,E)_C$ with kernel $D(M,E)_{C_2\setminus C}\subseteq D(M,E)_{C_2}$. For each pair of compact open subsets $C\subseteq C'$ of $C_2\setminus C_1$, we have a canonical projection map $D(M,E)_{C'}\cong D(M,E)_C\oplus D(M,E)_{C'\setminus C}\twoheadrightarrow D(M,E)_C$, and thus we obtain a continuous map $D(M,E)_{C_2}\to\varprojlim_C D(M,E)_C$ with dense image and kernel $\bigcap_C D(M,E)_{C_2\setminus C}$. As $C_2\setminus C_1$ is a union of its compact open subsets, the intersection of $C_2\setminus C$ for all possible compact open $C\subseteq C_2\setminus C_1$ is exactly C_1, which implies that $\bigcap_C D(M,E)_{C_2\setminus C}=D(M,E)_{C_1}$. Now we prove the surjectivity of $D(M,E)_{C_2}\to\varprojlim_C D(M,E)_C$ (which implies that $D(M,E)_{C_2}/D(M,E)_{C_1}\to\varprojlim_C D(M,E)_C$ is a topological isomorphism). As C_2 is compact, it suffices to replace M with a compact open neighborhood of C_2 and thus assume that M is compact, which implies that $C^{\mathrm{an}}(M,E)=\varinjlim_r C_r^{\mathrm{an}}(M,E)$ where $C_r^{\mathrm{an}}(M,E)\subseteq C^{\mathrm{an}}(M,E)$ is the subspace consisting of locally analytic functions that are analytic on each open disc of radius r. By taking dual, we have $D(M,E)=\varprojlim_r D_r(M,E)$ with $D_r(M,E)\stackrel{\mathrm{def}}{=}(C_r^{\mathrm{an}}(M,E))'$. For each compact subset $C\subseteq M$, we write $D_r(M,E)_C$ for the image of $D(M,E)_C\subseteq D(M,E)$ in $D_r(M,E)$, then we have $D(M,E)_C\cong\varprojlim_r D_r(M,E)_C$. If we write C_r for the union of all (finite number of) open disc of radius r that intersect C non-trivially, then C_r is a compact open subset of M that contains C and satisfies $D_r(M,E)_C=D_r(M,E)_{C_r}$. Now let C be a compact open subset of $C_2\setminus C_1$. For each fixed r, C_r stablizes when C is sufficiently large, and thus $\varprojlim_C D_r(M,E)_C=D_r(M,E)_C$ for some sufficiently large C. Hence, the canonical map $D_r(M,E)_{C_2}\to\varprojlim_C D_r(M,E)_C$ is a continuous surjection (of Banach spaces). For each $0<r'<r<1$, $D_{r'}(M,E)_{C_2}\to D_r(M,E)_{C_2}$ and $\varprojlim_C D_{r'}(M,E)_C\to\varprojlim_C D_r(M,E)_C$ are injective, and thus $\varprojlim_r D_r(M,E)_{C_2}\to\varprojlim_r\varprojlim_C D_r(M,E)_C$ is a continuous surjection. Note that we clearly have $\varprojlim_r\varprojlim_C D_r(M,E)_C=\varprojlim_C\varprojlim_r D_r(M,E)_C=\varprojlim_C D(M,E)_C$, and the proof is thus finished. □

Lemma 3.3. *Let M_1,M_2 two p-adic manifolds with closed subsets $C_1\subseteq M_1$ and $C_2\subseteq M_2$. Then we have a canonical isomorphism $D(M_1\times M_2,E)_{C_1\times C_2}\cong D(M_1,E)_{C_1}\widehat{\otimes}_{E,\iota}D(M_2,E)_{C_2}$.*

Proof. This is Lemma 3.2.12 of [BD19]. □

Let $G_{/\mathbb{Q}_p}$ be a reductive group and $P_0 \subseteq G$ be a fixed minimal parabolic subgroup containing a fixed maximal split torus L_0. Let $P_1, P_2 \subseteq G$ be two parabolic subgroups containing P_0. We can associate with P_1, P_2 the Bruhat stratification $\{P_1 w P_2\}_{W^{P_1,P_2}}$ where W^{P_1,P_2} is the corresponding set of minimal length representatives. We write $P_1' \stackrel{\mathrm{def}}{=} w^{-1}P_1 w$, $Y \stackrel{\mathrm{def}}{=} P_1' P_2 \subseteq G$, $\overline{Y}$ for the (reduced) Zariski closure of Y in G and $Z \stackrel{\mathrm{def}}{=} \overline{Y} \setminus Y$ with the reduced scheme structure. We write N_2 for the unipotent radical of P_2 and $L_2 \subseteq P_2$ its maximal Levi subgroup containing L_0. Then we set $L_2^\flat \stackrel{\mathrm{def}}{=} L_2 \cap P_1'$ and $N_2^\flat \stackrel{\mathrm{def}}{=} N_2 \cap P_1'$. Note that

$$Y \cong P_1' \times (P_1' \backslash Y) \cong P_1' \times (P_1' \cap P_2 \backslash P_2) \cong P_1' \times (L_2^\flat \backslash L_2) \times (N_2^\flat \backslash N_2)$$

and the isomorphisms preserve the natural left P_1'-action and right P_2-action on each term. Note that $L_2^\flat$ is a parabolic subgroup inside L_2, and thus $L_2^\flat \backslash L_2$ is proper.

We write $D(G)_{\overline{Y}} \stackrel{\mathrm{def}}{=} D(G(\mathbb{Q}_p), E)_{\overline{Y}(\mathbb{Q}_p)}$ for short and similarly for others (in particular, we omit E from all spaces of distributions). As G, $\overline{Y}$ and Z are left P_1'-stable and right P_2-stable, $D(G)_{\overline{Y}}$ and $D(G)_Z$ are naturally left $D(P_2)$-modules and right $D(P_1')$-modules. In the following, we will frequently abbreviate $P_1'(\mathbb{Q}_p)$ as P_1' and the others are similar.

We set $\widehat{D}(G)_Y \stackrel{\mathrm{def}}{=} \left(\varprojlim_C D(P_1' \backslash G)_C\right) \widehat{\otimes}_{E,\iota} D(P_1')$ where C running through all compact open subsets of $P_1' \backslash Y(\mathbb{Q}_p)$.

Lemma 3.4. *The limit $\widehat{D}(G)_Y = \left(\varprojlim_C D(P_1' \backslash G)_C\right) \widehat{\otimes}_{E,\iota} D(P_1')$ is canonically a left $D(P_2)$-module and a right $D(P_1')$-module.*

Proof. The right $D(P_1')$-module structure is obvious from the definition of $\widehat{D}(G)_Y$. Let $H_2 \subseteq P_2(\mathbb{Q}_p)$ be an arbitrary compact open subgroup. Then for each compact open subset $C \subseteq P_1' \backslash Y(\mathbb{Q}_p)$, $CH_2 \subseteq P_1' \backslash Y(\mathbb{Q}_p)$ is an open closed subset (containing C) which is right H_2-stable. Consequently, right H_2-stable compact open subsets of $P_1' \backslash Y(\mathbb{Q}_p)$ are cofinal among all compact open subsets of $P_1' \backslash Y(\mathbb{Q}_p)$, which implies that $\varprojlim_C D(P_1' \backslash G)_C$ is canonically a left $D(H_2)$-module. There clearly exist a discrete subset $J_2 \subseteq P_2(\mathbb{Q}_p)$ such that $P_2(\mathbb{Q}_p) = \bigsqcup_{g_2 \in J_2} H_2 g_2$ as a p-adic manifold. For each compact open subset C of $P_1' \backslash Y(\mathbb{Q}_p)$ which is right H_2-stable, we have canonical isomorphisms $\ell_{g_2^{-1}} : D(P_1' \backslash G)_C \to \delta_{g_2^{-1}} * D(P_1' \backslash G)_{P_1' C} = D(P_1' \backslash G)_{C \cdot g_2}$. If we take inverse limit among all such C, we obtain the canonical left $D(P_2)$-action on $\varprojlim_C D(P_1' \backslash G)_C$ and thus on $\widehat{D}(G)_Y = \left(\varprojlim_C D(P_1' \backslash G)_C\right) \widehat{\otimes}_{E,\iota} D(P_1')$ as well. □

Proposition 3.5. *There exists a canonical isomorphisms $D(G)_{\overline{Y}}/D(G)_Z \cong \widehat{D}(G)_Y$ which respects the left $D(P_2)$-action and right $D(P_1')$-action.*

Proof. It follows from Lemma 3.3 that

$$D(G)_{\overline{Y}} \cong D(P_1' \backslash G)_{P_1' \backslash \overline{Y}} \widehat{\otimes}_{E,\iota} D(P_1') \text{ and } D(G)_Z \cong D(P_1' \backslash G)_{P_1' \backslash Z} \widehat{\otimes}_{E,\iota} D(P_1'),$$

and thus

$$(D(G)_{\overline{Y}}/D(G)_Z \cong \left(D(P_1' \backslash G)_{P_1' \backslash \overline{Y}}/D(P_1' \backslash G)_{P_1' \backslash Z}\right) \widehat{\otimes}_{E,\iota} D(P_1').$$

Consequently, it remains to construct a canonical $D(P_2)$-equivariant isomorphism

$$D(P_1' \backslash G)_{P_1' \backslash \overline{Y}}/D(P_1' \backslash G)_{P_1' \backslash Z} \cong \varprojlim_C D(P_1' \backslash G)_C \tag{3.1}$$

according to the definition of $\widehat{D}(G)_Y$. This follows from Lemma 3.2 as well as the facts that both $(P_1'\backslash\overline{Y})(\mathbb{Q}_p)$ and $(P_1'\backslash Z)(\mathbb{Q}_p)$ are compact subsets of $(P_1'\backslash G)(\mathbb{Q}_p)$. Now we check the left $D(P_2)$-equivariance. We borrow notation from the proof of Lemma 3.4. If C is a right H_2-stable compact open subset of $P_1'\backslash Y(\mathbb{Q}_p)$, then the map $D(P_1'\backslash G)_{P_1'\backslash\overline{Y}}/D(P_1'\backslash G)_{P_1'\backslash Z}\to D(P_1'\backslash G)_C$ is clearly left $D(H_2)$-equivariant (as the $D(H_2)$-actions on both sides are compatible with that on $D(G)$). The extension to left $D(P_2)$-equivariance of (3.1) follows from the same argument as in the proof of Lemma 3.4. $\square$

Let $G_0\subseteq G(\mathbb{Q}_p)$ be a compact open subgroup and choose $H_1\subseteq G_0\cap P_1'(\mathbb{Q}_p)$ and $H_2\subseteq G_0\cap P_2(\mathbb{Q}_p)$. Then exactly the same proof as that of (4.87) of [Schr11] shows that we have a canonical topological isomorphism (with all the involved spaces being Fréchet)

$$D(G_0)_{H_1H_2}\cong D(H_2)\widehat{\otimes}_{D(H_2)_{H_2\cap H_1}}D(G_0)_{H_1}. \tag{3.2}$$

It is clearly harmless to replace G_0 with G as it does not change the corresponding space of distributions. So we drop G_0 from now on and consider compact open subgroups $H_1\subseteq P_1'(\mathbb{Q}_p)$ and $H_2\subseteq P_2(\mathbb{Q}_p)$. We fix discrete subset $J_1\subseteq P_1'(\mathbb{Q}_p)$ and $J_2\subseteq P_2(\mathbb{Q}_p)$ such that $P_1'(\mathbb{Q}_p)=\bigsqcup_{g_1\in J_1}g_1H_1$ and $P_2(\mathbb{Q}_p)=\bigsqcup_{g_2\in J_2}H_2g_2$ as p-adic manifolds. We set $H_2'\overset{\mathrm{def}}{=}H_2\cap L_2(\mathbb{Q}_p)$ and $H_2''\overset{\mathrm{def}}{=}H_2\cap N_2(\mathbb{Q}_p)$.

Definition 3.6. We say the pair (H_1,H_2) is *good* if

(i) $H_2=H_2'H_2''$, $H_2\cap P_1'(\mathbb{Q}_p)=H_2\cap H_1=(H_2'\cap H_1)(H_2''\cap H_1)$;
(ii) $(H_2'\cap H_1)\backslash H_2'\cong L_2^\flat(\mathbb{Q}_p)\backslash L_2(\mathbb{Q}_p)$; and
(iii) $P_1'(\mathbb{Q}_p)H_2=\bigsqcup_{g_1\in J_1}g_1H_1H_2$.

Item (i) clearly implies topological isomorphisms (of Fréchet spaces) $D(H_2)\cong D(H_2')\widehat{\otimes}_ED(H_2'')$ and $D(H_2)_{H_2\cap H_1}\cong D(H_2')_{H_2'\cap H_1}\widehat{\otimes}_ED(H_2'')_{H_2''\cap H_1}$ with the convolution having the form induced from semi-direct product. These topological isomorphisms together with (3.2) implies that

$$D(G)_{H_1H_2}\cong D(H_2')\widehat{\otimes}_{D(H_2')_{H_2'\cap H_1}}(D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap H_1}}D(G)_{H_1}).$$

By convolution with $\delta_{g_1^{-1}}$ from the right, we obtain

$$D(G)_{g_1H_1H_2}\cong D(H_2')\widehat{\otimes}_{D(H_2')_{H_2'\cap H_1}}(D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap H_1}}D(G)_{g_1H_1}).$$

for each $g_1\in P_1'(\mathbb{Q}_p)$. As $P_1'(\mathbb{Q}_p)$ is closed in $G(\mathbb{Q}_p)$ and H_2 is compact, $P_1'(\mathbb{Q}_p)H_2$ is closed in $G(\mathbb{Q}_p)$. By taking direct sum over all $g_1\in J_1$ and using item (iii) of Definition 3.6 (and the fact that $\widehat{\otimes}_\iota$ commutes with direct sum), we obtain (by abusing P_1' for $P_1'(\mathbb{Q}_p)$)

$$D(G)_{P_1'H_2}\cong D(H_2')\widehat{\otimes}_{D(H_2')_{H_2'\cap P_1'}}(D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}}D(G)_{P_1'}).$$

According to item (ii) of Definition 3.6, we can replace $D(H_2')\widehat{\otimes}_{D(H_2')_{H_2'\cap H_1}}$ with $D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}}$ and deduce

$$D(G)_{P_1'H_2}\cong D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}}(D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}}D(G)_{P_1'}). \tag{3.3}$$

(Note that $L_2^\flat\backslash L_2$ is compact and thus we do not distinguish between $D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat},\iota}$ with $D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat},\pi}$.) By factoring out $\widehat{\otimes}_{E,\iota}D(P_1')$ from both sides of (3.3), we obtain the following isomorphism (with $D(P_1'\backslash G)_{P_1'\backslash P_1'H_2}$, $D(H_2'')$ and $D(P_1'\backslash G)_1$ being Fréchet)

$$D(P_1'\backslash G)_{P_1'\backslash P_1'H_2}\cong D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}}(D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}}D(P_1'\backslash G)_1). \tag{3.4}$$

We have the following two observations

- If (H_1, H_2) is a good pair, then (gH_1g^{-1}, gH_2g^{-1}) is a good pair for any $g \in L_0(\mathbb{Q}_p)$.
- For each compact open subgroup $H \subseteq N_2(\mathbb{Q}_p)$, we have $N_2(\mathbb{Q}_p) = \bigcup_{g\in L_0(\mathbb{Q}_p)} gHg^{-1}$. In particular $\{gHg^{-1}\}_{g\in L_0(\mathbb{Q}_p)}$ is cofinal among all compact open subgroups of $N_2(\mathbb{Q}_p)$ under inclusion.

Consequently, if we take (countable) projective limit with respect all possible H_2 that appears in a good pair (H_1, H_2) (and then use Lemma 3.1 for $D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat},\pi})$, we conclude that

$$\varprojlim_{H_2} D(P_1'\backslash G)_{P_1'\backslash P_1'H_2} \cong D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}} \left(\varprojlim_{H_2''} D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}} D(P_1'\backslash G)_1\right). \tag{3.5}$$

We have $H_2''\cap P_1' = H_2''\cap N_2^\flat$, $(H_2''\cap N_2^\flat)\backslash H_2'' \cong N_2^\flat\backslash N_2^\flat H_2''$, as well as canonical topological isomorphisms

$$D(N_2^\flat H_2'') \cong D(N_2^\flat\backslash N_2^\flat H_2'')\widehat{\otimes}_{E,\iota} D(N_2^\flat)$$

and

$$D(N_2^\flat\backslash N_2^\flat H_2'')_1\widehat{\otimes}_{E,\iota} D(N_2^\flat) \cong D(N_2^\flat H_2'')_{N_2^\flat} \cong D(N_2)_{N_2^\flat} \cong D(N_2^\flat\backslash N_2)_1\widehat{\otimes}_{E,\iota} D(N_2^\flat).$$

Note that for each Fréchet space V and $* \in \{\pi, \iota\}$, $D(N_2^\flat H_2'')\widehat{\otimes}_{D(N_2)_{N_2^\flat},*} V$ is a quotient of $D(N_2^\flat\backslash N_2^\flat H_2'')\widehat{\otimes}_{E,*} V$ by a closed subspace with $D(N_2^\flat\backslash N_2^\flat H_2'')$ being Fréchet, so we do not distinguish between $D(N_2^\flat H_2'')\widehat{\otimes}_{D(N_2)_{N_2^\flat},\pi}$ and $D(N_2^\flat H_2'')\widehat{\otimes}_{D(N_2)_{N_2^\flat},\iota}$. Consequently, we have a topological isomorphism

$$D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}} D(P_1'\backslash G)_1 \cong D(N_2^\flat H_2'')\widehat{\otimes}_{D(N_2)_{N_2^\flat}} D(P_1'\backslash G)_1$$

for each H_2 that appears in a good pair (H_1, H_2), which implies that (using Lemma 3.1)

$$\varprojlim_{H_2''} D(H_2'')\widehat{\otimes}_{D(H_2'')_{H_2''\cap P_1'}} D(P_1'\backslash G)_1 \cong \varprojlim_{H_2''} D(N_2^\flat H_2'')\widehat{\otimes}_{D(N_2)_{N_2^\flat}} D(P_1'\backslash G)_1 \cong \widehat{D}(N_2)\widehat{\otimes}_{D(N_2)_{N_2^\flat}} D(P_1'\backslash G)_1$$

where $\widehat{D}(N_2) \overset{\text{def}}{=} \left(\varprojlim_{H_2''} D(N_2^\flat\backslash N_2^\flat H_2'')\right)\widehat{\otimes}_{E,\iota} D(N_2^\flat)$. This together with (3.5) implies that

$$\widehat{D}(G)_Y \cong D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}} (\widehat{D}(N_2)\widehat{\otimes}_{D(N_2)_{N_2^\flat}} D(P_1'\backslash G)_1)\widehat{\otimes}_{E,\iota} D(P_1'). \tag{3.6}$$

Now we consider the following condition

Condition 3.7. *There exists a p-adic field K, a split reductive group $\widetilde{G}_{/L}$ together with its two parabolic subgroups $\widetilde{P}_1'$ and $\widetilde{P}_2$ such that $G \cong \mathrm{Res}_{K/\mathbb{Q}_p}(\widetilde{G})$, $P_1' \cong \mathrm{Res}_{K/\mathbb{Q}_p}(\widetilde{P}_1')$ and $P_2 \cong \mathrm{Res}_{K/\mathbb{Q}_p}(\widetilde{P}_2)$.*

Lemma 3.8. *If Condition 3.7 holds, then a good pair (H_1, H_2) exists.*

Proof. We fix K, $\widetilde{G}$, $\widetilde{P}_1'$ and $\widetilde{P}_2$ as in Condition 3.7. As $\widetilde{G}$ is split, we abuse the same notation for the Chevalley group scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Similarly, $\widetilde{P}_1'$ and $\widetilde{P}_2$ can be made closed subgroup schemes of $\widetilde{G}_{/\mathcal{O}_K}$. Let $\widetilde{L}_0 \subseteq \widetilde{P}_1'\cap\widetilde{P}_2$ be a (split) maximal torus which always exists. Upon replacing L_0 with $\mathrm{Res}_{K/\mathbb{Q}_p}\widetilde{L}_0$, we may assume that $L_0 \cong \mathrm{Res}_{K/\mathbb{Q}_p}\widetilde{L}_0$ and moreover $\widetilde{L}_0$ extends to a closed subgroup scheme of $\widetilde{G}_{/\mathcal{O}_K}$. Given the pair $(\widetilde{G}, \widetilde{L}_0)$, we consider the corresponding set of roots Φ, and $\widetilde{P}_1'$ (resp. $\widetilde{P}_2$) corresponds to a (closed) subset Φ_1 (resp. Φ_2) of Φ. We write $\widetilde{N}_2$ for the unipotent radical of $\widetilde{P}_2$ and there exists a unique maximal Levi subgroup $\widetilde{L}_2 \subseteq \widetilde{P}_2$ containing $\widetilde{L}_0$, such that $\widetilde{P}_2 = \widetilde{L}_2\widetilde{N}_2$ is a semi direct product. In the following, we understand all the group schemes mentioned above to be over $\mathcal{O}_K$. The closed subgroup $\widetilde{L}_2$ (resp. $\widetilde{N}_2$) of $\widetilde{P}_2$ corresponds to a closed subset Φ_2' (resp. Φ_2'') of Φ_2 such that $\Phi_2 = \Phi_2' \sqcup \Phi_2''$. The closed subsets $\Phi_1 \cap \Phi_2$, $\Phi_1 \cap \Phi_2'$ and $\Phi_1 \cap \Phi_2''$ of Φ_2 corresponds to closed subgroup schemes $\widetilde{P}_1 \cap \widetilde{P}_2$, $\widetilde{P}_1 \cap \widetilde{L}_2$ and $\widetilde{P}_1 \cap \widetilde{N}_2$ such that $\widetilde{P}_1 \cap \widetilde{P}_2 = (\widetilde{P}_1 \cap \widetilde{L}_2)(\widetilde{P}_1 \cap \widetilde{N}_2)$ is a semi direct product. Now we claim that $H_1 \overset{\mathrm{def}}{=} \widetilde{P}_1'(\mathcal{O}_K)$ and $H_2 \overset{\mathrm{def}}{=} \widetilde{P}_2(\mathcal{O}_K)$ gives a good pair as in Definition 3.6. As we clearly have $H_2' = \widetilde{L}_2(\mathcal{O}_K)$ and $H_2'' = \widetilde{N}_2(\mathcal{O}_K)$, item (i) follows from the corresponding facts on group schemes, and the fact that

$$H_2 \cap P_1'(\mathbb{Q}_p) = \widetilde{P}_2(\mathcal{O}_K) \cap \widetilde{P}_1'(L) = \widetilde{P}_2(\mathcal{O}_K) \cap (\widetilde{P}_2 \cap \widetilde{P}_1')(L) = (\widetilde{P}_2 \cap \widetilde{P}_1')(\mathcal{O}_K),$$

as $\widetilde{P}_2 \cap \widetilde{P}_1'$ is a closed subgroup scheme of $\widetilde{P}_2$. Item (ii) follows from the fact that $\widetilde{L}_2 \cap \widetilde{P}_1 \backslash \widetilde{L}_2$ is proper over $\mathcal{O}_K$. For item (iii), we clearly have $P_1'(\mathbb{Q}_p)H_2 = \bigcup_{g_1 \in J_1} g_1 H_1 H_2$. Let $g_1, g_1' \in J_1$ be elements such that $g_1 H_1 H_2 \cap g_1' H_1 H_2 \neq \emptyset$, namely there exists $h_1, h_1' \in H_1$ and $h_2, h_2' \in H_2$ such that $g_1 h_1 h_2 = g_1' h_1' h_2'$. As we have a natural bijection $P_1'(\mathbb{Q}_p) \backslash P_1'(\mathbb{Q}_p) H_2 \cong (P_1'(\mathbb{Q}_p) \cap H_2) \backslash H_2 = (H_1 \cap H_2) \backslash H_2$, $P_1'(\mathbb{Q}_p) h_2 = P_1'(\mathbb{Q}_p) h_2'$ implies that $h_2' h_2^{-1} \in H_1 \cap H_2$, which together with $g_1 h_1 h_2 = g_1' h_1' h_2'$ forces $g_1 H_1 = g_1' H_1$ and thus $g_1 = g_1'$ by the definition of J_1. The proof is thus finished. □

Proposition 3.9. *If Condition 3.7 holds, then we have*

$$D(G)_{\overline{Y}}/D(G)_Z \cong \widehat{D}(G)_Y \cong D(L_2)\widehat{\otimes}_{D(L_2)_{L_2^\flat}}(\widehat{D}(N_2)\widehat{\otimes}_{D(N_2)_{N_2^\flat}} D(P_1'\backslash G)_1)\widehat{\otimes}_{E,\iota} D(P_1').$$

Proof. This follows directly from Proposition 3.5, (3.6) and Lemma 3.8. □

4. Extensions between locally analytic generalized Steinberg

In this section, we compute various Ext-groups between locally analytic generalized Steinberg representations (see Theorem 4.22 and Corollary 4.23). Based on computations of the spectral sequence $E_{\bullet,I_0,I_1}^{\bullet,\bullet}$ in Section 2 and the decomposition in Section 3 (see Proposition 3.9), the main extra technical ingredients are the computations of some N-(co)homologies (see Lemma 4.14 and Lemma 4.15) which, via a standard spectral sequence, compute the Ext-groups between locally analytic principal series in Proposition 4.16.

4.1. **Locally analytic geometric lemma.** In this section, we generalize the classical Bernstein–Zelevinsky geometric lemma in smooth representation theory (see Section 2.12 of [BZ77]) to Orlik-Strauch representations in Proposition 4.7. As an application, we compute the Ext-groups between various $i_{n,I}^{\mathrm{an}}(\lambda)$ in Proposition 4.16. We write $N_{n,I}$ for the unipotent radical of $P_{n,I}$, for each $I \subseteq \Delta_n$.

Let $I, I' \subseteq \Delta_n$ be two subsets, and we write $W_n^{I',I} \subseteq W(G_n)$ be a set of (minimal length) representative so that $\{P_{n,I'} w P_{n,I}\}_{w \in W_n^{I',I}}$ is a stratification on the group scheme G_n. For each $w \in W_n^{I',I}$, we write $C_w \overset{\mathrm{def}}{=} P_{n,I'} w P_{n,I}$ (which is a locally closed subscheme of G_n) and $\overline{C_w}$ for its Zariski closure in G_n. We will abuse C_w and $\overline{C_w}$ for their K-points from now on. As in Section 3, we use the shorted notation $D(G_n)_{\overline{C_w}} \overset{\mathrm{def}}{=} D(G_n, E)_{\overline{C_w}}$ and similarly for other spaces of distributions. Then $\{D(G_n)_{\overline{C_w}}\}_{w \in W_n^{I',I}}$ is an increasing $W_n^{I',I}$-filtration on $D(G_n)$ with respect to the Bruhat order on $W_n^{I',I}$. As $\overline{C_w} \setminus C_w = \bigcup_{w' \in W_n^{I',I}, w' < w} \overline{C_{w'}}$, we clearly have

$$D(G_n)_{\overline{C_w} \setminus C_w} = \sum_{w' \in W_n^{I',I}, w' < w} D(G_n)_{\overline{C_{w'}}},$$

and thus $D(G_n)_{\overline{C_w}}/D(G_n)_{\overline{C_w} \setminus C_w}$ is the w-graded piece associated with the filtration $\{D(G_n)_{\overline{C_w}}\}_{w \in W_n^{I',I}}$. For technical convenience, we set $P_{n,I'}^w \overset{\mathrm{def}}{=} w^{-1} P_{n,I'} w$, $L_{n,I,w} \overset{\mathrm{def}}{=} L_{n,I} \cap P_{n,I'}^w$, $N_{n,I,w} \overset{\mathrm{def}}{=} N_{n,I} \cap P_{n,I'}^w$,

$Y_w \stackrel{\mathrm{def}}{=} w^{-1}C_w$ whose closure in G_n is $\overline{Y_w} \stackrel{\mathrm{def}}{=} w^{-1}\overline{C_w}$. According to Proposition 3.5, there exists a canonical left $D(P_{n,I})$-equivariant and right $D(P^w_{n,I'})$-equivariant isomorphism

$$\widehat{D}(G_n)_{Y_w} \stackrel{\mathrm{def}}{=} \varprojlim_C D(P^w_{n,I'}\backslash G_n)_C \widehat{\otimes}_{E,\iota} D(P^w_{n,I'}) \cong D(G_n)_{\overline{Y_w}}/D(G_n)_{\overline{Y_w}\backslash Y_w}$$

where C runs through compact open subsets of $P^w_{n,I'}\backslash P^w_{n,I'}P_{n,I}$.

Given a coadmissible $D(L_{n,I'})$-module $\mathcal{M}_{I'}$ and an integer $k \geq 0$, it is a fundamental question to compute

$$H^k(N_{n,I}, \mathcal{M})$$

with $\mathcal{M} \stackrel{\mathrm{def}}{=} D(G_n)\widehat{\otimes}_{D(P_{n,I'}),\iota}\mathcal{M}_{I'}$. With the help of the filtration $\{D(G_n)_{\overline{C_w}}\}_{w\in W_n^{I',I}}$, we usually first compute

$$H^k(N_{n,I}, (D(G_n)_{\overline{C_w}}/D(G_n)_{\overline{C_w}\backslash C_w})\widehat{\otimes}_{D(P_{n,I'}),\iota}\mathcal{M}_{I'}) \tag{4.1}$$

for each $w \in W_n^{I',I}$. We clearly have a canonical isomorphism

$$\mathrm{gr}_w(\mathcal{M}) \stackrel{\mathrm{def}}{=} (D(G_n)_{\overline{C_w}}/D(G_n)_{\overline{C_w}\backslash C_w})\widehat{\otimes}_{D(P_{n,I'}),\iota}\mathcal{M}_{I'} \cong (D(G_n)_{\overline{Y_w}}/D(G_n)_{\overline{Y_w}\backslash Y_w})\widehat{\otimes}_{D(P^w_{n,I'}),\iota}\mathcal{M}^w_{I'} \tag{4.2}$$

with $\mathcal{M}^w_{I'}$ being the w-conjugate of $\mathcal{M}_{I'}$. Combine with Proposition 3.9, (4.1) can be reduced to

$$H^k(N_{n,I}, D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}}(\widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} D(P^w_{n,I'}\backslash G_n)_1\widehat{\otimes}_{E,\iota}\mathcal{M}^w_{I'})) \tag{4.3}$$

for each $w \in W_n^{I',I}$, where $\widehat{D}(N_{n,I}) \stackrel{\mathrm{def}}{=} \left(\varprojlim_C D(C)\right)\widehat{\otimes}_{E,\iota}D(N_{n,I,w})$ and C runs through compact open subsets of $N_{n,I,w}\backslash N_{n,I}$. See for example the discussion after Lemme 5.1.1 of [Bre19] that (4.3) admits a natural $D(L_{n,I})$-module structure (and in particular a $D(Z_{n,I})$-module structure).

We assume from now on that $\mathcal{M}_{I'} = L^{I'}(\mu)\otimes_E(\pi^\infty)'$ with $\mu \in X(T_{n,E})$ dominant with respect to $B_{n,E}\cap L_{n,I',E}$ and π^∞ an admissible smooth representation of $L_{n,I'}$. Now we set

$$\mathcal{M}^1 \stackrel{\mathrm{def}}{=} D(P_{n,I'}\backslash G_n)_1\widehat{\otimes}_E L^{I'}(\mu) \cong D(G_n)_1\widehat{\otimes}_{D(P_{n,I'})_1}L^{I'}(\mu)$$

$$\mathrm{gr}^1_w(\mathcal{M}) \stackrel{\mathrm{def}}{=} (\mathcal{M}^1)^w\widehat{\otimes}_E(\pi^{\infty,w})' \cong D(G_n)_1\widehat{\otimes}_{D(P^w_{n,I'})_1}(L^{I'}(\mu)^w\otimes_E(\pi^{\infty,w})'),$$

$$\mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}) \stackrel{\mathrm{def}}{=} \widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}}\mathrm{gr}^1_w(\mathcal{M})$$

and note that

$$\mathrm{gr}_w(\mathcal{M}) \cong D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}}\mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})$$

for each $w \in W_n^{I',I}$.

Let $\mathfrak{p}$ be a (finite dimensional) Lie algebra over $\mathbb{Q}_p$ with an ideal $\mathfrak{n}$ and a commutative sub Lie algebra $\mathfrak{t}$ such that $\mathfrak{t}\subseteq\mathfrak{p}$ induces an embedding $\mathfrak{t}\hookrightarrow\mathfrak{p}/\mathfrak{n}$. We set $\mathfrak{p}_E \stackrel{\mathrm{def}}{=} \mathfrak{p}\otimes_{\mathbb{Q}_p}E$ and similarly for others. We write $X(\mathfrak{t}_E)$ for the set of all eigencharacters of $\mathfrak{t}_E$ (which is countable).

Definition 4.1. Let V be a Fréchet unclear space equipped with a $U(\mathfrak{p}_E)$-action. We say the topological $U(\mathfrak{p}_E)$-module V is *nice* if there exists a dense sub $U(\mathfrak{p}_E)$-module $V^\flat\subseteq V$ such that the $\mathfrak{t}_E$-eigenspace $V^\flat_\eta$ is finite dimensional for each $\eta\in X(\mathfrak{t}_E)$ and $V^\flat = \bigoplus_{\eta\in X(\mathfrak{t}_E)}V^\flat_\eta$. Here $V^\flat_\eta$ for the η-eigenspace of $V^\flat$ under the $\mathfrak{t}_E$-action.

We endow $V^\flat$ with the subspace topology induced from V, and set $V^\flat_X \stackrel{\mathrm{def}}{=} \bigoplus_{\eta\in X} V^\flat_\eta$ for each subset $X \subseteq X(\mathfrak{t}_E)$. Now we consider the Chevalley-Eilenberg complex $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_E, V)$ that computes the Lie algebra cohomology $H^\bullet(\mathfrak{n}_E, V)$. Equipped with its natural topology induced from V, $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_E, V)$ contains a (dense) subcomplex $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_E, V^\flat)$ with each term endowed with the induced subspace topology. Then there exists a finite set $\Phi_{\mathfrak{p}} \subseteq X(\mathfrak{t}_E)$ depending only on our choice of $\mathfrak{p}$, such that the image of $\mathrm{Hom}_E(\wedge^k \mathfrak{n}_E, V^\flat_X)$ under

$$\varphi^k : \mathrm{Hom}_E(\wedge^k \mathfrak{n}_E, V^\flat) \to \mathrm{Hom}_E(\wedge^{k+1} \mathfrak{n}_E, V^\flat) \tag{4.4}$$

is contained in $\mathrm{Hom}_E(\wedge^{k+1} \mathfrak{n}_E, V^\flat_{X+\Phi_{\mathfrak{p}}})$ for each finite subset $X \subseteq X(\mathfrak{t}_E)$ (with $X + \Phi_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} \{\eta + \eta' \mid \eta \in X, \eta' \in \Phi_{\mathfrak{p}}\}$).

Lemma 4.2. *Let V be a nice topological $U(\mathfrak{p}_E)$-module as in Definition 4.1. Each map of the complex $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_E, V^\flat)$ is strict with closed image, with respect to the topology induced from $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_E, V)$.*

Proof. Let $k \geq 0$ and consider the map (4.4). Let

$$X_0 \subseteq X_1 \subseteq \cdots$$

be a sequence of subsets of $X(\mathfrak{t}_E)$ satisfying $X_i + \Phi_{\mathfrak{p}} \subseteq X_{i+1}$ for each $i \geq 0$ and $X(\mathfrak{t}_E) = \bigcup_{i=0}^\infty X_i$. As $V^\flat = \bigoplus_{\eta\in X(\mathfrak{t}_E)} V^\flat_\eta$, we have $V^\flat = \varinjlim_i V^\flat_{X_i}$. By the definition of $\Phi_{\mathfrak{p}}$, the map φ^k has the form $\varphi^k = \varinjlim_i \varphi^k_i$ where $\varphi^k_i : \mathrm{Hom}_E(\wedge^k \mathfrak{n}_E, V^\flat_{X_i}) \to \mathrm{Hom}_E(\wedge^{k+1} \mathfrak{n}_E, V^\flat_{X_{i+1}})$ is the restriction of φ^k. As φ^k_i is a map between finite dimensional E-vector spaces for each $i \geq 0$, we deduce that φ^k is strict with closed image. $\square$

Lemma 4.3. *Let V be a nice topological $U(\mathfrak{p}_E)$-module as in Definition 4.1. For each $k \geq 0$, the space $H^k(\mathfrak{n}_E, V)$ is a Fréchet nuclear space and the map $H^k(\mathfrak{n}_E, V^\flat) \to H^k(\mathfrak{n}_E, V)$ is an injection with dense image.*

Proof. This follows from Lemma 4.2 and Theorem 7.4 of [Koh11]. $\square$

For each p-adic manifold M (paracompact and finite dimensional), we write $C^\infty(M) = C^\infty(M, E)$ for the space of locally constant E-valued functions on M and set $D^\infty(M) \stackrel{\mathrm{def}}{=} C^\infty(M)'$. The closed embedding $C^\infty(M) \hookrightarrow C^{\mathrm{an}}(M)$ induces a canonical surjection

$$D(M) \twoheadrightarrow D^\infty(M). \tag{4.5}$$

For each closed subset $C \subseteq M$, we write $D^\infty(M)_C \subseteq D^\infty(M)$ for the closed subspace consists of locally constant distributions that are supported on C, and note that $D^\infty(M)_C$ is simply the image of $D(M)_C$ under (4.5). If C is a closed submanifold of M, then we actually have a canonical isomorphism

$$D^\infty(M)_C \cong D^\infty(C) : \delta \mapsto \delta|_C$$

which makes $D^\infty(M)_C$ a much simpler object than $D(M)_C$. Lemma 3.2 and Lemma 3.3 have their analogue for $D^\infty(-)$, and $D^\infty(M)$ is Fréchet nuclear if M is compact. In particular, we have a topological isomorphism

$$D^\infty(N_{n,I}) \cong D^\infty(N_{n,I,w}\backslash N_{n,I}) \widehat{\otimes}_{E,\iota} D^\infty(N_{n,I,w}).$$

We set $\widehat{D}^\infty(N_{n,I}) \stackrel{\mathrm{def}}{=} (\varprojlim_C D^\infty(C)) \widehat{\otimes}_{E,\iota} D^\infty(N_{n,I,w})$ where C runs through compact open subsets of $N_{n,I,w}\backslash N_{n,I}$. Note that there exists a canonical $D(N_{n,I})$-equivariant surjection

$$\widehat{D}(N_{n,I}) \twoheadrightarrow \widehat{D}^\infty(N_{n,I})$$

with the $D(N_{n,I})$-action on the target factoring through $D^\infty(N_{n,I})$.

Lemma 4.4. *For each $k \geq 0$ and each $w \in W_n^{I',I}$, the cohomology groups*

$$H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w), H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})),\ H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))$$

are Fréchet nuclear spaces. Moreover, we have canonical $D(L_{n,I})_{L_{n,I,w}}$-equivariant and $D^\infty(N_{n,I})$-equivariant topological isomorphisms

$$\begin{aligned} H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})) &\cong \widehat{D}^\infty(N_{n,I})\widehat{\otimes}_{D^\infty(N_{n,I,w})} H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})) \\ &\cong \left(\widehat{D}^\infty(N_{n,I})\widehat{\otimes}_{D^\infty(N_{n,I,w})}(\pi^{\infty,w})'\right)\widehat{\otimes}_E H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w). \end{aligned}$$

Proof. We fix an element $w \in W_n^{I',I}$ throughout. As $(\mathcal{M}^1)^w \cong D(G_n)_1\widehat{\otimes}_{D(P^w_{n,I'})_1} L^{I'}(\mu)^w$, we can consider the dense subspace $(\mathcal{M}^{1,\flat})^w$ with

$$\mathcal{M}^{1,\flat} \overset{\mathrm{def}}{=} U(\mathfrak{g}_{n,E}) \otimes_{U(\mathfrak{p}_{n,I',E})} L^{I'}(\mu)$$

and observe that $(\mathcal{M}^1)^w$ is a nice $U(\mathfrak{p}_{n,I,E})$-module (see Definition 4.1). Therefore it follows from Lemma 4.2 (with $\mathfrak{p} = \mathfrak{p}_{n,I}$, $\mathfrak{n} = \mathfrak{n}_{n,I}$ and $\mathfrak{t} = \mathfrak{t}_n$) that each map of the complex $\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w)$ is strict with closed image and $H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w)$ is a Fréchet nuclear space for each $k \geq 0$. As π^∞ is admissible smooth, it is a countable direct limit of finite dimensional subspaces, making $(\pi^{\infty,w})'$ a countable inverse limit of finite dimensional E-vector spaces. In particular, $(\pi^{\infty,w})'$ is Fréchet nuclear and for each strict exact sequence of Fréchet nuclear spaces $V_1 \hookrightarrow V_2 \twoheadrightarrow V_3$, we have a strict exact sequence of Fréchet nuclear spaces

$$V_1\widehat{\otimes}_E(\pi^{\infty,w})' \hookrightarrow V_2\widehat{\otimes}_E(\pi^{\infty,w})' \twoheadrightarrow V_3\widehat{\otimes}_E(\pi^{\infty,w})'.$$

Consequently, each map of the complex

$$\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})) \cong \mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w)\widehat{\otimes}_E(\pi^{\infty,w})'$$

is strict with closed image, making $H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M}))$ a Fréchet nuclear space for each $k \geq 0$.

Now we consider $D(N_{n,I})$-equivariant topological isomorphism of complexes

$$\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})) \cong \widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} \mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})). \tag{4.6}$$

For each Fréchet nuclear space V we have the following topological isomorphism of Fréchet nuclear spaces

$$\widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} V \cong \varprojlim_{C,r} D(N_{n,I,w}C)_r\widehat{\otimes}_{D(N_{n,I,w}C)_{N_{n,I,w},r}} V \tag{4.7}$$

with the radius $\frac{1}{p} < r < 1$ and C running through compact open subgroups of $N_{n,I}$. Following the argument of Lemme 4.27 of [Schr11], we observe that $D(N_{n,I,w}C)_r$ is finite free over $D(N_{n,I,w}C)_{N_{n,I,w},r}$ and that the system

$$\{D(N_{n,I,w}C)_r\widehat{\otimes}_{D(N_{n,I,w}C)_{N_{n,I,w},r}} V\}_{C,r}$$

is Mittag-Leffler. Consequently, for each strict exact sequence of Fréchet nuclear spaces $V_1 \hookrightarrow V_2 \twoheadrightarrow V_3$, we have a strict exact sequence of Fréchet nuclear spaces

$$\widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} V_1 \hookrightarrow \widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} V_2 \twoheadrightarrow \widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} V_3.$$

In particular, we deduces that each map of the complex (4.6) is strict with closed image, and $H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))$ is a Fréchet nuclear space for each $k \geq 0$. The above argument clearly produces a $D(N_{n,I})$-equivariant isomorphism

$$H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})) \cong \widehat{D}(N_{n,I})\widehat{\otimes}_{D(N_{n,I})_{N_{n,I,w}}} H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})) \tag{4.8}$$

for each $k \geq 0$. As $\mathfrak{n}_{n,I,E}$ acts trivially on $H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M}))$, the $D(N_{n,I})_{N_{n,I,w}}$-action on $H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M}))$ factors through $D^\infty(N_{n,I})_{N_{n,I,w}} \cong D^\infty(N_{n,I,w})$, and thus the $D(N_{n,I})$-equivariant isomorphism (4.8) factors through a $D^\infty(N_{n,I})$-equivariant isomorphism

$$H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})) \cong \widehat{D}^\infty(N_{n,I})\widehat{\otimes}_{D^\infty(N_{n,I,w})} H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})) \tag{4.9}$$

Then we observe that we have a $D^\infty(N_{n,I,w})$-equivariant isomorphism

$$H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^1_w(\mathcal{M})) \cong H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w)\widehat{\otimes}_E (\pi^{\infty,w})' \tag{4.10}$$

for each $k \geq 0$. The $D(L_{n,I})_{L_{n,I,w}}$-equivariance of both (4.9) and (4.10) are clear. □

Note that we have a canonical isomorphism

$$\widehat{D}^\infty(N_{n,I})\widehat{\otimes}_{D^\infty(N_{n,I,w})}(\pi^{\infty,w})' \cong \left((\text{c-Ind}_{\mathrm{N_{n,I,w}}}^{\mathrm{N_{n,I}}} \pi^{\infty,\mathrm{w}})^\infty\right)'$$

by the definition of the smooth compact induction functor $(\text{c-Ind}_{\mathrm{N_{n,I,w}}}^{\mathrm{N_{n,I}}} -)^\infty$ and that of $\widehat{D}^\infty(N_{n,I})$.

Lemma 4.5. *For each $k \geq 0$ and each $w \in W_n^{I',I}$, the cohomology group $H^k(N_{n,I}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))$ is a Fréchet nuclear space, and we have a canonical $D(L_{n,I})_{L_{n,I,w}}$-equivariant topological isomorphism*

$$H^k(N_{n,I}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})) \cong J_{n,I,I',w}(\pi^\infty)'\widehat{\otimes}_E H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w).$$

with $J_{n,I,I',w}(\pi^\infty) \overset{def}{=} \left((\text{c-Ind}_{\mathrm{N_{n,I,w}}}^{\mathrm{N_{n,I}}} \pi^{\infty,\mathrm{w}})^\infty\right)_{N_{n,I}}$ an admissible smooth E-representation of the Levi quotient of $L_{n,I,w}$.

Proof. This follows directly from Lemma 4.4 and Theorem 7.1 of [Koh11], namely the $D(L_{n,I})_{L_{n,I,w}}$-equivariant isomorphism

$$H^k(N_{n,I}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))' \cong \left(H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))'\right)_{N_{n,I}}.$$

□

Lemma 4.6. *For each $k \geq 0$ and each $w \in W_n^{I',I}$, the cohomology groups $H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}_w(\mathcal{M}))$ and $H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))$ are Fréchet nuclear spaces. Moreover, we have a canonical $D(L_{n,I})$-equivariant topological isomorphisms*

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M})) \cong D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}} H^k(N_{n,I}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})).$$

Proof. The same argument as Lemme 4.27 of [Schr11] shows that $D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}}$ sends a strict complex of Fréchet spaces to a strict complex of Fréchet spaces, and in particular

$$H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}_w(\mathcal{M})) \cong D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}} H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))$$

for each $k \geq 0$, using the isomorphism of complexes

$$\mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, \mathrm{gr}_w(\mathcal{M})) \cong D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}} \mathrm{Hom}_E(\wedge^\bullet \mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})).$$

The desired isomorphism thus follows from Theorem 7.1 of [Koh11], namely the $D(L_{n,I})_{L_{n,I,w}}$-equivariant isomorphism

$$H^k(N_{n,I}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))' \cong \left(H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}))'\right)_{N_{n,I}}$$

and the $D(L_{n,I})$-equivariant isomorphism

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))' \cong \left(H^k(\mathfrak{n}_{n,I,E}, \mathrm{gr}_w(\mathcal{M}))'\right)_{N_{n,I}}.$$

□

Given a split reductive group G over K and a parabolic subgroup P with Levi quotient L, we recall from [OS15] the so-called Orlik–Strauch functor $\mathcal{F}^G_P(\cdot,\cdot)$ which sends a pair (V, π^∞) to an admissible locally analytic representation $\mathcal{F}^G_P(V, \pi^\infty)$ of $G(K)$. Here V is a locally $\mathfrak{p}$-finite object in category $\mathcal{O}$ and π^∞ is an admissible smooth E-representation of $L(K)$. The main properties of $\mathcal{F}^G_P(\cdot,\cdot)$ are summarized in the main theorem of *loc.it.* and in particular $\mathcal{F}^G_P(\cdot,\cdot)$ is exact in both arguments.

Combining all the results above, we arrive at the following locally analytic geometric lemma (with the terminology borrowed from the classical geometric lemma in smooth representation theory).

Proposition 4.7. *For each $k \geq 0$ and each $w \in W_n^{I',I}$, we have a $D(L_{n,I})$-equivariant isomorphism of Fréchet nuclear spaces*

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M})) \cong \mathcal{F}^{L_{n,I}}_{L_{n,I,w}}(H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^{1,\flat})^w), J_{n,I,I',w}(\pi^\infty))'.$$

Proof. As $\mathcal{M}^{1,\flat}$ is $\mathfrak{t}_{n,E}$-semisimple and locally $\mathfrak{p}_{n,I',E}$-finite, $(\mathcal{M}^{1,\flat})^w$ is $\mathfrak{t}_{n,E}$-semisimple and locally $\mathfrak{p}^w_{n,I',E}$-finite, making $H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^{1,\flat})^w)$ $\mathfrak{t}_{n,E}$-semisimple and locally $\mathfrak{l}_{n,I,w} = \mathfrak{l}_{n,I,E} \cap \mathfrak{p}^w_{n,I',E}$-finite, and thus the Orlik-Strauch representation $\mathcal{F}^{L_{n,I}}_{L_{n,I,w}}(H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^{1,\flat})^w), J_{n,I,I',w}(\pi^\infty))$ is well-defined. We conclude by Lemma 4.5, Lemma 4.6 and the following $D(L_{n,I})$-equivariant isomorphism

$$\mathcal{F}^{L_{n,I}}_{L_{n,I,w}}(H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^{1,\flat})^w), J_{n,I,I',w}(\pi^\infty))' \cong D(L_{n,I})\widehat{\otimes}_{D(L_{n,I})_{L_{n,I,w}}} \left(J_{n,I,I',w}(\pi^\infty)'\widehat{\otimes}_E H^k(\mathfrak{n}_{n,I,E}, (\mathcal{M}^1)^w)\right).$$

□

For each locally algebraic character χ of $Z_{n,I}$ and each $D(Z_{n,I})$-module V, we write $V_{\chi'}$ for its maximal quotient on which $D(Z_{n,I})$ acts by χ'. There exists a finite set $\Sigma_{w,k}(\mathcal{M})$ of locally algebraic characters of $Z_{n,I}$ such that the χ'-isotypic component $H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))_{\chi'}$ is non-zero for some locally algebraic character χ of $Z_{n,I}$ if and only if $\chi \in \Sigma_{w,k}(\mathcal{M})$, and moreover

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M})) \cong \bigoplus_{\chi \in \Sigma_{w,k}(\mathcal{M})} H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))_{\chi'}.$$

Similarly, we write $\Sigma^\infty_w(\pi^\infty)$ for the set of smooth characters of $Z_{n,I}$ such that $J_{n,I,I',w}(\pi^\infty)_{\chi^\infty} \neq 0$ if and only if $\chi^\infty \in \Sigma^\infty_w(\pi^\infty)$. There clearly exists a natural surjection $\Sigma_{w,k}(\mathcal{M}) \twoheadrightarrow \Sigma^\infty_w(\pi^\infty)$ sending each $\chi \in \Sigma_{w,k}$ to its smooth part χ^∞.

Definition 4.8. Let $I, I' \subseteq \Delta_n$ be a pair of subsets and π^∞ an admissible smooth E-representation of $L_{n,I'}$. We say that π^∞ is (I,I')-*regular* if $\Sigma_w \cap \Sigma_{w'} = \emptyset$ for each $w \neq w' \in W_n^{I',I}$.

We have the following corollary of Proposition 4.7

Corollary 4.9. *Let $I, I' \subseteq \Delta_n$ be a pair of subsets and π^∞ a (I, I')-regular admissible smooth E-representation of $L_{n,I'}$. For each $k \geq 0$, the cohomology $H^k(N_{n,I}, \mathcal{M})$ is a Fréchet nuclear space and there exists a $D(L_{n,I})$-equivariant isomorphism*

$$H^k(N_{n,I}, \mathcal{M}) \cong \bigoplus_{w \in W_n^{I',I}} H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))$$

Proof. Note that $\mathcal{M}$ admits a canonical filtration $\{\mathrm{Fil}_w(\mathcal{M})\}_{w \in W_n^{I',I}}$. Our assumption implies that, for each smooth character χ^∞ of $Z_{n,I}$, there exists at most one $w \in W_n^{I',I}$ such that

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))_{\chi'} \neq 0$$

for some locally algebraic character χ of $Z_{n,I}$ whose smooth part is χ^∞.

We prove by induction on $w \in W_n^{I',I}$ with respect to Bruhat order. For each $w \in W_n^{I',I}$, we set $\mathrm{Fil}_{<w}(\mathcal{M}) \overset{\mathrm{def}}{=} \sum_{w'<w} \mathrm{Fil}_{w'}(\mathcal{M})$ and consider the following long exact sequence

$$\cdots \to H^k(N_{n,I}, \mathrm{Fil}_{<w}(\mathcal{M})) \to H^k(N_{n,I}, \mathrm{Fil}_w(\mathcal{M})) \to H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M})) \to H^{k+1}(N_{n,I}, \mathrm{Fil}_{<w}(\mathcal{M})) \to \cdots$$

induced from $\mathrm{Fil}_{<w}(\mathcal{M}) \hookrightarrow \mathrm{Fil}_w(\mathcal{M}) \twoheadrightarrow \mathrm{gr}_w(\mathcal{M})$. Our inductive assumption says that

$$H^k(N_{n,I}, \mathrm{Fil}_{w'}(\mathcal{M})) \cong \bigoplus_{w'' \in W_n^{I',I}, w'' \leq w'} H^k(N_{n,I}, \mathrm{gr}_{w''}(\mathcal{M}))$$

for each $w' < w$ and $k \geq 0$, which implies that

$$H^k(N_{n,I}, \mathrm{Fil}_{<w}(\mathcal{M})) \cong \bigoplus_{w'' \in W_n^{I',I}, w'' < w} H^k(N_{n,I}, \mathrm{gr}_{w''}(\mathcal{M})).$$

This together with the previous long exact sequence clearly finishes our induction step. The proof is thus finished. □

When π^∞ is (I, I')-regular, we can define $H_k(N_{n,I}, \mathcal{M}')$ (for each $k \geq 0$) as the unique (up to isomorphism) admissible locally analytic representation such that

$$H_k(N_{n,I}, \mathcal{M}')' \cong H^k(N_{n,I}, \mathcal{M}), \tag{4.11}$$

which exists by Proposition 4.7 and Corollary 4.9. For each locally algebraic character χ of $Z_{n,I}$, we write $H_k(N_{n,I}, \mathcal{M}')_\chi$ for the χ-isotypic component of $H_k(N_{n,I}, \mathcal{M}')$, namely the unique (up to isomorphism) direct summand of $H_k(N_{n,I}, \mathcal{M}')$ so that (4.11) induces an isomorphism

$$\big(H_k(N_{n,I}, \mathcal{M}')_\chi\big)' \cong H^k(N_{n,I}, \mathcal{M})_{\chi'}.$$

For each $w \in W(G_n)$, we set $\chi_w^\infty \overset{\mathrm{def}}{=} |\cdot| \circ \delta_{w\cdot 0}$ where $\delta_{w\cdot 0} : T_n \to E^\times$ is the algebraic character associated with the weight $w \cdot 0 \in X(T_n)$.

Lemma 4.10. *Let $I, I' \subseteq \Delta_n$ be a pair of subsets and π^∞ an admissible smooth E-representation of $L_{n,I'}$. Assume that π^∞ is the subquotient of $i^\infty_{n,\emptyset,I'}(\chi_w^\infty)$ for some $w \in W(G_n)$, then π^∞ is (I, I')-regular. In particular, the trivial representation $1_{L_{n,I'}}$ is (I, I')-regular.*

Proof. According to Definition 4.8, it is harmless to assume that $\pi^\infty = i^\infty_{n,\emptyset,I'}(\chi^\infty_w)$. According to the classical geometric lemma in representation theory (cf. Section VI.5.1 of [Ren]), we have

$$\left(i^\infty_{n,\emptyset}(\chi^\infty_w)\right)_{N_{n,\emptyset}} \cong \bigoplus_{w'\in W(G_n)} \chi^\infty_{w'} \tag{4.12}$$

and

$$\left(i^\infty_{n,\emptyset}(\chi^\infty_w)\right)_{N_{n,I}} \cong \left(i^\infty_{n,I'} i^\infty_{n,\emptyset,I'}(\chi^\infty_w)\right)_{N_{n,I}} \cong \bigoplus_{w'\in W_n^{I',I}} (\mathrm{Ind}_{L_{n,I,w}}^{L_{n,I}} J_{n,I,I',w}(\pi^\infty))^\infty. \tag{4.13}$$

Compare (4.13) with (4.12), we observe that the set $\Sigma^\infty_{w'}(\pi^\infty)$ consists of elements of the form $\chi^\infty_{w''}|_{Z_{n,I}}$ with $w'' \in wW(L_{n,I'})w'W(L_{n,I})$. Given two elements $w'', w''' \in W(G_n)$, we have $\chi^\infty_{w''}|_{Z_{n,I}} = \chi^\infty_{w'''}|_{Z_{n,I}}$ if and only if $w''W(L_{n,I}) = w'''W(L_{n,I})$, and thus

$$\Sigma^\infty_{w'}(\pi^\infty) = \{\chi^\infty_{w''}|_{Z_{n,I}} \mid w'' \in wW(L_{n,I'})w'W(L_{n,I})\}$$

are disjoint for different $w' \in W_n^{I',I}$. The last claim follows from the fact that $1_{L_{n,I'}}$ is a subrepresentation of $i^\infty_{n,\emptyset,I'}(\chi^\infty_1)$. □

Remark 4.11. Concerning the results up to Proposition 4.7 and Corollary 4.9, it is actually possible to treat $\mathcal{M}$ of the form $\mathcal{F}^{G_n}_{P_{n,I'}}(V,\pi^\infty)'$, namely the dual of an arbitrary Orlik-Strauch representation, without assuming that V has the form $M^{\Delta_n}_{I'}(\mu)$ for some $\mu \in X(T_{n,E})$ dominant for $L_{n,I',E}\cap B_{n,E}$. In fact, each $V \in \mathcal{O}^{I'}_{n,\Delta_n}$ naturally determines a $D(G_n)_1$-module $\mathcal{M}^1$ which contains V as the (dense) subset of $\mathfrak{p}_{n,I'}$-finite vectors and satisfies

$$\mathcal{M} \cong D(G_n)\widehat{\otimes}_{D(G_n)_{P_{n,I'}}}(\mathcal{M}^1\widehat{\otimes}_E(\pi^{\infty,w})').$$

We can thus define $\mathrm{gr}^1_w(\mathcal{M})$, $\mathrm{gr}^{\mathrm{u}}_w(\mathcal{M})$ and $\mathrm{gr}_w(\mathcal{M})$ by exactly the same formula as before. Note that the computation of such $H^k(N_{n,I},\mathcal{M})$ largely reduces to that of $H^k(\mathfrak{n}_{n,I,E},V^w)$ for $V \in \mathcal{O}^{I'}_{n,\Delta_n}$ and $w \in W_n^{I',I}$, which is complicated in general.

Remark 4.12. Proposition 4.7 is actually a true generalization of the classical geometric lemma in smooth representation theory. According to Remark 4.11, we can simply take $\mathcal{M}^1$ to be the trivial $D(G_n)_1$-module, in which case we have

$$\mathrm{gr}^1_w(\mathcal{M}) = (\pi^{\infty,w})',$$

$$\mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}) = \widehat{D}^\infty(N_{n,I})\widehat{\otimes}_{D^\infty(N_{n,I,w})}(\pi^{\infty,w})' \cong \left(\mathrm{c\text{-}Ind}^{\mathrm{N_{n,I}}}_{\mathrm{N_{n,I,w}}}\pi^{\infty,\mathrm{w}}\right)'$$

and

$$\mathrm{gr}_w(\mathcal{M}) \cong D^\infty(L_{n,I})\widehat{\otimes}_{D^\infty(L_{n,I,w})}\mathrm{gr}^{\mathrm{u}}_w(\mathcal{M}) \cong \left(\mathrm{c\text{-}Ind}^{\mathrm{P^w_{n,I'}P_{n,I}}}_{\mathrm{P^w_{n,I'}}}\pi^{\infty,\mathrm{w}}\right)'.$$

For each $I \subseteq \Delta_n$, we write $\mathcal{O}_{n,I} \overset{\mathrm{def}}{=} \mathcal{O}^{\mathfrak{l}_{n,I,E}\cap\mathfrak{b}_{n,E}}_{\mathrm{alg}}$ for the BGG category $\mathcal{O}$ attached to $\mathfrak{l}_{n,I,E}$ and $\mathfrak{l}_{n,I,E}\cap\mathfrak{b}_{n,E}$. For each $\mu \in X(T_{n,E})$ which is dominant for $L_{n,I,E}\cap B_{n,E}$, we write $L^I(\mu)$ for the unique simple object in $\mathcal{O}_{n,I}$ with highest weight μ, $M^{\Delta_n}_I(\mu) \overset{\mathrm{def}}{=} U(\mathfrak{g}_{n,E})\otimes_{U(\mathfrak{p}_{n,I,E})} L^I(\mu)$ for the generalized Verma module, and $\mathcal{O}_{n,I,\mu}$ for the maximal subcategory of $\mathcal{O}_{n,I}$ so that $U(\mathfrak{z}_{n,I,E})$ acts on each object of $\mathcal{O}_{n,I,\mu}$ by the same character as that of $M^{\Delta_n}_I(\mu)$. More generally, we write $\mathcal{O}^{I'}_{n,I} \subseteq \mathcal{O}_{n,I}$ for the parabolic BGG category corresponding to $\mathfrak{l}_{n,I,E}\cap\mathfrak{p}_{n,I',E} \supseteq \mathfrak{l}_{n,I,E}\cap\mathfrak{b}_{n,E}$ and $\mathcal{O}^{I'}_{n,I,\mu} \overset{\mathrm{def}}{=} \mathcal{O}^{I'}_{n,I}\cap\mathcal{O}_{n,I,\mu}$. For a $U(\mathfrak{z}_{n,I,E})$-module V, we write $V_{n,I,\mu}$ for its maximal quotient on which $U(\mathfrak{z}_{n,I,E})$ acts by the same character as that of $M_I^{\Delta_n}(\mu)$, and in particular $V \mapsto V_{n,I,\mu}$ gives a projection functor $\mathcal{O}_{n,I} \to \mathcal{O}_{n,I,\mu}$. We assume from now that $\mu = -\lambda$ where $\lambda \in X(T_{n,E})$ is dominant with respect to $B_{n,E}^+$.

Lemma 4.13. *Let $I, I' \subseteq \Delta_n$ be two subsets. Then we have*

(i) $H^k(\mathfrak{n}_{n,I,E}, V)_{n,I,\mu} = 0$ for each $V \in \mathcal{O}_{n,\Delta_n,\mu}$ and each $k \geq 1$;

(ii) $H^0(\mathfrak{n}_{n,I,E}, -)_{n,I,\mu}$ induces an exact functor $\mathcal{O}_{n,\Delta_n,\mu} \to \mathcal{O}_{n,I,\mu}$; and

(iii) $H^0(\mathfrak{n}_{n,I,E}, M_{I'}^{\Delta_n}(\mu))_{n,I,\mu} \cong F_{n,I}(\lambda)'$ for each $I \subseteq I' \subseteq \Delta_n$.

Proof. Let V be an object in $\mathcal{O}_{n,\Delta_n,\mu}$. We can compute $H^k(\mathfrak{n}_{n,I,E}, V)$ as the cohomology of the Chevalley–Eilenberg complex with the k-th term given by $\mathrm{Hom}_E(\wedge^k \mathfrak{n}_{n,I,E}, V)$. Note that $\mathrm{Hom}_E(\wedge^k \mathfrak{n}_{n,I,E}, V)$ is semisimple as a $U(\mathfrak{z}_{n,I,E})$-module and contains $H^k(\mathfrak{n}_{n,I,E}, V)$ as a $U(\mathfrak{z}_{n,I,E})$-submodule. Then we conclude by the observation that the $U(\mathfrak{z}_{n,I,E})$-module $\mathrm{Hom}_E(\wedge^k \mathfrak{n}_{n,I,E}, V) \cong V \otimes_E \wedge^k \mathfrak{n}_{n,I,E}^+$ (with $\mathfrak{n}_{n,I,E}^+$ the Lie algebra of the unipotent radical of the parabolic subgroup opposite to $P_{n,I,E} \subseteq G_{n,E}$) satisfies $(V \otimes_E \wedge^k \mathfrak{n}_{n,I,E}^+)_{n,I,\mu} = 0$ for each $k \geq 1$. The vanishing of $H^1(\mathfrak{n}_{n,I,E}, -)_{n,I,\mu}$ implies the exactness of $H^0(\mathfrak{n}_{n,I,E}, -)_{n,I,\mu}$. It is not difficult to see that, for each simple object $L^{\Delta_n}(\mu') \in \mathcal{O}_{n,\Delta_n,\mu}$, we have

$$H^0(\mathfrak{n}_{n,I,E}, L^{\Delta_n}(\mu')) \cong L^I(\mu')$$

and thus

$$H^0(\mathfrak{n}_{n,I,E}, L^{\Delta_n}(\mu'))_{n,I,\mu} \neq 0$$

if and only if $\mu' = w \cdot \mu$ for some element $w \in W(L_{n,I,E})$ (in the Weyl group of $L_{n,I,E}$). Now let $I \subseteq I' \subseteq \Delta_n$ be two subsets, then the surjection $M_{I'}^{\Delta_n}(\mu) \twoheadrightarrow L^{\Delta_n}(\mu)$ induces a surjection

$$H^0(\mathfrak{n}_{n,I,E}, M_{I'}^{\Delta_n}(\mu))_{n,I,\mu} \twoheadrightarrow H^0(\mathfrak{n}_{n,I,E}, L^{\Delta_n}(\mu))_{n,I,\mu} \cong L^I(\mu).$$

If this surjection is not an isomorphism, then there exists $1 \neq w \in W(L_{n,I,E})$ such that $w \cdot \mu$ is dominant with respect to $B_{n,E} \cap L_{n,I',E}$, but this is impossible as $I \subseteq I'$. □

We assume from now on that $\pi^\infty = 1_{L_{n,I'}}$, which implies that $\mathrm{gr}_w^1(\mathcal{M}) = (\mathcal{M}^1)^w$ for each $w \in W_n^{I',I}$. We continue to assume that $\mu = -\lambda$ where $\lambda \in X(T_{n,E})$ is dominant with respect to $B_{n,E}^+$. Note that $Z_{n,I}$ acts on $F_{n,I}(\lambda)$ by an algebraic character $\delta_{n,I,\lambda}$.

Lemma 4.14. *If $w = 1$ and thus $C_1 = \overline{C_1} = Y_1 = \overline{Y_1} = P_{n,I'}P_{n,I}$ is the closed cell, then we have*

(i) $H^k(N_{n,I}, \mathrm{gr}_1(\mathcal{M}))_{\delta'_{n,I,\lambda}} = 0$ if $k \geq 1$;

(ii) $H^0(N_{n,I}, \mathrm{gr}_1(\mathcal{M}))_{\delta'_{n,I,\lambda}} = F_{n,I}(\lambda)$ if $I \subseteq I'$; and

(iii) $H^0(N_{n,I}, \mathrm{gr}_1(\mathcal{M}))_{\delta'_{n,I,\lambda}} = \mathcal{F}_{L_{n,I} \cap P_{n,I \cap I'}}^{L_{n,I}}(V, 1_{L_{n,I\cap I'}})$ for some $V \in \mathcal{O}_{n,I,\mu}^{I \cap I'}$ if $I \not\subseteq I'$.

Proof. When $w = 1$, we have $N_{n,I} = N_{n,I,w}$, $L_{n,I,1} = L_{n,I} \cap P_{n,I'}$ and $\mathrm{gr}_1^{\mathrm{u}}(\mathcal{M}) = \mathrm{gr}_1^1(\mathcal{M}) = \mathcal{M}^1$. The key observation is Proposition 4.7 induces an isomorphism

$$H^k(N_{n,I}, \mathrm{gr}_1(\mathcal{M}))_{\delta'_{n,I,\lambda}} \cong \mathcal{F}_{L_{n,I,1}}^{L_{n,I}}(H^k(\mathfrak{n}_{n,I,E}, \mathcal{M}^{1,\flat})_{n,I,\mu}, (\pi^\infty)_{N_{n,I,1}})'$$

with $N_{n,I,1}$ being the unipotent radical of $L_{n,I,1}$. As $\mathcal{M}^{1,\flat} = M_{I'}^{\Delta_n}(\mu)$, the desired results follow directly from Lemma 4.13. □

Lemma 4.15. *If $1 \neq w \in W_n^{I',I}$, then we have*

$$H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))_{\delta'_{n,I,\lambda}} = 0$$

for all $k \geq 0$.

Proof. This follows directly from Proposition 4.7, Lemma 4.10 and Lemma 4.14. In fact, Lemma 4.10 already implies that, if $H^k(N_{n,I}, \mathrm{gr}_1(\mathcal{M}))_{\delta'_{n,I,\lambda}} \neq 0$ for certain $k \geq 0$, then $H^k(N_{n,I}, \mathrm{gr}_w(\mathcal{M}))_{\delta'_{n,I,\lambda}} \neq 0$ for all $k \geq 0$ and $1 \neq w \in W_n^{I',I}$. $\square$

Proposition 4.16. *Let $I, I' \subseteq \Delta_n$ be two subsets. Then we have*

(i) $\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$ *for each* $k \geq 0$, *if* $I \subseteq I'$; *and*

(ii) $\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) = 0$ *for each* $k \geq 0$, *if* $I \not\subseteq I'$.

Proof. Recall that we write $\mathrm{Ext}^\bullet_{D(G_n),\lambda}$ (resp. $\mathrm{Ext}^\bullet_{D(L_{n,I}),\lambda}$) for the Ext-groups between abstract $D(G_n)$-modules (resp. $D(L_{n,I})$-modules) fixing Z_n-character (which equals that of $F_{n,I}(\lambda)'$). There exists a standard spectral sequence (see Lemma 1.8)

$$\mathrm{Ext}^{k_1}_{D(L_{n,I}),\lambda}(F_{n,I}(\lambda)', H^{k_2}(N_{n,I}, i^{\mathrm{an}}_{n,I'}(\lambda)')) \Rightarrow \mathrm{Ext}^{k_1+k_2}_{D(G_n),\lambda}(i^{\mathrm{an}}_{n,I}(\lambda)', i^{\mathrm{an}}_{n,I'}(\lambda)') = \mathrm{Ext}^{k_1+k_2}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)).$$

It follows from Lemma 4.14 and Lemma 4.15 that

$$\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \cong \mathrm{Ext}^k_{L_{n,I},\lambda}(H_0(N_{n,I}, i^{\mathrm{an}}_{n,I'}(\lambda))_{\delta_{n,I,\lambda}}, F_{n,I}(\lambda)).$$

If $I \subseteq I'$, we have

$$\mathrm{Ext}^k_{L_{n,I},\lambda}(H_0(N_{n,I}, i^{\mathrm{an}}_{n,I'}(\lambda))_{\delta_{n,I,\lambda}}, F_{n,I}(\lambda)) \cong \mathrm{Ext}^k_{L_{n,I},\lambda}(F_{n,I}(\lambda), F_{n,I}(\lambda)) \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$$

where the second isomorphism follows from for example the translation functor in [JL21]. If $I \not\subseteq I'$, then it suffices to show that

$$\mathrm{Ext}^k_{L_{n,I},\lambda}(\mathcal{F}^{L_{n,I}}_{L_{n,I}\cap P_{n,I\cap I'}}(V, 1_{L_{n,I\cap I'}}), F_{n,I}(\lambda)) = 0$$

for each $V \in \mathcal{O}^{I\cap I'}_{n,I,\mu}$ and $k \geq 0$. As each $V \in \mathcal{O}^{I\cap I'}_{n,I,\mu}$ admits BGG resolution by generalized Verma modules in $\mathcal{O}^{I\cap I'}_{n,I,\mu}$, we reduce to the case when V has the form $M^I_{I\cap I'}(\omega) = U(\mathfrak{l}_{n,I}) \otimes_{U(\mathfrak{l}_{n,I}\cap\mathfrak{p}_{n,I\cap I'})} F_{n,I\cap I'}(\omega) \in \mathcal{O}^{I\cap I'}_{n,I,\lambda}$ where $\omega \in X(T_n)$ is a weight dominant for $\mathfrak{b}_n \cap \mathfrak{l}_{n,I\cap I'}$. Then it follows from Proposition 6.5 of [ST05] that

$$\begin{aligned}\mathrm{Ext}^k_{L_{n,I},\lambda}(\mathcal{F}^{L_{n,I}}_{L_{n,I}\cap P_{n,I\cap I'}}(V, 1_{L_{n,I\cap I'}}), F_{n,I}(\lambda)) &\cong \mathrm{Ext}^k_{L_{n,I},\lambda}(i^{\mathrm{an}}_{n,I,I\cap I'}(F_{n,I\cap I'}(\omega)), F_{n,I}(\lambda)) \\ &\cong \mathrm{Ext}^\ell_{L_{n,I},-\lambda}(F_{n,I}(\lambda)^\vee, i^{\mathrm{an}}_{n,I,I\cap I'}(F_{n,I\cap I'}(\omega)^\vee \otimes_E \mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}}))\end{aligned}$$

for each $k, \ell \geq 0$ satisfying $\ell = k + \dim L_{n,I} \cap P_{n,I\cap I'} - \dim L_{n,I}$. Here $\mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}}$ is locally algebraic character of $L_{n,I}\cap P_{n,I\cap I'}$ with non-trivial smooth part as $I \not\subseteq I'$. Hence, $F_{n,I\cap I'}(\omega)^\vee \otimes_E \mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}}$ and $H_{\ell_2}(N_{n,I\cap I'}, F_{n,I}(\lambda)^\vee)$ never share the same $Z_{n,I\cap I'}$-character for any $\ell_2 \geq 0$, which together with the spectral sequence

$$\begin{aligned}&\mathrm{Ext}^{\ell_1}_{L_{n,I},-\lambda}(H_{\ell_2}(N_{n,I\cap I'}, F_{n,I}(\lambda)^\vee), F_{n,I\cap I'}(\omega)^\vee \otimes_E \mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}}) \\ &\qquad \Rightarrow \mathrm{Ext}^{\ell_1+\ell_2}_{L_{n,I},-\lambda}(F_{n,I}(\lambda)^\vee, i^{\mathrm{an}}_{n,I,I\cap I'}(F_{n,I\cap I'}(\omega)^\vee \otimes_E \mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}}))\end{aligned}$$

implies that

$$\mathrm{Ext}^\ell_{L_{n,I},-\lambda}(F_{n,I}(\lambda)^\vee, i^{\mathrm{an}}_{n,I,I\cap I'}(F_{n,I\cap I'}(\omega)^\vee \otimes_E \mathfrak{d}_{L_{n,I}\cap P_{n,I\cap I'}})) = 0$$

for each $\ell \geq 0$. $\square$

Lemma 4.17. *Let $I \subseteq I' \subseteq I'' \subseteq \Delta_n$ be three subsets and $k, k' \geq 0$ be two integers. Then the following commutative diagram is commutative.*

$$\begin{array}{ccc}
\mathrm{Ext}^{k'}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I''}(\lambda), i^{\mathrm{an}}_{n,I'}(\lambda)) \times \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) & \xrightarrow{\sim} & H^{k'}(\overline{L}_{n,I'}, 1_{\overline{L}_{n,I'}}) \times H^{k}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \\
 & & \downarrow \\
\Big\downarrow \cup & & H^{k'}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \times H^{k}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \\
 & & \Big\downarrow \cup \\
\mathrm{Ext}^{k+k'}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I''}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) & \xrightarrow{\sim} & H^{k+k'}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})
\end{array}$$

Proof. It is clear that $\mathrm{Hom}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$ is one dimensional and spanned by the natural embedding $\iota : i^{\mathrm{an}}_{n,I'}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I}(\lambda)$. Moreover, Proposition 4.16 also implies that the cup product map

$$\mathrm{Hom}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \times \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \xrightarrow{\cup} \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$$

induces an isomorphism

$$\iota \cup \bullet : \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$$

which corresponds to identity map of $H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$. For each $x \in \mathrm{Ext}^{k'}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I''}(\lambda), i^{\mathrm{an}}_{n,I'}(\lambda))$ and $y \in \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$, there exists $y' \in \mathrm{Ext}^{k}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$ such that $y = \iota \cup y'$, and thus $x \cup y = (x \cup \iota) \cup y'$. We finish the proof by noting that the map

$$\bullet \cup \iota : \mathrm{Ext}^{k'}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I''}(\lambda), i^{\mathrm{an}}_{n,I'}(\lambda)) \to \mathrm{Ext}^{k'}_{G_n,\lambda}(i^{\mathrm{an}}_{n,I''}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$$

corresponds to the restriction map $H^{k'}(\overline{L}_{n,I'}, 1_{\overline{L}_{n,I'}}) \to H^{k'}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$ under Proposition 4.16. $\square$

We use the shortened notation $H^\bullet_I \overset{\mathrm{def}}{=} H^\bullet(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$, $H^\bullet_{I'} \overset{\mathrm{def}}{=} H^\bullet(\overline{L}_{n,I'}, 1_{\overline{L}_{n,I'}})$ and write $\mathrm{Res}^\bullet_{n,I,I'} : H^\bullet_I \to H^\bullet_{I'}$ for the restriction map, for each $I' \subseteq I \subseteq \Delta_n$. We consider two pair of subsets $I_0 \subseteq I_1 \subseteq \Delta_n$ and $I_2 \subseteq I_3 \subseteq \Delta_n$ satisfying $I_3 \subseteq I_1$ and $I_1 \setminus I_0 = I_3 \setminus I_2 = \{i\}$ for some $i \in \Delta_n$. Let $\theta, \theta' \in E^\times$ be two scalars. We define a complex $\mathbf{C}_\theta \overset{\mathrm{def}}{=} [i^{\mathrm{an}}_{n,I_1}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I_0}(\lambda)]$ (resp. $\mathbf{C}_{\theta'} \overset{\mathrm{def}}{=} [i^{\mathrm{an}}_{n,I_3}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I_2}(\lambda)]$) supported in degree $[-1, 0]$ with the injection given by $\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)$ (resp. $\theta'\kappa^{\mathrm{an}}_{I_3,I_2}(\lambda)$). We have canonical maps $p_\theta : i^{\mathrm{an}}_{n,I_0}(\lambda) \to \mathbf{C}_\theta$ and $q_\theta : \mathbf{C}_\theta \to i^{\mathrm{an}}_{n,I_1}(\lambda)[-1]$ and similarly for $p_{\theta'}$ and $q_{\theta'}$. In the following, we will use the isomorphisms in item (i) Proposition 4.16 without further explanation. It follows directly from Proposition 4.16 that

- $\mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda), i^{\mathrm{an}}_{n,I_3}(\lambda)) = 0$;
- $\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)$ induces an isomorphism $\mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda), i^{\mathrm{an}}_{n,I_2}(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda), i^{\mathrm{an}}_{n,I_2}(\lambda))$ which corresponds to identity map on $H^\bullet_{I_2}$;
- $\kappa^{\mathrm{an}}_{I_3,I_2}(\lambda)$ induces a map $\mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda), i^{\mathrm{an}}_{n,I_3}(\lambda)) \to \mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda), i^{\mathrm{an}}_{n,I_2}(\lambda))$ which corresponds to $\mathrm{Res}^\bullet_{n,I_3,I_2} : H^\bullet_{I_3} \to H^\bullet_{I_2}$.

Consequently, for each $\theta, \theta' \in E^\times$, we have

- $\mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_\theta, i^{\mathrm{an}}_{n,I_2}(\lambda)) = 0$ as we can compute it using a spectral sequence whose second page is identically zero;
- $q_{\theta'}$ and q_θ induce isomorphisms $\mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_\theta, \mathbf{C}_{\theta'}) \xrightarrow{q_{\theta',*}} \mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_\theta, i^{\mathrm{an}}_{n,I_3}(\lambda)[-1]) \xleftarrow{q^*_\theta} H^\bullet_{I_3}$;
- $p_{\theta'}$ induces a isomorphism $H^\bullet_{I_2} \xrightarrow{p_{\theta',*}} \mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda), \mathbf{C}_{\theta'})$.

The following is our main goal.

Lemma 4.18. *The composition*

$$H^\bullet_{I_3} \xrightarrow{(q_{\theta',*})^{-1}q_\theta^*} \mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_\theta, \mathbf{C}_{\theta'}) \xrightarrow{p_\theta^*} \mathrm{Ext}^\bullet_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda), \mathbf{C}_{\theta'}) \xrightarrow{p_{\theta',*}^{-1}} H^\bullet_{I_2}$$

is given by $\theta'\theta^{-1}\mathrm{Res}^\bullet_{n,I_3,I_2}$.

Proof. Let $k \geq 0$ be an integer. It follows from item (i) of Lemma 1.7 that we can compute $\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], \mathbf{C}_{\theta'})$ via a spectral sequence whose degenerate at its second page due to degree reason. Therefore we have a canonical isomorphism in the derived category of E-vector spaces

$$\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], \mathbf{C}_{\theta'}) \cong H^k_{\theta'} \overset{\mathrm{def}}{=} [H^k_{I_3} \to H^k_{I_2}]$$

with RHS supported in degree $[0,1]$ and the map given by $\theta'\mathrm{Res}^k_{n,I_3,I_2}$. The distinguished triangle

$$i^{\mathrm{an}}_{n,I_0}(\lambda) \xrightarrow{p_\theta} \mathbf{C}_\theta \xrightarrow{q_\theta} i^{\mathrm{an}}_{n,I_1}(\lambda)[-1] \to \tag{4.14}$$

determines the distinguish triangle

$$\mathbf{C}_\theta \xrightarrow{q_\theta} i^{\mathrm{an}}_{n,I_1}(\lambda)[-1] \xrightarrow{\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)[-1]} i^{\mathrm{an}}_{n,I_0}(\lambda)[-1] \xrightarrow{p_\theta[-1]} . \tag{4.15}$$

This induces a distinguish triangle
(4.16)

$$\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda)[-1], \mathbf{C}_{\theta'}) \xrightarrow{\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)[-1]^*} \mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], \mathbf{C}_{\theta'}) \xrightarrow{q_\theta^*} \mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_\theta, \mathbf{C}_{\theta'}) \xrightarrow{p_\theta^*} .$$

The commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda)[-1], \mathbf{C}_{\theta'}) & \xrightarrow{\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)[-1]^*} & \mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], \mathbf{C}_{\theta'}) \\
\cong\uparrow & & \cong\uparrow \\
\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_0}(\lambda)[-1], i^{\mathrm{an}}_{n,I_2}(\lambda)) & \xrightarrow{\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)[-1]^*} & \mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], i^{\mathrm{an}}_{n,I_2}(\lambda))
\end{array}$$

implies that $\theta\kappa^{\mathrm{an}}_{I_1,I_0}(\lambda)[-1]^*$ in (4.16) corresponds to the composition $H^k_{I_2}[1] \xrightarrow{\theta} H^k_{I_2}[1] \xrightarrow{f_{\theta'}} H^k_{\theta'}$ with the second map $f_{\theta'}$ being the standard truncation map. On the other hand, using the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], \mathbf{C}_{\theta'}) & \xrightarrow{q_\theta^*} & \mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_\theta, \mathbf{C}_{\theta'}) \\
\cong\uparrow & & \cong\uparrow \\
\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I_1}(\lambda)[-1], i^{\mathrm{an}}_{n,I_3}(\lambda)) & \xrightarrow{q_\theta^*} & \mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_\theta, i^{\mathrm{an}}_{n,I_3}(\lambda))
\end{array}$$

we know that the map q_θ^* in (4.16) corresponds to the truncation map $g_{\theta'} : H^k_{\theta'} \to H^k_{I_3}$. Consequently, we deduce from (4.16) the following distinguish triangle

$$H^k_{I_2}[1] \xrightarrow{\theta f_{\theta'}} H^k_{\theta'} \xrightarrow{g_{\theta'}} H^k_{I_3} \xrightarrow{(q_{\theta',*})^{-1}q_\theta^* p_\theta^* p_{\theta',*}^{-1}} .$$

If we compare it with the obvious distinguish triangle

$$H^k_{I_2}[1] \xrightarrow{f_{\theta'}} H^k_{\theta'} \xrightarrow{g_{\theta'}} H^k_{I_3} \xrightarrow{\theta'\mathrm{Res}^k_{n,I_3,I_2}}$$

from the definition of $H^k_{\theta'}$, we conclude that $(q_{\theta',*})^{-1}q^*_\theta p^*_\theta p^{-1}_{\theta',*} = \theta^{-1}\theta'\mathrm{Res}^k_{n,I_3,I_2}$, which finishes the proof. □

4.2. **Extensions between two complex.** In this section, we compute the Ext-groups between certain complexes (with each term given by direct sum of various $i^{\mathrm{an}}_{n,I}(\lambda)$) using the computation of $E^{\bullet,\bullet}_{\bullet,I_0,I_1}$ in Section 2 and Proposition 4.16.

We first introduce here some general complexes that will be essential in Section 4.3. Note that $\mathbf{C}_{I_0,I_1}(\lambda)$ (for various $I_0, I_1 \subseteq \Delta_n$) are special examples of the more general complexes we consider here.

We consider a tuple $\mathcal{P} = (\underline{I}, \underline{\ell}^+, \underline{\ell}^-)$ that satisfies the following conditions

- $\underline{I} = (I_{r'})_{1\leq r'\leq r}$ with $\Delta_n = \bigsqcup_{r'=1}^r I_{r'}$ a partition;
- $\underline{\ell}^+ = (\ell^+_{r'})_{1\leq r'\leq r}$ and $\underline{\ell}^- = (\ell^-_{r'})_{1\leq r'\leq r}$ with $0 \leq \ell^-_{r'} \leq \ell^+_{r'} \leq \#I_{r'}$ for each $1 \leq r' \leq r$.

Given a tuple $\mathcal{P}$ as above, we define a complex $\mathbf{C}_{\mathcal{P}}(\lambda)$ whose degree $-\ell$ term is the direct sum of all $i^{\mathrm{an}}_{n,I'}(\lambda)$ satisfying $\#I' \cap I_{r'} \in [\ell^-_{r'}, \ell^+_{r'}]$ for each $1 \leq r' \leq r$.

Given an extra $I \subseteq \Delta_n$, we can compute $\mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_{\mathcal{P}}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$ by a spectral sequence ${}_{\mathcal{P}}E^{\bullet,\bullet}_{\bullet,I}$ (see item (ii) of Lemma 1.7) with ${}_{\mathcal{P}}E^{-\ell,k}_{1,I}$ isomorphic to the direct sum of

$$\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$$

for $I \subseteq I' \subseteq \Delta_n$ satisfying $\#I' = \ell$ and $\#I' \cap I_{r'} \in [\ell^-_{r'}, \ell^+_{r'}]$ for each $1 \leq r' \leq r$. For three integers $\ell' \leq \ell' \leq \ell$, we use the shorted notation $c(\ell,\ell',\ell'') \overset{\mathrm{def}}{=} \frac{(\ell-\ell'')!}{(\ell-\ell')!(\ell'-\ell'')!}$. We fix a $1 \leq r' \leq r$ for now and set $\ell^-_{r',I} \overset{\mathrm{def}}{=} \max\{\ell^-_{r'}, \#I \cap I_{r'}\}$. We set $d^+_{r'} \overset{\mathrm{def}}{=} c(\#I_{r'}, \ell^+_{r'}, \#I \cap I_{r'})$ if $\ell^+_{r'} = \ell^-_{r',I}$, $d^+_{r'} \overset{\mathrm{def}}{=} 0$ if $\#I_{r'} = \ell^+_{r'} > \ell^-_{r',I}$, and $d^+_{r'} \overset{\mathrm{def}}{=} c(\#I_{r'}, \ell^+_{r'}, 1 + \#I \cap I_{r'})$ if $\#I_{r'} > \ell^+_{r'} > \ell^-_{r',I}$. We also set $d^-_{I,r'} \overset{\mathrm{def}}{=} 0$ if $\ell^+_{r'} > \ell^-_{r',I} = \#I \cap I_{r'}$, and $d^-_{I,r'} \overset{\mathrm{def}}{=} c(\#I_{r'} - 1, \ell^-_{r',I}, \#I \cap I_{r'})$ if $\ell^+_{r'} > \ell^-_{r',I} > \#I \cap I_{r'}$.

Then we consider the complex $\mathbf{C}_{\mathcal{P},I,r'}$ of E-vector spaces (having support in degree $[\ell^-_{r',I}, \ell^+_{r'}]$) with the dimension of the degree $\ell_{r'}$ term of $\mathbf{C}_{\mathcal{P},I,r'}$ counting the number of sets $I'_{r'}$ satisfying $I \cap I_{r'} \subseteq I'_{r'} \subseteq I_{r'}$ and $\#I'_{r'} = \ell_{r'} \in [\ell^-_{r',I}, \ell^+_{r'}]$. The differential of $\mathbf{C}_{\mathcal{P},I,r'}$ is naturally induced from inclusion between various $I'_{r'}$. We write $H^\bullet(\mathbf{C}_{\mathcal{P},I,r'})$ for its cohomology (understood to be a complex with zero differentials) and observe that $H^\bullet(\mathbf{C}_{\mathcal{P},I,r'}) = 0$ if $\ell^-_{r',I} > \ell^+_{r'}$, $H^\bullet(\mathbf{C}_{\mathcal{P},I,r'}) = E^{\oplus d^+_{r'}}[\ell^+_{r'}]$ if $\ell^-_{r',I} = \ell^+_{r'}$, and $H^\bullet(\mathbf{C}_{\mathcal{P},I,r'}) = E^{\oplus d^-_{I,r'}}[\ell^-_{r',I}] \oplus E^{\oplus d^+_{r'}}[\ell^+_{r'}]$ if $\ell^-_{r',I} < \ell^+_{r'}$.

Lemma 4.19. *We have an isomorphism of complexes*

$$ {}_{\mathcal{P}}E^{\bullet,k}_{1,I} \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \otimes \bigotimes_{r'=1}^r \mathbf{C}_{\mathcal{P},I,r'} \tag{4.17}$$

which induces an isomorphism of cohomologies

$$ {}_{\mathcal{P}}E^{\bullet,k}_{2,I} \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \otimes \bigotimes_{r'=1}^r H^\bullet(\mathbf{C}_{\mathcal{P},I,r'}),$$

and the spectral sequence ${}_{\mathcal{P}}E^{\bullet,\bullet}_{\bullet,I}$ degenerates at the second page. In particular, if $\ell^-_{r',I} = \ell^+_{r'}$ for each $1 \leq r' \leq r$, then ${}_{\mathcal{P}}E^{\bullet,k}_{2,I} \cong H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})^{\oplus d^+_{\mathcal{P}}}[\ell^+_{\mathcal{P}}]$ with $d^+_{\mathcal{P}} \overset{def}{=} \sum_{r'=1}^r d^+_{r'}$ and $\ell^+_{\mathcal{P}} \overset{def}{=} \sum_{r'=1}^r \ell^+_{r'}$.

Proof. We notice that the complex ${}_{\mathcal{P}}E^{\bullet,k}_{1,I}$ is isomorphic to the tensor of $H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$ with a complex $\mathbf{C}_{\mathcal{P},I}$ of vector spaces, whose degree ℓ term has dimension counting the number of $I \subseteq$

$I' \subseteq \Delta_n$ satisfying $\#I' = \ell$ and $\#I' \cap I_{r'} \in [\ell_{r'}^-, \ell_{r'}^+]$ for each $1 \leq r' \leq r$. As each such I' admits a partition $I' = \bigsqcup_{r'=1}^{r} I' \cap I_{r'}$, the complex $\mathbf{C}_{\mathcal{P},I}$ is isomorphic to the tensor product of complexes $\mathbf{C}_{\mathcal{P},I,r'}$ for $1 \leq r' \leq r$ with $\mathbf{C}_{\mathcal{P},I,r'}$ defined before this lemma. We also note that $H^\bullet(\mathbf{C}_{\mathcal{P},I})$ is clearly isomorphic to the tensor of various $H^\bullet(\mathbf{C}_{\mathcal{P},I,r'})$. The decomposition (4.17) actually indicates that ${}_{\mathcal{P}}E_{\bullet,I}^{\bullet,\bullet}$ is actually the tensor product of two spectral sequences, one supported inside the 0-th row and the other supported inside the 0-th column with both of them degenerate at the second page, which forces ${}_{\mathcal{P}}E_{\bullet,I}^{\bullet,\bullet}$ to degenerate at the second page as well. □

The following is a special case of Lemma 4.19 when we take $r = 3$ and consider the partition $\Delta_n = I_0 \sqcup (I_1 \setminus I_0) \sqcup (\Delta_n \setminus I_1)$ with $\ell_1^+ = \ell_1^- = \#I_0$, $\ell_2^+ = \#I_1 \setminus I_0$, $\ell_2^- = 0$ and $\ell_3^+ = \ell_3^- = 0$. Note that we have $\mathbf{C}_{\mathcal{P}}(\lambda) \cong \mathbf{C}_{I_0,I_1}(\lambda)$ in this case.

Lemma 4.20. *For each $k \in \mathbb{Z}$, we have*

$$\mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), i_{n,I}^{\mathrm{an}}(\lambda)) \xleftarrow{\sim} \mathrm{Ext}_{G_n,\lambda}^{k-\#I_1}(i_{n,I_1}^{\mathrm{an}}(\lambda), i_{n,I}^{\mathrm{an}}(\lambda)) \cong H^{k-\#I_1}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$$

if $I_0 \cup I = I_1$ and $\mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), i_{n,I}^{\mathrm{an}}(\lambda)) = 0$ otherwise.

Note that $I_0 \cup I = I_1$ if and only if $I_1 \setminus I_0 \subseteq I \subseteq I_1$. Now we consider a second pair of subsets $I_2 \subseteq I_3 \subseteq \Delta_n$. Using Lemma 4.20, we can compute $\mathrm{Ext}_{G_n,\lambda}^{\bullet}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda))$ by a spectral sequence $\{{}_{I_0,I_1}E_{r,I_2,I_3}^{-\ell,k}\}_{r\geq 1,\#I_2\leq\ell\leq\#I_3,k\geq 0}$ whose first page has $(-\ell, k)$ term given by

$$\bigoplus_{I_2\subseteq I\subseteq I_3,\#I=\ell} \mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), i_{n,I}^{\mathrm{an}}(\lambda))$$
$$\cong \bigoplus_{I_2\subseteq I\subseteq I_3,I_0\cup I=I_1,\#I=\ell} H^{k-\#I_1}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}) \cong \bigoplus_{I_2\cup(I_1\setminus I_0)\subseteq I\subseteq I_1\cap I_3,\#I=\ell} H^{k-\#I_1}(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}}),$$

which is precisely $E_{1,I_2\cup(I_1\setminus I_0),I_1\cap I_3}^{-\ell,k-\#I_1}$ as defined at the beginning of Section 2.2. The following cases are particularly simple.

Lemma 4.21. *(i) If $I_2 \not\subseteq I_1$ or $I_1 \not\subseteq I_0 \cup I_3$, then $\mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda)) = 0$ for each $k \in \mathbb{Z}$.*

(ii) If $I_0 \subseteq I_2 \subseteq I_1 \subseteq I_3$, then $\mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda)) \cong H^k(\overline{L}_{n,I_1}, 1_{\overline{L}_{n,I_1}})$ for each $k \in \mathbb{Z}$. In particular, $\mathrm{Hom}_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda))$ is spanned by the composition of two truncation morphisms $\mathbf{C}_{I_0,I_1}(\lambda) \to \mathbf{C}_{I_2,I_1}(\lambda) \to \mathbf{C}_{I_2,I_3}(\lambda)$.

(iii) The truncation map $\mathbf{C}_{I_2,I_1\cap I_3}(\lambda) \to \mathbf{C}_{I_2,I_3}(\lambda)$ induces an isomorphism of spectral sequences

$${}_{I_0,I_1}E_{\bullet,I_2,I_1\cap I_3}^{\bullet,\bullet} \xrightarrow{\sim} {}_{I_0,I_1}E_{\bullet,I_2,I_3}^{\bullet,\bullet}.$$

Here we understand $\mathbf{C}_{I_2,I_1\cap I_3}(\lambda)$ to be zero if $I_2 \not\subseteq I_1$.

(iv) If $I_2 \subseteq I_0 \subseteq I_1 = I_3$, then we have a canonical isomorphism of spectral sequences

$${}_{I_0,I_1}E_{\bullet,I_2,I_1}^{\bullet,\bullet} \cong {}_{I_0\setminus I,I_1}E_{\bullet,I_2\setminus I,I_1}^{\bullet,\bullet}$$

for each $I \subseteq I_2$.

Proof. The first part follows from the observation that $I_2 \cup (I_1 \setminus I_0) \subseteq I_1 \cap I_3$ is equivalent to $I_2 \subseteq I_1 \subseteq I_0 \cup I_3$. For the second part, it suffices to note that $I_0 \subseteq I_2 \subseteq I_1 \subseteq I_3$ implies that $I_2 \cup (I_1 \setminus I_0) = I_1 = I_1 \cap I_3$, and thus

$$\mathrm{Ext}_{G_n,\lambda}^{k}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda)) \cong {}_{I_0,I_1}E_{1,I_2,I_3}^{-\#I_1,k+\#I_1} \cong E_{1,I_2\cup(I_1\setminus I_0),I_1\cap I_3}^{-\#I_1,k} \cong H^k(\overline{L}_{n,I_1}, 1_{\overline{L}_{n,I_1}}).$$

In particular, $\mathrm{Hom}_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda),\mathbf{C}_{I_2,I_3}(\lambda))$ is one dimensional and thus spanned by $\mathbf{C}_{I_0,I_1}(\lambda)\to\mathbf{C}_{I_2,I_1}(\lambda)\to\mathbf{C}_{I_2,I_3}(\lambda)$. The third part follows from the following isomorphism of first page of spectral sequences

$$ {}_{I_0,I_1}E^{-\ell,k}_{1,I_2,I_3}\cong E^{-\ell,k-\#I_1}_{1,I_2\cup(I_1\setminus I_0),I_1\cap I_3}\cong_{I_0,I_1}E^{-\ell,k}_{1,I_2,I_1\cap I_3}. $$

The fourth part follows from the isomorphism of first page of spectral sequences

$$ {}_{I_0,I_1}E^{\bullet,\bullet}_{\bullet,I_2,I_1}\cong E^{\bullet,\bullet-\#I_1}_{1,I_2\cup(I_1\setminus I_0),I_1}\cong_{I_0\setminus I,I_1}E^{\bullet,\bullet}_{1,I_2\setminus I,I_1} $$

based on the equality $I_2\cup(I_1\setminus I_0)=(I_2\setminus I)\cup(I_1\setminus(I_0\setminus I))$. □

Now we apply computations from Section 2.2 to ${}_{I_0,I_1}E^{-\ell,k}_{r,I_2,I_3}\cong E^{-\ell,k-\#I_1}_{1,I_2\cup(I_1\setminus I_0),I_1\cap I_3}$. Lemma 2.27 implies that $\{{}_{I_0,I_1}E^{-\ell,k}_{r,I_2,I_3}\}_{r\geq 1,\#I_2\leq\ell\leq\#I_3,k\geq 0}$ degenerates at the second page. Hence, we deduce from Lemma 2.16 and Proposition 2.26 that

Theorem 4.22. *Let $I_0\subseteq I_1\subseteq\Delta_n$ and $I_2\subseteq I_3\subseteq\Delta_n$ be two pairs of subsets with $I_0^\sharp\overset{def}{=}I_2\cup(I_1\setminus I_0)$ and $I_1^\sharp\overset{def}{=}I_1\cap I_3$.*

- *If $\mathrm{Ext}^h_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda),\mathbf{C}_{I_2,I_3}(\lambda))\neq 0$, then we have $I_2\subseteq I_1\subseteq I_0\cup I_3$ and $\#I_1^\sharp-2\#I_0^\sharp+\#I_1\leq h\leq n^2-n+\#I_1$.*
- *For each $h\in\{\#I_1^\sharp-2\#I_0^\sharp+\#I_1,\#I_1^\sharp-2\#I_0^\sharp+\#I_1+1\}$, the space*

$$ M^h_{I_0,I_1,I_2,I_3}\overset{def}{=}\mathrm{Ext}^{\#I_1^\sharp-2\#I_0^\sharp+\#I_1}_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda),\mathbf{C}_{I_2,I_3}(\lambda)) $$

admits a canonical decreasing filtration

$$ 0=\mathrm{Fil}^{-\#I_0^\sharp+1}(M^h_{I_0,I_1,I_2,I_3})\subseteq\mathrm{Fil}^{-\#I_0^\sharp}(M^h_{I_0,I_1,I_2,I_3})\subseteq\cdots\subseteq\mathrm{Fil}^{-\#I_1^\sharp}(M^h_{I_0,I_1,I_2,I_3})=M^h_{I_0,I_1,I_2,I_3} $$

such that $\mathrm{Fil}^{-\ell}(M^h_{I_0,I_1,I_2,I_3})/\mathrm{Fil}^{-\ell+1}(M^h_{I_0,I_1,I_2,I_3})\cong E^{-\ell,\ell+h-\#I_1}_{2,I_0^\sharp,I_1^\sharp}$ admits a basis indexed by $\Psi^{-\ell,\ell+h-\#I_1}_{I_0^\sharp,I_1^\sharp}$ for each $\#I_0^\sharp\leq\ell\leq\#I_1^\sharp$.

Recall that $\mathbf{C}_{I,\Delta_n}(\lambda)\cong V^{\mathrm{an}}_{n,I}(\lambda)[-\#I]$ for each $I\subseteq\Delta_n$. If we take $I_1=I_3=\Delta_n$, we have $I_0^\sharp=I_2\cup(\Delta_n\setminus I_0)\subseteq\Delta_n=I_1^\sharp$. We set

$$ \begin{aligned} h_{I_0,I_2}&\overset{\mathrm{def}}{=}(\#I_1^\sharp-2\#I_0^\sharp+\#I_1)+\#I_2-\#I_0=2n-2-2\#I_2\cup(\Delta_n\setminus I_0)+\#I_2-\#I_0\\ &=2n-2+2\#I_2\cap(\Delta_n\setminus I_0)-2\#I_2-2\#\Delta_n\setminus I_0+\#I_2-\#I_0=2\#I_2\setminus I_0+\#I_0-\#I_2\\ &=\#I_0+\#I_2-2\#I_0\cap I_2=\#I_0\setminus I_2+\#I_2\setminus I_0. \end{aligned} $$

We observe that $I_2\subseteq I_0$ if and only if $h_{I_0,I_2}=\#I_0-\#I_2$, and $I_0\subseteq I_2$ if and only if $h_{I_0,I_2}=\#I_2-\#I_0$. As a consequence of Theorem 4.22, we obtain

Corollary 4.23. *Let $I_0,I_2\subseteq\Delta_n$ be two subsets.*

- *If $\mathrm{Ext}^h_{G_n,\lambda}(V^{\mathrm{an}}_{n,I_0}(\lambda),V^{\mathrm{an}}_{n,I_2}(\lambda))\neq 0$, then $h_{I_0,I_2}\leq h\leq n^2-1+\#I_2-\#I_0$.*
- *The space $\mathbf{E}_{I_0,I_2}\overset{def}{=}\mathrm{Ext}^{h_{I_0,I_2}}_{G_n,\lambda}(V^{\mathrm{an}}_{n,I_0}(\lambda),V^{\mathrm{an}}_{n,I_2}(\lambda))$ admits a canonical decreasing filtration*

$$ 0=\mathrm{Fil}^{-\#I_0^\sharp+1}(\mathbf{E}_{I_0,I_2})\subseteq\mathrm{Fil}^{-\#I_0^\sharp}(\mathbf{E}_{I_0,I_2})\subseteq\cdots\subseteq\mathrm{Fil}^{-n+1}(\mathbf{E}_{I_0,I_2})=\mathbf{E}_{I_0,I_2} $$

such that $\mathrm{Fil}^{-\ell}(\mathbf{E}_{I_0,I_2})/\mathrm{Fil}^{-\ell+1}(\mathbf{E}_{I_0,I_2})\cong E^{-\ell,\ell+n-1-2\#I_0^\sharp}_{2,I_0^\sharp,\Delta_n}$ admits a basis indexed by $\Psi^{-\ell,\ell+n-1-2\#I_0^\sharp}_{I_0^\sharp,\Delta_n}$ for each $\#I_0^\sharp\leq\ell\leq n-1$.

- *The space* $\mathbf{E}'_{I_0,I_2} \overset{def}{=} \mathrm{Ext}^{h_{I_0,I_2}+1}_{G_n,\lambda}(V^{\mathrm{an}}_{n,I_0}(\lambda), V^{\mathrm{an}}_{n,I_2}(\lambda))$ *admits a canonical decreasing filtration*
$$0 = \mathrm{Fil}^{-\#I_0^\sharp+1}(\mathbf{E}'_{I_0,I_2}) \subseteq \mathrm{Fil}^{-\#I_0^\sharp}(\mathbf{E}'_{I_0,I_2}) \subseteq \cdots \subseteq \mathrm{Fil}^{-n+1}(\mathbf{E}'_{I_0,I_2}) = \mathbf{E}'_{I_0,I_2}$$
such that $\mathrm{Fil}^{-\ell}(\mathbf{E}'_{I_0,I_2})/\mathrm{Fil}^{-\ell+1}(\mathbf{E}'_{I_0,I_2}) \cong E^{-\ell,\ell+n-2\#I_0^\sharp}_{2,I_0^\sharp,\Delta_n}$ *admits a basis indexed by* $\Psi^{-\ell,\ell+n-\#I_0^\sharp}_{I_0^\sharp,\Delta_n}$ *for each* $\#I_0^\sharp \leq \ell \leq n-1$.

Remark 4.24. Recall that $i^\infty_{n,I} = i^\infty_{n,I,\Delta_n}(1_{\overline{L}_{n,I}})$ and $V^\infty_{n,I} = i^\infty_{n,I}/\sum_{I\subsetneq I'\subseteq \Delta_n} i^\infty_{n,I'}$ for each $I \subseteq \Delta_n$. Then for each pair $I_0 \subseteq I_1 \subseteq \Delta_n$, we can define a complex $\mathbf{C}^\infty_{I_0,I_1}$ supported in degree $[-\#I_1, -\#I_0]$ whose degree $-\ell$ term is $\bigoplus_{I_0\subseteq I\subseteq I_1, \#I=\ell} i^\infty_{n,I}$. For each $\lambda \in X(T_{n,E})$ which is dominant with respect to $B_{n,E}$, we have $i^{\mathrm{alg}}_{n,I}(\lambda) = F_{n,\Delta_n}(\lambda) \otimes_E i^\infty_{n,I}$, $V^{\mathrm{alg}}_{n,I}(\lambda) = F_{n,\Delta_n}(\lambda) \otimes_E V^\infty_{n,I}$ and $\mathbf{C}^{\mathrm{alg}}_{I_0,I_1}(\lambda) \overset{\mathrm{def}}{=} F_{n,\Delta_n}(\lambda) \otimes_E \mathbf{C}^\infty_{I_0,I_1}$. It follows from Theorem 7.1 of [Koh11], Lemma 4.14 and Lemma 4.15 that the natural embedding $i^{\mathrm{alg}}_{n,I'}(\lambda) \hookrightarrow i^{\mathrm{an}}_{n,I'}(\lambda)$ induces isomorphisms
$$H^k(N_{n,I}, i^{\mathrm{an}}_{n,I'}(\lambda)')_{\delta'_{n,I,\lambda}} \cong H^k(N_{n,I}, i^{\mathrm{alg}}_{n,I'}(\lambda)')_{\delta'_{n,I,\lambda}}$$
for each $k \geq 0$, which together with standard spectral sequences implies the isomorphism
$$\mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{an}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^k_{G_n,\lambda}(i^{\mathrm{alg}}_{n,I'}(\lambda), i^{\mathrm{an}}_{n,I}(\lambda))$$
for each $I, I' \subseteq \Delta_n$ and $k \geq 0$. Consequently, if $I_2 \subseteq I_2 \subseteq \Delta_n$ is a second pair of subsets, then $\mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_2}(\lambda))$ and $\mathrm{Ext}^\bullet_{G_n,\lambda}(\mathbf{C}^{\mathrm{alg}}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_2}(\lambda))$ can be computed by exactly the same spectral sequence, and thus isomorphic as filtered spaces (equipped with the canonical filtrations coming from the same spectral sequence). As a special case, we conclude that the natural embedding $V^{\mathrm{alg}}_{n,I_0}(\lambda) \hookrightarrow V^{\mathrm{an}}_{n,I_0}(\lambda)$ induces an isomorphism
$$\mathrm{Ext}^h_{G_n,\lambda}(V^{\mathrm{an}}_{n,I_0}(\lambda), V^{\mathrm{an}}_{n,I_2}(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^h_{G_n,\lambda}(V^{\mathrm{alg}}_{n,I_0}(\lambda), V^{\mathrm{an}}_{n,I_2}(\lambda))$$
for each $I_0, I_2 \subseteq \Delta_n$ and $h \geq 0$.

4.3. **Explicit cup product.** In this section, we explicitly compute certain cup product maps (see (4.18)) using canonical filtration on eachExt-group as well as the basis of each graded piece established in Section 2 and Section 4.2. The key new ingredients are the commutative diagrams (4.20) and (4.23) which carefully record the effect on Ext-groups under various truncations of complexes.

Recall from Theorem 4.22 that we use the shortened notation $M^k_{I_0,I_1,I_2,I_3}$ for $\mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{I_2,I_3}(\lambda))$, and we will use similar notation for other Ext-groups (for example $M^k_{\mathcal{P},\mathcal{P}'}$ for $\mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_{\mathcal{P}}(\lambda), \mathbf{C}_{\mathcal{P}'}(\lambda))$ and $M^k_{I_0,I_1,\mathcal{P}'}$ for $\mathrm{Ext}^k_{G_n,\lambda}(\mathbf{C}_{I_0,I_1}(\lambda), \mathbf{C}_{\mathcal{P}'}(\lambda))$). We also set $H^k_S \overset{\mathrm{def}}{=} \bigoplus_{I\in S} H^k(\overline{L}_{n,I}, 1_{\overline{L}_{n,I}})$ for each set S of subsets of Δ_n and $k \geq 0$. Given two integers $k_0, k_1 \in \mathbb{Z}$ and three tuples $\mathcal{P}, \mathcal{P}', \mathcal{P}''$ as introduced at the beginning of Section 4.2, we have a canonical cup product map
$$M^{k_0}_{\mathcal{P},\mathcal{P}'} \otimes M^{k_1}_{\mathcal{P}',\mathcal{P}''} \xrightarrow{\cup} M^{k_0+k_1}_{\mathcal{P},\mathcal{P}''}.$$
In particular, given two integers $k_0, k_1 \in \mathbb{Z}$ and three pairs of subsets of Δ_n $I_0 \subseteq I_1$, $I_2 \subseteq I_3$ and $I_4 \subseteq I_5$, we have the following cup product map
$$M^{k_0}_{I_0,I_1,I_2,I_3} \otimes M^{k_1}_{I_2,I_3,I_4,I_5} \xrightarrow{\cup} M^{k_0+k_1}_{I_0,I_1,I_4,I_5}. \tag{4.18}$$
We assume from now the following simplifying condition which is already sufficient for our later application.

Condition 4.25. *We have* $I_1 = I_3 = I_5$ *and* $I_4 \subseteq I_2 \subseteq I_0$.

Let $J \subseteq [0, \#I_2 \setminus I_4]$ and $J' \subseteq [0, \#I_0 \setminus I_2]$ be two subintervals. Under Condition 4.25, we consider a list of tuples $\mathcal{P}_{J,J'}$ with a common (ordered) partition of set

$$\Delta_n = I_4 \sqcup (I_2 \setminus I_4) \sqcup (I_0 \setminus I_2) \sqcup (I_1 \setminus I_0) \sqcup (\Delta_n \setminus I_1)$$

that satisfies $[\ell^-_{J,J',1}, \ell^+_{J,J',1}] = \{\#I_4\}$, $[\ell^-_{J,J',2}, \ell^+_{J,J',2}] = J$, $[\ell^-_{J,J',3}, \ell^+_{J,J',3}] = J'$, $[\ell^-_{J,J',4}, \ell^+_{J,J',4}] = \{\#I_1 \setminus I_0\}$ and $[\ell^-_{J,J',5}, \ell^+_{J,J',5}] = \{0\}$. Given two intervals $J_i = [\ell^-_i, \ell^+_i]$ for $i = 1, 2$, we write $J_1 \leq J_2$ if $\ell^-_1 \leq \ell^-_2$ and $\ell^+_1 \leq \ell^+_2$. Given $J_1, J_2 \subseteq [0, \#I_2 \setminus I_4]$ and $J'_1, J'_2 \subseteq [0, \#I_0 \setminus I_2]$ satisfying $J_1 \leq J_2$ and $J'_1 \leq J'_2$, then there exists a canonical morphism

$$\mathbf{C}_{\mathcal{P}_{J_1,J'_1}}(\lambda) \to \mathbf{C}_{\mathcal{P}_{J_2,J'_2}}(\lambda). \tag{4.19}$$

We write $t_0 \overset{\mathrm{def}}{=} \#I_0 \setminus I_2$, $t_1 \overset{\mathrm{def}}{=} \#I_2 \setminus I_4$ and let $s_0 \in [0, t_0]$ and $s_1 \in [0, t_1]$ be two integers. We define the following three set of subsets of Δ_n.

- $S_0 \overset{\mathrm{def}}{=} \{I \subseteq \Delta_n \mid I_2 \cup (I_1 \setminus I_0) \subseteq I \subseteq I_1,\ \#I \cap (I_0 \setminus I_2) = s_0\}$.
- $S_1 \overset{\mathrm{def}}{=} \{I \subseteq \Delta_n \mid I_4 \cup (I_1 \setminus I_2) \subseteq I \subseteq I_1,\ \#I \cap (I_2 \setminus I_4) = s_1\}$.
- $S_2 \overset{\mathrm{def}}{=} \{I \subseteq \Delta_n \mid I_4 \cup (I_1 \setminus I_0) \subseteq I \subseteq I_1,\ \#I \cap (I_2 \setminus I_4) = s_1,\ \#I \cap (I_0 \setminus I_2) = s_0\}$.

For each $I \in S_0$ (resp. $I \in S_1$, resp. $I \in S_2$), we have $\#I = \ell_0 \overset{\mathrm{def}}{=} \#I_2 + \#I_1 \setminus I_0 + s_0$ (resp. $\#I = \ell_1 \overset{\mathrm{def}}{=} \#I_4 + \#I_1 \setminus I_2 + s_1$, resp. $\#I = \ell_2 \overset{\mathrm{def}}{=} \#I_4 + \#I_1 \setminus I_0 + s_0 + s_1$). Note that we have $\ell_0 + \ell_1 - \ell_2 = \#I_2 + \#I_1 \setminus I_2 = \#I_1$ and the map $S_0 \times S_1 \to S_2 :\ (I, I') \mapsto I \cap I'$ is a bijection.

Assuming Condition 4.25 and using maps of the form (4.19), we can extend (4.18) to the following commutative diagram

$$
\begin{array}{ccccc}
M^{k_0}_{I_0,I_1,I_2,I_1} & \otimes & M^{k_1}_{I_2,I_1,I_4,I_1} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,I_4,I_1} \\
\Vert & & \uparrow{\scriptstyle p_0} & & \uparrow{\scriptstyle p_1} \\
M^{k_0}_{I_0,I_1,I_2,I_1} & \otimes & M^{k_1}_{I_2,I_1,\mathcal{P}_{[0,s_1],[0,t_0]}} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{[0,s_1],[0,t_0]}} \\
\Vert & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\
M^{k_0}_{I_0,I_1,I_2,I_1} & \otimes & M^{k_1}_{I_2,I_1,\mathcal{P}_{\{s_1\},[0,t_0]}} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{\{s_1\},[0,t_0]}} \\
\Vert & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle p_5} \\
M^{k_0}_{I_0,I_1,I_2,I_1} & \otimes & M^{k_1}_{I_2,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \\
\uparrow{\scriptstyle p_6} & & \downarrow{\scriptstyle p_7} & & \uparrow{\scriptstyle p_8} \\
M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},[0,s_0]}} & \otimes & M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},\{s_0\}}} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{\{s_1\},\{s_0\}}} \\
\downarrow{\scriptstyle p_9} & & \uparrow{\scriptstyle p_{10}} & & \Vert \\
M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},\{s_0\}}} & \otimes & M^{k_1}_{\mathcal{P}_{\{t_1\},\{s_0\}},\mathcal{P}_{\{s_1\},\{s_0\}}} & \xrightarrow{\cup} & M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{\{s_1\},\{s_0\}}} \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
H^{k_0+\ell_0-\#I_1}_{S_0} & \otimes & H^{k_1+\ell_1-\#I_1}_{S_2} & & H^{k_0+k_1+\ell_2-\#I_1}_{S_2} \\
\downarrow{\scriptstyle p_{11}} & & \Vert & & \Vert \\
H^{k_0+\ell_0-\#I_1}_{S_2} & \otimes & H^{k_1+\ell_1-\#I_1}_{S_2} & \xrightarrow{\cup} & H^{k_0+k_1+\ell_2-\#I_1}_{S_2}
\end{array}
\tag{4.20}
$$

The commutativity of (4.20) (including the definition of each map inside) is clear except that between the fourth and the fifth rows. In fact, we have an isomorphism between spectral sequences ${}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{\bullet,\mathcal{P}_{\{s_1\},\{s_0\}}} \xrightarrow{\sim} {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{\bullet,\mathcal{P}_{\{s_1\},[s_0,t_0]}}$ from the isomorphism on their first pages. Consequently, there exists a canonical isomorphism $M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},\{s_0\}}} \xrightarrow{\sim} M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}}$ and we can define p_7 as the composition

$$M^{k_1}_{I_2,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \to M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}} \cong M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},\{s_0\}}}. \tag{4.21}$$

The commutativity between the fourth and the fifth row of (4.20) thus follows from the fact that the composition

$$M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},\{s_0\}}} \otimes M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},\{s_0\}}} \xrightarrow{\sim} M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},\{s_0\}}} \otimes M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}} \xrightarrow{\cup} M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}}$$

clearly factors through p_8. The isomorphism between the sixth and seventh rows is clear from the description of ${}_{I_0,I_1}E^{\bullet,\bullet}_{1,\mathcal{P}_{\{t_1\},\{s_0\}}}$ (resp. ${}_{\mathcal{P}_{\{t_1\},\{s_0\}}}E^{\bullet,\bullet}_{1,\mathcal{P}_{\{s_1\},\{s_0\}}}$, resp. ${}_{I_0,I_1}E^{\bullet,\bullet}_{1,\mathcal{P}_{\{s_1\},\{s_0\}}}$) which concentrate in bidegrees $(-\ell_0,\bullet)$ (resp. $(-\ell_1,\bullet)$, resp. $(-\ell_2,\bullet)$).

By the definition of the canonical filtration on each term of the first row of (4.20), we observe that $\mathrm{im}(p_4) = \mathrm{Fil}^{-\ell_0}(M^{k_0}_{I_0,I_1,I_2,I_1})$ and $\mathrm{im}(p_0) = \mathrm{Fil}^{-\ell_1}(M^{k_1}_{I_2,I_1,I_4,I_1})$. As $I_4 \cup (I_1 \setminus I_0) \subseteq I \subseteq I_1$

satisfying $\#I \cap (I_2 \setminus I_4) \leq s_1$ and $\#I \cap (I_0 \setminus I_2) \leq s_0$ must also satisfy $\#I \leq \ell_2$, we deduce that the image of $M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{[0,s_1],[0,s_0]}} \to M^{k_0+k_1}_{I_0,I_1,I_4,I_1}$ sits inside $\mathrm{Fil}^{-\ell_2}(M^{k_0+k_1}_{I_0,I_1,I_4,I_1})$. We consider the following composition of canonical maps

$$\begin{aligned} M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},[0,s_0]}} \otimes M^{k_1}_{I_2,I_1,\mathcal{P}_{[0,s_1],[0,t_0]}} &\to M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},[0,s_0]}} \otimes M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{[0,s_1],[0,t_0]}} \\ &\cong M^{k_0}_{I_0,I_1,\mathcal{P}_{\{t_1\},[0,s_0]}} \otimes M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{[0,s_1],[0,s_0]}} \xrightarrow{\cup} M^{k_0+k_1}_{I_0,I_1,\mathcal{P}_{[0,s_1],[0,s_0]}}, \end{aligned}$$

which implies that the first row of (4.20) restricts to a canonical map

$$\mathrm{Fil}^{-\ell_0}(M^{k_0}_{I_0,I_1,I_2,I_1}) \otimes \mathrm{Fil}^{-\ell_1}(M^{k_1}_{I_2,I_1,I_4,I_1}) \xrightarrow{\cup} \mathrm{Fil}^{-\ell_2}(M^{k_0+k_1}_{I_0,I_1,I_4,I_1}) \tag{4.22}$$

for each $s_0 \in [0, t_0]$ and $s_1 \in [0, t_1]$.

We write $s'_0 \overset{\mathrm{def}}{=} t_0 - s_0$ for short. Each map in (4.20) actually arises from a map between the corresponding spectral sequences. In particular, we obtain the following commutative diagram by considering specific terms of their second pages. Note that each term in the following diagram is naturally a graded piece of the canonical filtration on its corresponding term in (4.20).

$$\begin{array}{ccccc}
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} & \otimes & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I_4,I_1} & \xrightarrow{\cup} & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,I_4,I_1} \\
\Big\| & & \Big\uparrow{\scriptstyle q_0} & & \Big\uparrow{\scriptstyle q_1} \\
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} & \otimes & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}_{[0,s_1],[0,t_0]}} & & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,\mathcal{P}_{[0,s_1],[0,t_0]}} \\
\Big\| & & \Big\downarrow{\scriptstyle q_2} & & \Big\downarrow{\scriptstyle q_3} \\
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} & \otimes & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}_{\{s_1\},[0,t_0]}} & & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,\mathcal{P}_{\{s_1\},[0,t_0]}} \\
\Big\| & & \Big\downarrow{\scriptstyle q_4} & & \Big\downarrow{\scriptstyle q_5} \\
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} & \otimes & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \\
\Big\uparrow{\scriptstyle q_6} & & \Big\downarrow{\scriptstyle q_7} & & \Big\uparrow{\scriptstyle q_8} \\
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,\mathcal{P}_{\{t_1\},[0,s_0]}} & \otimes & {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s'_0,k_1+\ell_1-s'_0}_{2,\mathcal{P}_{\{s_1\},\{s_0\}}} & & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,\mathcal{P}_{\{s_1\},\{s_0\}}} \\
\Big\downarrow{\scriptstyle q_9} & & \Big\uparrow{\scriptstyle q_{10}} & & \Big\| \\
{}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,\mathcal{P}_{\{t_1\},\{s_0\}}} & \otimes & {}_{\mathcal{P}_{\{t_1\},\{s_0\}}}E^{-\ell_1+s'_0,k_1+\ell_1-s'_0}_{2,\mathcal{P}_{\{s_1\},\{s_0\}}} & & {}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,\mathcal{P}_{\{s_1\},\{s_0\}}} \\
\cong\Big\uparrow & & \cong\Big\uparrow & & \cong\Big\uparrow \\
H^{k_0+\ell_0-\#I_1}_{S_0} & \otimes & H^{k_1+\ell_1-\#I_1}_{S_2} & & H^{k_0+k_1+\ell_2-\#I_1}_{S_2} \\
\Big\downarrow{\scriptstyle p_{11}} & & \Big\| & & \Big\| \\
H^{k_0+\ell_0-\#I_1}_{S_2} & \otimes & H^{k_1+\ell_1-\#I_1}_{S_2} & \xrightarrow{\cup} & H^{k_0+k_1+\ell_2-\#I_1}_{S_2}
\end{array} \tag{4.23}$$

The main properties of maps in (4.23) (except q_7) are summarized as follows.

- For technical simplicity, we only keep track of the horizontal maps for the first and last rows induced from (4.20), although there should be a well-defined horizontal map for each row that makes the diagram commutative. Note that the horizontal map in the first row of (4.23) exists thanks to (4.22) for each $s_0 \in [0, t_0]$ and $s_1 \in [0, t_1]$.

- We have the following factorization for the map q_0

$$ {}_{I_2,I_1}E_{2,\mathcal{P}_{[0,s_1],[0,t_0]}}^{-\ell_1,k_1+\ell_1} \cong \ker({}_{I_2,I_1}d_{1,\mathcal{P}_{[0,s_1],[0,t_0]}}^{-\ell_1,k_1+\ell_1}) \hookrightarrow \ker({}_{I_2,I_1}d_{1,I_4,I_1}^{-\ell_1,k_1+\ell_1}) \twoheadrightarrow {}_{I_2,I_1} E_{2,I_4,I_1}^{-\ell_1,k_1+\ell_1} \tag{4.24} $$

and similarly for q_1, q_6 and q_8.

- The map q_2 is injective with the following factorization

(4.25)
$$ {}_{I_2,I_1}E_{2,\mathcal{P}_{[0,s_1],[0,t_0]}}^{-\ell_1,k_1+\ell_1} \cong \ker({}_{I_2,I_1}d_{1,\mathcal{P}_{[0,s_1],[0,t_0]}}^{-\ell_1,k_1+\ell_1}) \hookrightarrow {}_{I_2,I_1} E_{1,\mathcal{P}_{[0,s_1],[0,t_0]}}^{-\ell_1,k_1+\ell_1} \cong {}_{I_2,I_1} E_{1,\mathcal{P}_{\{s_1\},[0,t_0]}}^{-\ell_1,k_1+\ell_1} \cong {}_{I_2,I_1} E_{2,\mathcal{P}_{\{s_1\},[0,t_0]}}^{-\ell_1,k_1+\ell_1} $$

and so are q_3, q_4, q_5 and q_9. Here we use the following fact for (4.25) of q_2: if $I_4 \cup (I_1 \setminus I_2) \subseteq I \subseteq I_1$ satisfies $\#I = \ell_1$ and $\#I \cap (I_2 \setminus I_4) \leq s_1$, then we must have $\#I \cap (I_2 \setminus I_4) = s_1$ (namely $I \in S_1$). It is easy to check that q_4 is actually an isomorphism.

- The map q_{10} is an isomorphism as there is an isomorphism of complexes

$$ {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E_{1,\mathcal{P}_{\{s_1\},\{s_0\}}}^{\bullet,k_1+\ell_1-s_0'} \to {}_{\mathcal{P}_{\{t_1\},\{s_0\}}} E_{1,\mathcal{P}_{\{s_1\},\{s_0\}}}^{\bullet,k_1+\ell_1-s_0'}. $$

This isomorphism uses the following fact: for each $I_2 \cup (I_1 \setminus I_0) \subseteq I' \subseteq I_1$ and $I_4 \cup (I_1 \setminus I_0) \subseteq I \subseteq I_1$ satisfying $\#I' \cap (I_0 \setminus I_2) \leq s_0$, $\#I \cap (I_2 \setminus I_4) = s_1$ and $\#I \cap (I_0 \setminus I_2) = s_0$, we have $I_2 \cup I \subseteq I'$ if and only if $I_2 \cup I = I'$ and $\#I' \cap (I_0 \setminus I_2) = s_0$.

It follows from Lemma 4.19 that ${}_{I_2,I_1}E_{1,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{\bullet,\bullet}$ is supported in degree $(-\ell_1, \bullet)$ and thus

$$ M_{I_2,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{k_1} \cong {}_{I_2,I_1}E_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{-\ell_1,k_1+\ell_1} \cong H_{S_1}^{k_1+\ell_1-\#I_1} \tag{4.26} $$

for each $k \in \mathbb{Z}$. Similarly, we clearly have ${}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E_{1,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{\bullet,\bullet} \xrightarrow{\sim} {}_{\mathcal{P}_{\{t_1\},[0,s_0]}} E_{1,\mathcal{P}_{\{s_1\},\{s_0\}}}^{\bullet,\bullet}$ is supported in degree $(-\ell_1 + s_0', \bullet)$, which implies that

(4.27)
$$ M_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{k_1} \cong {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{-\ell_1+s_0',k_1+\ell_1-s_0'} \xrightarrow{\sim} {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E_{2,\mathcal{P}_{\{s_1\},\{s_0\}}}^{-\ell_1+s_0',k_1+\ell_1-s_0'} \cong H_{S_2}^{k_1+\ell_1-\#I_1}. $$

Lemma 4.26. *The truncation* $\mathbf{C}_{\mathcal{P}_{\{t_1\},[0,s_0]}}(\lambda) \to \mathbf{C}_{I_2,I_1}(\lambda)$ *induces a canonical map*

$$ {}_{I_2,I_1}E_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{-\ell_1,k_1+\ell_1} \to {}_{\mathcal{P}_{\{t_1\},[0,s_0]}} E_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{-\ell_1+s_0',k_1+\ell_1-s_0'} \tag{4.28} $$

whose composition with the middle isomorphism of (4.27) gives q_7*. Under the canonical isomorphisms (4.26) and (4.27), (4.28) is given by*

$$ \bigoplus_{I\in S_1, I'\in S_2} \mathrm{Res}_{n,I,I'}^{k_1+\ell_1-\#I_1} : H_{S_1}^{k_1+\ell_1-\#I_1} \to H_{S_2}^{k_1+\ell_1-\#I_1}. $$

Proof. The truncation $\mathbf{C}_{\mathcal{P}_{\{t_1\},[0,s_0]}}(\lambda) \to \mathbf{C}_{I_2,I_1}(\lambda)$ induces a canonical map

$$ M_{I_2,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{k_1} \to M_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}}^{k_1} $$

which together with (4.26) and (4.27) gives the canonical map (4.28).

Let $I_4 \cup (I_1 \setminus I_2) \subseteq I \subseteq I_1$ be a set satisfying $\#I \cap (I_2 \setminus I_4) = s_1$ and $I_4 \cup (I_1 \setminus I_0) \subseteq I' \subseteq I_1 \cap I$ be a set satisfying $I' \cap (I_2 \setminus I_4) = I \cap (I_2 \setminus I_4)$ and $\#I' \cap (I_0 \setminus I_2) = s_0$. We set $S_I \overset{\mathrm{def}}{=} \{I' \in S_2 \mid I' \subseteq I\}$. Then there exists a tuple $\mathcal{P}$ (of the kind introduced at the beginning of Section 4.2) such that $\mathbf{C}_{\mathcal{P}}(\lambda) \to \mathbf{C}_{\mathcal{P}_{\{s_1\},[s_0,t_0]}}(\lambda)$ is a minimal truncation with the degree $-\ell_1$ term of $\mathbf{C}_{\mathcal{P}}(\lambda)$ being $i_{n,I}^{\mathrm{an}}(\lambda)[-\ell_1]$, and moreover we also have canonical truncation map $\mathbf{C}_{\mathcal{P}}(\lambda) \to \mathbf{C}_{I',I}(\lambda)$. The diagram

$$ \mathbf{C}_{\mathcal{P}_{\{s_1\},[s_0,t_0]}}(\lambda) \leftarrow \mathbf{C}_{\mathcal{P}}(\lambda) \to \mathbf{C}_{I',I}(\lambda) $$

induces a commutative diagram

$$\begin{array}{ccc} M^{k_1}_{I_2,I_1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & \longrightarrow & M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}_{\{s_1\},[s_0,t_0]}} \\ \uparrow & & \uparrow \\ M^{k_1}_{I_2,I_1,\mathcal{P}} & \longrightarrow & M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},\mathcal{P}} \\ \downarrow & & \downarrow \\ M^{k_1}_{I_2,I_1,I',I} & \longrightarrow & M^{k_1}_{\mathcal{P}_{\{t_1\},[0,s_0]},I',I} \end{array} \tag{4.29}$$

Then we observe that

$$ {}_{I_2,I_1}E^{\bullet,\bullet}_{1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \leftarrow {}_{I_2,I_1}E^{\bullet,\bullet}_{1,\mathcal{P}} \xrightarrow{\sim} {}_{I_2,I_1}E^{\bullet,\bullet}_{1,I',I}$$

are supported in degree $(-\ell_1,\bullet)$, which implies that

$$ {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \leftarrow {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}} \xrightarrow{\sim} {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I',I} \cong H^{k_1+\ell_1-\#I_1}_{I}. \tag{4.30}$$

We also observe that

$$ {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{1,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \leftarrow {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{1,\mathcal{P}} \rightarrow {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{1,I',I}$$

are supported in degree $(-\ell_1+s_0',\bullet)$, which implies that

$$ {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} \leftarrow {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,\mathcal{P}} \rightarrow {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,I',I} \cong H^{k_1+\ell_1-\#I_1}_{I'}. \tag{4.31}$$

We combine (4.29) with (4.30) as well as (4.31), and obtain the following diagram

$$\begin{array}{ccccccc} H^{k_1+\ell_1-\#I_1}_{S_1} & \longleftarrow & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & \longrightarrow & {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & \longrightarrow & H^{k_1+\ell_1-\#I_1}_{S_2} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^{k_1+\ell_1-\#I_1}_{I} & \longleftarrow & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,\mathcal{P}} & \longrightarrow & {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,\mathcal{P}_{\{s_1\},[s_0,t_0]}} & \longrightarrow & H^{k_1+\ell_1-\#I_1}_{S_I} \\ \Vert & & \downarrow & & \downarrow & & \downarrow \\ H^{k_1+\ell_1-\#I_1}_{I} & \longleftarrow & {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I',I} & \longrightarrow & {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,I',I} & \longrightarrow & H^{k_1+\ell_1-\#I_1}_{I'} \end{array} \tag{4.32}$$

with all horizontal maps towards group cohomologies being isomorphisms. The isomorphism of first pages ${}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{\bullet,\bullet}_{1,I',I} \xrightarrow{\sim} {}_{I_2,I_2\cup I'}E^{\bullet,\bullet}_{1,I',I}$ (which are both supported in degree $(-\ell_1+s_0',\bullet)$) induces an isomorphism

$$ {}_{\mathcal{P}_{\{t_1\},[0,s_0]}}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,I',I} \xrightarrow{\sim} {}_{I_2,I_2\cup I'}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,I',I} \cong H^{k_1+\ell_1-\#I_1}_{I'}. \tag{4.33}$$

Combining (4.32) with (4.33), it suffices to show that the canonical map

$$H^{k_1+\ell_1-\#I_1}_{I} \cong {}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I',I} \rightarrow {}_{I_2,I_2\cup I'}E^{-\ell_1+s_0',k_1+\ell_1-s_0'}_{2,I',I} \cong H^{k_1+\ell_1-\#I_1}_{I'}$$

is given $\mathrm{Res}^{k_1+\ell_1-\#I_1}_{n,I,I'}$. Now we choose a sequence of subsets

$$I' = I[s_0] \subsetneq I[s_0+1] \subsetneq \cdots \subsetneq I[t_0] = I$$

which necessarily satisfies $\#I[s] \cap (I_2 \setminus I_4) = s$ for each $s \in [s_0, t_0]$, and induces another sequence

$$I_2 \cup I' = I_2 \cup I[s_0] \subsetneq I_2 \cup I[s_0+1] \subsetneq \cdots \subsetneq I_2 \cup I[t_0] = I_2 \cup I = I_1.$$

Using induction on $s \in [s_0, t_0]$ with $s' \overset{\text{def}}{=} t_0 - s$ and the fact that

$$\mathrm{Res}_{n,I,I'}^{k_1+\ell_1-\#I_1} = \mathrm{Res}_{n,I[s_0+1],I[s_0]}^{k_1+\ell_1-\#I_1} \circ \cdots \circ \mathrm{Res}_{n,I[t_0],I[t_0-1]}^{k_1+\ell_1-\#I_1},$$

it suffices to show that the composition of

$$H_{I[s]}^{k_1+\ell_1-\#I_1} \cong_{I_2, I_2 \cup I[s]} E_{2,I',I}^{-\ell_1+s',k_1+\ell_1-s'} \to_{I_2, I_2 \cup I[s-1]} E_{2,I',I}^{-\ell_1+s'+1,k_1+\ell_1-s'-1} \cong H_{I[s-1]}^{k_1+\ell_1-\#I_1}$$

is given $\mathrm{Res}_{n,I[s],I[s-1]}^{k_1+\ell_1-\#I_1}$ for each $s \in [s_0+1, t_0]$. We finish the proof by Lemma 4.18 (with $\theta = \theta'$ there) and the following commutative diagram

$$\begin{array}{ccc}
{}_{I_2, I_2 \cup I[s]}E_{2,I',I}^{-\ell_1+s',k_1+\ell_1-s'} & \longrightarrow & {}_{I_2, I_2 \cup I[s-1]}E_{2,I',I}^{-\ell_1+s'+1,k_1+\ell_1-s'-1} \\
\downarrow & & \downarrow \\
{}_{I_2 \cup I[s-1], I_2 \cup I[s]}E_{2,I[s-1],I[s]}^{-\ell_1+s',k_1+\ell_1-s'} & \longrightarrow & {}_{I_2 \cup I[s-1], I_2 \cup I[s-1]}E_{2,I[s-1],I[s]}^{-\ell_1+s'+1,k_1+\ell_1-s'-1}
\end{array}$$

with vertical maps being isomorphisms. □

Let $v_0 \subseteq \mathcal{B}_{n,I_2 \cup (I_1 \setminus I_0)}$ and $v_1 \subseteq \mathcal{B}_{n,I_4 \cup (I_1 \setminus I_2)}$ be subsets. Let Ω_0 (resp. Ω_1) be a set of tuples $\Theta' = (v_0, I', \underline{k}', \underline{\lambda}')$ with bidegree $(-\ell_0, k_0+\ell_0)$ (resp. $\Theta'' = (v_1, I'', \underline{k}'', \underline{\lambda}'')$ with bidegree $(-\ell_1, k_1+\ell_1)$) that satisfies $I_2 \cup (I_1 \setminus I_0) \subseteq I \subseteq I_1$ (resp. that satisfies $I_4 \cup (I_1 \setminus I_2) \subseteq I \subseteq I_1$). In particular, we observe that $\Theta' \in \Omega_0$ (resp. $\Theta'' \in \Omega_1$) forces $I' \in S_0$ (resp. forces $I'' \in S_1$). As usual, we can define

$$x_{\Omega_0} = \sum_{\Theta \in \Omega_0} \varepsilon(\Theta) x_\Theta \in H_{S_0}^{k_0+\ell_0-\#I_1} \cong_{I_0,I_1} E_{1,I_2,I_1}^{-\ell_0,k_0+\ell_0}$$

and

$$x_{\Omega_1} = \sum_{\Theta \in \Omega_1} \varepsilon(\Theta) x_\Theta \in H_{S_1}^{k_1+\ell_1-\#I_1} \cong_{I_2,I_1} E_{1,I_4,I_1}^{-\ell_1,k_1+\ell_1}$$

with $\varepsilon(\Theta)$ defined in Definition 2.13. We define

$$x_{\Omega_0} \cup x_{\Omega_1} \in H_{S_2}^{k_0+k_1+\ell_2-\#I_1} \hookrightarrow_{I_0,I_1} E_{1,I_4,I_1}^{-\ell_2,k_0+k_1+\ell_2}$$

as the image of $(x_{\Omega_0}, x_{\Omega_1})$ under the composition

$$H_{S_0}^{k_0+\ell_0-\#I_1} \otimes H_{S_1}^{k_1+\ell_1-\#I_1} \to H_{S_2}^{k_0+\ell_0-\#I_1} \otimes H_{S_2}^{k_1+\ell_1-\#I_1} \xrightarrow{\cup} H_{S_2}^{k_0+k_1+\ell_2-\#I_1}.$$

The following is the main outcome of diagram (4.23).

Lemma 4.27. *Assume that* ${}_{I_0,I_1}d_{1,I_2,I_1}^{-\ell_0,k_0+\ell_0}(x_0) = 0$, ${}_{I_2,I_1}d_{1,I_4,I_1}^{-\ell_1,k_1+\ell_1}(x_{\Omega_1}) = 0$ *and* ${}_{I_0,I_1}d_{1,I_4,I_1}^{-\ell_2,k_0+k_1+\ell_2}(x_{\Omega_0} \cup x_{\Omega_1}) = 0$. *If we abuse* x_{Ω_0}, x_{Ω_1} *and* $x_{\Omega_0} \cup x_{\Omega_1}$ *for their images in the second page of the corresponding spectral sequences, then* $x_{\Omega_0} \cup x_{\Omega_1}$ *is the image of* $(x_{\Omega_0}, x_{\Omega_1})$ *under*

$$(4.34) \qquad {}_{I_0,I_1}E_{2,I_2,I_1}^{-\ell_0,k_0+\ell_0} \otimes_{I_2,I_1} E_{2,I_4,I_1}^{-\ell_1,k_1+\ell_1} \xrightarrow{\cup}_{I_0,I_1} E_{2,I_4,I_1}^{-\ell_2,k_0+k_1+\ell_2}.$$

Proof. It suffices to observe that x_{Ω_0} (resp. x_{Ω_1}, resp. $x_{\Omega_0} \cup x_{\Omega_1}$) can be naturally understood as elements of each term of the first (resp. second, resp. third) column of diagram (4.23). □

We assume from now on the following condition

Condition 4.28. *Exactly one of the following holds*

- $k_0 = 2\#I_0 - 2\#I_2$ *and* $k_1 = 2\#I_2 - 2\#I_4$;
- $k_0 = 2\#I_0 - 2\#I_2 + 1$ *and* $k_1 = 2\#I_2 - 2\#I_4$;
- $k_0 = 2\#I_0 - 2\#I_2$ *and* $k_1 = 2\#I_2 - 2\#I_4 + 1$.

Note that we may always assume that $M^{k_0}_{I_0,I_1,I_2,I_1} \neq 0$ (resp. $M^{k_1}_{I_2,I_1,I_4,I_1} \neq 0$) so that the map

$$M^{k_0}_{I_0,I_1,I_2,I_1} \otimes M^{k_1}_{I_2,I_3,I_4,I_1} \xrightarrow{\cup} M^{k_0+k_1}_{I_0,I_1,I_4,I_1} \tag{4.35}$$

is interesting, which together with first part of Theorem 4.22 implies that $k_0 \geq 2\#I_0 - 2\#I_2$ (resp. $k_1 \geq 2\#I_2 - 2\#I_4$). In other words, Condition 4.28 is equivalent to saying that $k_0 \leq 2\#I_0 - 2\#I_2 + 1$, $k_1 \leq 2\#I_2 - 2\#I_4 + 1$ and $k_0 + k_1 \leq 2\#I_0 - 2\#I_4 + 1$.

Given $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$ as above, we write $I_2' \overset{\mathrm{def}}{=} I_4 \cup (I_0 \setminus I_2)$. The following result makes crucial use of the construction in Section 2.6.

Proposition 4.29. *Assume that Condition 4.28 holds and* $\max\{i' \mid i' \in I_0 \setminus I_2\} < \min\{i' \mid i' \in I_2 \setminus I_4\}$.

(i) For each $\Omega_0 \in \Psi^{-\ell_0,\ell_0+k_0-\#I_1}_{I_2\cup(I_1\setminus I_0),I_1}$ *and* $\Omega_1 \in \Psi^{-\ell_1,\ell_1+k_1-\#I_1}_{I_4\cup(I_1\setminus I_2),I_1}$*, the image of* $(x_{\Omega_0}, x_{\Omega_1})$ *under (4.34) is* x_{Ω_2} *for a* $\Omega_2 \in \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1}$ *uniquely determined by the pair* (Ω_0, Ω_1)*. Moreover, the map*

$$\Psi^{-\ell_0,\ell_0+k_0-\#I_1}_{I_2\cup(I_1\setminus I_0),I_1} \times \Psi^{-\ell_1,\ell_1+k_1-\#I_1}_{I_4\cup(I_1\setminus I_2),I_1} \to \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1} : (\Omega_0, \Omega_1) \mapsto \Omega_2 \tag{4.36}$$

is injective.

(ii) We have canonical isomorphisms ${}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} \cong_{I_2',I_1} E^{-\ell_0,k_0+\ell_0}_{2,I_4,I_1}$ *and* ${}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I_4,I_1} \cong_{I_0,I_1} E^{-\ell_1,k_1+\ell_1}_{2,I_2',I_1}$*. Using the notation in item (i) above, the map*

$${}_{I_0,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I_2',I_1} \otimes_{I_2',I_1} E^{-\ell_0,k_0+\ell_0}_{2,I_4,I_1} \xrightarrow{\cup}_{I_0,I_1} E^{-\ell_2,k_0+k_1+\ell_2}_{2,I_4,I_1}$$

sends $(x_{\Omega_1}, x_{\Omega_0})$ *to* $(-1)^{k_0k_1}x_{\Omega_2}$.

(iii) The map (4.34) is injective for each pair (ℓ_0, ℓ_1) *and thus (4.35) is injective and compatible with the canonical filtration on both the source and the target. If furthermore there exists* $i \in \Delta_n$ *such that* $\max\{i' \mid i' \in I_0 \setminus I_2\} < i < \min\{i' \mid i' \in I_2 \setminus I_4\}$*, then (4.34) is an isomorphism for each pair* (ℓ_0, ℓ_1) *and thus (4.35) is an isomorphism.*

Proof. It is harmless to assume $I_4 \subsetneq I_2 \subsetneq I_0$ throughout the proof. The main idea of the proof of item (i) can be divided into the following steps.

- Construct a set of Ω_0' of tuples with bidegree $(-\ell_0, \ell_0 + k_0 - \#I_1)$ and prove that $x_{\Omega_0'}$ induce the same element as x_{Ω_0} in ${}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1}$.
- Check that Ω_0' and Ω_1 satisfies the assumption of Lemma 4.27, so that $x_{\Omega_0'} \cup x_{\Omega_1}$ is defined and has the form $x_{\Omega_2'}$ with Ω_2' a set of tuples with bidegree $(-\ell_2, \ell_2 + k_0 + k_1 - \#I_1)$ which induces the same element as x_{Ω_2} in ${}_{I_0,I_1}E^{-\ell_2,k_0+k_1+\ell_2}_{2,I_4,I_1}$ for some $\Omega_2 \in \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1}$.
- Check that the map $(\Omega_0, \Omega_1) \mapsto \Omega_2$ is an injection.

We write $\Theta_0 = (v, I, \underline{k}, \underline{\Lambda})$ (resp. $\Theta_1 = (v', I', \underline{k}', \underline{\Lambda}')$) for the maximal element in Ω_0 (resp. Ω_1). Note that $\max\{i' \mid i' \in I_0 \setminus I_2\} < i < \min\{i' \mid i' \in I_2 \setminus I_4\}$ implies that $I_v \cup I_{v'} = \Delta_n$ and thus $v \cup v' \subseteq \mathcal{B}_{n,\emptyset}$ with $I_4 \cup (I_1 \setminus I_0) \subseteq I \cap I' \subseteq I_{v\cup v'}$. We set $d_0 \overset{\mathrm{def}}{=} r_I$ and $s_0 \overset{\mathrm{def}}{=} r_{I_v \cap I_1}$. According to the definition of $\Psi^{-\ell_0,\ell_0+k_0-\#I_1}_{I_2\cup(I_1\setminus I_0),I_1}$ (based on Lemma 2.16 and Lemma 2.25) we have exactly two possibilities

- We have $\Lambda_{r^{s_0-1}_{v,I_1,I}+1} = \emptyset$ and s_0 satisfies Condition 2.29, in which case we set $\Omega'_0 \overset{\mathrm{def}}{=} \Omega_0^{s_0,d_0}$ which is the (s_0,d_0)-twist of Ω_0 as defined before Condition 2.30. We also set $\Theta'_0 \overset{\mathrm{def}}{=} \Theta_0^{s_0,d_0}$. Note that $x_{\Omega'_0}$ and x_{Ω_0} induces the same element in ${}_{I_0,I_1}E_{2,I_2,I_1}^{-\ell_0,k_0+\ell_0}$ by Proposition 2.32.
- We have $I^d \cap (I_2 \cup (I_1 \setminus I_0)) = \emptyset$ and $\Lambda_d = \{(2n_d-1,\iota_d)\}$ for some $\iota_d \in S$, for each $r^{s_0-1}_{v,I_1,I}+1 \leq d \leq d_0$. This is impossible as we can deduce from $\max\{i' \mid i' \in I_0 \setminus I_2\} < i < \min\{i' \mid i' \in I_2 \setminus I_4\}$ that $I_2 \setminus I_4 \subseteq I^{d_0}$.

Our assumption $\max\{i' \mid i' \in I_0 \setminus I_2\} < i < \min\{i' \mid i' \in I_2 \setminus I_4\}$ implies that $I_0 \setminus I_2 \subseteq (I')^1$, which together with Lemma 2.16 and Lemma 2.25 forces $\Lambda'_1 = \emptyset$. We claim that $x_{\Omega'_0} \cup x_{\Omega_1} = x_{\Omega'_2}$ with $\Omega'_2 = \Omega_2^{s_0,d_0}$ the (s_0,d_0)-twist of some $\Omega_2 \in \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1}$. This follows from the following two observations.

- We have $x_{\Theta'_0} \cup x_{\Theta_1} = x_{\Theta'_2}$ for some maximally (s_0,d_0)-twisted $(I_4 \cup (I_1 \setminus I_0), I_1)$-atomic tuple Θ'_2. The tuple $\Theta'_2 = (v \cup v', I \cap I', \underline{k}'', \underline{\Lambda}'')$ is characterized by $\Lambda''_d = \Lambda_{d-1}$ for each $2 \leq d \leq d_0$, $\Lambda''_{d_0} = \emptyset$ and $\Lambda''_d = \Lambda'_{d-d_0+1}$ for each $d_0+1 \leq d \leq r_{I\cap I'}$. There clearly exists a maximally $(I_4 \cup (I_1 \setminus I_0), I_1)$-atomic tuple Θ_2 such that $\Theta'_2 = \Theta_2^{s_0,d_0}$. We define Ω'_2 as the (s_0,d_0)-twisted equivalence class of Θ'_2 and $\Omega_2 \in \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1}$ as the equivalence class of Θ_2.
- Similar construction as above actually produces a bijection $\Omega'_0 \times \Omega_1 \to \Omega'_2 : (\Theta,\Theta') \mapsto \Theta''$ with $\varepsilon(\Theta'') = \varepsilon_\sharp \varepsilon(\Theta)\varepsilon(\Theta')$ for some $\varepsilon_\sharp$ depending only on $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$ and the pair (ℓ_0,ℓ_1). This implies that
$$x_{\Omega'_0} \cup x_{\Omega_1} = \varepsilon_\sharp x_{\Omega'_2}.$$

Now we check that Ω_0 and Ω_1 can be recovered from Ω_2, $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$ and the pair (ℓ_0,ℓ_1). This is because we can recover d_0 (if exists, namely if Ω_2 actually arises from some (Ω_0,Ω_1)), s_0, $\Omega_2^{s_0,d_0}$ and then $\Omega_0^{s_0,d_0}$ and Ω_1 in order.

Item (ii) follows from item (i) by a comparison with the symmetric construction for $I_4 \subseteq I'_2 \subseteq I_0 \subseteq I_1$ which satisfies the condition $\min\{i' \mid i' \in I_0 \setminus I'_2\} > \max\{i' \mid i' \in I'_2 \setminus I_4\}$.

The first part of item (iii) follows from item (i) and Theorem 4.22, as we obtain an injective map $x_{\Omega_0} \otimes x_{\Omega_1} \mapsto x_{\Omega_2}$ from a basis of the source to a basis of the target. For the second part of item (iii), it suffices to check that the map (4.36) is bijective if there exists $i \in \Delta_n$ such that $\max\{i' \mid i' \in I_0 \setminus I_2\} < i < \min\{i' \mid i' \in I_2 \setminus I_4\}$. In fact, if $\Theta_2 = (v'', I'', \underline{k}'', \underline{\Lambda}'') \in \Omega_2$ is the maximal element, then as $i \in \Delta_n \setminus (I_0 \setminus I_4)$, either $i \notin I_1$ or there exists a unique $1 \leq d_0 \leq r_{I''}$ such that $i \in (I'')^{d_0} \cap (I_4 \cup (I_1 \setminus I_0))$. We give explicit construction of maximal element $\Theta_0 \in \Omega_0$ (resp. $\Theta_1 \in \Omega_1$) in both cases.

- Assume that $i \notin I_1$, then there exists a unique $1 \leq d_1 \leq r_{I''}$ such that $i = \sum_{d'=1}^{d_1} n''_{d'}$. Then Θ_0 and Θ_1 can be uniquely characterized by $I = I'' \cup (I_2 \setminus I_4)$, $I' = I'' \cup (I_0 \setminus I_2)$, $i \in I_v$, $\Lambda_d = \Lambda''_d$ for each $1 \leq d \leq d_1$, $\Lambda'_1 = \emptyset$ and $\Lambda'_d = \Lambda_{d+d_1-1}$ for each $2 \leq d \leq r_{I'}$.
- Assume that there exists a unique $1 \leq d_0 \leq r_{I''}$ such that $i \in (I'')^{d_0} \cap (I_4 \cup (I_1 \setminus I_0))$. There exists a unique $1 \leq s_0 \leq r_{I_{v''}\cap I_1}$ such that $r^{s_0-1}_{v'',I_1,I''}+1 \leq d_0 \leq r^{s_0}_{v'',I_1,I''}$. Note that $i \in (I'')^{d_0} \cap (I_4 \cup (I_1 \setminus I_0))$ (together with Lemma 2.16 and Lemma 2.25) forces $\Lambda_{r^{s_0-1}_{v'',I_1,I''}+1} = \emptyset$. Then Θ_0 and Θ_1 can be uniquely characterized by $I = I'' \cup (I_2 \setminus I_4)$, $I' = I'' \cup (I_0 \setminus I_2)$, $\Lambda_d = \Lambda''_d$ for each $1 \leq d \leq d_0$, $\Lambda'_1 = \emptyset$ and $\Lambda'_d = \Lambda_{d+d_0-1}$ for each $2 \leq d \leq r_{I'}$.

□

Definition 4.30. Let $I, I' \subseteq \Delta_n$ be two subsets. We say that I *and* I' *do not connect* if $|i-i'| \geq 2$ for any $i \in I$ and $i' \in I'$.

If we understand I, I' as two sets of positive simple roots, then I and I' do not connect if and only if $\alpha + \alpha'$ is not a root for any $\alpha \in I$ and $\alpha' \in I'$. This is intuitive from Dynkin diagram.

Theorem 4.31. *Assume that Condition 4.28 holds.*

(i) For each $\Omega_0 \in \Psi^{-\ell_0,\ell_0+k_0-\#I_1}_{I_2\cup(I_1\setminus I_0),I_1}$ and $\Omega_1 \in \Psi^{-\ell_1,\ell_1+k_1-\#I_1}_{I_4\cup(I_1\setminus I_2),I_1}$, the image of $(x_{\Omega_0}, x_{\Omega_1})$ under (4.34) is εx_{Ω_2} for a $\Omega_2 \in \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1}$ uniquely determined by the pair (Ω_0, Ω_1) and a sign $\varepsilon \in \{1,-1\}$ uniquely determined by $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$ and the pair (ℓ_0, ℓ_1). Moreover, the map

$$\Psi^{-\ell_0,\ell_0+k_0-\#I_1}_{I_2\cup(I_1\setminus I_0),I_1} \times \Psi^{-\ell_1,\ell_1+k_1-\#I_1}_{I_4\cup(I_1\setminus I_2),I_1} \to \Psi^{-\ell_2,\ell_2+k_0+k_1-\#I_1}_{I_4\cup(I_1\setminus I_0),I_1} : (\Omega_0, \Omega_1) \mapsto \Omega_2 \tag{4.37}$$

is injective.

(ii) We have canonical isomorphisms ${}_{I_0,I_1}E^{-\ell_0,k_0+\ell_0}_{2,I_2,I_1} \cong_{I'_2,I_1} E^{-\ell_0,k_0+\ell_0}_{2,I_4,I_1}$ and ${}_{I_2,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I_4,I_1} \cong_{I_0,I_1} E^{-\ell_1,k_1+\ell_1}_{2,I'_2,I_1}$. Using the notation in item (i) above, the map

$${}_{I_0,I_1}E^{-\ell_1,k_1+\ell_1}_{2,I'_2,I_1} \otimes_{I'_2,I_1} E^{-\ell_0,k_0+\ell_0}_{2,I_4,I_1} \xrightarrow{\cup}_{I_0,I_1} E^{-\ell_2,k_0+k_1+\ell_2}_{2,I_4,I_1}$$

sends $(x_{\Omega_1}, x_{\Omega_0})$ to $(-1)^{k_0k_1}\varepsilon x_{\Omega_2}$.

(iii) The map (4.34) is injective for each pair (ℓ_0, ℓ_1) and thus (4.35) is injective and compatible with the canonical filtration on both the source and the target. If furthermore $I_0 \setminus I_2$ and $I_2 \setminus I_4$ do not connect, then (4.34) is an isomorphism for each pair (ℓ_0, ℓ_1) and thus (4.35) is an isomorphism.

Proof. Given two subsets $I, I' \subseteq \Delta_n$, we use the shortened notation $I < I'$ for $\max\{i' \mid i' \in I\} < \min\{i' \mid i' \in I'\}$. Note that if $I, I' \subseteq \Delta_n$ are two non-empty subintervals satisfying $I \cap I' = \emptyset$, then we have either $I < I'$ or $I' < I$. It is harmless to assume that $I_4 \subsetneq I_2 \subsetneq I_0$ otherwise the claims are easy. We write

$$I_0 \setminus I_2 = \bigsqcup_{t'=1}^{t_0} I_{0,t'} \text{ and } I_2 \setminus I_4 = \bigsqcup_{t'=1}^{t_2} I_{2,t'}$$

as disjoint union of non-empty maximal subintervals satisfying $I_{0,1} < \cdots < I_{0,t_0}$ and $I_{2,1} < \cdots < I_{2,t_2}$. As $(I_0 \setminus I_2) \cap (I_2 \setminus I_4) = \emptyset$, we have $I_{0,t'} \cap I_{2,t''} = \emptyset$ for each $1 \leq t' \leq t_0$ and $1 \leq t'' \leq t_2$. We define the defect of the triple $I_4 \subseteq I_2 \subseteq I_0$ as

$$\delta_{I_0,I_2,I_4} = \#\{(t',t'') \mid 1 \leq t' \leq t_0,\ 1 \leq t'' \leq t_2,\ I_{0,t'} > I_{2,t''}\}.$$

We prove item (i), item (ii) and item (iii) using Proposition 4.29 and an induction on the defect δ_{I_0,I_2,I_4}.

If $\delta_{I_0,I_2,I_4} = 0$, then we have $I_0 \setminus I_2 < I_2 \setminus I_4$ and the result follows entirely from Proposition 4.29. Now we assume that $\delta_{I_0,I_2,I_4} \geq 1$ and that all three items hold for any triple $I_7 \subseteq I_6 \subseteq I_5 \subseteq I_1$ satisfying $\delta_{I_5,I_6,I_7} < \delta_{I_0,I_2,I_4}$. Note that $\delta \geq 1$ is equivalent to $I_{0,t_0} > I_{2,1}$, and we define $I''_2 \overset{\mathrm{def}}{=} (I_2 \setminus I_{2,1}) \sqcup I_{0,t_0}$ and observe that $\delta_{I_0,I''_2,I_4} < \delta_{I_0,I_2,I_4}$. We also write $1 \leq t'_0 \leq t_0$ (resp. $1 \leq t'_2 \leq t_2$) for the minimal (resp. maximal) integer such that $I_{2,1} < I_{0,t'_0}$ (resp. such that $I_{2,t'_2} < I_{0,t_0}$) and then write $I^+_2 \overset{\mathrm{def}}{=} I_2 \sqcup I_{0,t_0}$, $I^{++}_2 \overset{\mathrm{def}}{=} I_2 \sqcup \bigsqcup_{t'=t'_0}^{t_0} I_{0,t'}$, $I^\sharp_2 \overset{\mathrm{def}}{=} I^{++}_2 \setminus I_{2,1}$, $I^-_2 \overset{\mathrm{def}}{=} I_2 \setminus I_{2,1}$, $I^{--}_2 \overset{\mathrm{def}}{=} I_2 \setminus \bigsqcup_{t''=1}^{t'_2} I_{2,t''}$ and $I^\flat_2 \overset{\mathrm{def}}{=} I^{--}_2 \sqcup I_{0,t_0}$. Then we have the following constructions using item (iii) of Proposition 4.29.

- As I_{0,t_0} is a maximal subinterval of $I_0 \setminus I_2$ satisfying $I_0 \setminus I_2^{++} < I_2^{++} \setminus I_2^{+} < I_2^{+} \setminus I_2 = I_{0,t_0}$, we obtain an isomorphism $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_0,I_2^{++}} \otimes \mathbf{E}_{I_2^{++},I_2^{+}} \otimes \mathbf{E}_{I_2^{+},I_2}$ which is compatible with canonical filtration on both source and target. Consequently, Ω_0 determines a triple $(\Omega_0^{++}, \Omega_0^{+}, \Omega_0^{-})$ where Ω_0^{++} is an equivalence class of $(I_2^{++} \cup (I_1 \setminus I_0), I_1)$-atomic tuples, Ω_0^{+} is an equivalence class of $(I_2^{+} \cup (I_1 \setminus I_2^{++}), I_1)$-atomic tuples and Ω_0^{-} is an equivalence class of $(I_2 \cup (I_1 \setminus I_2^{+}), I_1)$-atomic tuples.
- As $I_{2,1}$ is a maximal subinterval of $I_2 \setminus I_4$ satisfying $I_{2,1} = I_2 \setminus I_2^{-} < I_2^{-} \setminus I_2^{-}- < I_2^{-}- \setminus I_4$, we obtain an isomorphism $\mathbf{E}_{I_2,I_4} \cong \mathbf{E}_{I_2,I_2^{-}} \otimes \mathbf{E}_{I_2^{-},I_2^{--}} \otimes \mathbf{E}_{I_2^{--},I_4}$ which is compatible with canonical filtration on both source and target. Consequently, Ω_1 determines a pair $(\Omega_1^{+}, \Omega_1^{-}, \Omega_1^{--})$ where Ω_1^{+} is an equivalence class of $(I_2^{-} \cup (I_1 \setminus I_2), I_1)$-atomic tuples, Ω_1^{-} is an equivalence class of $(I_2^{--} \cup (I_1 \setminus I_2^{-}), I_1)$-atomic tuples and Ω_1^{--} is an equivalence class of $(I_4 \cup (I_1 \setminus I_2^{--}), I_1)$-atomic tuples.

Now we have the following observations from item (ii).

- We have a canonical isomorphism $\mathbf{E}_{I_2^{+},I_2} \otimes \mathbf{E}_{I_2,I_2^{-}} \cong \mathbf{E}_{I_2^{+},I_2''} \otimes \mathbf{E}_{I_2'',I_2^{-}}$ which exchange Ω_0^{-} and Ω_1^{+}.
- We have a canonical isomorphism $\mathbf{E}_{I_2^{++},I_2^{+}} \otimes \mathbf{E}_{I_2^{+},I_2''} \cong \mathbf{E}_{I_2^{++},I_2^{\sharp}} \otimes \mathbf{E}_{I_2^{\sharp},I_2''}$ which exchange Ω_0^{+} and Ω_1^{+}, and $\mathbf{E}_{I_2'',I_2^{-}} \otimes \mathbf{E}_{I_2^{-},I_2^{--}} \cong \mathbf{E}_{I_2'',I_2^{\flat}} \otimes \mathbf{E}_{I_2^{\flat},I_2^{--}}$ which exchange Ω_0^{-} and Ω_1^{-}.

The we deduce again from item (iii) of Proposition 4.29 a canonical isomorphism

$$\mathbf{E}_{I_0,I_2^{++}} \otimes \mathbf{E}_{I_2^{++},I_2^{\sharp}} \otimes \mathbf{E}_{I_2^{\sharp},I_2''} \cong \mathbf{E}_{I_0,I_2''}$$

which determines an equivalence class Ω_0'' of $(I_2'' \cup (I_1 \setminus I_0), I_1)$-atomic tuples from $(\Omega_0^{++}, \Omega_1^{+}, \Omega_0^{+})$, and similarly a canonical isomorphism

$$\mathbf{E}_{I_2'',I_2^{\flat}} \otimes \mathbf{E}_{I_2^{\flat},I_2^{--}} \otimes \mathbf{E}_{I_2^{--},I_4} \cong \mathbf{E}_{I_2'',I_4}$$

which determines an equivalence class Ω_1'' of $(I_4 \cup (I_1 \setminus I_2''), I_1)$-atomic tuples from $(\Omega_1^{-}, \Omega_0^{-}, \Omega_1^{--})$. As $\delta_{I_0,I_2'',I_4} < \delta_{I_0,I_2,I_4}$, the map $\mathbf{E}_{I_0,I_2''} \otimes \mathbf{E}_{I_2'',I_4} \to \mathbf{E}_{I_0,I_4}$ together with our inductive assumption determines an equivalence class Ω_2 of $(I_4 \cup (I_1 \setminus I_0), I_1)$-atomic tuples. To summary, we define Ω_2 via the following composition

$$\begin{aligned}
\mathbf{E}_{I_0,I_2} \otimes \mathbf{E}_{I_2,I_4} &\cong \mathbf{E}_{I_0,I_2^{++}} \otimes \mathbf{E}_{I_2^{++},I_2^{+}} \otimes \mathbf{E}_{I_2^{+},I_2} \otimes \mathbf{E}_{I_2,I_2^{-}} \otimes \mathbf{E}_{I_2^{-},I_2^{--}} \otimes \mathbf{E}_{I_2^{--},I_4} \\
&\cong \mathbf{E}_{I_0,I_2^{++}} \otimes \mathbf{E}_{I_2^{++},I_2^{+}} \otimes \mathbf{E}_{I_2^{+},I_2''} \otimes \mathbf{E}_{I_2'',I_2^{-}} \otimes \mathbf{E}_{I_2^{-},I_2^{--}} \otimes \mathbf{E}_{I_2^{--},I_4} \\
&\cong \mathbf{E}_{I_0,I_2^{++}} \otimes \mathbf{E}_{I_2^{++},I_2^{\sharp}} \otimes \mathbf{E}_{I_2^{\sharp},I_2''} \otimes \mathbf{E}_{I_2'',I_2^{\flat}} \otimes \mathbf{E}_{I_2^{\flat},I_2^{--}} \otimes \mathbf{E}_{I_2^{--},I_4} \\
&\cong \mathbf{E}_{I_0,I_2''} \otimes \mathbf{E}_{I_2'',I_4} \to \mathbf{E}_{I_0,I_4}.
\end{aligned}$$

Note that this composition differs from the cup product map $\mathbf{E}_{I_0,I_2} \otimes \mathbf{E}_{I_2,I_4} \xrightarrow{\cup} \mathbf{E}_{I_0,I_4}$ by a sign depending only on $I_0 \subseteq I_2 \subseteq I_0 \subseteq I_1$ and (ℓ_0, ℓ_1), by applying item (ii) and item (iii) to the composition. Note that all the subsets of Δ_n involved in the construction of Ω_2 depends only on $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$. Moreover, given $I_4 \subseteq I_2 \subseteq I_0 \subseteq I_1$ and (ℓ_0, ℓ_1), we can read off $\Omega_0^{+}+$, Ω_1^{+}, Ω_0^{+}, Ω_1^{-}, Ω_0^{-} and Ω_1^{--} from Ω_2, and thus Ω_0 and Ω_1 as well. This implies that (4.37) is injective. Item (ii) of this theorem follows from constructing another composition symmetric to one above and apply our inductive assumption to the isomorphism $\mathbf{E}_{I_0,I_2''} \otimes \mathbf{E}_{I_2'',I_4} \cong \mathbf{E}_{I_0,I_2'''} \otimes \mathbf{E}_{I_2''',I_4}$ where $I_2''' \stackrel{\text{def}}{=} I_4 \cup (I_0 \setminus I_2'')$. The first part of item (iii) follows from the injectivity of (4.37) as it induces an injection from a basis of ${}_{I_0,I_1}E_{2,I_2,I_1}^{-\ell_0,k_0+\ell_0} \otimes_{I_2,I_1} E_{2,I_4,I_1}^{-\ell_1,k_1+\ell_1}$ to a basis of ${}_{I_0,I_1}E_{2,I_4,I_1}^{-\ell_2,k_0+k_1+\ell_2}$. The second part of item (iii) follows from our inductive assumption and the observation that $I_0 \setminus I_2$ and $I_2 \setminus I_4$

do not connect if and only if $I_0 \setminus I_2''$ and $I_2'' \setminus I_4$ do not connect if and only if $I_{0,t'}$ and $I_{2,t''}$ do not connect for each $1 \leq t' \leq t_0$ and $1 \leq t'' \leq t_2$. The proof is thus completed. □

5. Breuil-Schraen $\mathscr{L}$-invariant

5.1. **Definition of Breuil-Schraen $\mathscr{L}$-invariant.**

In this section, we define *Breuil-Schraen $\mathscr{L}$-invariant* in Definition 5.6 and study its moduli space in Theorem 5.9. Then we formulate Conjecture 5.11 that relates Breuil-Schraen $\mathscr{L}$-invariants with Breuil-Ding's approach of higher $\mathscr{L}$-invariants.

We recall from Corollary 4.23 the definition of $\mathbf{E}_{I_0,I_2}$ and $\mathbf{E}'_{I_0,I_2}$ for each pair of subsets $I_2 \subseteq I_0 \subseteq \Delta_n$. We consider a triple $I_4 \subseteq I_2 \subseteq I_0 \subseteq \Delta_n$ and set $I_2' \stackrel{\text{def}}{=} I_4 \cup (I_0 \setminus I_2)$ as usual. By taking $I_1 = \Delta_n$ in Theorem 4.31, we obtain the following result.

Corollary 5.1. *(i) For each $I_4 \subseteq I_2 \subseteq I_0 \subseteq \Delta_n$, the following three maps are injective*

- $\mathbf{E}_{I_0,I_2} \otimes \mathbf{E}_{I_2,I_4} \xrightarrow{\cup} \mathbf{E}_{I_0,I_4}$;
- $\mathbf{E}_{I_0,I_2} \otimes \mathbf{E}'_{I_2,I_4} \xrightarrow{\cup} \mathbf{E}'_{I_0,I_4}$;
- $\mathbf{E}'_{I_0,I_2} \otimes \mathbf{E}_{I_2,I_4} \xrightarrow{\cup} \mathbf{E}'_{I_0,I_4}$.

(ii) We have canonical isomorphisms $\mathbf{E}^*_{I_0,I_2} \cong \mathbf{E}^*_{I_2',I_4}$ and $\mathbf{E}^*_{I_2,I_4} \cong \mathbf{E}^*_{I_0,I_2'}$ for each $* \in \{\ ,'\}$. If we abuse the notation x for an element of $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_2',I_4}$ and y for an element of $\mathbf{E}_{I_2,I_4} \cong \mathbf{E}_{I_0,I_2'}$, then $\mathbf{E}_{I_0,I_2'} \otimes \mathbf{E}_{I_2',I_4} \xrightarrow{\cup} \mathbf{E}_{I_0,I_4}$ sends (y,x) to $(-1)^{(\#I_0-\#I_2)(\#I_2-\#I_4)}x \cup y$. Similar facts hold for other two kinds of maps in item (i).

(iii) If $I_0 \setminus I_2$ and $I_2 \setminus I_4$ do not connect (see Definition 4.30), then the three maps in item (i) are isomorphisms.

Thanks to item (i), we can identify $\mathbf{E}_{I_0,I_2} \otimes \mathbf{E}_{I_2,I_4}$ with a subspace of $\mathbf{E}_{I_0,I_4}$ for any $I_4 \subseteq I_2 \subseteq I_0 \subseteq \Delta_n$ from now on.

Lemma 5.2. *For each $I_2 \subseteq I_0 \subseteq \Delta_n$ satisfying $\#I_0 \setminus I_2 = 1$, we have a canonical isomorphism*

$$\mathbf{E}_{I_0,I_2} \cong \mathrm{Hom}_{\mathrm{cont}}(K^\times, E),$$

which is compatible with $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_0\setminus I_2,\emptyset}$.

Proof. We write $I_0 \setminus I_2 = \{i\}$ for some $1 \leq i \leq n-1$. Note from Corollary 4.23 that $\mathbf{E}_{I_0,I_2}$ admits a canonical filtration

$$0 = \mathrm{Fil}^{-n+3}(\mathbf{E}_{I_0,I_2}) \subseteq \mathrm{Fil}^{-n+2}(\mathbf{E}_{I_0,I_2}) \subseteq \mathrm{Fil}^{-n+1}(\mathbf{E}_{I_0,I_2}) = \mathbf{E}_{I_0,I_2}.$$

We observe that $\Psi^{-n+1,2}_{\Delta_n\setminus\{i\},\Delta_n} = 0$ and $\Psi^{-n+2,1}_{\Delta_n\setminus\{i\},\Delta_n} = \{(v, \Delta_n \setminus \{i\}, \underline{k}, \underline{\Lambda}) \mid v \in \mathcal{B}_{n,\Delta_n\setminus\{i\}}\}$ with $\Lambda_1 = \Lambda_2 = \emptyset$, based on Lemma 2.16. In other words, we have canonical isomorphisms

$$\mathbf{E}_{I_0,I_2} \cong_{I_0,\Delta_n} E^{-n+2,n}_{2,I_2,\Delta_n} \cong E^{-n+2,1}_{2,\Delta_n\setminus\{i\},\Delta_n} \cong \mathrm{Hom}_{\mathrm{cont}}(\overline{Z}_{n,\Delta_n\setminus\{i\}}, E) \cong \mathrm{Hom}_{\mathrm{cont}}(K^\times, E)$$

where $\overline{Z}_{n,\Delta_n\setminus\{i\}} \cong K^\times$ is the center of $\overline{L}_{n,\Delta_n\setminus\{i\}}$. The compatibility with $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_0\setminus I_2,\emptyset}$ is obvious from the argument above. □

For each $I_2 \subseteq I_0 \subseteq \Delta_n$, we set $\mathbf{E}^<_{I_0,I_2} \stackrel{\text{def}}{=} \sum_{I_2\subsetneq I\subsetneq I_0} \mathbf{E}_{I_0,I} \otimes \mathbf{E}_{I,I_2} \subseteq \mathbf{E}_{I_0,I_2}$.

Lemma 5.3. *Let $I_2 \subseteq I_0 \subseteq \Delta_n$ be subsets with $\#I_0 \setminus I_2 \geq 2$. We have $\dim_E \mathbf{E}_{I_0,I_2}/\mathbf{E}^<_{I_0,I_2} \in \{0, [K : \mathbb{Q}_p]\}$, and it is non-zero if and only if $I_0 \setminus I_2$ is an interval.*

Proof. If $I_0 \setminus I_2$ is not an interval, then we can choose $I_2 \subsetneq I \subsetneq I_0$ such that $I_0 \setminus I$ is a maximal interval of $I_0 \setminus I_2$, which together with item (iii) implies that $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_0,I} \otimes \mathbf{E}_{I,I_2} \subseteq \mathbf{E}^{<}_{I_0,I_2}$ and thus $\mathbf{E}^{<}_{I_0,I_2} = \mathbf{E}_{I_0,I_2}$. It remains to treat the case when $I_0 \setminus I_2$ is an interval of the form $\{i, i+1, \dots, j\}$. Note from Corollary 4.23 that $\mathbf{E}_{I_0,I_2}$ admits a canonical filtration

$$0 = \mathrm{Fil}^{-n+2+\#I_0-\#I_2}(\mathbf{E}_{I_0,I_2}) \subseteq \mathrm{Fil}^{-n+1+\#I_0-\#I_2}(\mathbf{E}_{I_0,I_2}) \subseteq \cdots \subseteq \mathrm{Fil}^{-n+1}(\mathbf{E}_{I_0,I_2}) = \mathbf{E}_{I_0,I_2}.$$

We finish the proof by the following two claims.

(i) We have $\dim_E \mathbf{E}_{I_0,I_2}/\mathrm{Fil}^{-n+3}(\mathbf{E}_{I_0,I_2}) = \#S = [K : \mathbb{Q}_p]$. According to Corollary 4.23, it suffices to observe that $\Psi^{-n+1,2\#I_0-\#I_2}_{I_2\cup(\Delta_n\setminus I_0),\Delta_n} = \emptyset$ and $\Psi^{-n+2,2\#I_0-2\#I_2-1}_{I_2\cup(\Delta_n\setminus I_0),\Delta_n}$ consists of those equivalent classes whose maximal elements $\Theta = (v, I, \underline{k}, \underline{\Lambda})$ satisfies $v = \emptyset$, $I = \Delta_n \setminus \{i\}$, $\Lambda_1 = \emptyset$ and $\Lambda_2 = \{(2\#I_0 \setminus I_2 - 1, \iota)\}$ for some $\iota \in S$. In particular, there exists a natural bijection between $\Psi^{-n+2,2\#I_0-2\#I_2-1}_{I_2\cup(\Delta_n\setminus I_0),\Delta_n}$ and S.

(ii) We have $\mathbf{E}^{<}_{I_0,I_2} = \mathrm{Fil}^{-n+3}(\mathbf{E}_{I_0,I_2})$. For each $\#I_0-\#I_2 \leq \ell \leq n-3$, let $\Omega' \in \Psi^{-\ell,\ell+2\#I_0-2\#I_2-n+1}_{I_2\cup(\Delta_n\setminus I_0),\Delta_n}$ be an equivalence class and $\Theta' = (v', I', \underline{k}', \underline{\Lambda}')$ be the maximal element inside. We always have $\Lambda_1 = \emptyset$ thanks to Lemma 2.16. As $r_{I'} = n - \ell \geq 3$, we have the following two possibilities.
   - We have $v = \emptyset$ and there exists $2 \leq d \leq r_{I'} - 1$ such that $\Lambda'_d \neq \emptyset \neq \Lambda'_{d+1}$. We write $i' \stackrel{\text{def}}{=} \sum_{d'=1}^{d} n_{d'}$ and set $I \stackrel{\text{def}}{=} \{i', i'+1, \dots, j\} \cup I_2$.
   - We have $I_v \subsetneq \Delta_n$ and thus there exists $2 \leq d = r^1_{v',\Delta_n,I'} + 1 \leq r_{I'}$ such that $\Lambda'_d = \emptyset$. We write $i' \stackrel{\text{def}}{=} \sum_{d'=1}^{d-1} n_{d'}$ and set $I \stackrel{\text{def}}{=} \{i', i'+1, \dots, j\} \cup I_2$.

   In both possibilities above, we define $\Theta'' = (\emptyset, I'', \underline{k}'', \underline{\Lambda}'')$ by $I'' = I \cup (\Delta_n \setminus I_0)$ and $\Lambda''_{d'} \stackrel{\text{def}}{=} \Lambda'_{d'}$ for each $1 \leq d' \leq r_{I''} = d$. We also define $\Theta''' = (\emptyset, I''', \underline{k}''', \underline{\Lambda}''')$ by $I''' = I_2 \cup (\Delta_n \setminus I)$, $\Lambda'''_1 \stackrel{\text{def}}{=} \emptyset$ and $\Lambda'''_{d'} \stackrel{\text{def}}{=} \Lambda'_{d'+d-1}$ for each $2 \leq d' \leq r_{I'''} = r_{I'} - d + 1$. We write Ω'' (resp. Ω''') for the equivalence class of Θ'' (resp. of Θ''') and claim that

$$x_{\Omega''} \cup x_{\Omega'''} = x_{\Omega'} \in \mathrm{Fil}^{-\ell}(\mathbf{E}_{I_0,I_2})/\mathrm{Fil}^{-\ell+1}(\mathbf{E}_{I_0,I_2})$$

   from the proof of item (i) of Theorem 4.31. In other words, we have

$$x_{\Omega'} \in \Big(\mathbf{E}_{I_0,I} \otimes \mathbf{E}_{I,I_2} + \mathrm{Fil}^{-\ell+1}(\mathbf{E}_{I_0,I_2})\Big)/\mathrm{Fil}^{-\ell+1}(\mathbf{E}_{I_0,I_2})$$

   Let Ω' run through $\Psi^{-\ell,\ell+2\#I_0-2\#I_2-n+1}_{I_2\cup(\Delta_n\setminus I_2),\Delta_n}$, we have thus shown that

$$\mathbf{E}^{<}_{I_0,I_2} \cap \mathrm{Fil}^{-\ell}(\mathbf{E}_{I_0,I_2}) + \mathrm{Fil}^{-\ell+1}(\mathbf{E}_{I_0,I_2}) = \mathrm{Fil}^{-\ell}(\mathbf{E}_{I_0,I_2})$$

   for each $\#I_0 - \#I_2 \leq \ell \leq n-3$, which is clearly sufficient to conclude.

□

For each positive root $(i,j) \in \Phi^+$ (with $1 \leq i < j \leq n$), we can clearly attach a subinterval $I_\alpha \stackrel{\text{def}}{=} \{i, i+1, \dots, j-1\} \subseteq \Delta_n$ and this induces a bijection between the set of positive roots (with respect to (B_n^+, T_n)) and the set of (non-empty) subintervals of Δ_n. More generally, for each subset of $I \subseteq \Delta_n$, we can clearly attach an element in the root lattice $\alpha_I \stackrel{\text{def}}{=} \sum_{i\in I}(i, i+1)$. For each $\alpha \in \Phi^+$ with $\#I_\alpha \geq 2$, we choose a set $\overline{X}_\alpha \stackrel{\text{def}}{=} \{x_{\alpha,\iota} \mid \iota \in S\} \subseteq \mathbf{E}_{I_\alpha,\emptyset}$ which image in $\mathbf{E}_{I_\alpha,\emptyset}/\mathbf{E}^{<}_{I_\alpha,\emptyset}$ naturally corresponds to $\{x_\Omega \mid \Omega \in \Psi^{-n+2,2\#I_\alpha-1}_{\Delta_n\setminus I_\alpha,\Delta_n}\}$ via the bijection described in item (i) in the proof of Lemma 5.3. For each $\alpha \in \Phi^+$ with $\#I_\alpha = 1$, we write x^∞_α (resp. $x_{\alpha,\iota}$) for the elements in $\mathbf{E}_{I_\alpha,\emptyset}$ corresponding to val (resp. $\log_\iota$) under the isomorphism $\mathbf{E}_{I_\alpha,\emptyset} \cong \mathrm{Hom}_{\mathrm{cont}}(K^\times, E)$ (see

Lemma 5.2), and then set $\overline{X}_\alpha \overset{\text{def}}{=} \{x_\alpha^\infty\} \sqcup \{x_{\alpha,\iota} \mid \iota \in S\}$. For each $I_2 \subseteq I_0 \subseteq \Delta_n$ with $I_0 \setminus I_2 = I_\alpha$, we abuse $x_{\alpha,\iota}$ (and possibly x_α^∞) for the vector in $\mathbf{E}_{I_0,I_2}$ obtained from the isomorphism $\mathbf{E}_{I_0,I_2} \cong \mathbf{E}_{I_\alpha,\emptyset}$. Consequently, for each $I_2 \subseteq I_0 \subseteq \Delta_n$ and each partition into positive roots $\alpha_{I_0\setminus I_2} = \alpha_1 + \cdots + \alpha_t$, we obtain a well defined element

$$x_{\alpha_1} \cup x_{\alpha_2} \cup \cdots \cup x_{\alpha_t} \in \mathbf{E}_{I_0,I_2}$$

for each $x_{\alpha_{t'}} \in \overline{X}_{\alpha_{t'}}$ and $1 \leq t' \leq t$.

Lemma 5.4. *For each $I_2 \subseteq I_0 \subseteq \Delta_n$, $\mathbf{E}_{I_0,I_2}$ admits a basis of the form*

$$X_{I_0,I_2} \overset{\text{def}}{=} \{x_{\alpha_1} \cup x_{\alpha_2} \cup \cdots \cup x_{\alpha_t}\}_{\alpha_{I_0\setminus I_2} = \alpha_1 + \cdots + \alpha_t} \tag{5.1}$$

where $x_{\alpha_{t'}} \in \overline{X}_{\alpha_{t'}}$ for each $1 \leq t' \leq t$ and $\{\alpha_1, \ldots, \alpha_t\}$ runs through all the (unordered) partition of $\alpha_{I_0\setminus I_2}$.

Proof. We prove by an increasing induction on $\#I_0 \setminus I_2$. The case when $\#I_0 \setminus I_2 = 1$ is clear. Thanks to item (iii) we may assume without loss of generality that $I_0 \setminus I_2 = I_\alpha$ for some $\alpha \in \Phi^+$ with $\#I_\alpha \geq 2$. According to Lemma 5.3, it suffices to show that $X_{I_\alpha,\emptyset} \setminus \overline{X}_\alpha$ forms a basis of $\mathbf{E}_{I_\alpha,\emptyset}^<$. We write $\alpha = (i,j)$ for some $1 \leq i < j \leq n$ and note that $\mathbf{E}_{I_\alpha,\emptyset}^< = \sum_{i<k<j} \mathbf{E}_{I_\alpha,I_{(k,j)}} \otimes \mathbf{E}_{I_{(k,j)},\emptyset}$ admits an increasing filtration $\mathrm{Fil}_\ell \mathbf{E}_{I_\alpha,\emptyset}^< = \sum_{i<k\leq \ell} \mathbf{E}_{I_\alpha,I_{(k,j)}} \otimes \mathbf{E}_{I_{(k,j)},\emptyset}$ with $i \leq \ell \leq j-1$. Then we observe that

$$\mathrm{Fil}_\ell \mathbf{E}_{I_\alpha,\emptyset}^< / \mathrm{Fil}_{\ell-1} \mathbf{E}_{I_\alpha,\emptyset}^< = (\mathbf{E}_{I_{(i,\ell)},\emptyset} / \mathbf{E}_{I_{(i,\ell)},\emptyset}^<) \otimes \mathbf{E}_{I_{(\ell,j)},\emptyset}$$

which admits a basis induced from

$$X_{I_\alpha,\emptyset}^\ell \overset{\text{def}}{=} \{x_{(i,\ell)} \otimes x'_{(\ell,j)} \mid x_{(i,\ell)} \in \overline{X}_{(i,\ell)}, x'_{(\ell,j)} \in X_{I_{(\ell,j)},\emptyset}\}$$

for each $i+1 \leq \ell \leq j-1$. We conclude by the observation that $X_{I_\alpha,\emptyset} \setminus \overline{X}_\alpha = \bigsqcup_{\ell=i+1}^{j-1} X_{I_\alpha,\emptyset}^\ell$. □

Note that val spans a canonical line in $\mathrm{Hom}_{\mathrm{cont}}(K^\times, E)$, and thus induces a canonical line $\mathbf{E}_{I_0,I_2}^\infty \subseteq \mathbf{E}_{I_0,I_2}$ for each $I_2 \subseteq I_0 \subseteq \Delta_n$ with $\#I_0 \setminus I_2 = 1$ according to Lemma 5.2. For a general pair $I_2 \subseteq I_0 \subseteq \Delta_n$, we choose a sequence $I_2 = I_{2,0} \subsetneq I_{2,1} \subsetneq \cdots \subsetneq I_{2,t} = I_0$ for $t = \#I_0 - \#I_2$ and thus $\#I_{2,t'} \setminus I_{2,t'-1} = 1$ for each $1 \leq t' \leq t$. Then we define $\mathbf{E}_{I_0,I_2}^\infty$ as the image of the composition

$$\mathbf{E}_{I_{2,t},I_{2,t-1}}^\infty \otimes \cdots \otimes \mathbf{E}_{I_{2,1},I_{2,0}}^\infty \hookrightarrow \mathbf{E}_{I_{2,t},I_{2,t-1}} \otimes \cdots \otimes \mathbf{E}_{I_{2,1},I_{2,0}} \xrightarrow{\cup} \mathbf{E}_{I_0,I_2},$$

which gives a canonical line in $\mathbf{E}_{I_0,I_2}$. Note that item (ii) of Corollary 5.1 implies that $\mathbf{E}_{I_0,I_2}^\infty$ is independent of the choice of $I_2 = I_{2,0} \subsetneq I_{2,1} \subsetneq \cdots \subsetneq I_{2,t} = I_0$.

We write $\widehat{\mathbf{E}}_n \overset{\text{def}}{=} \mathbf{E}_{\Delta_n,\emptyset}$. For each $I_2 \subseteq I_0 \subseteq \Delta_n$ we define $\widehat{\mathbf{E}}_{I_0,I_2}$ as the image of

$$\mathbf{E}_{\Delta_n,I_0}^\infty \otimes \mathbf{E}_{I_0,I_2} \otimes \mathbf{E}_{I_2,\emptyset}^\infty \xrightarrow{\cup} \mathbf{E}_{\Delta_n,\emptyset} = \widehat{\mathbf{E}}_n$$

which gives a canonical subspace of $\widehat{\mathbf{E}}_n$. There exists clearly a non-canonical isomorphism ι_{I_0,I_2} : $\mathbf{E}_{I_0,I_2} \xrightarrow{\sim} \widehat{\mathbf{E}}_{I_0,I_2}$ by item (i) of Corollary 5.1 and the definition of $\widehat{\mathbf{E}}_{I_0,I_2}$, depending on our choice of val $\in \mathrm{Hom}_{\mathrm{cont}}(K^\times, E)$. Note that we use the convention $\mathbf{E}_{I_0,I_0} \cong \mathbf{E}_{\emptyset,\emptyset} \cong E$, and thus $\widehat{\mathbf{E}}_{I_0,I_0} = \mathbf{E}_{\Delta_n,\emptyset}^\infty \subseteq \widehat{\mathbf{E}}_n$ for each $I_0 \subseteq \Delta_n$.

Lemma 5.5.
- *We have $\widehat{\mathbf{E}}_{I_0,I_2} = \widehat{\mathbf{E}}_{I_0\setminus I_2,\emptyset}$ for each $I_2 \subseteq I_0 \subseteq \Delta_n$.*
- *We have $\widehat{\mathbf{E}}_{I'_0,I'_2} \subseteq \widehat{\mathbf{E}}_{I_0,I_2}$ for each $I_2 \subseteq I_0 \subseteq \Delta_n$ and $I'_2 \subseteq I'_0 \subseteq \Delta_n$ satisfying $I'_0 \setminus I'_2 \subseteq I_0 \setminus I_2$.*

Proof. The first part follows immediately from item (ii) of Corollary 5.1 as the image of $\mathbf{E}_{I_0,I_2}\otimes\mathbf{E}^{\infty}_{I_2,\emptyset}\xrightarrow{\cup}\mathbf{E}_{I_0,\emptyset}$ clearly equals that of $\mathbf{E}^{\infty}_{I_0,I_0\setminus I_2}\otimes\mathbf{E}_{I_0\setminus I_2,\emptyset}\xrightarrow{\cup}\mathbf{E}_{I_0,\emptyset}$. Using the first part, we may assume that $I_2=I_2'=\emptyset$ while checking the second part. We finish the proof by the observation that the map $\mathbf{E}^{\infty}_{\Delta_n,I_0'}\otimes\mathbf{E}_{I_0',\emptyset}\xrightarrow{\cup}\widehat{\mathbf{E}}_n$ factors as

$$\mathbf{E}^{\infty}_{\Delta_n,I_0'}\otimes\mathbf{E}_{I_0',\emptyset}\cong\mathbf{E}^{\infty}_{\Delta_n,I_0}\otimes\mathbf{E}^{\infty}_{I_0,I_0'}\otimes\mathbf{E}_{I_0',\emptyset}\to\mathbf{E}^{\infty}_{\Delta_n,I_0}\otimes\mathbf{E}_{I_0,\emptyset}\xrightarrow{\cup}\widehat{\mathbf{E}}_n.$$

□

Definition 5.6. A *Breuil-Schraen $\mathscr{L}$-invariant* is a codimension one subspace $W\subseteq\widehat{\mathbf{E}}_n$ such that

(i) $W\cap\widehat{\mathbf{E}}_{I_0,I_2}\subsetneq\widehat{\mathbf{E}}_{I_0,I_2}$ and thus $W_{I_0,I_2}\overset{\mathrm{def}}{=}\iota^{-1}_{I_0,I_2}(W\cap\widehat{\mathbf{E}}_{I_0,I_2})$ satisfies $\dim_E\mathbf{E}_{I_0,I_2}/W_{I_0,I_2}=1$ for each $I_2\subseteq I_0\subseteq\Delta_n$;

(ii) the composition

$$\mathbf{E}_{I_0,I_2}\otimes\mathbf{E}_{I_2,I_4}\xrightarrow{\cup}\mathbf{E}_{I_0,I_4}\twoheadrightarrow\mathbf{E}_{I_0,I_4}/W_{I_0,I_4}$$

factors through an isomorphism of lines

$$(\mathbf{E}_{I_0,I_2}/W_{I_0,I_2})\otimes(\mathbf{E}_{I_2,I_4}/W_{I_2,I_4})\xrightarrow{\sim}\mathbf{E}_{I_0,I_4}/W_{I_0,I_4}$$

for each $I_4\subseteq I_2\subseteq I_0\subseteq\Delta_n$.

Remark 5.7. Based on our Corollary 5.1, one can immediately generalize the definition of automorphic (simple) $\mathscr{L}$-invariants in Section 3.3 of [Geh21] to all automorphic higher $\mathscr{L}$-invariants, at least when the fixed global set up is locally GL_n in nature. One key idea in [Geh21] is to define a $\mathscr{L}$-invariant as kernel of certain cup product map, and condition (ii) is very natural from this point of view. In other words, our definition of Breuil-Schraen $\mathscr{L}$-invariants come from an attempt to combine [Geh21] with representation theoretic computations in [Schr11].

Note that codimension one subspaces $W\subseteq\widehat{\mathbf{E}}_n$ satisfying condition (i) of Definition 5.6 are clearly parameterized by a Zariski open subvariety $\mathbb{P}(\widehat{\mathbf{E}}_n)^{\circ}$ of the projective space $\mathbb{P}(\widehat{\mathbf{E}}_n)$. Adding the condition (ii) of Definition 5.6 cut out a closed subvariety $\mathcal{BS}\subseteq\mathbb{P}(\widehat{\mathbf{E}}_n)^{\circ}$.

Lemma 5.8. *Let $W\subseteq\widehat{\mathbf{E}}_n$ be a hyperplane. Assume that*

- $\dim_E\mathbf{E}_{I_0,\emptyset}/W_{I_0,\emptyset}=1$ *for each $I_0\subseteq\Delta_n$ which is an (possibly empty) interval (see Definition 2.10);*
- $W_{I_0,\emptyset}$ *contains the image of $\mathbf{E}_{I_0,I_2}\otimes W_{I_2,\emptyset}\xrightarrow{\cup}\mathbf{E}_{I_0,\emptyset}$ for each pair of (possibly empty) subintervals $I_2\subseteq I_0\subseteq\Delta_n$ such that $I_0\setminus I_2$ is also an interval.*

Then W is a Breuil-Schraen $\mathscr{L}$-invariant.

Proof. To check condition (i) of Definition 5.6, we may assume without loss of generality that $I_2=\emptyset$ thanks to Lemma 5.5. We consider $I_{0,1}$ which is a maximal subinterval of I_0. If $I_{0,1}=I_0$, then we have nothing to prove. Otherwise we have $\widehat{\mathbf{E}}_{I_{0,1},\emptyset}\subseteq\widehat{\mathbf{E}}_{I_0,\emptyset}$. As $\widehat{\mathbf{E}}_{I_{0,1},\emptyset}\not\subseteq W$, we clearly have $\widehat{\mathbf{E}}_{I_0,\emptyset}\not\subseteq W$, which finishes the proof of condition (i) of Definition 5.6.

Now we check condition (ii) of Definition 5.6. Again we may assume using Lemma 5.5 that $I_4=\emptyset$. If $I_0\setminus I_2$ is not an interval, then there exists $I_2\subsetneq I\subsetneq I_0$ such that $I\setminus I_2$ is a maximal subinterval of $I_0\setminus I_2$ and $\mathbf{E}_{I_0,I_2}\cong\mathbf{E}_{I_0,I}\otimes\mathbf{E}_{I,I_2}$, and thus we can reduce this case to that of the pair $I\subseteq I_0$ and the pair $I_2\subseteq I$. We assume that $I_0\setminus I_2$ is an interval from now on. We observe that the kernel of

$$\mathbf{E}_{I_0,I_2}\otimes\mathbf{E}_{I_2,\emptyset}\twoheadrightarrow(\mathbf{E}_{I_0,I_2}/W_{I_0,I_2})\otimes(\mathbf{E}_{I_2,\emptyset}/W_{I_2,\emptyset})$$

is simply

$$\mathbf{E}_{I_0,I_2}\otimes W_{I_2,\emptyset}+W_{I_0,I_2}\otimes \mathbf{E}_{I_2,\emptyset}\cong \mathbf{E}_{I_0,I_2}\otimes W_{I_2,\emptyset}+\mathbf{E}_{I_0,I_0\setminus I_2}\otimes W_{I_0\setminus I_2,\emptyset}\subseteq W_{I_0,\emptyset}$$

by our assumption. Here we identify $\mathbf{E}_{I_0,I_2}\otimes \mathbf{E}_{I_2,\emptyset}$ with a subspace of $\mathbf{E}_{I_0,\emptyset}$ using item (i) of Corollary 5.1, and then use item (ii) of Corollary 5.1 to transform the second direct summand. We also use the fact that W_{I_0,I_2} is sent to $W_{I_0\setminus I_2,\emptyset}$ under the isomorphism $\mathbf{E}_{I_0,I_2}\cong \mathbf{E}_{I_0\setminus I_2,\emptyset}$. □

For each $I=\{i_1<i_2<\cdots<i_\ell\}\subseteq \Delta_n$, we set

$$x_{\alpha_I}^{\infty}\overset{\mathrm{def}}{=}x_{(i_1,i_1+1)}^{\infty}\cup x_{(i_2,i_2+1)}^{\infty}\cup\cdots\cup x_{(i_\ell,i_\ell+1)}^{\infty}\in \mathbf{E}_{I,\emptyset}^{\infty}.$$

Theorem 5.9. *(i) Let $W\subseteq \widehat{\mathbf{E}}_n$ be a Breuil-Schraen $\mathscr{L}$-invariant. For each $\alpha\in\Phi^+$ and each $\iota\in S$, there exists a unique $\mathscr{L}_{\alpha,\iota}\in E$ such that $x_{\alpha,\iota}-\mathscr{L}_{\alpha,\iota}x_\alpha^{\infty}\in W_{I_\alpha,\emptyset}$.*

(ii) The map

$$\mathcal{BS}\cong U_{n,E}^{+}:\ W\mapsto (\mathscr{L}_{\alpha,\iota})_{\alpha\in\Phi^+,\iota\in S}$$

is an isomorphism, where $U_{n,E}^{+}$ is the unipotent radical of the Borel subgroup $B_{n,E}^{+}\subseteq G_{n,E}$.

Proof. Let $W\subseteq \widehat{\mathbf{E}}_n$ be a Breuil-Schraen $\mathscr{L}$-invariant, $\alpha\in\Phi^+$ be a positive root and $\iota\in S$ an embedding. If there exists $\mathscr{L}_{\alpha,\iota}\neq \mathscr{L}'_{\alpha,\iota}$ such that $x_{\alpha,\iota}-\mathscr{L}_{\alpha,\iota}x_\alpha^{\infty}, x_{\alpha,\iota}-\mathscr{L}'_{\alpha,\iota}x_\alpha^{\infty}\in W_{I,\emptyset}$, then we have $x_\alpha^{\infty}\in W_{I,\emptyset}$ and thus $\mathbf{E}_{I,\emptyset}^{\infty}\subseteq W_{I,\emptyset}$. This forces $\widehat{\mathbf{E}}_{\emptyset,\emptyset}=\mathbf{E}_{\Delta_n,I}^{\infty}\otimes \mathbf{E}_{I,\emptyset}^{\infty}\subseteq \iota_{I,\emptyset}(W_{I,\emptyset})\subseteq W$ and contradict condition (i) of Definition 5.6. Hence $\mathscr{L}_{\alpha,\iota}\in E$, if exists, is unique. Now we prove the existence by induction on the natural partial order on Φ^+. If $\#I_\alpha=1$, then $W_{I_\alpha,\emptyset}$ is a hyperplane in $\mathbf{E}_{I_\alpha,\emptyset}$ not containing Ex_α^{∞} (as $\widehat{\mathbf{E}}_{\emptyset,\emptyset}\not\subseteq W$), and thus $W_{I_\alpha,\emptyset}\cap(Ex_\alpha^{\infty}\oplus Ex_{\alpha,\iota})$ is a hyperplane in $Ex_\alpha^{\infty}\oplus Ex_{\alpha,\iota}$ not containing Ex_α^{∞}, which implies the existence of a unique $\mathscr{L}_{\alpha,\iota}\in E$ such that $x_{\alpha,\iota}-\mathscr{L}_{\alpha,\iota}x_\alpha^{\infty}\in W_{I_\alpha,\emptyset}$. Now assume that $\#I_\alpha\geq 2$ and $\mathscr{L}_{\alpha',\iota}\in E$ exists for each $\alpha'<\alpha$ and each $\iota\in S$. Note that $\mathbf{E}_{I_\alpha,\emptyset}^{<}\not\subseteq W_{I_\alpha,\emptyset}$ as we clearly have $W_{I_\alpha,\emptyset}+Ex_\alpha^{\infty}=\mathbf{E}_{I_\alpha,\emptyset}$. Recall from (the proof of) Lemma 5.4 that $X_{I_\alpha,\emptyset}\setminus\{x_\alpha\}$ forms a basis of $\mathbf{E}_{I_\alpha,\emptyset}^{<}$. For each partition $\alpha=\alpha_1+\cdots+\alpha_t$ with $t\geq 2$ and $x_{\alpha_1}\cup x_{\alpha_2}\cup\cdots\cup x_{\alpha_t}\in X_{I_\alpha,\emptyset}\setminus\{x_\alpha\}$ (with $x_{\alpha_{t'}}\in \overline{X}_{\alpha_{t'}}$ for each $1\leq t'\leq t$), we define a new element

$$y_{\alpha_1}\cup y_{\alpha_2}\cup\cdots\cup y_{\alpha_t}\in \mathbf{E}_{I_\alpha,\emptyset}^{<}$$

by taking $y_{\alpha_{t'}}\overset{\mathrm{def}}{=}x_{\alpha_{t'}}-\mathscr{L}_{\alpha_{t'},\iota}x_{\alpha_{t'}}^{\infty}$ if $x_{\alpha_{t'}}=x_{\alpha_{t'},\iota}$ and $y_{\alpha_{t'}}\overset{\mathrm{def}}{=}x_{\alpha_{t'}}^{\infty}$ if $x_{\alpha_{t'}}=x_{\alpha_{t'}}^{\infty}$. Hence, we obtain a new set of vectors $Y_{I_\alpha,\emptyset}^{<}$ which is clearly a basis of $\mathbf{E}_{I_\alpha,\emptyset}^{<}$ as it differs from $X_{I_\alpha,\emptyset}\setminus\{x_\alpha\}$ by a triangular matrix. As $x_{\alpha',\iota}-\mathscr{L}_{\alpha',\iota}x_{\alpha'}^{\infty}\in W_{I_{\alpha'},\emptyset}$ for each $\alpha'<\alpha$ and $\iota\in S$, we deduce that $Y_{I_\alpha,\emptyset}^{<}\setminus\{x_\alpha^{\infty}\}\subseteq W_{I_\alpha,\emptyset}\cap \mathbf{E}_{I_\alpha,\emptyset}^{<}$, which implies that $Y_{I_\alpha,\emptyset}^{<}\setminus\{x_\alpha^{\infty}\}$ is a basis of $W_{I_\alpha,\emptyset}\cap\mathbf{E}_{I_\alpha,\emptyset}^{<}$ as

$$\#Y_{I_\alpha,\emptyset}^{<}\setminus\{x_\alpha^{\infty}\}=\#X_{I_\alpha,\emptyset}\setminus\{x_\alpha\}-1=\dim_E\mathbf{E}_{I_\alpha,\emptyset}^{<}-1=\dim_E W_{I_\alpha,\emptyset}\cap\mathbf{E}_{I_\alpha,\emptyset}^{<}.$$

As $\mathbf{E}_{I_\alpha,\emptyset}^{<}\not\subseteq W_{I_\alpha,\emptyset}$, the inclusion $W_{I_\alpha,\emptyset}\subseteq \mathbf{E}_{I_\alpha,\emptyset}$ induces an isomorphism

$$W_{I_\alpha,\emptyset}/W_{I_\alpha,\emptyset}\cap\mathbf{E}_{I_\alpha,\emptyset}^{<}\xrightarrow{\sim}\mathbf{E}_{I_\alpha,\emptyset}/\mathbf{E}_{I_\alpha,\emptyset}^{<}.$$

Consequently, for each $\iota\in S$, $W_{I_\alpha,\emptyset}$ contains a vector of the form $x_{\alpha,\iota}-x'$ for x' a linear combination of vectors in $X_{I_\alpha,\emptyset}\setminus\{x_\alpha\}$, or equivalently a linear combination of vectors in $Y_{I_\alpha,\emptyset}^{<}$. However, as $Y_{I_\alpha,\emptyset}^{<}\setminus\{x_\alpha^{\infty}\}\subseteq W_{I_\alpha,\emptyset}$, we may choose x' to have the form $\mathscr{L}_{\alpha,\iota}x_\alpha^{\infty}$ for some $\mathscr{L}_{\alpha,\iota}\in E$. Consequently, $\mathscr{L}_{\alpha,\iota}$ exists, and the proof of item (i) is finished by an induction.

Conversely, given a tuple $(\mathscr{L}_{\alpha,\iota})_{\alpha\in\Phi^+,\iota\in S}\in U_{n,E}^{+}(E)$, we can define $\overline{Y}_{I_\alpha,\emptyset}\overset{\mathrm{def}}{=}\{x_{\alpha,\iota}-\mathscr{L}_{\alpha,\iota}x_\alpha^{\infty}\mid \iota\in S\}$ for each $\alpha\in\Phi^+$. Then we consider the set $Z_{I_\alpha,\emptyset}\overset{\mathrm{def}}{=}\bigsqcup_{\alpha'\leq\alpha}\overline{Y}_{I_\alpha,\emptyset}$ and define $W_{I_\alpha,\emptyset}$ as the span of $Z_{I_\alpha,\emptyset}$ for each $\alpha \in \Phi^+$. It suffices to check the second condition in Lemma 5.8 to conclude that $W_{\Delta_n,\emptyset}$ is a Breuil-Schraen $\mathscr{L}$-invariant. This is clear as for each pair of subintervals $\emptyset \neq I_2 \subsetneq I_0 \subseteq \Delta_n$ with $I_0 \setminus I_2$ being an interval, $\mathbf{E}_{I_0,I_2} \otimes W_{I_2,\emptyset}$ admits a basis of the form

$$(Z_{I_0\setminus I_2,\emptyset} \sqcup \{x^\infty_{\alpha_{I_0\setminus I_2}}\}) \otimes Z_{I_2,\emptyset} \subseteq Z_{I_0,\emptyset}.$$

The proof is thus finished. □

Remark 5.10. In the proof of Theorem 5.9, we have shown that $Y^<_{I_\alpha,\emptyset}\setminus\{x^\infty_\alpha\}$ is a basis of $W_{I_\alpha,\emptyset}\cap\mathbf{E}^<_{I_\alpha,\emptyset}$ for each $\alpha \in \Phi^+$ with $\#I_\alpha \geq 2$. This actually implies the following equality

$$\mathbf{E}^<_{I_\alpha,\emptyset} \cap W_{I_\alpha,\emptyset} = \sum_{\alpha'<\alpha} \mathbf{E}_{I_\alpha,I_{\alpha'}} \otimes W_{I_{\alpha'},\emptyset}. \tag{5.2}$$

The RHS is clearly inside LHS by condition (ii). To see the LHS is inside RHS, it sufficient to check an arbitrary element

$$y_{\alpha_1} \cup y_{\alpha_2} \cup \cdots \cup y_{\alpha_t} \in Y^<_{I_\alpha,\emptyset} \setminus \{x^\infty_\alpha\}$$

for some partition $\alpha = \alpha_1 + \cdots + \alpha_t$ with $t \geq 2$. Then there clearly exists some $1 \leq t \leq t'$ such that $y_{\alpha_{t'}} \neq x^\infty_{\alpha_{t'}}$, which implies that $y_{\alpha_{t'}} \in W_{I_{\alpha_{t'}},\emptyset}$ and thus

$$y_{\alpha_1} \cup y_{\alpha_2} \cup \cdots \cup y_{\alpha_t} \in \mathbf{E}_{I_\alpha,I_{\alpha_{t'}}} \otimes W_{I_{\alpha_{t'}},\emptyset}.$$

Now we are ready to formulate our first main conjecture on the existence of certain family of locally analytic representations parameterized by Breuil-Schraen $\mathscr{L}$-invariants. Note that $\mathcal{BS}$ is an affine scheme isomorphic to $U^+_{n,E}$, and we write $\mathcal{O}(\mathcal{BS})$ for its ring of global sections. Each closed point x of $\mathcal{BS}$ corresponds to a maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}(\mathcal{BS})$ with residual field $E_x \overset{\text{def}}{=} \mathcal{O}(\mathcal{BS})/\mathfrak{m}_x$.

Conjecture 5.11. *We fix a weight $\lambda \in X(T_{n,E})$ which is dominant with respect to $B^+_{n,E}$. There exists a $\mathcal{O}(\mathcal{BS}) \otimes_E D(G_n)$-module $\mathcal{M}(\lambda)$, such that for each closed point x of $\mathcal{BS}$, we have*

$$\mathcal{M}(\lambda)/\mathfrak{m}\mathcal{M}(\lambda) \cong \mathcal{W}_x(\lambda)'$$

for some admissible locally analytic representation $\mathcal{W}_x(\lambda)$ of G_n satisfying

- *(i) $\mathcal{W}_x(\lambda)$ is of finite length and each of its Jordan–Hölder factor is Orlik-Strauch;*
- *(ii) both the socle and the maximal locally algebraic subrepresentation of $\mathcal{W}_x(\lambda)$ are isomorphic to $\mathrm{St}^{\mathrm{alg}}_n(\lambda)$, and $\mathrm{St}^{\mathrm{alg}}_n(\lambda)$ has multiplicity one inside $\mathcal{W}_x(\lambda)$;*
- *(iii) $\dim_E \mathrm{Hom}_{G_n,\lambda}(\mathrm{St}^{\mathrm{an}}_n(\lambda), \mathcal{W}_x(\lambda)) = 1$, and any embedding $\mathrm{St}^{\mathrm{an}}_n(\lambda) \hookrightarrow \mathcal{W}_x(\lambda)$ induces a surjection*

$$\widehat{\mathbf{E}}_n = \mathrm{Ext}^{n-1}_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), \mathrm{St}^{\mathrm{an}}_n(\lambda)) \twoheadrightarrow \mathrm{Ext}^{n-1}_{G_n,\lambda}(F_{n,\Delta_n}(\lambda), \mathcal{W}_x(\lambda))$$

with kernel W_x, where $W_x \subseteq \widehat{\mathbf{E}}_n$ is the Breuil-Schraen $\mathscr{L}$-invariant attached to x.

Remark 5.12. Conjecture 5.11 is known for $n = 2$ with $K = \mathbb{Q}_p$ by Breuil in [Bre04] and [Bre10]), by Schraen and Ding for $n = 2$ with general K in [Schr10] and [Ding16]), and for $n = 3$ with $K = \mathbb{Q}_p$ by [Schr11], [Bre19], [BD20] and [Qian21]. We refer further details to Remark 1.4.

Remark 5.13. As in [Bre10], [Ding16], [Bre19], [BD20] and [Qian21], the representation $\mathcal{W}_x(\lambda)$ is expected to satisfy certain p-adic local-global compatibility. Let F be a number field, $v|p$ a finite place of p, $G_{/F}$ a reductive group satisfying $G(F_v) \cong \mathrm{GL}_n(F_v)$ and $U^v \subseteq G(\mathbf{A}^{\infty,v})$ a compact open subgroup. We define $\widehat{S}(U^v, \mathcal{O})$ to be the space of $\mathcal{O}$-valued p-adic continuous functions on the profinite set $(G(F)\backslash G(\mathbf{A}^\infty))/U^v$ and then define $\widehat{S}(U^v, E) \overset{\text{def}}{=} S(U^v, \mathcal{O}) \otimes_{\mathcal{O}} E$. The space $\widehat{S}(U^v, \mathcal{O})$ admits commuting action of $G(F_v) \cong \mathrm{GL}_n(F_v)$ and a Hecke algebra $\mathbb{T}(U^v)_{\mathcal{O}}$. Let $r : \mathrm{Gal}(\overline{F}/F) \to G^\vee(E)$

be a Galois representation with certain unramified conditions so that it determines a maximal ideal $\mathfrak{m}_r \subseteq \mathbb{T}(U^v)\otimes_{\mathcal{O}} E$. We consider the $\mathfrak{m}_r$-isotypic space $\widehat{S}(U^v,E)[\mathfrak{m}_r]$ which is an admissible unitary Banach representation of $\mathrm{GL}_n(F_v)$, whose set of locally analytic vectors $\widehat{S}(U^v,E)[\mathfrak{m}_r]^{\mathrm{an}}$ is an admissible locally analytic representation of $\mathrm{GL}_n(F_v)$. Suppose that

$$\mathrm{Hom}_{\mathrm{GL}_n(F_v)}\left(\mathrm{St}_n^{\mathrm{alg}}(\lambda),\widehat{S}(U^v,E)[\mathfrak{m}_r]^{\mathrm{an}}\right)\neq 0$$

for some dominant weight $\lambda\in X(T_{n,E})$, which under favorable conditions on G and r might imply that $\rho \overset{\mathrm{def}}{=} r|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ is semi-stable with $N^{n-1}\neq 0$. Then we would expect the existence of a $x\in\mathcal{BS}(E)$ uniquely determined by ρ such that any embedding $\mathrm{St}_n^{\mathrm{alg}}(\lambda)\hookrightarrow\mathcal{W}_x(\lambda)$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_n(F_v)}\left(\mathcal{W}_x(\lambda),\widehat{S}(U^v,E)[\mathfrak{m}_r]^{\mathrm{an}}\right)\cong\mathrm{Hom}_{\mathrm{GL}_n(F_v)}\left(\mathrm{St}_n^{\mathrm{alg}}(\lambda),\widehat{S}(U^v,E)[\mathfrak{m}_r]^{\mathrm{an}}\right).$$

5.2. **Breuil-Schraen $\mathscr{L}$-invariant and Galois representations.** In this section, we conjecture an isomorphism (see Conjecture 5.18) between the moduli of Breuil-Schraen $\mathscr{L}$-invariants and certain moduli of Galois representations *of Steinberg type* (see Definition 5.16) via a universal Galois representation. For simplicity of presentation, we only treat the ordinary case, namely $\lambda=0$ and $V_{n,I}^{\mathrm{an}}\overset{\mathrm{def}}{=}V_{n,I}^{\mathrm{an}}(0)$ is the locally analytic vector of a continuous generalized Steinberg $V_{n,I}^{\mathrm{cont}}$ defined in a way similar to (1.15). This saves us from considering (φ,Γ)-modules over the Robba ring. We write $G_K\overset{\mathrm{def}}{=}\mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K and $\varepsilon: G_K\hookrightarrow G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ for the cyclotomic character.

We recall the following standard lemma.

Lemma 5.14. *Let ℓ_1,ℓ_2 be two integers. Then we have*
- *(i)* $\mathrm{Hom}_{G_K}(\varepsilon^{\ell_1},\varepsilon^{\ell_2})=0$ *if $\ell_1\neq\ell_2$ and is one dimensional otherwise;*
- *(ii)* $\mathrm{Ext}^1_{G_K}(\varepsilon^{\ell_1},\varepsilon^{\ell_2})$ *has dimension $[K:\mathbb{Q}_p]+1$ if $\ell_1\in\{\ell_2,\ell_2-1\}$ and has dimension $[K:\mathbb{Q}_p]$ otherwise;*
- *(iii)* $\mathrm{Ext}^2_{G_K}(\varepsilon^{\ell_1},\varepsilon^{\ell_2})=0$ *if $\ell_1\neq\ell_2-1$ and is one dimensional otherwise.*

Given a filtered E-vector space V, we write $V^\vee\overset{\mathrm{def}}{=}\mathrm{Hom}_E(V,E)$ for its algebraic dual with the induced filtration.

Lemma 5.15. *There exists a unique (up to isomorphism) indecomposable continuous E-representation $\mathbf{V}_n$ of G_K that fits into the following short exact sequence*

$$(\varepsilon^{n-1})^{\oplus\dim_E\widehat{\mathbf{E}}_n}\hookrightarrow\mathbf{V}_n\twoheadrightarrow\mathbf{V}_{n-1}$$

for each $n\geq 2$. Here we understand $\mathbf{V}_1=1_{G_K}$ to be the trivial representation of G_K.

Proof. It suffices to prove that

$$\dim_E\mathrm{Ext}^1_{G_K}(\mathbf{V}_{n-1},\varepsilon^{n-1})=\dim_E\widehat{\mathbf{E}}_n \tag{5.3}$$

by induction on $n\geq 2$. Our inductive assumption (namely the existence of $\mathbf{V}_1,\dots\mathbf{V}_{n-1}$) gives a increasing filtration $0=\mathrm{Fil}_0(\mathbf{V}_{n-1})\subsetneq\mathrm{Fil}_1(\mathbf{V}_{n-1})\subsetneq\cdots\subsetneq\mathrm{Fil}_{n-1}(\mathbf{V}_{n-1})=\mathbf{V}_{n-1}$ such that $\mathbf{V}_{n-1}/\mathrm{Fil}_{\ell-1}(\mathbf{V}_{n-1})\cong\mathbf{V}_{n-\ell}$ and

$$\mathrm{Fil}_\ell(\mathbf{V}_{n-1})/\mathrm{Fil}_{\ell-1}(\mathbf{V}_{n-1})\cong(\varepsilon^{n-\ell-1})^{\oplus\dim_E\widehat{\mathbf{E}}_{n-\ell}} \tag{5.4}$$

for each $1\leq\ell\leq n-1$. Item (iii) of Lemma 5.14 together with a simple dévissage shows that

$$\mathrm{Ext}^2_{G_K}(\mathbf{V}_{n-1}/\mathrm{Fil}_\ell(\mathbf{V}_{n-1}),\varepsilon^{n-1})=0 \tag{5.5}$$

for each $1 \leq \ell \leq n-1$. Item (ii) of *loc.it.* implies that

$$\dim_E \mathrm{Ext}^1_{G_K}(\mathrm{Fil}_\ell(\mathbf{V}_{n-1})/\mathrm{Fil}_{\ell-1}(\mathbf{V}_{n-1}), \varepsilon^{n-1}) = [K:\mathbb{Q}_p] \dim_E \widehat{\mathbf{E}}_{n-\ell}$$

for each $2 \leq \ell \leq n-1$ and

$$\dim_E \mathrm{Ext}^1_{G_K}(\mathrm{Fil}_1(\mathbf{V}_{n-1}), \varepsilon^{n-1}) = (1+[K:\mathbb{Q}_p]) \dim_E \widehat{\mathbf{E}}_{n-1},$$

which together with item (i) of *loc.it.* and (5.5) inductively shows that

$$\dim_E \mathrm{Ext}^1_{G_K}(\mathbf{V}_{n-1}, \varepsilon^{n-1}) = (1+[K:\mathbb{Q}_p]) \dim_E \widehat{\mathbf{E}}_{n-1} + [K:\mathbb{Q}_p] \sum_{\ell=2}^{n-1} \dim_E \widehat{\mathbf{E}}_{n-\ell}. \tag{5.6}$$

Now we recall the partition

$$X_{I_\alpha,\emptyset} = \overline{X}_\alpha \sqcup \bigsqcup_{\ell=i+1}^{j-1} X^\ell_{I_\alpha,\emptyset} \tag{5.7}$$

from the proof of Lemma 5.4 and take $\alpha = \alpha_{\Delta_n} = (1,n)$ (namely $i=1$ and $j=n$). The definition of $X^\ell_{\Delta_n,\emptyset}$ forces $\#X^\ell_{\Delta_n,\emptyset} = \#\overline{X}_{(i,\ell)} \#X_{I_{(\ell,j)},\emptyset}$. As $\#\overline{X}_\beta = 1+[K:\mathbb{Q}_p]$ if β is simple and $\#\overline{X}_\beta = [K:\mathbb{Q}_p]$ otherwise, we deduce from (5.7) (and Lemma 5.4) that

$$\begin{aligned}\dim_E \widehat{\mathbf{E}}_n = \#X_{\Delta_n,\emptyset} = \#\overline{X}_{(1,n)} + \sum_{\ell=2}^{n-1} \#\overline{X}_{(1,\ell)} \#X_{I_{(\ell,n)},\emptyset} \\ = (1+[K:\mathbb{Q}_p]) \dim_E \widehat{\mathbf{E}}_{n-1} + [K:\mathbb{Q}_p] \sum_{\ell=2}^{n-1} \dim_E \widehat{\mathbf{E}}_{n-\ell},\end{aligned}$$

which together with (5.6) clearly implies (5.3). □

Definition 5.16. For each $n \geq 1$, we call $\mathbf{V}_n$ the *n-th universal Steinberg representation of G_K*. A continuous $\rho: G_K \to \mathrm{GL}_n(E)$ is called *of Steinberg type* if it does not have crystalline subquotient of dimension ≥ 2 and it admits a increasing filtration $0 = \mathrm{Fil}_0(\rho) \subsetneq \mathrm{Fil}_1(\rho) \subsetneq \cdots \subsetneq \mathrm{Fil}_n(\rho) = \rho$ such that $\mathrm{Fil}_\ell(\rho)/\mathrm{Fil}_{\ell-1}(\rho) \cong \varepsilon^{n-\ell}$ for each $1 \leq \ell \leq n$.

The following result justifies our terminology in Definition 5.16.

Proposition 5.17. *(i) We have* $\dim_E \mathrm{Hom}_{G_K}(\mathbf{V}_n, \rho) = 1$ *for each* $\rho: G_K \to \mathrm{GL}_n(E)$ *which is of Steinberg type.*

(ii) If $\rho: G_K \to \mathrm{GL}_n(E)$ *does not have crystalline subquotient of dimension* ≥ 2 *and satisfies* $\mathrm{Hom}_{G_K}(\mathbf{V}_n, \rho) \neq 0$*, then* ρ *is of Steinberg type.*

Proof. We first treat item (i). Let $\rho: G_K \to \mathrm{GL}_n(E)$ be of Steinberg type, then it is maximally non-split and there exists a unique $(n-1)$-dimensional quotient ρ' of ρ which is of Steinberg type. By induction on dimension we may assume that

$$\dim_E \mathrm{Hom}_{G_K}(\mathbf{V}_{n-1}, \rho') = 1.$$

Note that $\mathrm{Ext}^1_{G_K}(\rho', \varepsilon^{n-1}) \to \mathrm{Ext}^1_{G_K}(\mathbf{V}_{n-1}, \varepsilon^{n-1})$ is an embedding (which is unique up to a scalar). Any quotient of $\mathbf{V}_n$ isomorphic to ρ necessarily determines a E-line in $\mathrm{Ext}^1_{G_K}(\mathbf{V}_{n-1}, \varepsilon^{n-1})$ which must land in $\mathrm{Ext}^1_{G_K}(\rho', \varepsilon^{n-1})$. Such a E-line clearly exists and is unique, which implies that $\dim_E \mathrm{Hom}_{G_K}(\mathbf{V}_n, \rho) = 1$.

For item (ii), it suffices to find the filtration as in Definition 5.16. The standard increasing filtration on $\mathbf{V}_n$ induces a n-step filtration on ρ. Our ρ is clear maximally non-split by assumption, which forces $\mathrm{Fil}_\ell(\rho)/\mathrm{Fil}_{\ell-1}(\rho) \cong \varepsilon^{n-\ell}$ for each $1 \leq \ell \leq n$. □

We set $\mathcal{E}_n \overset{\mathrm{def}}{=} \mathrm{Ext}^1_{G_K}(\mathbf{V}_{n-1}, \varepsilon^{n-1})^\vee$ for each $n \geq 2$. Then Lemma 5.15 (together with its proof) can be summarized as the following

- The Galois representation $\mathbf{V}_n$ is defined inductively (for each $n \geq 2$) by the universal extension
$$\mathcal{E}_n \otimes \varepsilon^{n-1} \hookrightarrow \mathbf{V}_n \twoheadrightarrow \mathbf{V}_{n-1}$$
where G_K acts trivially on $\mathcal{E}_n$.
- The space $\mathcal{E}_n$ admits a canonical filtration
$$0 = \mathrm{Fil}_0(\mathcal{E}_n) \subsetneq \mathrm{Fil}_1(\mathcal{E}_n) \subsetneq \cdots \subsetneq \mathrm{Fil}_{n-1}(\mathcal{E}_n) = \mathcal{E}_n$$
with $\mathrm{Fil}_\ell(\mathcal{E}_n) \overset{\mathrm{def}}{=} \mathrm{Ext}^1_{G_K}(\mathrm{Fil}_\ell(\mathbf{V}_{n-1}), \varepsilon^{n-1})^\vee$ for each $1 \leq \ell \leq n-1$. Moreover, we have a canonical isomorphism
$$\mathrm{Fil}_\ell(\mathcal{E}_n)/\mathrm{Fil}_{\ell-1}(\mathcal{E}_n) \cong \mathrm{Ext}^1_{G_K}(\varepsilon^{n-\ell-1}, \varepsilon^{n-1})^\vee \otimes \mathcal{E}_{n-\ell} \cong \mathrm{Ext}^1_{G_K}(1, \varepsilon^{\ell})^\vee \otimes \mathcal{E}_{n-\ell}$$
for each $1 \leq \ell \leq n-1$.

Recall that $\widehat{\mathbf{E}}_n$ satisfies conditions that are parallel to those of $\mathcal{E}_n$ above. More precisely, the space $\widehat{\mathbf{E}}_n$ admits a canonical filtration
$$0 = \mathrm{Fil}_0(\widehat{\mathbf{E}}_n) \subsetneq \mathrm{Fil}_1(\widehat{\mathbf{E}}_n) \subsetneq \cdots \subsetneq \mathrm{Fil}_{n-1}(\widehat{\mathbf{E}}_n) = \widehat{\mathbf{E}}_n$$
as defined in the proof of Lemma 5.4. Moreover, we have a canonical isomorphism
$$\mathrm{Fil}_\ell(\widehat{\mathbf{E}}_n)/\mathrm{Fil}_{\ell-1}(\widehat{\mathbf{E}}_n) \cong P^{2\ell-1}(\mathfrak{g}_{\ell,E}) \otimes \widehat{\mathbf{E}}_{n-\ell}$$
for each $2 \leq \ell \leq n-1$ (with $P^{2\ell-1}(\mathfrak{g}_{\ell,E})$ as in Theorem 2.3), and a canonical isomorphism $\mathrm{Fil}_1(\widehat{\mathbf{E}}_n) \cong \mathrm{Hom}(K^\times, E) \otimes \widehat{\mathbf{E}}_{n-1}$. Therefore it seems plausible that there should be a natural isomorphism $\widehat{\mathbf{E}}_n \cong \mathcal{E}_n$ of filtered E-vector spaces for each $n \geq 1$, and moreover such isomorphism should be of geometric nature.

Following [Bre04] and [Schr11], it is natural to expect that such isomorphisms $\widehat{\mathbf{E}}_n \cong \mathcal{E}_n$ might be realized via the so-called Drinfeld upper half spaces $\mathcal{X}$. Recall that $\mathcal{X}$ is a rigid K-analytic space satisfying $\mathcal{X}(\mathbb{C}_p) = \mathbb{P}^{n-1}(\mathbb{C}_p) \setminus \bigcup_{H \in \mathscr{H}} H(\mathbb{C}_p)$ where $\mathscr{H}$ is the set of hyperplanes of $\mathbb{P}^{n-1}_K$ defined over K. The $\mathrm{GL}_{n/K}$-action on $\mathbb{P}^{n-1}_K$ clearly induces a $G_n = \mathrm{GL}_n(K)$-action on $\mathcal{X}(\mathbb{C}_p)$. We write
$$R\Gamma_{\mathrm{dR}}(\mathcal{X}) \overset{\mathrm{def}}{=} [\mathcal{O}(\mathcal{X}) \to \Omega^1(\mathcal{X}) \to \cdots \to \Omega^{n-1}(\mathcal{X})]$$
for the de Rham complex of $\mathcal{X}$ (with coefficients E), which is an object in the derived category $\mathcal{M}(G_n)$ attached to the abelian category $\mathrm{Mod}_{D(G_n)}$ of (abstract) $D(G_n)$-modules. (As $\mathcal{X}$ is Stein, we only need to consider global sections of various Ω^i.) We write K_0 for the maximal unramified subfield of K. According to Hyodo-Kato isomorphism (see the ι_{HK} in Theorem 1.8 of [CDN20] and [GK05]), there exists a complex $R\Gamma_{\mathrm{HK}}(\mathcal{X})$ of K_0-vector spaces with suitable (φ, N)-action (on the complex) satisfying $N\varphi = p\varphi N$, as well as a canonical isomorphism $R\Gamma_{\mathrm{HK}}(\mathcal{X}) \otimes_{K_0} E \cong R\Gamma_{\mathrm{dR}}(\mathcal{X})$ in $\mathcal{M}(G_n)$. Consequently, $R\Gamma_{\mathrm{dR}}(\mathcal{X})$ is an object in $\mathcal{M}(G_n)$ equipped with a (φ, N)-action (that commutes with $D(G_n)$-action), which induces a (φ, N)-action on the functor $\mathrm{Hom}_{\mathcal{M}(G_n)}(-, R\Gamma_{\mathrm{dR}}(\mathcal{X}))$. We write $\mathbf{D}_{\mathrm{st}}$ for Fontaine's (covariant) functor (see [Fon94]) that sends a semi-stable G_K-representation to a filtered (φ, N)-module (with coefficients extended to E). Motivated by Proposition 6.21, Théorème 6.23 and Remarque 6.24 of [Schr11], we have the following conjecture.

Conjecture 5.18. *There exists an isomorphism of filtered (φ, N)-modules*

$$\mathrm{Hom}_{\mathcal{M}(G_n)}((\mathrm{St}_n^{\mathrm{an}})'[1-n], R\Gamma_{\mathrm{dR}}(\mathcal{X})) \cong \mathbf{D}_{\mathrm{st}}(\varepsilon^{1-n} \otimes_E \mathbf{V}_n). \tag{5.8}$$

In the following, we use the term "motivic" to indicate that certain map is compatible with the conjectural p-adic Langlands correspondence.

Remark 5.19. (i) The existence of isomorphism (5.8) follows from [Bre04] if $n = 2$ and $K = \mathbb{Q}_p$, from Théorèm 01 of [Schr10] if $n = 2$ with general K, and from Proposition 6.21 of [Schr11] if $n = 3$ and $K = \mathbb{Q}_p$. In a forthcoming work, we plan to prove the *existence* of one such isomorphism, which depends on detailed computations of Ext groups in $\mathcal{M}(G_n)$.

(ii) As $\mathbf{V}_n$ has lots of automorphisms, such an isomorphism (5.8), if exists, has many choices. But we expect that there exists a unique such (5.8) which is "motivic" (for example, can be interpreted as certain p-adic regulator map).

(iii) For each $x \in \mathcal{BS}(E)$, we recall $\mathcal{W}_x(\lambda)$ from Conjecture 5.11 and write $\mathcal{W}_x \stackrel{\mathrm{def}}{=} \mathcal{W}_x(0)$ for short. Each choice of (5.8) would induce a bijection $x \mapsto \rho_x$ between $\mathcal{BS}(E)$ and the set of $\rho_x : G_K \to \mathrm{GL}_n(E)$ which are of Steinberg type, such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{D}_{\mathrm{st}}(\varepsilon^{1-n} \otimes_E \mathbf{V}_n) & \twoheadrightarrow & \mathbf{D}_{\mathrm{st}}(\varepsilon^{1-n} \otimes_E \rho_x) \\ \downarrow \cong & & \downarrow \\ \mathrm{Hom}_{\mathcal{M}(G_n)}((\mathrm{St}_n^{\mathrm{an}})'[1-n], R\Gamma_{\mathrm{dR}}(\mathcal{X})) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}(G_n)}(\mathcal{W}_x'[1-n], R\Gamma_{\mathrm{dR}}(\mathcal{X})). \end{array}$$

We expect that the "motivic" choice of (5.8) should induce a bijection $x \mapsto \rho_x$ which is compatible with p-adic local-global compatibility (see Remark 5.13).

(iv) We have $H^\ell_{\mathrm{dR}}(\mathcal{X}) \cong (V^\infty_{n,\{1,2,\cdots,n-1-\ell\}})'$ for each $0 \leq \ell \leq n-1$ by [SS91]. Using the same argument as in Section 6.1 of [Schr11] based on [Dat06] and [Or05], there exists a splitting

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}) \cong \bigoplus_{\ell=0}^{n-1} H^\ell_{\mathrm{dR}}(\mathcal{X})[-\ell]. \tag{5.9}$$

As the endormorphism algebra of $\bigoplus_{\ell=0}^{n-1} H^\ell_{\mathrm{dR}}(\mathcal{X})[-\ell]$ is easily shown to be isomorphic to the algebra of size n upper triangular nilpotent matrices (see Corollaire 6.2 of [Schr11]), the splitting (5.9) is far from being canonical. Nevertheless, (5.9) induces an isomorphism of E-vector spaces

$$\mathrm{Hom}_{\mathcal{M}(G_n)}((\mathrm{St}_n^{\mathrm{an}})'[1-n], R\Gamma_{\mathrm{dR}}(\mathcal{X})) \cong \bigoplus_{\ell=0}^{n-1} \mathrm{Hom}_{\mathcal{M}(G_n)}((\mathrm{St}_n^{\mathrm{an}})'[1-n], H^\ell_{\mathrm{dR}}(\mathcal{X})[-\ell]) \cong \bigoplus_{\ell=0}^{n-1} \widehat{\mathbf{E}}_{n-\ell},$$

which together with (5.8) induces an isomorphism $\widehat{\mathbf{E}}_{n-\ell} \cong \mathcal{E}_{n-\ell}$ for each $0 \leq \ell \leq n-1$ (where we identify $\mathcal{E}_{n-\ell}$ with the canonical $\varepsilon^{n-\ell-1}$-isotypic sub-quotient of $\mathbf{V}_n$ by their definition). We expect such an isomorphism $\widehat{\mathbf{E}}_{n-\ell} \cong \mathcal{E}_{n-\ell}$ to respect filtration on each side, regardless of the choice of the splitting (5.9). This suggests that there should be a "motivic" isomorphism

$$P^{2n-1}(\mathfrak{g}_{n,E}) \cong \mathrm{Ext}^1_{G_K}(1, \varepsilon^n)^\vee \tag{5.10}$$

for each $n \geq 2$. There already exists a natural candidate for (5.10), see [HK11] and [Sou81].

(v) Let $\Lambda(G_n)$ be the dual of p-adic continuous functions on G_n and we write $M(G_n)$ for the derived category attached to the abelian category of (abstract) modules over $\Lambda(G_n)$. Motivated by [CDN20], one may wish that there exists a "geometrically constructed" object $\mathbf{M} \in M(G_n)$, equipped with a commuting continuous action of G_K, such that there exists a unique "motivic" G_K-equivariant isomorphism

$$\mathrm{Hom}_{M(G_n)}((\mathrm{St}_n^{\mathrm{cont}})'[1-n], \mathbf{M}) \cong \varepsilon^{1-n} \otimes_E \mathbf{V}_n.$$

(vi) Given a central division algebra D over K with invariant $\frac{1}{n}$, Scholze ([Sch18]) has constructed a cohomological covariant δ-functor $\{\mathcal{S}^i, i \geq 0\}$ from the category of smooth $\mathcal{O}$-torsion $\mathcal{O}[G_n]$-modules to smooth $\mathcal{O}$-torsion $\mathcal{O}[D^\times]$-modules which carry a continuous and commuting action of G_K. More precisely, for each π, $\mathcal{S}^i(\pi)$ is defined as the cohomology group $H^i_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$, where $\mathcal{F}_\pi$ is a certain Weil-equivariant sheaf on the adic space $\mathbb{P}^{n-1}_{\mathbb{C}_p}$. His construction is expected to realize both p-adic local Langlands and Jacquet-Langlands correspondence. Moreover, Scholze has computed $\mathcal{S}^0(\pi)$ and showed that $\mathcal{S}^i(\pi) = 0$ for each $i > 2(n-1)$. Given an admissible unitary E-Banach representation Π of G_n, one is particularly interested in the following limit

$$\varprojlim_r \mathcal{S}^{n-1}(\Pi/p^r\Pi)$$

which is an admissible unitary E-Banach representation of $D^\times$, carrying a continuous and commutinng action of G_K. Concerning the relation between Scholze's functor and cohomology of Drinfeld space, our Conjecture 5.18 seems to suggest that we could have a G_K-equivariant isomorphism

$$\varprojlim_r \mathcal{S}^{n-1}(\mathrm{St}_n^{\mathrm{cont}}/p^r\mathrm{St}_n^{\mathrm{cont}})^{D^\times} \cong \varepsilon^{1-n} \otimes_E \mathbf{V}_n \tag{5.11}$$

where we take $D^\times$-invariant on the LHS. Again, there should be many isomorphisms of the form (5.11), but there should be a unique one which is "motivic". The mod p version of this isomorphism holds when $n = 2$ and $K = \mathbb{Q}_p$ according to Theorem 8.34 of [HW22].

(vii) The Galois representation $\mathbf{V}_n$ might seem to be a p-adic counterpart of the mixed Tate motives considered in [De89] and [DG05] (with thanks to Ma Luo, Liang Xiao and Daxin Xu for guiding me to those references). Note that Deligne–Goncharov mentioned in [DG05] a result of Beilinson (see Proposition 3.4 of *loc.it.*) for general connected Hausdorff topological spaces, which has a similar form compared with (5.8). In particular, we expect certain versions of p-adic polylogarithm functions to appear in an explicit description of the desired "motivic" isomorphism (5.8).

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