

Generalized Hukuhara Weak Subdifferential and its Application on Identifying Optimality Conditions for Nonsmooth Interval-valued Functions

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Abstract In this article, we introduce the idea of gH -weak subdifferential for interval-valued functions (IVFs) and show how to calculate gH -weak subgradients. It is observed that a nonempty gH -weak subdifferential set is closed and convex. In characterizing the class of functions for which the gH -weak subdifferential set is nonempty, it is identified that this class is the collection of gH -lower Lipschitz IVFs. In checking the validity of sum rule of gH -weak subdifferential for a pair of IVFs, a counterexample is obtained, which reflects that the sum rule does not hold. However, under a mild restriction on one of the IVFs, one-sided inclusion for the sum rule holds. Next, as applications, we employ gH -weak subdifferential to provide a few optimality conditions for nonsmooth IVFs. Further, a necessary optimality condition for interval optimization problems with difference of two nonsmooth IVFs as objective is established. Lastly, a necessary and sufficient condition via augmented normal cone and gH -weak subdifferential of IVFs for finding weak efficient point is presented.

Keywords Interval optimization · Nonsmooth interval-valued functions · gH -weak subgradient · gH -Fréchet lower subdifferential · Difference of two IVFs

1 Introduction

Interval arithmetic of Moore [26] is the foundation stone in interval analysis. Realistic applicability of Moore's method is relevant till today. We can currently discover several papers in the community of interval-valued optimization problems (IOPs) where Moore's interval analysis is applied extensively. To find optimality conditions for IOPs, ideas of derivatives for interval-valued function (IVF) have been proposed [5, 14, 25, 28, 33]. In [25], the concept of gH -differentiability for IVFs was introduced. Chalco-Cano [6] addressed the algebraic property of gH -differentiable interval-valued functions. Ghosh et al. [14] proved the existence of gH -directional derivative for convex IVFs and presented optimality conditions for IOPs.

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It is a familiar fact that in nonsmooth optimization, the classical gradient algorithm fails: even in finding the optimum point, as there is no derivative, the conventional optimality condition $\nabla f(x) = 0$ does not hold. More crucially, it is observed that optima of an almost everywhere differentiable function categorically arise at nondifferentiable points—for instance, take the minimization of $f(x) = |x|$. The notion of subdifferential, defined by Rockafellar [31], is a crucial factor in the body of optimization theory that perfectly replaces the role of the gradient to identify optima for convex functions. However, subdifferential is inadequate in developing optimality conditions for nonconvex optimization problems. Due to this insufficiency, the idea of subdifferential has been generalized. Most common of such generalizations is weak subdifferential [3]. Based on this notion, a strong duality theorem for nonconvex inequality constrained problem has been found by defining a weak conjugate function [37]. A substantial application of this notion in duality theory with the help of a weak subdifferentiable perturbation function is given in [34].

In the context of nonsmooth calculus for nondifferentiable convex IVFs, Ghosh et al. [12] has recently proposed the idea of gH -subgradient and gH -subdifferential. The same article [12] found that gH -directional derivative is the maximum of all the products of the direction and gH -subgradients. Afterward, Anshika et al. [1] characterized weak efficiency for nonconvex composite optimization problems with the subdifferential sets of convex interval-valued functions. In [1], by formulating supremum and infimum of an IVF, a Fermat-type, a Fritz-John-type, and a KKT-type weak efficiency condition for nonsmooth IOPs have been derived. Anshika et al. [2] introduced gH -subdifferential of interval-valued value function. Furthermore, Chauhan et al. [8] derived the notion of gH -Clarke derivative for IVFs and IOPs. Under the Clarke subdifferentiability assumption, Chen and Li [7] provided KKT conditions for efficient solutions. Additionally, Karaman et al. [22] presented two subdifferentials for interval-valued functions and some optimality criteria, which were obtained by using subdifferentials.

From the available literature on nonsmooth IOPs, it is found that the study of gH -weak subdifferential notion has not yet been addressed. However, the notion of gH -weak subdifferential might be effective to characterize and capture the efficient solutions of IOPs with nonconvex and nonsmooth IVFs. By using subgradient, one may face difficulties to solve problem which does not satisfy convexity assumption because subgradient refers to the slope of a supporting hyperplane to the graph of convex functions in convex analysis. Thus, in this study, we introduce the notion of weak subgradient, which does not need any kind of convexity.

In this article, we attempt to show various properties of weak-subdifferential and its use in nonsmooth nonconvex IOPs. As an application of the proposed gH -weak subdifferential, we propose a necessary and sufficient optimality condition for finding weak efficient points of difference of two IVFs.

The rest of article is presented as follows. Section 2 is devoted to the conventional properties of intervals, followed by calculus of IVFs. Section 3 introduces the notion of gH -weak subdifferential for IVFs and discusses their properties such as convexity, closedness and nonemptiness. Additionally, the role of gH -weak subdifferential to derive the necessary condition for weak efficiency for gH -weak subdifferentiable IVFs is shown in Section 3. In Section 4, we analyze the necessary condition for obtaining efficient solution of difference of two IVFs. Finally, we draw conclusion with future directions to extend the present study.

2 Preliminaries

In this section, required terminologies and notions on intervals including calculus of IVFs are given. Throughout the paper, we extensively use the following notations.

- \mathbb{R} is the set of real numbers
- \mathbb{R}_+ represents the set of nonnegative real numbers
- $I(\mathbb{R})$ is the collection of all compact intervals
- $\widehat{I}(\mathbb{R}) = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathbf{0} = [0, 0]$
- Elements of $I(\mathbb{R})$ are presented by bold capital letters: $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \dots$
- $I(\mathbb{R})^n = I(\mathbb{R}) \times I(\mathbb{R}) \times I(\mathbb{R}) \times \dots \times I(\mathbb{R})$ (n times)
- Interval vectors in $I(\mathbb{R})^n$ are denoted by $\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}, \widehat{\mathbf{Z}}, \dots$
- $B_\alpha(\bar{u})$ is the open ball with center at $\bar{u} \in \mathbb{R}^n$ and radius $\alpha \geq 0$
- $\mathcal{N}(\bar{x})$ is a neighbourhood of $\bar{x} \in \mathbb{R}^n$
- $\|\cdot\|_{I(\mathbb{R})}$ denotes the norm on $I(\mathbb{R})$

2.1 Arithmetic and dominance of intervals

Throughout the text, we represent an element \mathbf{X} of $I(\mathbb{R})$ by the corresponding small letter:

$$\mathbf{X} = [\underline{x}, \bar{x}], \text{ where } \underline{x} \text{ and } \bar{x} \text{ are in } \mathbb{R} \text{ with } \underline{x} \leq \bar{x}.$$

Recall that Moore's interval addition (\oplus), subtraction (\ominus), and multiplication (\odot) [26, 27] are given by

$$\begin{aligned} \mathbf{X} \oplus \mathbf{Y} &= [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \quad \mathbf{X} \ominus \mathbf{Y} = [\underline{x} - \bar{y}, \bar{x} - \underline{y}], \text{ and} \\ \mathbf{X} \odot \mathbf{Y} &= [\min\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}]. \end{aligned}$$

Definition 1 (*gH -difference of intervals* [32]). The gH -difference for a pair of intervals \mathbf{P} and \mathbf{Q} , denoted by $\mathbf{P} \ominus_{gH} \mathbf{Q}$, is the interval \mathbf{Y} such that

$$\mathbf{P} = \mathbf{Q} \oplus \mathbf{Y} \text{ or } \mathbf{Q} = \mathbf{P} \ominus \mathbf{Y}.$$

It is well-known that for $\mathbf{P} = [p, \bar{p}]$ and $\mathbf{Q} = [q, \bar{q}]$,

$$\mathbf{P} \ominus_{gH} \mathbf{Q} = [\min\{p - q, \bar{p} - \bar{q}\}, \max\{p - q, \bar{p} - \bar{q}\}] \text{ and } \mathbf{P} \ominus_{gH} \mathbf{P} = \mathbf{0}.$$

For two elements $\widehat{\mathbf{I}} = (\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n)$ and $\widehat{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \dots, \mathbf{J}_n)$ of $I(\mathbb{R})^n$, the algebraic operation $\widehat{\mathbf{I}} \star \widehat{\mathbf{J}}$ is defined by

$$\widehat{\mathbf{I}} \star \widehat{\mathbf{J}} = (\mathbf{I}_1 \star \mathbf{J}_1, \mathbf{I}_2 \star \mathbf{J}_2, \dots, \mathbf{I}_n \star \mathbf{J}_n),$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

Definition 2 (*Dominance of intervals* [36]). Let \mathbf{Z} and \mathbf{W} be in $I(\mathbb{R})$.

- (i) \mathbf{W} is called *dominated* by \mathbf{Z} if $\underline{z} \leq \underline{w}$ and $\bar{z} \leq \bar{w}$, and then we express it by $\mathbf{Z} \preceq \mathbf{W}$.
- (ii) \mathbf{W} is said to be *strictly dominated* by \mathbf{Z} if either ' $\underline{z} \leq \underline{w}$ and $\bar{z} < \bar{w}$ ' or ' $\underline{z} < \underline{w}$ and $\bar{z} \leq \bar{w}$ ', and then we express it by $\mathbf{Z} \prec \mathbf{W}$.
- (iii) If \mathbf{W} is not dominated by \mathbf{Z} , then we write $\mathbf{Z} \not\preceq \mathbf{W}$. If \mathbf{W} is not strictly dominated by \mathbf{Z} , then we write $\mathbf{Z} \not\prec \mathbf{W}$.
- (iv) If $\mathbf{W} \not\preceq \mathbf{Z}$ and $\mathbf{Z} \not\preceq \mathbf{W}$, then it is called that *none of \mathbf{W} and \mathbf{Z} dominates the other*, or *\mathbf{W} and \mathbf{Z} are not comparable*.

For any two elements $\widehat{\mathbf{I}} = (\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n)^\top$ and $\widehat{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \dots, \mathbf{J}_n)^\top$ in $I(\mathbb{R})^n$,

$$\widehat{\mathbf{I}} \preceq \widehat{\mathbf{J}} \iff \mathbf{I}_j \preceq \mathbf{J}_j \text{ for all } j = 1, 2, \dots, n.$$

2.2 Concavity and differential calculus of IVFs

Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let an IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ be presented by

$$\Phi(y) = [\underline{\phi}(y), \overline{\phi}(y)] \quad \forall y \in \mathcal{Y},$$

where $\underline{\phi}(y) \leq \overline{\phi}(y)$ for all $y \in \mathcal{Y}$ and $\underline{\phi}$ and $\overline{\phi}$ are called lower and upper real-valued functions on \mathcal{Y} .

Definition 3 (*Concave IVF*). If \mathcal{Y} is convex, then an IVF Φ is said to be a *concave* IVF on \mathcal{Y} if for any $y_1, y_2 \in \mathcal{Y}, \beta_1, \beta_2 \in [0, 1]$, and $\beta_1 + \beta_2 = 1$,

$$\beta_1 \odot \Phi(y_1) \oplus \beta_2 \odot \Phi(y_2) \preceq \Phi(\beta_1 y_1 + \beta_2 y_2).$$

Lemma 1 *If Φ is a concave IVF on a convex set $\mathcal{Y} \subseteq \mathbb{R}^n$, then $\underline{\phi}$ and $\overline{\phi}$ are concave on \mathcal{Y} and vice-versa.*

Proof The proof is similar with the proof of Proposition 6.1 in [36].

Example 1 Let \mathcal{Y} be the Euclidean space \mathbb{R}^n . Then, the IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ which is defined by

$$\Phi(y) = \widehat{\mathbf{M}}^\top \odot y \ominus_{gH} \|y\|, \quad \text{where } \widehat{\mathbf{M}} = (\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_n) \in I(\mathbb{R})^n,$$

and for all $y = (y_1, y_2, \dots, y_n) \in \mathcal{Y}$ is a concave IVF on \mathcal{Y} . The reason is as follows.

Without loss of generality, first p components of y are assumed to be non-negative and rest $n - p$ be negative. Then, letting $\mathbf{M}_i = [\underline{m}_i, \overline{m}_i]$ for all $i = 1, 2, \dots, n$,

$$\Phi(y) = \bigoplus_{i=1}^p [\underline{m}_i y_i, \overline{m}_i y_i] \oplus \bigoplus_{j=p+1}^n [\overline{m}_j y_j, \underline{m}_j y_j] \ominus_{gH} \|y\|.$$

It is evident that $\sum_{i=1}^p \underline{m}_i y_i + \sum_{j=p+1}^n \overline{m}_j y_j$ and $\sum_{i=1}^p \overline{m}_i y_i + \sum_{j=p+1}^n \underline{m}_j y_j$, being linear, are concave functions. Also, $-\|y\|$ is a concave function. Therefore, $\sum_{i=1}^p \underline{m}_i y_i + \sum_{j=p+1}^n \overline{m}_j y_j - \|y\|$ and $\sum_{i=1}^p \overline{m}_i y_i + \sum_{j=p+1}^n \underline{m}_j y_j - \|y\|$ are concave functions. Hence, by Lemma 1, Φ is a concave IVF.

Definition 4 (*gH-continuity* [13]). An IVF Φ is said to be *gH-continuous* at $u \in \mathcal{Y}$ if

$$\lim_{\|d\| \rightarrow 0} (\Phi(u+d) \ominus_{gH} \Phi(u)) = \mathbf{0}.$$

If at every $u \in \mathcal{Y}$, Φ is *gH-continuous*, then Φ is called *gH-continuous* on \mathcal{Y} .

Lemma 2 (See [15]). *For a gH-continuous IVF Φ its $\underline{\phi}$ and $\overline{\phi}$ are continuous and vice-versa.*

Definition 5 (*gH-derivative* [4]). Let $\mathcal{Y} \subseteq \mathbb{R}^n$. The *gH-derivative* of an IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ at $u \in \mathcal{Y}$ is the limit

$$\Phi'(u) := \lim_{d \rightarrow 0} \frac{1}{d} \odot \{\Phi(u+d) \ominus_{gH} \Phi(u)\}.$$

Definition 6 (*gH-Gâteaux derivative* [14]). Let an IVF Φ be defined on a nonempty open subset \mathcal{Y} of \mathbb{R}^n . Then, Φ is known to be *gH-Gâteaux differentiable* with *gH-Gâteaux derivative* $\Phi_{\mathcal{G}}(u)$ at $u \in \mathcal{Y}$ if the following limit

$$\Phi_{\mathcal{G}}(u)(h) := \lim_{\beta \rightarrow 0^+} \frac{1}{\beta} \odot (\Phi(u + \beta h) \ominus_{gH} \Phi(u))$$

is finite for all $h \in \mathbb{R}^n$ and $\Phi_{\mathcal{G}}(u)$ is a *gH-continuous* and linear IVF from \mathbb{R}^n to $I(\mathbb{R})$.

Definition 7 (*gH -Fréchet derivative [14]*). Let an IVF Φ be defined on a nonempty open subset \mathcal{Y} of \mathbb{R}^n . Then, Φ is said to be *gH -Fréchet differentiable* at $u \in \mathcal{Y}$ if there exists a gH -continuous and linear mapping $\mathbf{G} : \mathcal{Y} \rightarrow I(\mathbb{R})$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{1}{\|h\|} \odot (\|\Phi(u+h) \ominus_{gH} \Phi(u) \ominus_{gH} \mathbf{G}(h)\|_{I(\mathbb{R})}) = 0,$$

where \mathbf{G} will be referred to as $\Phi_{\mathcal{F}}(u)$.

Definition 8 (*Efficient point [14]*). Let $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \rightarrow I(\mathbb{R})$ be an IVF. A point $u \in \mathcal{Y}$ is said to be an *efficient point* of the IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ if $\Phi(y) \not\prec \Phi(u)$ for all $y \in \mathcal{Y}$.

Definition 9 (*Weak efficient point [1]*). Let $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \rightarrow I(\mathbb{R})$ be an IVF. A point $u \in \mathcal{Y}$ is said to be a *weak efficient point* of the IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ if $\Phi(u) \preceq \Phi(y)$ for all $y \in \mathcal{Y}$.

2.3 Few properties of the elements in $I(\mathbb{R})$

Let $\mathbf{Y} = [y, \bar{y}]$ and $\widehat{\mathbf{Y}} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n)$ be elements in $I(\mathbb{R})$ and $I(\mathbb{R})^n$, respectively. The following two functions $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \rightarrow \mathbb{R}_+$ and $\|\cdot\|_{I(\mathbb{R})^n} : I(\mathbb{R})^n \rightarrow \mathbb{R}_+$ are referred to as norm [26, 27] on $I(\mathbb{R})$ and $I(\mathbb{R})^n$, respectively:

$$\|\mathbf{Y}\|_{I(\mathbb{R})} = \max\{|y|, |\bar{y}|\}, \text{ and } \|\widehat{\mathbf{Y}}\|_{I(\mathbb{R})^n} = \sum_{j=1}^n \|\mathbf{Y}_j\|_{I(\mathbb{R})}.$$

Lemma 3 For any $\mathbf{W}, \mathbf{Y}, \mathbf{Z} \in I(\mathbb{R})$ and $\epsilon \geq 0$, we have

$$\epsilon \preceq (\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} \implies \mathbf{Z} \oplus \epsilon \preceq \mathbf{W} \ominus_{gH} \mathbf{Y}.$$

Proof See Appendix A.

Lemma 4 For any $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W} \in I(\mathbb{R})$, we have

$$(\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \subseteq (\mathbf{X} \ominus_{gH} \mathbf{Z}) \oplus (\mathbf{Y} \ominus_{gH} \mathbf{W}).$$

Proof See Appendix B.

Lemma 5 For any $\mathbf{W}, \mathbf{Y}, \mathbf{Z} \in I(\mathbb{R})$,

$$\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = ((\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}).$$

Proof See Appendix C.

Lemma 6 For all \mathbf{X}, \mathbf{Y} , and \mathbf{Z} of $I(\mathbb{R})$,

- (i) if $\mathbf{0} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, then $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$,
- (ii) if $\mathbf{Z} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, then $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$, for all $\mathbf{W} \in I(\mathbb{R})$,
- (iii) if $\mathbf{X} \ominus_{gH} \mathbf{Y} \preceq [L, L]$, then $[-L, -L] \preceq \mathbf{Y} \ominus_{gH} \mathbf{X}$, where $L \in \mathbb{R}$,
- (iv) if $[-\gamma, -\gamma] \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, then $\mathbf{Y} \ominus_{gH} [\gamma, \gamma] \preceq \mathbf{X}$, where $\gamma \in \mathbb{R}$, and
- (v) if $\mathbf{Z} \preceq \mathbf{X} \oplus \mathbf{Y}$, then $\mathbf{Z} \ominus_{gH} \mathbf{Y} \preceq \mathbf{X}$.

Proof See Appendix D.

Definition 10 (*Sequence in $I(\mathbb{R})^n$ [12]*). A function $\widehat{\Phi} : \mathbb{N} \rightarrow I(\mathbb{R})^n$ is called a sequence in $I(\mathbb{R})^n$, where \mathbb{N} is the set of natural numbers.

Definition 11 (*Closed set in $I(\mathbb{R})^n$ [1]*). A nonempty subset $\mathcal{U} \subseteq I(\mathbb{R})^n$ is known to be *closed* if for every convergent sequence $\{\widehat{\mathbf{M}}_k\}$ in \mathcal{U} converging to $\widehat{\mathbf{M}}$, $\widehat{\mathbf{M}}$ must belong to \mathcal{U} .

Definition 12 (*Closure of a set in $I(\mathbb{R})^n$*). Let $\mathcal{Y} \subseteq I(\mathbb{R})^n$. The intersection of all closed sets containing \mathcal{Y} is called the closure of \mathcal{Y} , abbreviated by $cl(\mathcal{Y})$.

Definition 13 (*Convergent sequence in $I(\mathbb{R})^n$ [12]*). Let $\{\widehat{\mathbf{M}}_k\}$ be a sequence in $I(\mathbb{R})^n$. If there exists $\widehat{\mathbf{M}} \in I(\mathbb{R})^n$ for which for any $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that

$$\|\widehat{\mathbf{M}}_k \ominus_{gH} \widehat{\mathbf{M}}\|_{I(\mathbb{R})^n} < \epsilon \text{ for all } k \geq p,$$

then $\{\widehat{\mathbf{M}}_k\}$ is said to be convergent and converges to $\widehat{\mathbf{M}}$.

Remark 1 It is to note that if a sequence $\{\widehat{\mathbf{M}}_k\} = (\mathbf{M}_{k1}, \mathbf{M}_{k2}, \dots, \mathbf{M}_{kn})^\top$ in $I(\mathbb{R})^n$ converges to $\widehat{\mathbf{M}} = (\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_n)^\top \in I(\mathbb{R})^n$, then by the definition of norm on $I(\mathbb{R})^n$, the sequence \mathbf{M}_{kj} in $I(\mathbb{R})$ converges to $\mathbf{M}_j \in I(\mathbb{R})$ for all $j = 1, 2, \dots, n$. Also, according to the definition of norm on $I(\mathbb{R})$, the sequences $\{\underline{m}_{kj}\}$ and $\{\overline{m}_{kj}\}$ in \mathbb{R} converge to $\{\underline{m}_j\}$ and $\{\overline{m}_j\}$, respectively, for all j .

Definition 14 (*Infimum and supremum of a subset of $I(\mathbb{R})$ [24]*). Let $\mathcal{U} \subseteq \overline{I(\mathbb{R})}$. We call an interval $\mathbf{X} \in I(\mathbb{R})$ a lower bound (respectively, an upper bound) of \mathcal{U} if $\mathbf{U} \in \mathcal{U}$ implies $\mathbf{X} \preceq \mathbf{U}$ (respectively, $\mathbf{U} \preceq \mathbf{X}$).

A lower bound \mathbf{X} of \mathcal{U} is called *infimum* of \mathcal{U} , denoted by $\inf \mathcal{U}$, if for any lower bound \mathbf{Z} of \mathcal{U} , $\mathbf{Z} \preceq \mathbf{X}$.

An upper bound \mathbf{X} of \mathcal{U} is called *supremum* of \mathcal{U} , denoted by $\sup \mathcal{U}$, if for any upper bound \mathbf{Z} of \mathcal{U} , $\mathbf{X} \preceq \mathbf{Z}$.

Remark 2 [24] Let $\mathcal{S} = \{[a_\mu, b_\mu] \in \overline{I(\mathbb{R})} : \mu \in \Lambda \text{ and } \Lambda \text{ being an index set}\}$. Then, by Definition 14, it follows that $\inf \mathcal{S} = \left[\inf_{\mu \in \Lambda} a_\mu, \inf_{\mu \in \Lambda} b_\mu \right]$ and $\sup \mathcal{S} = \left[\sup_{\mu \in \Lambda} a_\mu, \sup_{\mu \in \Lambda} b_\mu \right]$.

3 gH -weak subdifferential calculus for IVFs

In this section, we introduce the ideas of gH -weak subgradient and gH -weak subdifferential for IVFs. Some properties of gH -weak subdifferential and an inclusion for sum rule are provided. Its relation with gH -Fréchet lower subdifferential is also discussed.

Definition 15 (*gH -weak subdifferential*). Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$ and Φ be an IVF defined on \mathcal{Y} . A pair $(\widehat{\mathbf{G}}^w, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ is said to be a gH -weak subgradient of Φ at $u \in \mathcal{Y}$ if for every $y \in \mathcal{Y}$,

$$\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \quad (1)$$

The set of all gH -weak subgradients of Φ at $u \in \mathcal{Y}$, i.e.,

$$\partial^w \Phi(u) = \left\{ (\widehat{\mathbf{G}}^w, c) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \forall y \in \mathcal{Y} \right\}$$

is said to be gH -weak subdifferential of Φ at $u \in \mathcal{Y}$.

Example 2 Let an IVF $\Phi : [-1, 1] \rightarrow I(\mathbb{R})$ be defined by

$$\Phi(y) = [y^2, |y|], \text{ where } y \in [-1, 1].$$

Let us compute the gH -weak subdifferential of Φ at 0 and 1, i.e., $\partial^w \Phi(0)$ and $\partial^w \Phi(1)$, respectively. Note that

$$\begin{aligned} \partial^w \Phi(0) &= \{(\mathbf{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \mathbf{G}_1^w \odot y \ominus_{gH} c|y| \preceq [y^2, |y|] \ \forall y \in [-1, 1]\} \\ &= \left\{ \left([\underline{g}_1^w, \overline{g}_1^w], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g}_1^w, \overline{g}_1^w] \odot y \ominus_{gH} c|y| \preceq [y^2, |y|] \ \forall y \in [-1, 1] \right\}, \end{aligned}$$

which yields the following two cases corresponding to $y \in [0, 1]$ and $y \in [-1, 0]$.

• **Case 1.**

$$\begin{aligned} \partial^w \Phi(0) &= \left\{ \left([\underline{g}_1^w, \overline{g}_1^w], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g}_1^w, \overline{g}_1^w] \odot y \ominus_{gH} c|y| \preceq [y^2, |y|] \ \forall y \in [0, 1] \right\} \\ &= \left\{ \left([\underline{g}_1^w, \overline{g}_1^w], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g}_1^w y - cy \leq y^2 \text{ and } \overline{g}_1^w y - cy \leq y \ \forall y \in [0, 1] \right\} \\ &= \left\{ \left([\underline{g}_1^w, \overline{g}_1^w], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g}_1^w - c \leq 0 \text{ and } \overline{g}_1^w - c \leq 1 \right\}. \end{aligned}$$

• **Case 2.** Likewise,

$$\partial^w \Phi(0) = \left\{ \left([\underline{g}_1^w, \overline{g}_1^w], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : -1 \leq \underline{g}_1^w + c \text{ and } 0 \leq \overline{g}_1^w + c \right\}.$$

Hence, by combining **Case 1** and **Case 2**, we obtain

$$\partial^w \Phi(0) = \left\{ (\mathbf{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1 - c, -c] \preceq \mathbf{G}_1^w \preceq [c, 1 + c] \right\}.$$

Similarly,

$$\partial^w \Phi(1) = \left\{ (\mathbf{G}_2^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [1 - c, 2 - c] \preceq \mathbf{G}_2^w \right\}.$$

Remark 3 To understand the geometric interpretation of the gH -weak subdifferential of an IVF Φ , let $(\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u)$. This means that $(\widehat{\mathbf{G}}^w, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$, for every $c \geq 0$, is a gH -weak subgradient of Φ at $u \in \mathcal{Y}$ if and only if there exists a concave and gH -continuous IVF $\mathbf{H} : \mathcal{Y} \rightarrow I(\mathbb{R})$, which is defined by

$$\mathbf{H}(y) = \Phi(u) \oplus \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c\|y - u\| \ \forall y \in \mathcal{Y},$$

that satisfies

$$(\forall y \in \mathcal{Y}) \ \mathbf{H}(y) \preceq \Phi(y) \text{ and } \mathbf{H}(u) = \Phi(u).$$

This condition shows that \mathbf{H} must intersect Φ at least at the point $(u, \Phi(u))$ from bottom. Hence, it concludes that if Φ is gH -weak subdifferentiable at u and $(\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u)$, then the graph of IVF \mathbf{H} , that is,

$$Gr(\mathbf{H}) = \{(y, \mathbf{Y}) \in \mathcal{Y} \times I(\mathbb{R}) : \mathbf{Y} = \mathbf{H}(y)\}$$

always lie below the epigraph of Φ , i.e.,

$$Epi(\Phi) = \{(y, \mathbf{Y}) \in \mathcal{Y} \times I(\mathbb{R}) : \Phi(y) \preceq \mathbf{Y}\},$$

such that

$$Epi(\Phi) \subset Epi(\mathbf{H}) \text{ and } cl(Epi(\Phi)) \cap Gr(\mathbf{H}) \text{ is nonempty.}$$

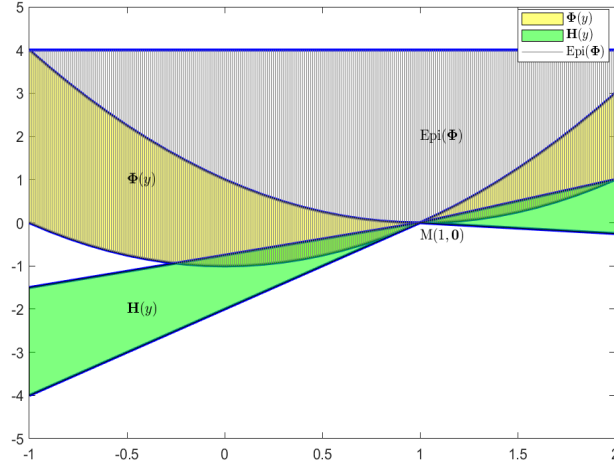


Figure 1 The geometrical view of the graph of IVF Φ (yellow) and the graph of IVF \mathbf{H} (green), which intersects the gray shaded region ($\text{Epi}(\Phi)$) from below.

For example, Let $\mathcal{Y} = [-1, 2]$. Consider an IVF $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ which is given by

$$\Phi(y) = \begin{cases} [y^2 - 1, (y - 1)^2] & \text{if } y \in [-1, 1] \\ [(y - 1)^2, y^2 - 1] & \text{if } y \in (1, 2]. \end{cases}$$

The gH -weak subdifferential of Φ at $u = 1$ is

$$\partial^w \Phi(1) = \{(\mathbf{G}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-c, 2 - c] \preceq \mathbf{G}^w \preceq [c, 2 + c]\}.$$

For instance, $(\mathbf{G}^w, c) = ([0.25, 1.5], 0.5) \in \partial^w \Phi(1)$, geometrically indicates that the IVF

$$\mathbf{H}(y) = \Phi(1) \oplus [0.25, 1.5] \odot (y - 1) \ominus_{gH} 0.5|y - 1|$$

intersects

$$\text{Epi}(\Phi) = \{(y, 4) \in \mathcal{Y} \times \mathbb{R} : \Phi(y) \preceq 4\}$$

at the point $M(1, 0)$ from below as shown in Figure 1. We also observe from the figure that

$$\text{Epi}(\Phi) \subset \text{Epi}(\mathbf{H}), \text{ and } cl(\text{Epi}(\Phi)) \cap Gr(\mathbf{H}) \text{ is nonempty.}$$

Theorem 1 (Convexity of gH -weak subdifferential). *Let $\mathcal{Y} \subset \mathbb{R}^n$. Let the gH -weak subdifferential of $\Phi : \mathcal{Y} \rightarrow \mathbf{I}(\mathbb{R})$ at u be nonempty. Then, the set $\partial^w \Phi(u)$ is convex.*

Proof Let $(\widehat{\mathbf{G}}_1^w, c_1)$ and $(\widehat{\mathbf{G}}_2^w, c_2) \in \partial^w \Phi(u)$, where $\widehat{\mathbf{G}}_1^w = (\mathbf{G}_{11}^w, \mathbf{G}_{12}^w, \dots, \mathbf{G}_{1n}^w)^\top$, $\widehat{\mathbf{G}}_2^w = (\mathbf{G}_{21}^w, \mathbf{G}_{22}^w, \dots, \mathbf{G}_{2n}^w)^\top$. Let $\beta \in [0, 1]$. From the definition of $\partial^w \Phi(u)$, we have

$$\widehat{\mathbf{G}}_1^{w\top} \odot (y - u) \ominus_{gH} c_1 \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \text{ and} \quad (2)$$

$$\widehat{\mathbf{G}}_2^{w\top} \odot (y - u) \ominus_{gH} c_2 \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u), \quad (3)$$

for all $y \in \mathcal{Y}$. Up to a rearrangement of terms, let the first m components of $(y-u)$ be non-negative and the rest be negative. Then, from the inequalities (2) and (3), we get

$$\bigoplus_{i=1}^m (y_i - u_i) \odot \mathbf{G}_{1i}^w \bigoplus_{j=m+1}^n (y_j - u_j) \odot \mathbf{G}_{1j}^w \ominus_{gH} c_1 \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u)$$

and

$$\bigoplus_{i=1}^m (y_i - u_i) \odot \mathbf{G}_{2i}^w \bigoplus_{j=m+1}^n (y_j - u_j) \odot \mathbf{G}_{2j}^w \ominus_{gH} c_2 \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u).$$

Thus,

$$\bigoplus_{i=1}^m \beta \odot ((y_i - u_i) \odot \mathbf{G}_{1i}^w) \bigoplus_{j=m+1}^n \beta \odot ((y_j - u_j) \odot \mathbf{G}_{1j}^w) \ominus_{gH} \beta c_1 \|y - u\| \preceq \beta \odot (\Phi(y) \ominus \Phi(u)) \quad (4)$$

and

$$\begin{aligned} & \bigoplus_{i=1}^m (1 - \beta) \odot ((y_i - u_i) \odot \mathbf{G}_{2i}^w) \bigoplus_{j=m+1}^n (1 - \beta) \odot ((y_j - u_j) \odot \mathbf{G}_{2j}^w) \ominus_{gH} (1 - \beta)c_2 \|y - u\| \\ & \preceq (1 - \beta) \odot (\Phi(y) \ominus \Phi(u)). \end{aligned} \quad (5)$$

By adding (4) and (5), we obtain

$$\begin{aligned} & \bigoplus_{i=1}^m (y_i - u_i) \odot \{\beta \odot \mathbf{G}_{1i}^w \oplus (1 - \beta) \odot \mathbf{G}_{2i}^w\} \bigoplus_{j=m+1}^n (y_j - u_j) \odot \{\beta \odot \mathbf{G}_{1j}^w \oplus (1 - \beta) \odot \mathbf{G}_{2j}^w\} \\ & \ominus_{gH} (\beta c_1 \oplus (1 - \beta)c_2) \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \end{aligned} \quad (6)$$

Therefore, we have

$$\{\beta \odot \widehat{\mathbf{G}}_1^w \oplus (1 - \beta) \odot \widehat{\mathbf{G}}_2^w\}^\top \odot (y - u) \ominus_{gH} (\beta c_1 \oplus (1 - \beta)c_2) \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u),$$

i.e., $(\beta \odot \widehat{\mathbf{G}}_1^w \oplus (1 - \beta) \odot \widehat{\mathbf{G}}_2^w, \beta c_1 \oplus (1 - \beta)c_2) \in \partial^w \Phi(u)$, which proves the convexity of $\partial^w \Phi(u)$.

Theorem 2 (Closedness of gH -weak subdifferential). *Let $\emptyset \neq \mathcal{Y} \subseteq I(\mathbb{R})^n$. If for an IVF $\Psi : \mathcal{Y} \rightarrow \mathbf{I}(\mathbb{R})$, the set $\partial^w \Psi(u)$ is nonempty at $u \in \mathcal{Y}$, then $\partial^w \Psi(u)$ is closed.*

Proof Let $\{(\widehat{\mathbf{G}}_k^w, c_k)\}$ be an arbitrary sequence in $\partial^w \Psi(y)$ converging to $(\widehat{\mathbf{G}}^w, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$, where $\widehat{\mathbf{G}}_k^w = (\mathbf{G}_{k1}^w, \mathbf{G}_{k2}^w, \dots, \mathbf{G}_{kn}^w)^\top$ and $\widehat{\mathbf{G}}^w = (\mathbf{G}_1^w, \mathbf{G}_2^w, \dots, \mathbf{G}_n^w)^\top$. Since $(\widehat{\mathbf{G}}_k^w, c) \in \partial^w \mathbf{T}(y)$ for all $d \in \mathcal{Y}$, we obtain

$$\widehat{\mathbf{G}}_k^w{}^\top \odot d \ominus_{gH} c_k \|d\| \preceq \Psi(u + d) \ominus_{gH} \Psi(u),$$

which implies

$$\bigoplus_{i=1}^n d_i \odot \mathbf{G}_{ki}^w \ominus_{gH} c_k \|d\| \preceq \Psi(u + d) \ominus_{gH} \Psi(u). \quad (7)$$

Up to a rearrangement of terms, let the first p components of d be non-negative and the rest be negative. Then, from (7), we get

$$\bigoplus_{i=1}^p d_i \odot \mathbf{G}_{ki}^w \bigoplus_{j=p+1}^n d_j \odot \mathbf{G}_{kj}^w \ominus_{gH} c_k \|d\| \preceq \Psi(u + d) \ominus_{gH} \Psi(u)$$

$$\implies \bigoplus_{i=1}^p d_i \odot [\underline{g}_{ki}^w, \overline{g}_{ki}^w] \bigoplus_{j=p+1}^n d_j \odot [\underline{g}_{kj}^w, \overline{g}_{kj}^w] \ominus_{gH} c_k \|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u).$$

Therefore,

$$\sum_{i=1}^p \underline{g}_{ki}^w d_i + \sum_{j=p+1}^n \overline{g}_{kj}^w d_j - c_k \|d\| \preceq \min \{ \underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u) \} \quad (8)$$

and

$$\sum_{i=1}^p \overline{g}_{ki}^w d_i + \sum_{j=p+1}^n \underline{g}_{kj}^w d_j - c_k \|d\| \preceq \max \{ \underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u) \}. \quad (9)$$

Since the sequence $\widehat{\mathbf{G}}_k^w$ converges to $\widehat{\mathbf{G}}^w$, the sequences $\{g_{ki}^w\}$ and $\{\overline{g}_{ki}^w\}$ converge to $\{g_i^w\}$ and $\{\overline{g}_i^w\}$, respectively for all i . Thus, by (8) and (9), we have

$$\begin{aligned} \sum_{i=1}^p \underline{g}_{ki}^w d_i + \sum_{j=p+1}^n \overline{g}_{kj}^w d_j - c_k \|d\| &\rightarrow \sum_{i=1}^p \underline{g}_i^w d_i + \sum_{j=p+1}^n \overline{g}_j^w d_j - c \|d\| \\ &\preceq \min \left\{ \underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u) \right\} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^p \overline{g}_{ki}^w d_i + \sum_{j=p+1}^n \underline{g}_{kj}^w d_j - c_k \|d\| &\rightarrow \sum_{i=1}^p \overline{g}_i^w d_i + \sum_{j=p+1}^n \underline{g}_j^w d_j - c \|d\| \\ &\preceq \max \left\{ \underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u) \right\}. \end{aligned}$$

Hence, for any $u \in \mathcal{Y}$,

$$\begin{aligned} &\left[\sum_{i=1}^p \underline{g}_i^w d_i + \sum_{j=p+1}^n \overline{g}_j^w d_j - c \|d\|, \sum_{i=1}^p \overline{g}_i^w d_i + \sum_{j=p+1}^n \underline{g}_j^w d_j - c \|d\| \right] \preceq \Psi(u+d) \ominus_{gH} \Psi(u) \\ \implies &\bigoplus_{i=1}^p [\underline{g}_i^w d_i, \overline{g}_i^w d_i] \bigoplus_{j=p+1}^n [\overline{g}_j^w d_j, \underline{g}_j^w d_j] \ominus_{gH} c \|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u) \\ \implies &\bigoplus_{i=1}^p d_i \odot \mathbf{G}_i^w \bigoplus_{j=p+1}^n d_j \odot \mathbf{G}_j^w \ominus_{gH} c \|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u) \\ \implies &\widehat{\mathbf{G}}^w \odot d \ominus_{gH} c \|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u). \end{aligned}$$

Therefore, $\widehat{\mathbf{G}}^w \in \partial^w \Psi(u)$, and hence $\partial^w \Psi(u)$ is closed.

Definition 16 (*gH-Fréchet lower subdifferential*). Let $\Phi : \mathcal{Y} \rightarrow \mathbf{I}(\mathbb{R}) \cup \{-\infty, +\infty\}$ be an IVF that is finite at an $u \in \mathcal{Y}$. Then, the gH-Fréchet lower subdifferential of Φ at u is defined by

$$\begin{aligned} \partial_{\mathcal{F}}^- \Phi(u) &= \left\{ \widehat{\mathbf{G}} : \mathbf{0} \preceq \liminf_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{ \Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y-u) \}, \right. \\ &\quad \left. \text{where } \widehat{\mathbf{G}} : \mathcal{Y} \rightarrow I(\mathbb{R}) \text{ is } gH\text{-continuous and linear IVF} \right\}. \end{aligned}$$

One important fact is that gH -weak subdifferential is an immediate consequence of gH -Fréchet lower subdifferential.

Theorem 3 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. If $\Phi : \mathcal{Y} \rightarrow \mathbf{I}(\mathbb{R})$ has gH -Fréchet lower subdifferential $\widehat{\mathbf{G}}$ at the point u , then $(\widehat{\mathbf{G}}, \epsilon)$ is a gH -weak subgradient of Φ at u for any $\epsilon \in \mathbb{R}_+$.*

Proof Let $\widehat{\mathbf{G}} \in \partial_{\mathcal{F}}^- \Phi(u)$. Due to Definition 16, we can write

$$\mathbf{0} \preceq \liminf_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y - u\|} \odot \{\Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y - u)\}.$$

Then, for the $\epsilon > 0$ in the hypothesis there exists $\delta > 0$ such that

$$-\epsilon \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y - u) \quad \forall y \in B_\delta(u),$$

Then, from Lemma 3, we have

$$\widehat{\mathbf{G}}^\top \odot (y - u) \ominus_{gH} \epsilon \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u).$$

By Definition 15, $(\widehat{\mathbf{G}}, \epsilon)$ is a gH -weak subdifferential of \mathbf{T} at u .

Lemma 7 *For any $y \in \mathbb{R}^n$ and $\widehat{\mathbf{C}} = (\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \dots, \mathbf{C}_n) \in I(\mathbb{R})^n$,*

$$-\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \preceq \|y\|^\top \odot \widehat{\mathbf{C}}\|_{I(\mathbb{R})}$$

Proof See Appendix E.

To investigate the class of interval-valued functions for which weak subgradients always exist, we need the following definition.

Definition 17 (*gH -lower Lipschitz IVF*). Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. An IVF $\Phi : \mathcal{Y} \rightarrow \overline{I(\mathbb{R})}$ is called gH -lower locally Lipschitz at $u \in \mathcal{Y}$ if $\exists L \geq 0$ and a neighbourhood $\mathcal{N}(u)$ of u such that

$$-L \|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \quad \forall y \in \mathcal{N}(u). \quad (10)$$

If the inequality (10) satisfies for all $y \in \mathcal{Y}$, then Φ is called gH -lower Lipschitz at $u \in \mathcal{Y}$ with Lipschitz constant L .

Example 3 Let $\Phi : [1, \infty) \rightarrow I(\mathbb{R})$ be an IVF, defined by $\Phi(y) = \ln y \odot \mathbf{C}$ for all $y \in [1, \infty)$, where $\mathbf{0} \preceq \mathbf{C} = [\underline{c}, \bar{c}]$. Let $\delta > 0$. We choose the neighbourhood of u , $\mathcal{N}_\delta(u) = \{y : |y - u| < \delta\}$. If $0 < y - u < \delta$, then $u < y$ and also then $\frac{y}{u} > 1$ and then

$$\begin{aligned} 0 < \ln \frac{y}{u} < \frac{y}{u} - 1, \text{ since } \ln(1 + p) < p \text{ if } p > 0 \\ &\leq y - u. \end{aligned} \quad (11)$$

Since $\underline{c}, \bar{c} \geq 0$, we have

$$(\ln y - \ln u)\underline{c} \leq (y - u)\underline{c} \text{ and } (\ln y - \ln u)\bar{c} \leq (y - u)\bar{c}.$$

Then,

$$(\ln y - \ln u) \odot \mathbf{C} \preceq (y - u) \odot \mathbf{C}. \quad (12)$$

If $-\delta < y - u < 0$, then $y < u$ and also then $\frac{u}{y} > 1$ and then

$$\begin{aligned} 0 < \ln \frac{u}{y} < \frac{u}{y} - 1, \text{ since } \ln(1+p) < p \text{ if } p > 0 \\ &\leq u - y. \end{aligned} \quad (13)$$

Then, similarly, as seen in (12),

$$(\ln u - \ln y) \odot \mathbf{C} \preceq (u - y) \odot \mathbf{C}. \quad (14)$$

Combining (12) and (14), we have

$$\begin{aligned} &|\ln y - \ln u| \odot \mathbf{C} \preceq |y - u| \odot \mathbf{C} \\ \implies &\ln u \odot \mathbf{C} \ominus_{gH} \ln y \odot \mathbf{C} \preceq |y - u| \odot \mathbf{C} \\ \implies &-|y - u| \odot \mathbf{C} \preceq \ln y \odot \mathbf{C} \ominus_{gH} \ln u \odot \mathbf{C} \\ \implies &-\bar{c}|y - u| \preceq \Phi(y) \ominus_{gH} \Phi(u). \end{aligned}$$

This shows that Φ is gH -lower locally Lipschitz on $\mathcal{N}_\delta(u)$ with $L = \bar{c}$. From arbitrariness of y, u in $[1, \infty)$, we conclude that Φ is gH -lower Lipschitz on $[1, \infty)$.

Theorem 4 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\Phi : \mathcal{Y} \rightarrow \overline{I(\mathbb{R})}$ be an IVF, where $\Phi(u)$ is finite for some $u \in \mathcal{Y}$. Then, the following three statements are equivalent:

- (a) Φ is gH -weak subdifferentiable at u .
- (b) Φ is gH -lower Lipschitz at u .
- (c) Φ is gH -lower locally Lipschitz at u , and there exists a number $p \geq 0$ and an interval \mathcal{Q} such that

$$-p\|y\| \oplus \mathcal{Q} \preceq \Phi(y) \quad \forall y \in \mathcal{Y}. \quad (15)$$

Proof (a) implies (b) : Suppose Φ is gH -weak subdifferentiable at u . Then, there exists $(\widehat{\mathbf{G}}^w, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ such that for any $y \in \mathcal{Y}$, we have

$$\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c\|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \quad (16)$$

From Lemma 7, we have $-\|\widehat{\mathbf{G}}^w\|_{I(\mathbb{R})^n}\|y - u\| - c\|y - u\| \preceq \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c\|y - u\|$. Hence, the inequality (16) yields

$$-(\|\widehat{\mathbf{G}}^w\| + c)\|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \text{ by Lemma 2.3 (ii) of [1].}$$

By choosing $L = (\|\widehat{\mathbf{G}}^w\| + c)$, we obtain

$$-L\|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \quad \forall y \in \mathcal{Y}. \quad (17)$$

So, Φ is gH -lower Lipschitz at u .

(b) implies (c) : Suppose that (b) is satisfied. It needs to prove that the inequality (15) holds. Then, there exists an $L \geq 0$ such that

$$-L\|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \quad (18)$$

Note that $-L\|y\| - L\|u\| \leq -L\|y - u\|$. So, the inequality (18) gives

$$-L\|y\| - L\|u\| \preceq \Phi(y) \ominus_{gH} \Phi(u),$$

which gives $\Phi(u) \ominus_{gH} L\|u\| - L\|y\| \preceq \Phi(y)$ by (iv) of Lemma 6. Taking $\mathbf{Q} = \Phi(u) \ominus_{gH} L\|u\|$ and $p = L$, we obtain $-p\|y\| \oplus \mathbf{Q} \preceq \Phi(y)$ for all $y \in \mathcal{Y}$.

(c) implies (a) : Let $\mathcal{N}(u)$ be an ϵ -neighbourhood of u such that (10) holds. Then, we get

$$-L\|y - u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \quad \forall y \in \mathcal{N}(u) \quad (19)$$

and

$$-p\|y\| \oplus \mathbf{Q} \preceq \Phi(y) \quad \forall y \in \mathbb{R}^n. \quad (20)$$

Assume to the contrary that Φ is not gH -weak subdifferentiable at u . Then, for any $(\widehat{\mathbf{G}}_n^w, c_n) \in I(\mathbb{R}^n) \times \mathbb{R}_+$, there exists y_n such that

$$\Phi(y_n) \ominus_{gH} \Phi(u) \prec \widehat{\mathbf{G}}_n^w \odot (y_n - u) \ominus_{gH} c_n \|y_n - y\|.$$

If the sequence $\{\widehat{\mathbf{G}}_n^w\}$ is assumed to be converging to $\widehat{\mathbf{G}}^w$, then we get

$$\begin{aligned} \Phi(y_n) \ominus_{gH} \Phi(u) &\preceq \widehat{\mathbf{G}}^w \odot (y_n - u) \ominus_{gH} c_n \|y_n - y\| \\ &\preceq \|\widehat{\mathbf{G}}^w\| \|y_n - u\| - c_n \|y_n - u\|, \text{ by Theorem 3.1 of [12].} \end{aligned} \quad (21)$$

By putting $y = y_n$ in (20), we get

$$-p\|y_n - u\| - p\|y\| \oplus \mathbf{Q} \preceq -p\|y_n\| \oplus \mathbf{Q} \preceq \Phi(y_n),$$

which implies

$$(-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \Phi(u) \preceq \Phi(y_n) \ominus_{gH} \Phi(u) \text{ by Note 2 of [1].} \quad (22)$$

From (21) and (22), by Lemma 2.3 (ii) of [1], we deduce that

$$\begin{aligned} (-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \Phi(u) &\preceq \|\widehat{\mathbf{G}}^w\| \|y_n - u\| - c_n \|y_n - u\|, \\ \text{or, } (c_n - p - \|\widehat{\mathbf{G}}^w\|) \|y_n - u\| &\preceq \Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q} \text{ by (iii) of Lemma 6.} \end{aligned} \quad (23)$$

Assume, without loss of generality, that $c_n - p - \|\widehat{\mathbf{G}}^w\| \neq 0$. Then, from (6), we obtain

$$\|y_n - u\| \preceq \frac{1}{c_n - p - \|\widehat{\mathbf{G}}^w\|} \odot \{\Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q}\}.$$

As $(\Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q})$ is bounded below on $\mathcal{N}(u)$, we get $y_n \rightarrow u$ as $c_n \rightarrow \infty$. Thus, $y_n \in \mathcal{N}(u)$ for large n . Then, from (19) it follows that

$$-L\|y_n - u\| \preceq \Phi(y_n) \ominus_{gH} \Phi(u). \quad (24)$$

In view of (21), we obtain

$$\Phi(y_n) \ominus_{gH} \Phi(u) \preceq \|\widehat{\mathbf{G}}^w\| \|y_n - u\| - c_n \|y_n - u\| = -(c_n - \|\widehat{\mathbf{G}}^w\|) \|y_n - u\|.$$

Since $c_n \rightarrow +\infty$ and $L \geq 0$, we can pick c_n sufficiently large so that $c_n - \|\widehat{\mathbf{G}}^w\| \geq L$. So,

$$\Phi(y_n) \ominus_{gH} \Phi(u) \preceq -L\|y_n - u\|.$$

This inequality leads to a contradiction. So, the result follows.

Theorem 5 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\Psi : \mathcal{Y} \rightarrow I(\mathbb{R})$ be gH -Fréchet differentiable at u with gH -Fréchet derivative $\Psi_{\mathcal{F}}(u)$. Then,

$$\{(\Psi_{\mathcal{F}}(u), c) : c \geq 0\} \subset \partial^w \Psi(u).$$

Proof Since Ψ is gH -Fréchet differentiable at u with gH -Fréchet derivative $\Psi_{\mathcal{F}}(u)$, we get

$$\begin{aligned} & \lim_{\substack{y \rightarrow u \\ \|y-u\|}} \frac{1}{\|y-u\|} \odot \{\Psi(y) \ominus_{gH} \Psi(u) \ominus_{gH} \Psi_{\mathcal{F}}(u)^\top \odot (y-u)\} = \mathbf{0} \\ \implies & \liminf_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{\Psi(y) \ominus_{gH} \Psi(u) \ominus_{gH} \Psi_{\mathcal{F}}(u)^\top \odot (y-u)\} = \mathbf{0}. \end{aligned}$$

Therefore, by Definition 16, $\Psi_{\mathcal{F}}(u) \in \partial_{\mathcal{F}}^- \Psi(u)$. So,

$$\begin{aligned} & \Psi_{\mathcal{F}}(u)^\top \odot (y-u) \preceq \Psi(y) \ominus_{gH} \Psi(u) \quad \forall y \in \mathcal{Y} \\ \implies & \Psi_{\mathcal{F}}(u)^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Psi(y) \ominus_{gH} \Psi(u), \text{ for any } c \geq 0. \end{aligned}$$

Hence, $(\Psi_{\mathcal{F}}(u), c) \in \partial^w \Psi(u)$.

Lemma 8 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R})$ be gH -Fréchet differentiable at u with gH -Fréchet derivative $\Phi_{\mathcal{F}}(u)$. Then, $-1 \odot \Phi_{\mathcal{F}}(u) \in \partial_{\mathcal{F}}^- (-1 \odot \Phi)(u)$.

Proof Since Φ is gH -Fréchet differentiable at u with gH -Fréchet derivative $\Phi_{\mathcal{F}}(u)$, one gets

$$\lim_{y \rightarrow u} \frac{1}{\|y-u\|} \odot \{\Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \Phi_{\mathcal{F}}(u)^\top \odot (y-u)\} = \mathbf{0}.$$

By applying Lemma 5, we have

$$\begin{aligned} & \lim_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \left\{ \mathbf{0} \ominus_{gH} \{(-1 \odot \Phi)(y) \ominus_{gH} (-1 \odot \Phi)(u) \ominus_{gH} (-1 \odot \Phi_{\mathcal{F}}(u)^\top) \odot (y-u)\} \right\} = \mathbf{0} \\ \implies & \lim_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \left\{ (-1 \odot \Phi)(y) \ominus_{gH} (-1 \odot \Phi)(u) \ominus_{gH} (-1 \odot \Phi_{\mathcal{F}}(u)^\top) \odot (y-u) \right\} = \mathbf{0} \\ \implies & \liminf_{\substack{y \rightarrow u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{(-1 \odot \Phi)(y) \ominus_{gH} (-1 \odot \Phi)(u) \ominus_{gH} (-1 \odot \Phi_{\mathcal{F}}(u)^\top) \odot (y-u)\} = \mathbf{0}. \end{aligned}$$

Hence, $-1 \odot \Phi_{\mathcal{F}}(u) \in \partial_{\mathcal{F}}^- (-1 \odot \Phi)(u)$.

Next, we focus on investigating the sum rule of two functions in terms of gH -weak subdifferential. For two real-valued functions f_1 and f_2 , the sum rule [21] for their weak subdifferential is $\partial^w(f_1 + f_2)(x) = \partial^w f_1(x) + \partial^w f_2(x)$. However, this sum rule does not hold for interval-valued functions. In the following, we provide such an example.

Consider the interval-valued functions $\Phi_1 : [-1, 1] \rightarrow I(\mathbb{R})$ and $\Phi_2 : [-1, 1] \rightarrow I(\mathbb{R})$, defined by

$$\Phi_1(y) = \begin{cases} [-y, \frac{1}{2}y] & \text{if } y \in [0, 1] \\ [-\frac{1}{2}y - y] & \text{if } y \in [-1, 0] \end{cases} \quad \text{and} \quad \Phi_2(y) = [y^2, -y + 3],$$

respectively. For these two functions, the gH -weak subdifferential at $u = 0$ are given by

$$\partial^w \Phi_1(0) = \{(\mathbf{G}_1^w, c_1) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1, -\frac{1}{2}] \preceq \mathbf{G}_1^w \oplus c_1, \mathbf{G}_1^w \ominus_{gH} c_1 \preceq [-1, \frac{1}{2}] \quad \forall y \in [-1, 1]\}$$

and

$$\partial^w \Phi_2(0) = \{(\mathbf{G}_2^w, c_2) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1, 0] \preceq \mathbf{G}_2^w \oplus c_2, \mathbf{G}_2^w \ominus_{gH} c_2 \preceq [-1, 0] \quad \forall y \in [-1, 1]\}.$$

Thus, we have

$$\begin{aligned} & \partial^w \Phi_1(0) \oplus \partial^w \Phi_2(0) \\ &= \{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-2, -\frac{1}{2}] \preceq \mathbf{H}^w \oplus c, \mathbf{H}^w \ominus_{gH} c \preceq [-2, \frac{1}{2}] \ \forall y \in [-1, 1]\}. \end{aligned} \quad (25)$$

Now, let $(\mathbf{H}^w, c) \in \partial^w(\Phi_1 \oplus \Phi_2)(0)$, where

$$(\Phi_1 \oplus \Phi_2)(y) = \begin{cases} [y^2 - y, -\frac{1}{2}y + 3] & \text{if } y \in [0, 1] \\ [y^2 - \frac{1}{2}y, -2y + 3] & \text{if } y \in [-1, 0]. \end{cases}$$

There are the following two cases corresponding to $y \in [0, 1]$ and $y \in [-1, 0]$.

• **Case 1.** As $y \geq 0$, we have

$$\begin{aligned} & \mathbf{H}^w \odot y \ominus_{gH} c \odot y \preceq (\Phi_1 \oplus \Phi_2)(y) \ominus_{gH} (\Phi_1 \oplus \Phi_2)(0) \\ \implies & [\underline{h}^w - c, \overline{h}^w - c] \odot y \preceq [y^2 - y, -\frac{1}{2}y] \\ \implies & \underline{h}^w - c \leq -1 \text{ and } \overline{h}^w - c \leq -\frac{1}{2}. \end{aligned}$$

• **Case 2.** As $-1 \leq y \leq 0$, we have

$$\begin{aligned} & [(\overline{h}^w + c)y, (\underline{h}^w + c)y] \preceq [y^2 - \frac{1}{2}y, -2y + 3] \ominus_{gH} [0, 3] \\ \implies & [(\overline{h}^w + c)y, (\underline{h}^w + c)y] \preceq [y^2 - \frac{1}{2}y, -2y] \\ \implies & -2 - c \leq \underline{h}^w \text{ and } -\frac{1}{2} - c \leq \overline{h}^w. \end{aligned}$$

Therefore, from **Case 1** and **Case 2**, we have

$$\begin{aligned} & \partial^w(\Phi_1 \oplus \Phi_2)(0) \\ &= \{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-2, -\frac{1}{2}] \preceq (\mathbf{H}^w \oplus c), (\mathbf{H}^w \ominus_{gH} c) \preceq [-1, -\frac{1}{2}] \ \forall y \in [0, 1]\}. \end{aligned} \quad (26)$$

Thus, (25) and (26) are not equal.

In the following Theorem 6, we show that under some restriction on Φ_1 and Φ_2 one-sided inclusion for the sum rule holds.

Theorem 6 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\Phi_1 : \mathcal{Y} \rightarrow I(\mathbb{R})$ be gH -weak subdifferential at u and $\Phi_2 : \mathcal{Y} \rightarrow \mathbb{R}$ be gH -Fréchet differentiable at u . Then,*

$$\partial^w(\Phi_1 \oplus \Phi_2)(u) \subset \partial^w \Phi_1(u) \oplus \partial^w \Phi_2(u),$$

provided that $w(\widehat{\mathbf{G}}_1^w) \leq w(\widehat{\mathbf{G}}_2^w)$ for all $\widehat{\mathbf{G}}_1^w \in \partial \Phi_2(y)$ and $\widehat{\mathbf{G}}_2^w \in \partial(\Phi_1 \oplus \Phi_2)(y)$, where $w(\mathbf{A})$ is the width of the interval $\mathbf{A} \in I(\mathbb{R})$.

Proof If $(\widehat{\mathbf{G}}^w, c) \in \partial^w(\Phi_1 \oplus \Phi_2)(u)$, then

$$\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq (\Phi_1 \oplus \Phi_2)(y) \ominus_{gH} (\Phi_1 \oplus \Phi_2)(u). \quad (27)$$

We know that $\Phi_2 : \mathcal{Y} \rightarrow I(\mathbb{R})$ is gH -Fréchet differentiable at u with the gH -Fréchet derivative $\Phi_{2\mathcal{F}}(u)$. Hence, $\Phi_{2\mathcal{F}}(u) \in \partial_{\mathcal{F}}^- \Phi_2(u)$ implies $-1 \odot \Phi_{2\mathcal{F}}(u) \in \partial_{\mathcal{F}}^-(-1 \odot \Phi_2)(u)$. We can then write

$$\begin{aligned} & -1 \odot \Phi_{2\mathcal{F}}(u) \odot (y - u) \preceq (-1 \odot \Phi_2)(u) \ominus_{gH} (-1 \odot \Phi_2)(u) \\ \implies & -1 \odot \Phi_{2\mathcal{F}}(u) \odot (y - u) \preceq -1 \odot (\Phi_2(y) \ominus_{gH} \Phi_2(u)) \\ & \text{by properties of } gH\text{-difference (iv) of [35].} \end{aligned} \quad (28)$$

In view of Lemma 4, (27) becomes

$$\widehat{\mathbf{G}}^w \circledast (y - u) \ominus_{gH} c \|y - u\| \preceq (\Phi_1(y) \ominus_{gH} \Phi_1(u)) \oplus (\Phi_2(y) \ominus_{gH} \Phi_2(u)).$$

Using (v) of Lemma 6, this inequality reduces to

$$\widehat{\mathbf{G}}^w \circledast (y - u) \ominus_{gH} (\Phi_2(y) \ominus_{gH} \Phi_2(u)) \ominus_{gH} c \|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Now, from the inequality (28), we see that

$$\widehat{\mathbf{G}}^w \circledast (y - u) \ominus_{gH} \Phi_{2\mathcal{F}}(u) \circledast (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Thus,

$$(\widehat{\mathbf{G}}^w \ominus_{gH} \Phi_{2\mathcal{F}}(u)) \circledast (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Then, $(\widehat{\mathbf{G}}^w \ominus_{gH} \Phi_{2\mathcal{F}}(u), c) \in \partial^w \Phi_1(u)$ and $(\Phi_{2\mathcal{F}}(u), 0) \in \partial^w \Phi_2(u)$. Therefore, $(\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi_1(u) \oplus \partial^w \Phi_2(u)$. Hence, the result follows.

Theorem 7 Let \mathcal{Y} be a nonempty set of \mathbb{R}^n . Let $\Phi_1 : \mathcal{Y} \rightarrow I(\mathbb{R})$ be gH -Fréchet differentiable at u . Let $\Phi_2 : \mathcal{Y} \rightarrow I(\mathbb{R})$ be an IVF. If u is a weak efficient point of $\Phi_1 \oplus \Phi_2$, then $(-1 \circledast \Phi_{1\mathcal{F}}(u), 0) \in \partial^w \Phi_2(u)$.

Proof Since u is a weak efficient point of $\Phi_1 \oplus \Phi_2$, for any $y \in \mathcal{Y}$,

$$\begin{aligned} & (\Phi_1 \oplus \Phi_2)(u) \preceq (\Phi_1 \oplus \Phi_2)(y) \\ \implies & \Phi_1(u) \oplus \Phi_2(u) \preceq \Phi_1(y) \oplus \Phi_2(y) \\ \implies & \Phi_1(u) \ominus_{gH} \Phi_1(y) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \text{ using Lemma 2.3 of [1]} \\ \implies & (-1) \circledast \{\Phi_1(y) \ominus_{gH} \Phi_1(u)\} \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \text{ by } \ominus_{gH} \text{ property in (iv) of [35]} \\ \implies & (-1 \circledast \Phi_1)(y) \ominus_{gH} (-1 \circledast \Phi_1)(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \\ & \text{by } \ominus_{gH} \text{ property in (iv) of [35].} \end{aligned} \tag{29}$$

By the Lemma 8, we also obtain that

$$(-1) \circledast \Phi_{1\mathcal{F}}(u) \circledast (y - u) \preceq (-1 \circledast \Phi_1)(y) \ominus_{gH} (-1 \circledast \Phi_1)(u) \quad \forall y \in \mathcal{Y}. \tag{30}$$

We get, from (29) and (30) that

$$(-1) \circledast \Phi_{1\mathcal{F}}(u) \circledast (y - u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ by Lemma 2.3 (ii) of [1].}$$

This means that $((-1) \circledast \Phi_{1\mathcal{F}}(u), 0) \in \partial^w \Phi_2(u)$.

Theorem 8 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let Ψ be gH -Fréchet differentiable at u with the gH -Fréchet derivative $\Psi_{\mathcal{F}}(u)$. Then, Ψ has weak efficient solution at u if and only if for any $y \in \mathcal{Y}$,

$$\Psi_{\mathcal{F}}(u) \circledast (y - u) = \mathbf{0}.$$

Proof If Ψ has a weak efficient point at u , then

$$\begin{aligned} & \Psi(u) \preceq \Psi(y) \\ \text{or, } & \mathbf{0} \preceq \Psi(y) \ominus_{gH} \Psi(u), \text{ by Lemma 2.1 of [14].} \end{aligned}$$

By gH -Fréchet differentiability of Ψ at u , we get

$$\lim_{\|h\| \rightarrow 0} \frac{\|(\Psi(u+h) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathcal{F}}(u) \circledast h\|_{I(\mathbb{R})}}{\|h\|} = 0.$$

If we take $h = \lambda(y - u)$, then

$$\lim_{\lambda \rightarrow 0} \frac{\|(\Psi(u + \lambda(y - u)) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathcal{F}}(u)^{\top} \odot \{\lambda(y - u)\}\|_{I(\mathbb{R})}}{\|\lambda(y - u)\|} = 0. \quad (31)$$

Since u is a weak efficient point of Ψ , from (31) we have

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{\|\mathbf{0} \ominus_{gH} \lambda \odot \{\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u)\}\|_{I(\mathbb{R})}}{\|\lambda(y - u)\|} \leq 0 \text{ by (i) of Lemma 6} \\ \implies & \lim_{\lambda \rightarrow 0} \frac{\|\lambda \odot \{\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u)\}\|_{I(\mathbb{R})}}{\|\lambda(y - u)\|} \leq 0 \\ \implies & \lim_{\lambda \rightarrow 0} \frac{\lambda \|\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u)\|_{I(\mathbb{R})}}{\lambda \|(y - u)\|} \leq 0. \end{aligned}$$

Since norm gives non-negative value,

$$\frac{1}{\|(y - u)\|} \odot \{\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u)\} = \mathbf{0}.$$

Thus, we obtain

$$\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u) = \mathbf{0} \text{ for any } y \in \mathcal{Y}.$$

To show the reverse part, we suppose that $\Psi_{\mathcal{F}}(u)^{\top} \odot (y - u) = \mathbf{0}$ for all y . Then, we have $\Psi_{\mathcal{F}}(u) \in \partial_{\mathcal{F}}^{-} \Psi(u)$ and this clearly yields

$$\begin{aligned} \mathbf{0} &= \Psi_{\mathcal{F}}(u)^{\top} \odot (y - u) \preceq \Psi(y) \ominus_{gH} \Psi(u) \\ \implies \Psi(u) &\preceq \Psi(y) \text{ by (ii) of Lemma 2.1 in [14],} \end{aligned}$$

and this means that u is weak efficient point of \mathbf{T} .

Theorem 9 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. If Ψ is gH -Fréchet differentiable at u , then Ψ is gH -weak subdifferentiable at u if and only if $\Psi_{\mathcal{F}}(u)$ is gH -weak subdifferentiable at $0 \in \mathcal{Y}$, and*

$$\partial^w(\Psi(u)) = \partial^w(\Psi_{\mathcal{F}}(u)(0)).$$

Proof By the gH -Fréchet differentiability of Ψ at u , we have

$$\lim_{\|h\| \rightarrow 0} \frac{1}{\|h\|} \odot \{(\Psi(u + h) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathcal{F}}(u)^{\top} \odot h\} = \mathbf{0}.$$

Inserting $h = \lambda \odot (y - u)$, by gH -weak subdifferentiability of Ψ at u , there exists $(\widehat{\mathbf{G}}^w, c) \in \partial^w \Psi(u)$ such that for any $y \in \mathcal{Y}$,

$$\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Psi(y) \ominus_{gH} \mathbf{T}(u).$$

Hence,

$$\lim_{\lambda \rightarrow 0} \frac{1}{\|\lambda(y - u)\|} \odot \{(\Psi(u + \lambda(y - u)) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathcal{F}}(u)^{\top} \odot \lambda(y - u)\} = \mathbf{0}$$

and by gH -weak subdifferentiability of Ψ at u , we get, for any $y \in \mathcal{Y}$ that

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{\|\lambda(y - u)\|} \odot \left\{ (\widehat{\mathbf{G}}^w \odot \lambda(y - u) \ominus_{gH} \lambda c \|y - u\|) \ominus_{gH} \Psi_{\mathcal{F}}(u)^{\top} \odot \lambda(y - u) \right\} \preceq \mathbf{0}, \\ & \text{(by (ii) of Lemma 6)} \\ \implies & \frac{1}{\|(y - u)\|} \odot \{(\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\|) \ominus_{gH} \Psi_{\mathcal{F}}(u)^{\top} \odot (y - u)\} \preceq \mathbf{0}. \end{aligned}$$

Therefore,

$$\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \ominus_{gH} \Psi_{\mathcal{F}}(u) \odot (y - u) \preceq \mathbf{0} \quad \forall y \in \mathcal{Y}$$

and so by letting $z = y - u$, we obtain

$$\widehat{\mathbf{G}}^w \odot z \ominus_{gH} c \|z\| \preceq \Psi_{\mathcal{F}}(u) \odot z \quad \forall z \in \mathcal{Y}. \quad (32)$$

Note that the gH -Fréchet derivative $\Psi_{\mathcal{F}}(u)$ is also gH -Gâteaux derivative as in (see Theorem 5.2 of [14]). Hence, it is a linear IVF as in Definition 4.1 of [14]. By this fact, we have $\Psi_{\mathcal{F}}(u) \odot (0) = \mathbf{0}$. Then, the inequality (32) implies that $(\widehat{\mathbf{G}}^w, c) \in \partial^w(\Psi_{\mathcal{F}}(u)(0))$. Conversely, let $(\widehat{\mathbf{G}}^w, c) \in \partial^w(\Psi_{\mathcal{F}}(u)(0))$. Then, we can write

$$\begin{aligned} & \widehat{\mathbf{G}}^w \odot y \ominus_{gH} c \|y\| \preceq \Psi_{\mathcal{F}}(u) \odot y \quad \forall y \in \mathcal{Y} \\ \implies & \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Psi_{\mathcal{F}}(u) \odot (y - u) \quad \forall y \in \mathcal{Y}. \end{aligned}$$

Since Ψ has gH -Fréchet derivative $\Psi_{\mathcal{F}}(u)$ and it is also a gH -subgradient, it follows that

$$\Psi_{\mathcal{F}}(y) \odot (y - u) \preceq \Psi(y) \ominus_{gH} \Psi(u) \quad \forall y \in \mathcal{Y}.$$

Then, $\widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Psi(y) \ominus_{gH} \Psi(u)$. Hence the proof is complete.

Theorem 10 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let Φ is gH -Fréchet differentiable at u . If u is a weak efficient point of Φ , then

$$\sup \left\{ \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| : (\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u) \right\} = \mathbf{0}.$$

Proof First, we show that

$$\Phi_{\mathcal{F}}(u) \odot (y - u) = \sup \left\{ \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| : (\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u) \right\}$$

by which the desired equality can be easily proved. By gH -Fréchet differentiability of \mathbf{T} and by taking the supremum on the inequality (32), we obtain

$$\begin{aligned} \sup_{(\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u)} \left\{ \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \right\} & \preceq \sup_{(\widehat{\mathbf{G}}^w, c) \in \partial^w \mathbf{T}(u)} \left\{ \Phi_{\mathcal{F}}(u) \odot (y - u) \right\} \\ & = \Phi_{\mathcal{F}}(u) \odot (y - u). \end{aligned}$$

Since $(\Phi_{\mathcal{F}}(u), 0) \in \partial^w \mathbf{T}(y)$,

$$\Phi_{\mathcal{F}}(u) \odot (y - u) \in \left\{ \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| : (\widehat{\mathbf{G}}^w, c) \in \partial^w \Phi(u) \right\}$$

and hence the result follows.

4 Optimality for the difference of two IVFs

In this section, we consider the constrained IOP as below:

$$\min_{y \in \mathcal{Y}} \{ \Phi_2(y) \ominus_{gH} \Phi_1(y) \}, \quad (33)$$

where $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$ and $\Phi_1, \Phi_2 : \mathcal{Y} \rightarrow I(\mathbb{R})$ are two IVFs. We are going to study of weak efficiency condition for the IOP (33) under some additional assumptions.

Theorem 11 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\Phi_1, \Phi_2 : \mathcal{Y} \rightarrow I(\mathbb{R})$ be gH -weak subdifferentiable at u , which is a weak-efficient point of $\Phi_2 \ominus_{gH} \Phi_1$. If $\Phi_1(u) = \Phi_2(u)$, then*

$$\partial^w \Phi_1(u) \subset \partial^w \Phi_2(u).$$

Proof The gH -weak subdifferentiability of Φ_1 at u implies that $\partial^w \Phi_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u) \text{ for all } y \in \mathcal{Y}. \quad (34)$$

Since $\Phi_2 \ominus_{gH} \Phi_1$ gets the weak efficiency value $\mathbf{0}$ at u , for any $y \in \mathcal{Y}$, we have

$$\begin{aligned} \mathbf{0} &\preceq (\Phi_2 \ominus_{gH} \Phi_1)(y) \\ \implies \mathbf{0} &\preceq \Phi_2(y) \ominus_{gH} \Phi_1(y) \\ \implies \Phi_1(y) &\preceq \Phi_2(y) \text{ by Lemma 2.1(ii) of [14]} \\ \implies \Phi_1(y) \ominus_{gH} \Phi_1(u) &\preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ by Note 2 of [1]}. \end{aligned} \quad (35)$$

Consequently, the inequality (35) implies that

$$\widehat{\mathbf{U}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u).$$

This means $(\widehat{\mathbf{U}}^w, c) \in \partial^w \Phi_2(y)$. Hence, the result follows.

Note 1 If we had taken an efficient solution of $\Phi_2 \ominus_{gH} \Phi_1$ instead of a weak efficient solution, the additional condition $\Phi_1(u) = \Phi_2(u)$ becomes essential for Theorem 11 to hold. For instance, let two IVFs $\Phi_1 : [-\frac{1}{2}, \frac{1}{2}] \rightarrow I(\mathbb{R})$ and $\Phi_2 : [-\frac{1}{2}, \frac{1}{2}] \rightarrow I(\mathbb{R})$ be defined as

$$\Phi_1(y) = [2|y|, |y| + 1] \text{ and } \Phi_2(y) = [|y|, 2y^2 + |y|],$$

respectively. Now, according to Theorem 11, $(\Phi_2 \ominus_{gH} \Phi_1)(y) = [2y^2 - 1, -|y|]$, and 0 is an efficient point of $(\Phi_2 \ominus_{gH} \Phi_1)$ because $(\Phi_2 \ominus_{gH} \Phi_1)(y)$ and $(\Phi_2 \ominus_{gH} \Phi_1)(0)$ are not comparable for all $y \in [-\frac{1}{2}, \frac{1}{2}]$. Note that

$$\begin{aligned} \partial^w \Phi_1(0) &= \{(\mathbf{K}_1^w, c_1) : [-2, -1] \preceq (\mathbf{K}_1^w \oplus c_1), (\mathbf{K}_1^w \ominus_{gH} c_1) \preceq [1, 2]\} \\ \text{and } \partial^w \Phi_2(0) &= \{(\mathbf{K}_2^w, c_2) : [-1, -1] \preceq (\mathbf{K}_2^w \oplus c_2), (\mathbf{K}_2^w \ominus_{gH} c_2) \preceq [1, 1]\} \end{aligned}$$

Hence, $\partial^w \Phi_1(0) \not\subset \partial^w \Phi_2(0)$. So, $\Phi_1(u) = \Phi_2(u)$ is an essential condition.

As the restriction $\Phi_1(u) = \Phi_2(u)$ is a bit restrictive, in the next result, we give more flexible condition for which the inclusion in Theorem 11 holds.

Theorem 12 Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let Φ_1, Φ_2 have gH -weak subdifferential at $u \in \mathcal{Y}$, and $\Phi_2 \ominus_{gH} \Phi_1$ attains weak efficient solution at u . Then,

$$\partial^w \Phi_1(u) \subset \partial^w \Phi_2(u) \quad (36)$$

provided that $w(\Phi_1(y)) \geq w(\Phi_2(y))$ for $y \in \mathcal{Y}$ or $w(\Phi_1(y)) \leq w(\Phi_2(y))$ for $y \in \mathcal{Y}$, where $w(\mathbf{A})$ is the width of the interval $\mathbf{A} \in I(\mathbb{R})$.

Proof The gH -weak subdifferentiability of Φ_1 at u implies that $\partial^w \Phi_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u) \text{ for all } y \in \mathcal{Y}. \quad (37)$$

Since u is a weak efficient point of $(\Phi_2 \ominus_{gH} \Phi_1)$,

$$(\Phi_2 \ominus_{gH} \Phi_1)(u) \preceq (\Phi_2 \ominus_{gH} \Phi_1)(y) \quad \forall y \in \mathcal{Y}. \quad (38)$$

• **Case 1.** If $w(\Phi_1(u)) \geq w(\Phi_2(u))$, then from the inequality (38), for all $y \in \mathcal{Y}$, we have

$$\begin{aligned} & [\bar{\phi}_2(u) - \bar{\phi}_1(u), \underline{\phi}_2(u) - \underline{\phi}_1(u)] \preceq [\bar{\phi}_2(y) - \bar{\phi}_1(y), \underline{\phi}_2(y) - \underline{\phi}_1(y)] \\ \implies & \bar{\phi}_1(y) - \bar{\phi}_1(u) \leq \bar{\phi}_2(y) - \bar{\phi}_2(u), \text{ \& } \underline{\phi}_1(u) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \end{aligned} \quad (39)$$

Now there arises two subcases.

• **Subcase 1.** If $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \bar{\phi}_1(y) - \bar{\phi}_1(u)$,

$$\begin{aligned} & \underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\} \text{ and} \\ & \bar{\phi}_1(y) - \bar{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}. \text{ Clearly we have} \\ & [\underline{\phi}_1(y) - \underline{\phi}_1(u), \bar{\phi}_1(y) - \bar{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}, \\ & \quad \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}]. \end{aligned}$$

• **Subcase 2.** If $\bar{\phi}_1(y) - \bar{\phi}_1(u) \leq \underline{\phi}_1(y) - \underline{\phi}_1(u)$,

$$\begin{aligned} & \bar{\phi}_1(y) - \bar{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\} \text{ and} \\ & \underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}. \text{ Clearly we have} \\ & [\bar{\phi}_1(y) - \bar{\phi}_1(u), \underline{\phi}_1(y) - \underline{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}, \\ & \quad \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \bar{\phi}_2(y) - \bar{\phi}_2(u)\}]. \end{aligned}$$

Combining **Subcase 1** and **Subcase 2**, we have

$$\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u). \quad (40)$$

• **Case 2.** If $w(\Phi_2(u)) \geq w(\Phi_1(u))$, then from the inequality (38), for all $y \in \mathcal{Y}$, we have

$$\begin{aligned} & [\bar{\phi}_2(u) - \bar{\phi}_1(u), \underline{\phi}_2(u) - \underline{\phi}_1(u)] \preceq [\bar{\phi}_2(y) - \bar{\phi}_1(y), \underline{\phi}_2(y) - \underline{\phi}_1(y)] \\ \implies & \underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \text{ \& } \bar{\phi}_1(y) - \bar{\phi}_1(u) \leq \bar{\phi}_2(y) - \bar{\phi}_2(u). \end{aligned} \quad (41)$$

By a similar manner as in **Case 1**, we have

$$\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u).$$

Hence, in all cases, we have

$$\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u). \quad (42)$$

In view of (37) and from (42), we get

$$\widehat{\mathbf{U}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ for all } y \in \mathcal{Y}, \text{ by Lemma 2.3 (ii) of [1].}$$

which implies $(\widehat{\mathbf{U}}^w, c) \in \partial^w \Phi_2(u)$. Hence, the result follows.

Note 2 If we had taken an efficient solution of $\Phi_2 \ominus_{gH} \Phi_1$ instead of a weak efficient solution, the additional condition $w(\Phi_1(y)) \geq w(\Phi_2(y))$ or $w(\Phi_1(y)) \leq w(\Phi_2(y))$ becomes essential for Theorem 12 to hold. For instance, consider the IVFs $\Phi_1 : [-1, 1] \rightarrow I(\mathbb{R})$ and $\Phi_2 : [-1, 1] \rightarrow I(\mathbb{R})$ which are defined by

$$\Phi_1(y) = \begin{cases} [y^3, y] & \text{if } 0 \leq y \leq 1 \\ [4y, y] & \text{if } -1 \leq y < 0 \end{cases} \quad \text{and} \quad \Phi_2(y) = \begin{cases} [y^3, 5y] & \text{if } 0 \leq y \leq 1 \\ [3y, 2y] & \text{if } -1 \leq y < 0, \end{cases}$$

respectively. Now, according to Theorem 12,

$$(\Phi_2 \ominus_{gH} \Phi_1)(y) = \begin{cases} [0, 4y] & \text{if } 0 \leq y \leq 1 \\ [y, -y] & \text{if } -1 \leq y < 0 \end{cases}$$

gets efficient solution at 0 because $(\Phi_2 \ominus_{gH} \Phi_1)(0) \preceq (\Phi_2 \ominus_{gH} \Phi_1)(y)$ for all $y \in [0, 1]$ and $(\Phi_2 \ominus_{gH} \Phi_1)(0)$ is not comparable with the values $(\Phi_2 \ominus_{gH} \Phi_1)(y)$ for all $y \in [-1, 0]$. It is not difficult to check that

$$\begin{aligned} \partial^w \Phi_1(0) &= \{(\mathbf{K}_1^w, c_1) : [1, 4] \preceq (\mathbf{K}_1^w \oplus c_1), \mathbf{K}_1^w \ominus_{gH} c_1 \preceq [0, 1]\} \\ \text{and } \partial^w \Phi_2(0) &= \{(\mathbf{K}_2^w, c_2) : [2, 3] \preceq \mathbf{K}_2^w \oplus c_2, \mathbf{K}_2^w \ominus_{gH} c_2 \preceq [0, 5]\}. \end{aligned}$$

Here, we see that $\partial^w \Phi_1(0)$ and $\partial^w \Phi_2(0)$ are not comparable and at same time, we notice that $w(\Phi_2(y)) \geq w(\Phi_1(y))$ on $[0, 1]$ and $w(\Phi_1(y)) \geq w(\Phi_2(y))$ on $[-1, 0]$.

Remark 4 In Theorem 12, the inclusion (36) is necessary but not sufficient condition for weak efficient point of $\Phi_2 \ominus_{gH} \Phi_1$. For instance, consider the IVFs $\Phi_1 : [-1, 1] \rightarrow I(\mathbb{R})$ and $\Phi_2 : [-1, 1] \rightarrow I(\mathbb{R})$ that are defined by

$$\Phi_1(y) = \begin{cases} [y^3, y] & \text{if } 0 \leq y \leq 1 \\ [3y, 1.5y] & \text{if } -1 \leq y < 0 \end{cases} \quad \text{and} \quad \Phi_2(y) = \begin{cases} [y^3 + y^2, 2y^2 + y] & \text{if } 0 \leq y \leq 1 \\ [3y, 2y] & \text{if } -1 \leq y < 0. \end{cases}$$

We notice that $w(\Phi_2(y)) \geq w(\Phi_1(y))$ on $[0, 1]$ and $w(\Phi_2(y)) \leq w(\Phi_1(y))$ on $[-1, 0]$. Note that

$$\begin{aligned} \partial^w \Phi_1(0) &= \{(\mathbf{K}_1^w, c_1) : [1.5, 3] \preceq \mathbf{K}_1^w \oplus c_1, \mathbf{K}_1^w \ominus_{gH} c_1 \preceq [0, 1]\} \\ \text{and } \partial^w \Phi_2(0) &= \{(\mathbf{K}_2^w, c_2) : [2, 3] \preceq \mathbf{K}_2^w \oplus c_2, \mathbf{K}_2^w \ominus_{gH} c_2 \preceq [0, 1]\}. \end{aligned}$$

Hence, $\partial^w \Phi_1(0) \subset \partial^w \Phi_2(0)$ but 0 is not a weak efficient point of $\Phi_2 \ominus_{gH} \Phi_1$ on $[-1, 1]$.

Next, we study a relation between augmented normal cone and gH -weak subdifferential. So, let us define the augmented normal cone to \mathcal{Y} as below.

Definition 18 (*Augmented normal cone*). An *augmented normal cone* to \mathcal{Y} at u is

$$\mathcal{N}_{\mathcal{Y}}^c(u) = \left\{ (\hat{\mathbf{G}}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \hat{\mathbf{G}}^\top \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \mathbf{0} \forall y \in \mathcal{Y} \right\}.$$

Theorem 13 (Optimality condition via augmented normal cone). *An IVF $\Psi : \mathcal{Y} \rightarrow I(\mathbb{R})$ attains weak efficient solution at u if and only if $(\mathbf{0}, 0) \in \partial^w \Psi(u) \oplus \mathcal{N}_{\mathcal{Y}}^c(u)$, where $(\mathbf{0}, 0)$ denotes the zero of $I(\mathbb{R}) \times \mathbb{R}_+$.*

Proof Since u is a weak efficient point of Ψ on \mathcal{Y} ,

$$\begin{aligned} \Psi(u) &\preceq \Psi(y) \forall y \in \mathcal{Y} \\ \implies \mathbf{0} &\preceq \Psi(y) \ominus_{gH} \Psi(u) \forall y \in \mathcal{Y} \text{ by Lemma 2.1(ii) of [14]} \\ \implies (\mathbf{0}, 0) &\in \partial^w \Psi(u). \end{aligned}$$

Let $\delta_{\mathcal{Y}} : \mathcal{Y} \rightarrow I(\mathbb{R})$ be an indicator function, defined by $\delta_{\mathcal{Y}}(y) = \begin{cases} \mathbf{0}, & \text{for } y \in \mathcal{Y} \\ \infty, & \text{for } y \notin \mathcal{Y} \end{cases}$. Since

$$(\Psi \oplus \delta_{\mathcal{Y}})(y) = \begin{cases} \Psi(y) & \text{if } y \in \mathcal{Y} \\ \infty & \text{if } y \notin \mathcal{Y}, \end{cases}$$

$(\mathbf{0}, 0) \in \partial^w \Psi(u) = \partial^w(\Psi \oplus \delta_{\mathcal{Y}})(u)$. It needs to show that $\partial^w(\Psi \oplus \delta_{\mathcal{Y}})(u) \subset \partial^w \Psi(u) \oplus \mathcal{N}_{\mathcal{Y}}^c(u)$. To prove this, let $\widehat{\mathbf{G}}^w \in \partial^w(\Psi \oplus \delta_{\mathcal{Y}})(u)$. Then,

$$\begin{aligned} & \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq (\Psi \oplus \delta_{\mathcal{Y}})(y) \ominus_{gH} (\Psi \oplus \delta_{\mathcal{Y}})(u) \\ \implies & \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq (\Psi(y) \oplus \delta_{\mathcal{Y}}(y)) \ominus_{gH} (\Psi(u) \oplus \delta_{\mathcal{Y}}(u)) \\ \implies & \widehat{\mathbf{G}}^w \odot (y - u) \ominus_{gH} c \|y - u\| \preceq \Psi(y) \ominus_{gH} \Psi(u), \end{aligned}$$

which implies $\widehat{\mathbf{G}}^w \in \partial^w \Psi(u) \subset \partial^w \Psi(u) \oplus \partial^w \delta_{\mathcal{Y}}(u)$, where $\{(\mathbf{0}, 0)\} \subset \partial^w \delta_{\mathcal{Y}}(u)$. Hence, $\widehat{\mathbf{G}}^w \in \partial^w \Psi(u) \oplus \partial^w \delta_{\mathcal{Y}}(u) = \partial^w \Psi(u) \oplus \mathcal{N}_{\mathcal{Y}}^c(u)$.

To show the converse part, let $(\mathbf{0}, 0) \in \partial^w \Psi(u) \oplus \mathcal{N}_{\mathcal{Y}}^c(u) = \partial^w(\Psi(u) \oplus \delta_{\mathcal{Y}}(u))$. Now, for any $y \in \mathcal{Y}$,

$$\begin{aligned} & \mathbf{0} \odot (y - u) \ominus_{gH} 0 \|y - u\| \preceq (\Psi(y) \oplus \delta_{\mathcal{Y}}(y)) \ominus_{gH} (\Psi(u) \oplus \delta_{\mathcal{Y}}(u)) \\ \text{or, } & \mathbf{0} \preceq \Psi(y) \ominus_{gH} \Psi(u) \\ \text{or, } & \Psi(u) \preceq \Psi(y) \text{ by Lemma 2.1(ii) of [14].} \end{aligned}$$

So, u is a weak efficient solution of Ψ .

5 Conclusion

In this paper, the concepts of gH -weak subdifferentials and gH -weak subgradients (Definition 15) for IVFs with illustrative examples have been provided. The gH -weak subdifferential set of an IVF has been found to be convex (Theorem 1) and closed (Theorem 2). We have further introduced a necessary and sufficient condition (Theorem 4) for the set of gH -weak subgradients to be nonempty. We have derived the necessary optimality condition (Theorem 10) involving gH -Fréchet differential and gH -weak subdifferential for IVFs. We have derived a necessary optimality criterion for difference of two IVFs (Theorem 11 and Theorem 13). Towards the end of the paper, we have provided a necessary and sufficient condition for weak efficient solution in terms of two notions of augmented normal cone and gH -weak subdifferential.

Continuing the present study, in the forthcoming work we will attempt to solve the following three problems.

- Introducing a gH -weak subgradient algorithm which characterizes efficient solutions for nonsmooth nonconvex interval optimization problems.
- In future, we will take up the practical optimization problems to be solved by gH -weak subgradient algorithm.
- Analogous to the notion of weak-stability for conventional optimization problems [34], in future, one may attempt to extend the notion for the following IOP (P):

$$\begin{aligned} & \min \Phi(y) \\ \text{subject to } & g_j(y) \leq 0, \quad j = 1, 2, \dots, p \\ & y \in \mathcal{Y}, \end{aligned}$$

where $\Phi : \mathcal{Y} \rightarrow I(\mathbb{R}) \cup \{-\infty, +\infty\}$ is an IVF and $g_j : \mathcal{Y} \rightarrow \mathbb{R}$ is a real-valued constraint, $j = 1, 2, \dots, p$, and the feasible set C is

$$C = \{y \in \mathbb{R}^n : y \in \mathcal{Y}, g_j(y) \leq 0, j = 1, 2, \dots, p\}.$$

To establish an interrelation between strong duality and weak-stability for (P), one may define the augmented Lagrange interval-valued function for (P) as follows. Let J be an arbitrary index set, for which define

$$\mathbb{R}_\lambda^{(J)} := \{e \in \mathbb{R}^{(J)} : |e_j| \leq 1, j \in J(\lambda)\}$$

and $\Lambda := \left\{(\lambda, k) \in \mathbb{R}^{(J)} : \exists e \in \mathbb{R}_\lambda^{(J)}, ke - \lambda \in \mathbb{R}_+^{(J)}\right\},$

where

$$\mathbb{R}^{(J)} := \{\lambda = (\lambda_j)_{j \in J} : \lambda_j = 0 \text{ for all } j \in J \text{ but only finitely many } \lambda_j \neq 0\},$$

$$J(\lambda) := \{j \in J : \lambda_j \neq 0\}, \text{ is a finite subset of } J$$

and $\mathbb{R}_+^J := \{\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}^{(J)} : \lambda_j \geq 0, j \in J\}.$

For each $j \in J$, the augmented Lagrange interval-valued function for (P) can be defined by

$$\mathbf{L}(y, \lambda, k) = \mathbf{T}(y) \ominus_{gH} \langle \lambda, (g_j(y))_j \rangle \oplus \beta((g_j(y)), \lambda, k),$$

where $\beta(u, \lambda, k) : \mathbb{R}^J \times \mathbb{R}^{(J)} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that

$$\beta(y, \lambda, k) = \begin{cases} \sup_{e \in \mathbb{R}_\lambda^{(J)}} \{ \langle ke, u \rangle : ke - \lambda \in \mathbb{R}_+^{(J)} \} & \text{if } J(\lambda) \neq \emptyset, \\ 0 & \text{if } J(\lambda) = \emptyset. \end{cases}$$

The dual of (P) can be found as

$$\begin{aligned} & \max \quad \inf \mathbf{L}(x, \lambda, k) \\ & \text{subject to} \quad (\lambda, k) \in \Lambda. \end{aligned}$$

We will make an effort to reduce the duality gap by weak-stability property of the following perturbation function $\Psi : \mathcal{Y} \times \mathbb{R}^n \rightarrow I(\mathbb{R}) \cup \{+\infty\}$ associated to the IOP (P):

$$\Psi(y, u) = \begin{cases} \Phi(y) & \text{if } y \in \mathcal{Y} \subset \mathbb{R}^n \text{ and } g_j(y) \leq u_j, \forall j = 1, 2, 3, \dots, p \\ +\infty & \text{otherwise,} \end{cases}$$

where $u = (u_1, u_2, \dots, u_n)$ is called the perturbation vector.

- One may also try to apply of the gH -weak subdifferential in the context of zero duality gap in IOPs and interval-valued differential equations. The method for eliminating duality gap will be immediately applicable in the following areas:
 - two person zero-sum game [29]
 - optimal solutions of control problems with first order differential equations [30]
 - Hamilton-Jacobi field theory [30]
 - difference of convex programming [10].
- The newly defined augmented normal cone and gH -weak subdifferential together lead to the thought of introducing supporting cones for set of intervals in the future. This new concept may be used later to describe conic gap, which may be a crucial property to capturing the geometry of nonconvex set of intervals.

A Proof of Lemma 3

Proof Let $\mathbf{W} = [\underline{w}, \overline{w}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$. From the gH -difference, we have the following four possible cases:

- **Case 1.** Given that $\epsilon \preceq (\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$. Since $\underline{w} - \underline{y} \geq \underline{z} + \epsilon$ and $\overline{w} - \overline{y} \geq \overline{z} + \epsilon$, we have $\underline{z} + \epsilon \leq \overline{z} + \epsilon \leq \overline{w} - \overline{y}$. This implies $\underline{z} + \epsilon \leq \min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Also, $\overline{z} + \epsilon \leq \overline{w} - \overline{y} \leq \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Clearly we have $[\underline{z} + \epsilon, \overline{z} + \epsilon] \preceq [\min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}, \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}]$ and hence $\mathbf{Z} \oplus \epsilon \preceq \mathbf{W} \ominus_{gH} \mathbf{Y}$.
- **Case 2.** $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$. Thus, the proof is straightforward and identical to **Case 1**.
- **Case 3.** $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$. Since $\overline{w} - \overline{y} \geq \underline{z} + \epsilon$, $\underline{w} - \underline{y} \geq \overline{z} + \epsilon$, we have $\underline{z} + \epsilon \leq \overline{z} + \epsilon \leq \underline{w} - \underline{y}$. This implies $\underline{z} + \epsilon \leq \min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Also, $\overline{z} + \epsilon \leq \underline{w} - \underline{y} \leq \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Clearly we have $[\underline{z} + \epsilon, \overline{z} + \epsilon] \preceq [\min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}, \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}]$ and hence $\mathbf{Z} \oplus \epsilon \preceq \mathbf{W} \ominus_{gH} \mathbf{Y}$.
- **Case 4.** $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$. Thus, the proof is identical to **Case 3**.

B Proof of Lemma 4

Proof Let $\mathbf{X} = [\underline{x}, \overline{x}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$, $\mathbf{Z} = [\underline{z}, \overline{z}]$ and $\mathbf{W} = [\underline{w}, \overline{w}]$. Then,

$$\begin{aligned} & (\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \\ &= [\min\{\underline{x} + \underline{y} - \underline{z} - \underline{w}, \overline{x} + \overline{y} - \overline{z} - \overline{w}\}, \max\{\underline{x} + \underline{y} - \underline{z} - \underline{w}, \overline{x} + \overline{y} - \overline{z} - \overline{w}\}] \\ &= [\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}]. \end{aligned} \quad (43)$$

We have

$$\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\} \geq \min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\} \quad (44)$$

$$\text{and } \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\} \leq \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}. \quad (45)$$

By (44) and (45), from (43), we write

$$\begin{aligned} & (\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \\ &= [\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}] \\ &\subseteq [\min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}] \\ &= [\min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\}, \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\}] + [\min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}, \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}] \\ &= (\mathbf{X} \ominus_{gH} \mathbf{Z}) \oplus (\mathbf{Y} \ominus_{gH} \mathbf{W}). \end{aligned}$$

C Proof of Lemma 5

Proof Let $\mathbf{W} = [\underline{w}, \overline{w}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$. Then, $-1 \odot \mathbf{W} = [-\overline{w}, -\underline{w}]$, $-1 \odot \mathbf{Y} = [-\overline{y}, -\underline{y}]$, $-1 \odot \mathbf{Z} = [-\overline{z}, -\underline{z}]$. From Definition of gH -difference of two intervals, we have

either $-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\overline{y} - \overline{w}, \underline{y} - \underline{w}]$ or $-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\underline{y} - \underline{w}, \overline{y} - \overline{w}]$.

Then, one of the following holds true :

- (a) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y} - \overline{w} + \overline{z}, \underline{y} - \underline{w} + \underline{z}]$
- (b) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\underline{y} - \underline{w} + \underline{z}, \overline{y} - \overline{w} + \overline{z}]$
- (c) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y} - \underline{w} + \overline{z}, \overline{y} - \overline{w} + \underline{z}]$
- (d) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y} - \overline{w} + \underline{z}, \underline{y} - \underline{w} + \overline{z}]$.

From this, we have

- (a) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$
- (b) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$
- (c) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$
- (d) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$.

On the other hand,

- (a) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$
- (b) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$
- (c) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$
- (d) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$.

Hence, the desired result follows.

D Proof of Lemma 6

Proof Let $\mathbf{X} = [\underline{x}, \bar{x}]$, $\mathbf{Y} = [\underline{y}, \bar{y}]$ and $\mathbf{Z} = [\underline{z}, \bar{z}]$.

(i) Let us consider the following four representations:

- (a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \underline{z}, \bar{x} - \bar{y} - \bar{z}]$,
- (b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \bar{z}, \bar{x} - \bar{y} - \underline{z}]$,
- (c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\bar{x} - \bar{y} - \underline{z}, \underline{x} - \underline{y} - \bar{z}]$,
- (d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\bar{x} - \bar{y} - \bar{z}, \underline{x} - \underline{y} - \underline{z}]$.

• **Case 1.** Given that $\mathbf{0} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. Then we have

$$\begin{aligned} 0 &\leq \underline{x} - \underline{y} \text{ and } 0 \leq \bar{x} - \bar{y} \\ \implies 0 - \underline{z} &\leq \underline{x} - \underline{y} - \underline{z} \text{ and } 0 - \bar{z} \leq \bar{x} - \bar{y} - \bar{z} \\ \implies [0 - \underline{z}, 0 - \bar{z}] &\preceq [\underline{x} - \underline{y} - \underline{z}, \bar{x} - \bar{y} - \bar{z}]. \end{aligned} \quad (46)$$

So, from (46), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.

• **Case 2.** Similarly, we will arrive at this conclusion (46). So, from (46), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.

• **Case 3.** This case can be proved by using same steps as **Case 1**.

• **Case 4.** This case can be proved by using same steps as **Case 2**.

(ii) Let $\mathbf{W} = [\underline{w}, \bar{w}]$. By the definition of gH -difference, there may be the following four cases.

- (a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \underline{w}, \bar{x} - \bar{y} - \bar{w}]$
- (b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \bar{w}, \bar{x} - \bar{y} - \underline{w}]$
- (c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\bar{x} - \bar{y} - \underline{w}, \underline{x} - \underline{y} - \bar{w}]$
- (d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\bar{x} - \bar{y} - \bar{w}, \underline{x} - \underline{y} - \underline{w}]$.

The following two cases are needed to consider for the representation of these above four cases.

• **Case 1.** Since $\mathbf{Z} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, we have

$$\begin{aligned} \underline{z} &\leq \underline{x} - \underline{y} \text{ and } \bar{z} \leq \bar{x} - \bar{y} \\ \implies \underline{z} - \underline{w} &\leq \underline{x} - \underline{y} - \underline{w} \text{ and } \bar{z} - \bar{w} \leq \bar{x} - \bar{y} - \bar{w} \\ \implies \text{either } [\underline{z} - \underline{w}, \bar{z} - \bar{w}] &\preceq [\underline{x} - \underline{y} - \underline{w}, \bar{x} - \bar{y} - \bar{w}] \end{aligned} \quad (47)$$

$$\text{or } [\bar{z} - \bar{w}, \underline{z} - \underline{w}] \preceq [\bar{x} - \bar{y} - \bar{w}, \underline{x} - \underline{y} - \underline{w}] \quad (48)$$

From (47) and (48), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

• **Case 2.** Similarly, at the last step, we have

$$\text{either } [\underline{z} - \underline{w}, \bar{z} - \bar{w}] \preceq [\bar{x} - \bar{y} - \underline{w}, \underline{x} - \underline{y} - \bar{w}] \quad (49)$$

$$\text{or } [\bar{z} - \bar{w}, \underline{z} - \underline{w}] \preceq [\underline{x} - \underline{y} - \bar{w}, \bar{x} - \bar{y} - \underline{w}] \quad (50)$$

From (49) and (50), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

(iii) Given that $\mathbf{X} \ominus_{gH} \mathbf{Y} \preceq [L, L]$. From the formula of gH -difference of intervals,

$$\begin{aligned} \underline{x} - \underline{y} &\leq L \text{ and } \bar{x} - \bar{y} \leq L \\ \implies -L &\leq \underline{y} - \underline{x}, -L \leq \bar{y} - \bar{x} \\ \implies \text{either } [-L, -L] &\preceq [\underline{y} - \underline{x}, \bar{y} - \bar{x}] \\ \text{or } [-L, -L] &\preceq [\bar{y} - \bar{x}, \underline{y} - \underline{x}]. \end{aligned}$$

Hence, $[-L, -L] \preceq \mathbf{Y} \ominus_{gH} \mathbf{X}$.

(iv) Given that $[-\gamma, -\gamma] \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. From the formula of gH -difference of intervals,

$$\begin{aligned} -\gamma &\leq \underline{x} - \underline{y} \text{ and } -\gamma \leq \bar{x} - \bar{y} \\ \implies \underline{y} - \gamma &\leq \underline{x} \text{ and } \bar{y} - \gamma \leq \bar{x} \\ \implies [\underline{y} - \gamma, \bar{y} - \gamma] &\preceq [\underline{x}, \bar{x}]. \end{aligned}$$

Hence, $\mathbf{Y} \ominus_{gH} [\gamma, \gamma] \preceq \mathbf{X}$.

(v) Given that $\mathbf{Z} \preceq \mathbf{X} \oplus \mathbf{Y}$. Then,

$$\begin{aligned} [\underline{z}, \bar{z}] &\preceq [\underline{x}, \bar{x}] \oplus [\underline{y}, \bar{y}] \\ \implies \underline{z} &\leq \underline{x} + \underline{y}, \bar{z} \leq \bar{x} + \bar{y} \\ \implies \underline{z} - \underline{y} &\leq \underline{x}, \bar{z} - \bar{y} \leq \bar{x} \\ \implies [\underline{z} - \underline{y}, \bar{z} - \bar{y}] &\preceq [\underline{x}, \bar{x}]. \end{aligned}$$

Hence, $\mathbf{Z} \ominus_{gH} \mathbf{Y} \preceq \mathbf{X}$.

E Proof of Lemma 7

Proof Let $y^\top \odot \widehat{\mathbf{C}} = \mathbf{D}$ and $\mathbf{D} = [d, \bar{d}]$. Note that

$$\|\mathbf{D}\|_{I(\mathbb{R})} = \max\{|d|, |\bar{d}|\}. \quad (51)$$

On the other hand,

$$\begin{aligned} \|\mathbf{D}\|_{I(\mathbb{R})} &= \|y_1 \odot \mathbf{C}_1 \oplus y_2 \odot \mathbf{C}_2 \oplus \cdots \oplus y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\ &\leq \|y_1 \odot \mathbf{C}_1\|_{I(\mathbb{R})} + \|y_2 \odot \mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + \|y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\ &= |y_1| \|\mathbf{C}_1\|_{I(\mathbb{R})} \oplus |y_2| \|\mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + |y_n| \|\mathbf{C}_n\|_{I(\mathbb{R})} \\ &\leq \|y\| \sum_{i=1}^n \|\mathbf{C}_i\|_{I(\mathbb{R})} \\ &= \|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n. \end{aligned} \quad (52)$$

Then, taking into account (51) and (52), we obtain

$$\begin{aligned} |d| &\leq \|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n \text{ and } |\bar{d}| \leq \|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n \\ \implies -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n &\leq d \text{ and } -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n \leq \bar{d} \\ \implies -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n &\leq |d| \text{ and } -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n \leq |\bar{d}| \\ \implies -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n &\leq \max\{|d|, |\bar{d}|\} \\ \implies -\|y\| \|\widehat{\mathbf{C}}\|_{I(\mathbb{R})}^n &\leq \|\mathbf{D}\|_{I(\mathbb{R})} \end{aligned}$$

Thus, we arrived at the desired result.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Not applicable.

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Authors' contributions

All authors contributed to the study conception and analysis. All authors read and approved the final manuscript.

References

1. Anshika, Ghosh D, Chauhan R S and Mesiar R, Generalized-Hukuhara subdifferential analysis and its application in nonconvex composite optimization problems with interval-valued functions, (2021) arXiv preprint, arXiv:2109.14586
2. Anshika and Ghosh D, Interval-valued value function and its application in interval optimization problems, *J. Comput. Appl. Math.* **41(4)** (2022) 1–26
3. Azimov A Y and Gasimov R N, On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, *Int. J. Appl. Math. Comput. Sci.* **1(2)** (1999) 171–192
4. Chalco-Cano Y, Lodwick W A and Rufián-Lizana A, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, *Fuzzy Optim. Decis. Mak.* **12(3)** (2013) 305–322
5. Chalco-Cano Y, Rufián-Lizana A, Román-Flores H and Jiménez-Gamero M D, Calculus for interval-valued functions using generalized-Hukuhara derivative and applications, *Fuzzy Sets Syst.* **219** (2013) 49–67
6. Chalco-Cano Y, Maqui-Huamán G G, Silva G N, and Jiménez-Gamero M D, Algebra of generalized-Hukuhara differentiable interval-valued functions: review and new properties, *Fuzzy Sets Syst.* **375** (2019) 53–69
7. Chen X and Li Z, On optimality conditions and duality for non-differentiable interval-valued programming problems with the generalized (F, ρ) -convexity, *J. Ind. Manag.* **14(3)** (2018) 895–912
8. Chauhan R S, Ghosh D, Ramik J and Debnath A K, Generalized Hukuhara-Clarke derivative of interval-valued functions and its properties, *Soft Comput.* **25(23)** (2021) 14629–14643
9. Clarke F H, Optimization and Nonsmooth Analysis, (1983) (New York: Wiley-Interscience)
10. Gao D Y, Duality theory: biduality in nonconvex optimization, in: Floudas C., Pardalos P. (eds) Encyclopedia of Optimization (2008) (Boston, MA: Springer) pp. 477–482
11. Gasimov R N, Augmented Lagrangian duality and nondifferentiable optimization methods in nonconvex programming, *J Glob Optim.* **24(2)** (2002) 187–203
12. Ghosh D, Debnath, A K, Mesiar R and Chauhan R S, Generalized-Hukuhara subgradient and its application in optimization problem with interval-valued functions, *Sadhana* **47(2)** (2022) 1–16
13. Ghosh D, Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions, *J. Appl. Math. Comput.* **53(1-2)** (2017) 709–731
14. Ghosh D, Chauhan R S, Mesiar R and Debnath A K, Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions, *Inf. Sci.* **510** (2020) 317–340
15. Ghosh D, Debnath A K, Chauhan R S and Castillo O, Generalized-Hukuhara-Gradient efficient-direction method to solve optimization problems with interval-valued functions and its application in least squares problems, *Int. J. Fuzzy Syst.* **24(3)** (2021) 1275–1300
16. Hai S and Gong Z, The differential and subdifferential for fuzzy mappings based on the generalized difference of n -cell fuzzy-numbers, *J. Comput. Anal. Appl.* **24(1)** (2018) pp. 184–195
17. Hassani S, Mammadov M and Jamshidi M, Optimality conditions via weak subdifferentials in reflexive Banach spaces, *Turkish J. Math.* **41(1)** (2017) 1–8
18. Hassani S and Mammadov M A, Optimality conditions in infinite horizon optimization by contingent derivative, *Pac. J. Optim.* **14(3)** (2018) 451–462
19. Hiriart-Urruty J B, Generalized differentiability, duality and optimization for problems dealing with differences of convex functions, in: Ponstein J. (eds) Convexity and Duality in Optimization, Lect. Notes Econ. Math. Syst. **256** (1985) (Berlin, Heidelberg: Springer) pp. 37–70
20. Karaman E, Optimality conditions of interval-valued optimization problems by using subdifferentials, *Gazi Univ. J. Sci.* **33(4)** (2020) 835–841
21. Kasimbeyli R, Inceoglu G, The properties of the weak subdifferentials, *Gazi Univ. J. Sci.* **23(1)** (2010) 49–52
22. Karaman, E. (2021), A generalization of interval-valued optimization problems and optimality conditions by using scalarization and subdifferentials, *Kuwait J. Sci.* **48(2)** (2021) 1–11
23. Kasimbeyli R and Mammadov M, On weak subdifferentials, directional derivatives, and radial epiderivatives for nonconvex functions, *SIAM J. Optim.* **20(2)** (2009) 841–855
24. Kumar G, and Ghosh D, Ekeland’s variational principle for interval-valued functions, (2021) arXiv:2104.11167
25. Markov S, Calculus for interval functions of a real variable, *Computing*, **22(4)** (1979) 325–337
26. Moore R E, Interval Analysis, Vol. 4 (1966) (Englewood Cliffs: Prentice-Hall)
27. Moore, R E, Methods and Applications of Interval Analysis (1979) SIAM Philadelphia.
28. Osuna-Gómez R, Chalco-Cano Y, Hernández-Jiménez B and Ruiz-Garzón G, Optimality conditions for generalized differentiable interval-valued functions, *Inf. Sci.* **321** (2015) 136–146
29. Pappas P and Rustem B, Duality gaps in nonconvex optimization, in: Floudas C., Pardalos P. (eds) Encyclopedia of Optimization (2008) (Boston, MA: Springer) pp. 802–805
30. Pickenhain S, Duality in optimal control with first order differential equations, in: Floudas C., Pardalos P. (eds) Encyclopedia of Optimization (2008) (Boston, MA : Springer) pp. 805–811
31. Rockafellar R T, The Theory of Subgradients and its Applications to Problems of Optimization: Convex and Nonconvex Functions, Research and Education in Mathematics, Vol. 1 (1981) (Berlin: Heldermann Verlag)
32. Stefanini L, A Generalization of Hukuhara difference, in: Dubois D., Lubiano M. A., Prade H., Gil M. A., Grzegorzewski P., Hryniewicz O. (eds) Soft Methods for Handling Variability and Imprecision, adv. soft comput. **48** (2008) (Berlin, Heidelberg: Springer) pp. 203–210

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33. Stefanini L and Bede B, Generalized-Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Anal. Theory Methods Appl.* **71(3-4)** (2009) 1311–1328
 34. Son T Q, Kim D S, and Tam N N, Weak stability and strong duality of a class of nonconvex infinite programs via augmented Lagrangian, *J Glob Optim.* **53(2)** (2012) 165–184
 35. Tao J, and Zhang Z, Properties of interval-valued function space under the gH-difference and their application to semi-linear interval differential equations, *Adv. Differ. Equ.* **2016(1)** (2016) 1–28
 36. Wu H C (2007), The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, *Eur. J. Oper. Res.* **176(1)** 46–59
 37. Yalcin G D and Kasimbeyli R (2020), On weak conjugacy, augmented Lagrangians and duality in nonconvex optimization, *Math. Methods Oper. Res.* **92(1)** (2020) 199–228