

NONPOSITIVE TOWERS IN BING’S NEIGHBOURHOOD

MAX CHEMTOV AND DANIEL T. WISE

ABSTRACT. Every 2-dimensional spine of an aspherical 3-manifold has the nonpositive towers property, but every collapsed 2-dimensional spine of a 3-ball containing a 2-cell has an immersed sphere.

1. INTRODUCTION

Definition 1.1. A 2-complex X has *nonpositive immersions* if for every combinatorial immersion $Y \rightarrow X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

A 2-complex X has *nonpositive towers* if for every tower map $Y \rightarrow X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

There are many variations: For instance, one can generalize to combinatorial near-immersions, or relax to $\pi_1 Y = 1$ or $\chi(Y) \leq 1$, and there also variations requiring $\chi(Y) \leq -c|Y|$ for some “size” $|Y|$ of Y . Note that nonpositive immersions implies nonpositive towers. The main consequences of nonpositive immersions hold for nonpositive towers. E.g. if X has nonpositive towers then $\pi_1 X$ is locally indicable. These ideas have promise as a contextualizing framework towards Whitehead’s asphericity conjecture, as well as towards understanding coherence.

In [8] it was shown that every aspherical 3-manifold with nonempty boundary has a spine with nonpositive immersions. This utilized that there exists a spine with no near-immersion of a 2-sphere [1].

In this note, we observe the following failure, which is a special case of Proposition 3.5:

Theorem 1.2. *Every collapsed spine of a simply-connected 3-manifold containing a disc has an immersed sphere.*

Bing’s “house with two rooms” provides such a spine. It is obtained from a 3-ball divided into two rooms by a pair of collapses, corresponding to entering the left room from the right side of the house and entering the right room from the left side. See Figure 1.

W. Fisher also found examples of the failure of nonpositive immersions in other contractible 2-complexes: the Miller-Schupp balanced presentations of the trivial group [2]. There are thus two sources of counter-examples to the conjecture that contractible 2-complexes have nonpositive immersions [8, Conj 1.7].

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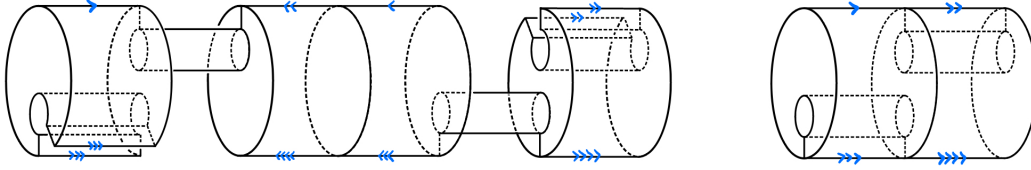


FIGURE 1. An immersed sphere (left) in Bing's house (right).

In [3] it is shown that for $n \geq 3$, every PL n -manifold M with $\partial M \neq \emptyset$ has a spine X such that $\partial M \rightarrow X$ is an immersion, and moreover, such spines are generic among all spines. So Theorem 1.2 is a variant of the simplest instance of their result. However it is a counterpoint to the following statement (see Theorem 2.11) which is our main motive:

Theorem 1.3. *Every 2-dimensional spine of an aspherical 3-manifold has nonpositive towers.*

Thus nonpositive immersions does not always hold for a natural family of contractible complexes which nevertheless have nonpositive towers, so we are motivated to refocus on:

Conjecture 1.4. *Every contractible 2-complex has nonpositive towers.*

For instance:

Proposition 1.5. *A contractible 2-complex with two 2-cells has nonpositive towers.*

Proof. As a contractible 2-complex X admits no nontrivial connected covering map, any tower map $Y \rightarrow X$ must begin with a subcomplex $X' \subset X$, so X' has at most one 2-cell. But X' has nonpositive immersions (hence nonpositive towers) by [7, 5]. \square

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2. THICKENINGS AND NONPOSITIVE TOWERS

Definition 2.1. Let X be a connected 2-complex. A *tower map* $Y \rightarrow X$ is a finite composition of covering maps and subcomplex embeddings, such that Y and the domain of each embedding are compact connected 2-complexes.

Tower maps arose in Papakyriakopolous' classical 3-manifold proofs, and also arose naturally in one-relator group theory [6].

Definition 2.2. A 2-complex X has *nonpositive towers* if, for any tower map $Y \rightarrow X$, either $\chi(Y) \leq 0$ or Y is contractible.

The goal of this section is to prove the nonpositive tower property for an aspherical 2-complex X embedded in a 3-manifold M . The idea of the proof is to consider a manifold "thickening" T of X in M , which deformation retracts to X . The asphericity of X ensures

that ∂T has no 2-sphere, which in turn ensures $\chi(X) \leq 0$. Asphericity is preserved by towers, since it is preserved by both covering maps and subcomplexes. The latter is ensured by a simple argument using the Sphere Theorem.

Convention 2.3. We work in the category of PL-manifolds: all submanifolds, as well as cells of complexes embedded in a manifold, are assumed to be PL-embedded. This avoids pathologies like Alexander's horned sphere.

Construction 2.4. Let X be a locally finite 2-complex. A *thickening* of X is a 3-manifold $T = T(X)$ with boundary, and a continuous map $\Theta : T \rightarrow X$, constructed as follows:

- Let T^0 be a disjoint union of closed 3-balls, one for each vertex in X , and let Θ map each ball to its corresponding vertex.
- For each edge e in X , define $T(e) \cong [0, 1] \times D^2$, and identify $\{0\} \times D^2$ and $\{1\} \times D^2$ with discs on the boundary of the components of T^0 corresponding to the endpoints of e . Let Θ map $(0, 1) \times D^2$ onto $\text{int}(e)$. The resulting complex is T^1 .

We require that each $T(e)$ embeds and that $T(e_1) \cap T(e_2) = \emptyset$ for $e_1 \neq e_2$.

- For each disc F in X , define $T(F) \cong D^2 \times [0, 1]$, and identify the outer cylinder $S^1 \times [0, 1]$ with an embedded cylinder on the boundary of T^1 which is mapped by Θ to the attaching loop of F in X . Let Θ map $\text{int}(D^2) \times [0, 1]$ onto $\text{int}(F)$. The resulting complex is T .

We require that each $T(F)$ embeds and that $T(F_1) \cap T(F_2) = \emptyset$ for $F_1 \neq F_2$.

For a subcomplex $A \subseteq X$, we use $T(A) \subseteq T(X)$ to denote the thickening of A to $\Theta^{-1}(A)$ induced by the thickening of X .

Remark 2.5. If a thickening T of X exists, then by construction, T is a 3-manifold with boundary. Furthermore, there is a PL-embedding $X \hookrightarrow \text{int}(T(X))$ such that T deformation retracts to X , with $T(A)$ retracting to A for any subcomplex $A \subseteq X$.

Remark 2.6. If X PL-embeds in a 3-manifold M , then we can take T to be a small closed neighbourhood of X in M . Let $\Theta : T(X) \rightarrow X$ be a retraction homotopic to the identity map $T(X) \rightarrow T(X)$. Then T and Θ give a thickening of X .

Lemma 2.7. *Let X be a locally finite aspherical connected 2-complex that PL-embeds in a 3-manifold. Then removing a collection of 2-cells $\{D_i\}$ in X results in a new 2-complex Y which is also aspherical.*

Proof. If $T(X)$ is non-orientable, we can consider an orientable double-cover $\widehat{T(X)}$. This induces a double-cover \widehat{X} of X , which is locally finite, aspherical, connected, and PL-embeds in $T(\widehat{X}) = \widehat{T(X)}$. Consider the induced orientable double-cover $\widehat{Y} \subseteq \widehat{X}$ of Y , and note that \widehat{Y} can be obtained from \widehat{X} by deleting a collection of 2-cells. It suffices to prove that \widehat{Y} is aspherical, since this would imply that Y is aspherical. Since the non-orientable case with X and Y reduces to the orientable case with \widehat{X} and \widehat{Y} , we can assume without loss of generality that $T(X)$ is orientable.

Let Y be a 2-complex, and let $X = Y \cup \left(\bigcup_i D_i \right)$, where each 2-cell D_i is attached to Y along ∂D_i . We assume X is a locally finite aspherical connected 2-complex that PL-embeds in an orientable manifold, and prove that Y is aspherical.

Suppose for contradiction that Y is not aspherical. Let $(T(X), \Theta)$ be an orientable thickening of X . Then $\pi_2(\text{int}(T(Y))) \neq 0$. Since $\text{int}(T(Y))$ is orientable, $\text{int}(T(Y))$ has an embedded essential 2-sphere S by the Sphere Theorem [4, Thm 3.8]. Since X is aspherical, S bounds a contractible submanifold $B \subseteq \text{int}(T(X))$ by [4, Prop 3.10]. Note that $T(X) - S$ has two connected components: $\text{int}(B)$ and the component C containing $\partial T(X)$.

For any D_i , we have $\Theta^{-1}(\text{int}(D_i)) \cap S = \emptyset$, since $S \subseteq T(Y)$ and $\text{int}(D_i) \cap Y = \emptyset$. Since $\Theta^{-1}(\text{int}(D_i))$ is connected, it must lie either entirely in $\text{int}(B)$ or entirely in C . By construction of $T(X)$, we know that $\Theta^{-1}(\text{int}(D_i)) \cap \partial T(X) \neq \emptyset$. So $\Theta^{-1}(\text{int}(D_i)) \subset C$.

Since this is true for all D_i , we have $\text{int}(B) \subseteq T(X) - \bigcup_i \Theta^{-1}(\text{int}(D_i)) = T(Y)$. Since $\text{int}(B)$ is an open submanifold of $T(Y)$, it is contained in $\text{int}(T(Y))$. Therefore, S bounds a contractible submanifold B of $\text{int}(T(Y))$, contradicting that S is essential in $\text{int}(T(Y))$. \square

Corollary 2.8. *Let X be a locally finite aspherical connected 2-complex that PL-embeds in a 3-manifold. Then every subcomplex $Y \subseteq X$ is aspherical.*

Proof. By Lemma 2.7, removing 2-cells from X yields another aspherical 2-complex. Since removing 0- and 1-cells also preserves asphericity, every subcomplex of X is aspherical. \square

Lemma 2.9. *Let M be a compact orientable 3-manifold with boundary. Then $\chi(M) = \frac{1}{2}\chi(\partial M)$. And if ∂M does not contain a 2-sphere then $\chi(M) \leq 0$.*

Proof. Let \widetilde{M} be the manifold obtained by gluing two copies of M along ∂M . Since \widetilde{M} is a closed 3-manifold, $2\chi(M) - \chi(\partial M) = \chi(\widetilde{M}) = 0$. So $\chi(M) = \frac{1}{2}\chi(\partial M)$. Since M is orientable, each component of ∂M is an orientable surface. Since ∂M contains no 2-sphere, every component of ∂M has nonpositive χ . So $\chi(M) = \frac{1}{2}\chi(\partial M) \leq 0$. \square

Lemma 2.10. *Let $X' \rightarrow X$ be a finite-sheeted cover of a compact connected 2-complex X . Then X has nonpositive towers if and only if X' has nonpositive towers.*

Proof. Suppose X has nonpositive towers. Let $Y \rightarrow X'$ be a tower map. Then $Y \rightarrow X' \rightarrow X$ is a tower map, so either $\chi(Y) \leq 0$ or Y is contractible.

Suppose that X' has nonpositive towers. Let $t : Y \rightarrow X$ be a tower map. Let n be the degree of the cover $p : X' \rightarrow X$. Then there is an induced tower map $Y' \rightarrow X'$, where Y' is an n -sheeted cover of Y , and maps to $p^{-1}(t(Y))$. Either $\chi(Y') \leq 0$ or Y' is contractible.

If $\chi(Y') \leq 0$, then $\chi(Y) = \frac{1}{n}\chi(Y') \leq 0$. If Y' is contractible, then Y' is the universal cover of Y , so Y is a $K(\pi_1(Y), 1)$ complex with $|\pi_1 Y| = n$. A nontrivial finite group does not have a compact $K(\pi, 1)$, so $\pi_1 Y = 1$. Thus $Y = Y'$ is contractible. \square

Theorem 2.11. *Let X be an aspherical compact connected 2-complex that PL-embeds in a 3-manifold M . Then X has nonpositive towers.*

Proof. Since X is compact, it PL-embeds in a thickening $T(X)$ in M that is a compact sub-3-manifold with boundary. So without loss of generality, we can assume that $M = T(X)$ is a compact 3-manifold with boundary that deformation retracts to X .

If M is non-orientable, we consider an orientable double cover of M that deformation retracts to a double cover X' of X . By Lemma 2.10, it suffices to show that X' has nonpositive towers. So without loss of generality, we can assume that M is orientable.

X itself is either contractible or has $\chi(X) \leq 0$. Indeed, if ∂M includes a 2-sphere S , then asphericity of M ensures that S bounds a contractible submanifold of M [4, Prop 3.10]. Since M is connected, this submanifold must be M itself, and so M and X are contractible. If ∂M does not include any 2-spheres, then $\chi(X) = \chi(M) \leq 0$ by Lemma 2.9.

Note that any covering map $\widehat{X} \rightarrow X$ (with \widehat{X} connected) extends to a covering map $\widehat{M} \rightarrow M$, where \widehat{X} PL-embeds in $\text{int}(\widehat{M})$. Then \widehat{X} is locally finite, aspherical, connected, and PL-embeds in a 3-manifold.

Let X' be a compact connected subcomplex of \widehat{X} . By Corollary 2.8, X' is aspherical. And X' also PL-embeds in $\text{int}(\widehat{M})$. Therefore, X' satisfies the same hypotheses as X , and is also either contractible or has $\chi(X') \leq 0$.

Any tower map $Y \rightarrow X$ is a composition of maps $X' \hookrightarrow \widehat{X} \twoheadrightarrow X$, so we are done. \square

3. BING'S NEIGHBOURS

In this section, we give an alternate construction of the thickening T of a 2-complex X embedded in a 3-manifold. The cell structure on ∂T “follows” X and provides an immersion $\partial T \rightarrow X$. When X is simply-connected, collapsed, and contains a 2-cell, ∂T is a union of 2-spheres. Thus examples like Bing’s house fail to have nonpositive immersions.

Definition 3.1. A 2-complex is *collapsed* if no cell has a free face - i.e. no 0-cell has degree 1, and no 1-cell is incident to a single side of a 2-cell.

Construction 3.2. Let X be a compact connected collapsed 2-complex with no isolated vertex or edge that PL-embeds in a 3-manifold M . We give a construction for the thickening $T = T(X)$, yielding an explicit cell structure on ∂T related to the cell structure on X :

- ∂T^0 For each vertex v in X , consider a small neighbourhood $N(v) \subseteq M$ of v . Add a 0-cell in each component of $N(v) - X$. The union of these 0-cells is ∂T^0 .
- ∂T^1 For each edge e in X , consider a small neighbourhood $N(e) \subseteq M$ of e containing the 0-cells in ∂T^0 associated with the endpoints of e . Add a 1-cell in each component of $N(e) - X$ that contains a 0-cell associated to each endpoint of e . That 1-cell joins those 0-cells. If n sides of discs are incident with e in X , then this process yields n 1-cells parallel to e . The union of ∂T^0 with these 1-cells is ∂T^1 .
- ∂T^2 For each disc d of X , consider a small neighbourhood $N(d) \subseteq M$ of d containing the 1-cells in ∂T^1 associated with the ∂d . Add a 2-cell in each component of $N(d) - X$ containing 1-cells associated to all edges of ∂d . Attach this 2-cell to those 1-cells according to ∂d . This yields two 2-cells on opposite sides of d in M . The union of ∂T^1 with these 2-cells is ∂T .

The submanifold T is the union of ∂T and the component of $M - \partial T$ containing X .

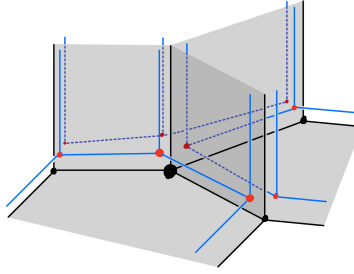


FIGURE 2. Part of $\partial T(X)$ for a complex X . The 0-cells of ∂T are red, and the 1-cells of ∂T are blue. The 2-cells of ∂T run parallel to the grey discs in X , forming a “bubble” hovering around X .

Lemma 3.3. *$T = T(X)$ deformation retracts to X . The retraction $r : T \rightarrow X$ induces an immersion $\partial T \rightarrow X$. If X is simply-connected, then ∂T is a union of 2-spheres.*

Proof. Consider the map $\partial T \rightarrow X$ sending i -cells in ∂T to their associated i -cells in X . T is homeomorphic to the mapping cylinder of $\partial T \rightarrow X$, yielding a deformation retraction.

Let c_1 and c_2 be closed i -cells in ∂T with $c_1 \cap c_2 \neq \emptyset$. Suppose $r(c_1) = r(c_2) = c$. Then $c_1 \cup c_2$ is a connected subset of the neighbourhood $N(c)$ used in the construction of ∂T . At most one i -cell was added for each component of $N(c)$, so $c_1 = c_2$. Thus $r|_{\partial T} : \partial T \rightarrow X$ is an immersion.

The map $\partial T \rightarrow X$ can also be seen geometrically to be an immersion: in terms of Figure 2, each cycle of 2-cells around a vertex in ∂T is mapped to a cycle of discs in X associated to a corner of $M - X$.

Suppose $\pi_1 X = 1$. Then T is a compact orientable 3-manifold with $\text{rank}(\mathbf{H}_1(T)) = 0$, so $\text{rank}(\mathbf{H}_1(\partial T)) = 0$ by “half lives, half dies” [4, Lem 3.5]. Thus ∂T is a union of 2-spheres. \square

Lemma 3.4. *Let X be a compact 2-complex with a collapsed subcomplex Y containing a disc. Then X has a connected collapsed subcomplex Y' with no isolated vertex or edge, whose inclusion map $Y' \hookrightarrow X$ is π_1 -injective.*

Proof. If X contains a free i -face, delete that face and its attached $(i+1)$ -cell. This deletion does not affect π_1 . Repeat this process until the remaining subcomplex is collapsed. Delete all isolated edges leaving a disjoint union X' of collapsed components X'_i without isolated vertices or edges. Each $X'_i \hookrightarrow X$ is π_1 -injective. As the disc from Y must be contained in some X'_i , we let $Y' = X'_i$. \square

Proposition 3.5. *Let X be a 2-spine of a simply-connected 3-manifold. Then X has an immersed 2-sphere if and only if X has a collapsed subcomplex containing a disc.*

Proof. Suppose $S^2 \looparrowright X$ is a combinatorial immersion. Then $\text{im}(S^2 \looparrowright X)$ is a collapsed subcomplex containing a disc.

Conversely, suppose X has a collapsed subcomplex Y containing a disc. By Lemma 3.4, we can assume Y has no isolated vertex or edge, and $\pi_1 Y \leq \pi_1 X = 1$. Then by Lemma 3.3, $\partial T(X)$ is a union of 2-spheres that immerses in X . \square

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DEPT. OF MATH. & STATS., MCGILL UNIV., MONTREAL, QC, CANADA H3A 0B9
Email address: max.chemtov@mail.mcgill.ca

DEPT. OF MATH. & STATS., MCGILL UNIV., MONTREAL, QC, CANADA H3A 0B9
Email address: wise@math.mcgill.ca