

NONLINEAR STABILITY OF SINUSOIDAL EULER FLOWS ON A FLAT TWO-TORUS

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ABSTRACT. Sinusoidal flows are an important class of explicit stationary solutions of the two-dimensional incompressible Euler equations on a flat torus. For such flows, the stream functions are eigenfunctions of the negative Laplacian. In this paper, we prove that any sinusoidal flow related to some least eigenfunction is, up to phase translations, nonlinearly stable under L^p norm of the vorticity for any $1 < p < +\infty$, which improves a classical stability result by Arnold based on the energy-Casimir method. The key point of the proof is to distinguish least eigenstates with fixed amplitude from others by using isovortical property of the Euler equations.

1. INTRODUCTION AND MAIN RESULT

1.1. **Two-dimensional Euler equations on a flat torus.** Let \mathbb{T}^2 be a flat two-torus whose fundamental domain is

$$\mathbb{T}^2 = [0, 2\pi\nu_1] \times [0, 2\pi\nu_2],$$

where ν_1, ν_2 are fixed positive constants. The motion of an ideal fluid of unit density on \mathbb{T}^2 is described by the following Euler equations:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P, & \mathbf{x} = (x_1, x_2) \in \mathbb{T}^2, t > 0, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \quad (1.1)$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity field, and P is the scalar pressure. The *scalar vorticity* ω of the fluid is given by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1. \quad (1.2)$$

For smooth solutions of (1.1), the following quantities are conserved for all time (see [21, 22]):

(C1) The total flux \mathbf{F} of velocity,

$$\mathbf{F} = \int_{\mathbb{T}^2} \mathbf{v} d\mathbf{x}, \quad (1.3)$$

(C2) The kinetic energy \mathcal{E} ,

$$\mathcal{E} = \frac{1}{2} \int_{\mathbb{T}^2} |\mathbf{v}|^2 d\mathbf{x}, \quad (1.4)$$

(C3) The distribution function of vorticity $\mathbf{d}_{\omega(t, \cdot)}$,

$$\mathbf{d}_{\omega(t, \cdot)}(s) = |\{\mathbf{x} \in \mathbb{T}^2 \mid \omega(t, \mathbf{x}) > s\}|, \quad s \in \mathbb{R},$$

where $|\cdot|$ denotes the two-dimensional Lebesgue measure.

If we denote by $\mathcal{R}(f)$ the set of all equimeasurable rearrangements of some $f \in L^1_{\text{loc}}(\mathbb{T}^2)$, i.e.,

$$\mathcal{R}(f) = \{g \in L^1_{\text{loc}}(\mathbb{T}^2) \mid d_g = d_f\},$$

then the conservation of the distribution function of vorticity can also be expressed as

$$\omega(t, \cdot) \in \mathcal{R}(\omega(0, \cdot)), \quad \forall t \geq 0. \quad (1.5)$$

As a consequence, the L^p norm of vorticity is conserved for any $1 \leq p \leq +\infty$. In particular, the *enstrophy* Z of the fluid, defined by

$$Z(\omega) = \frac{1}{2} \int_{\mathbb{T}^2} \omega^2 d\mathbf{x}, \quad (1.6)$$

is conserved.

Below we introduce the vorticity-stream formulation of Euler equations (1.1). Define the *normalized velocity* $\tilde{\mathbf{v}}$ as

$$\tilde{\mathbf{v}} = \mathbf{v} - \frac{1}{|\mathbb{T}^2|} \mathbf{F}, \quad (1.7)$$

where \mathbf{F} is the total flux given by (1.3). Note that \mathbf{F} a constant vector not depending on the time variable. It is clear that $\tilde{\mathbf{v}}$ is divergence-free and has zero integral over \mathbb{T}^2 . By the discussion on p. 50 of [21], there is a function $\tilde{\psi} : \mathbb{T}^2 \rightarrow \mathbb{R}$, called the *normalized stream function*, such that

$$\tilde{\mathbf{v}} = \nabla^\perp \tilde{\psi}. \quad (1.8)$$

Here and henceforth, $\mathbf{b}^\perp = (b_2, -b_1)$ denotes the clockwise rotation through $\pi/2$ of some planar vector $\mathbf{b} = (b_1, b_2)$, and $\nabla^\perp f = (\nabla f)^\perp$ for some function f . Without loss of generality, by adding a suitable constant, we always assume that the normalized stream function has zero integral over \mathbb{T}^2 . Then $\tilde{\psi}$ satisfies

$$\begin{cases} -\Delta \tilde{\psi} = \omega, & \mathbf{x} \in \mathbb{T}^2, \\ \int_{\mathbb{T}^2} \tilde{\psi} d\mathbf{x} = 0. \end{cases} \quad (1.9)$$

By our Lemma 2.1 in Section 2, the Poisson equation (1.9) has a unique solution, denoted by $\tilde{\psi} = K\omega$. Then the velocity \mathbf{v} can be determined by the vorticity ω and the total flux \mathbf{F} as follows:

$$\mathbf{v} = \nabla^\perp K\omega + \frac{1}{|\mathbb{T}^2|} \mathbf{F}, \quad (1.10)$$

which is usually called the Biot-Savart law. The *stream function* ψ of the fluid is defined by

$$\psi = K\omega - \frac{1}{|\mathbb{T}^2|} \mathbf{F}^\perp \cdot \mathbf{x} \quad (1.11)$$

such that $\mathbf{v} = \nabla^\perp \psi$. According to the above discussion, at any time the state of the fluid can be described by \mathbf{v} , or ψ , or the pair (ω, \mathbf{F}) .

The kinetic energy can be expressed in terms of ω and \mathbf{F} as follows:

$$\mathcal{E} = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla K \omega|^2 d\mathbf{x} + \frac{1}{2|\mathbb{T}^2|} |\mathbf{F}|^2. \tag{1.12}$$

Define

$$E(\omega) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla K \omega|^2 d\mathbf{x} = \frac{1}{2} \int_{\mathbb{T}^2} \omega K \omega d\mathbf{x}. \tag{1.13}$$

Then by energy conservation, E is also conserved:

$$E(\omega(t, \cdot)) = E(\omega(0, \cdot)), \quad \forall t \geq 0. \tag{1.14}$$

1.2. Sinusoidal flows and Arnold’s stability result. Stationary solutions of the Euler equations (1.1) are characterized by having $\nabla\psi$ and $\nabla\omega$ collinear. For this to hold, a sufficient condition is that ψ and ω satisfy the relation

$$\omega = f(\psi) \tag{1.15}$$

for some function $f : \mathbb{R} \rightarrow \mathbb{R}$. In particular, if f is linear, then (1.15) becomes the eigenvalue problem of $-\Delta$ on \mathbb{T}^2 :

$$-\Delta\psi = \lambda\psi. \tag{1.16}$$

For (1.16), any eigenvalue λ has the form (see [24], Chapter 1)

$$\lambda = \left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \tag{1.17}$$

for some integers k_1, k_2 such that $k_1^2 + k_2^2 \neq 0$, and the corresponding eigenspace is spanned by

$$\begin{aligned} & \sin\left(\frac{j_1}{\nu_1}x_1\right) \sin\left(\frac{j_2}{\nu_2}x_2\right), \quad \sin\left(\frac{j_1}{\nu_1}x_1\right) \cos\left(\frac{j_2}{\nu_2}x_2\right), \\ & \cos\left(\frac{j_1}{\nu_1}x_1\right) \sin\left(\frac{j_2}{\nu_2}x_2\right), \quad \cos\left(\frac{j_1}{\nu_1}x_1\right) \cos\left(\frac{j_2}{\nu_2}x_2\right), \end{aligned}$$

for all integers j_1, j_2 such that

$$\left(\frac{j_1}{\nu_1}\right)^2 + \left(\frac{j_2}{\nu_2}\right)^2 = \lambda. \tag{1.18}$$

A *sinusoidal flow*, or an *eigenstate*, is an Euler flow whose stream function is some eigenfunction of $-\Delta$ on \mathbb{T}^2 . For a sinusoidal flow related to some eigenvalue λ , the stream function can be written in the following form:

$$\sum_{(j_1, j_2) \in J_\lambda} A_{j_1, j_2} \sin\left(\frac{j_1}{\nu_1}x_1 + \frac{j_2}{\nu_2}x_2 + \alpha_{j_1, j_2}\right) + \sum_{(j_1, j_2) \in J_\lambda} B_{j_1, j_2} \sin\left(\frac{j_1}{\nu_1}x_1 - \frac{j_2}{\nu_2}x_2 + \beta_{j_1, j_2}\right),$$

where $A_{j_1, j_2}, B_{j_1, j_2} \geq 0$, $\alpha_{j_1, j_2}, \beta_{j_1, j_2} \in \mathbb{R}$, and J_λ is defined by

$$J_\lambda = \left\{ (j_1, j_2) \in \mathbb{Z}^2 \mid \left(\frac{j_1}{\nu_1}\right)^2 + \left(\frac{j_2}{\nu_2}\right)^2 = \lambda \right\}. \tag{1.19}$$

Note that for any sinusoidal flow, the total flux of velocity is $\mathbf{0}$, hence the normalized velocity is equal to the velocity, and the normalized stream function is equal to the stream function.

The stability of sinusoidal flows is a fundamental problem in fluid dynamics and has been extensively studied in the literature. For the linear theory, the results are quite rich, although many open questions still remain. See [6, 11, 16, 19, 20, 23] and the references therein. As to nonlinear stability, the first rigorous result was obtained by Arnold. In the 1960s, Arnold [2, 3] proved two nonlinear stability criteria for plane ideal flows, now usually referred to as Arnold's first and second stability theorems. See also [25, 28, 29] for some of their extensions. As a straightforward application of Arnold's second stability theorem, partial nonlinear stability can be proved for sinusoidal flows related to least eigenfunctions of $-\Delta$ on \mathbb{T}^2 .

To state Arnold's result, we briefly analyze the least eigenvalue λ_1 of $-\Delta$ on \mathbb{T}^2 . By (1.17), there are three cases:

- (i) If \mathbb{T}^2 is a short torus, i.e., $\nu_1 < \nu_2$, then $\lambda_1 = \nu_2^{-2}$. In this case, $J_{\lambda_1} = \{(0, 1), (0, -1)\}$, thus any least eigenfunction ψ_1 takes the form

$$\psi_1(x_1, x_2) = A \sin\left(\frac{x_2}{\nu_2} + \alpha\right), \quad (1.20)$$

where $A \geq 0$ is called the amplitude, and $\alpha \in \mathbb{R}$ is called the phase parameter.

- (ii) If \mathbb{T}^2 is a long torus, i.e., $\nu_1 > \nu_2$, then $\lambda_1 = \nu_1^{-2}$. In this case, $J_{\lambda_1} = \{(1, 0), (-1, 0)\}$, hence any least eigenfunction ψ_1 takes the form

$$\psi_1(x_1, x_2) = A \sin\left(\frac{x_1}{\nu_1} + \alpha\right) \quad (1.21)$$

for some $A \geq 0$ and $\alpha \in \mathbb{R}$.

- (iii) If \mathbb{T}^2 is a square torus, i.e., $\nu_1 = \nu_2 = \nu$, then $\lambda_1 = \nu^{-2}$. In this case, $J_{\lambda_1} = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$, hence any least eigenfunction ψ_1 takes the form

$$\psi_1(x_1, x_2) = A \sin\left(\frac{x_1}{\nu} + \alpha\right) + B \sin\left(\frac{x_2}{\nu} + \beta\right) \quad (1.22)$$

for some $A, B \geq 0$ and $\alpha, \beta \in \mathbb{R}$.

For convenience of subsequent presentation, we call a sinusoidal flow of the form (1.20) an x_2 -mode, and a sinusoidal flow of the form (1.21) an x_1 -mode. It is clear that all x_1 -modes, as well as all x_2 -modes, form a two-dimensional vector space, and all sinusoidal flows of the form (1.22) form a four-dimensional vector space.

Since it is more convenient to express Arnold's result in terms of vorticity, we denote by \mathcal{V}_i the set of vorticity functions of all x_i -modes, i.e.,

$$\mathcal{V}_i = \left\{ A \sin\left(\frac{x_i}{\nu_i} + \alpha\right) \mid A \geq 0, \alpha \in \mathbb{R} \right\}, \quad i = 1, 2. \quad (1.23)$$

If $\nu_1 = \nu_2 = \nu$, we denote by \mathcal{V} the set of vorticity functions of all sinusoidal flows of the form (1.22), i.e.,

$$\mathcal{V} = \left\{ A \sin\left(\frac{x_1}{\nu} + \alpha\right) + B \sin\left(\frac{x_2}{\nu} + \beta\right) \mid A, B \geq 0, \alpha, \beta \in \mathbb{R} \right\}. \quad (1.24)$$

Arnold’s result can be stated as follows.

Theorem 1.1 (Arnold, [2, 3]). *Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}$ be defined by (1.23), (1.24). Then the following assertions hold:*

- (i) *If $\nu_1 < \nu_2$, then \mathcal{V}_2 is nonlinearly stable in L^2 norm, i.e., for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any smooth Euler flow on \mathbb{T}^2 with vorticity ω , we have that*

$$\min_{v \in \mathcal{V}_2} \|\omega(0, \cdot) - v\|_{L^2(\mathbb{T}^2)} < \delta \implies \min_{v \in \mathcal{V}_2} \|\omega(t, \cdot) - v\|_{L^2(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0. \quad (1.25)$$

- (ii) *If $\nu_1 > \nu_2$, then \mathcal{V}_1 is nonlinearly stable in L^2 norm, i.e., (1.25) holds with \mathcal{V}_2 replaced by \mathcal{V}_1 .*
- (iii) *If $\nu_1 = \nu_2 = \nu$, then \mathcal{V} is nonlinearly stable in L^2 norm, i.e., (1.25) holds with \mathcal{V}_2 replaced by \mathcal{V} .*

Remark 1.2. By rotational symmetry, items (i) and (ii) in Theorem 1.1 in fact tell the same thing.

Remark 1.3. If $\nu_1 < \nu_2$ (or $\nu_1 > \nu_2$), then any x_1 -mode (x_2 -mode, accordingly) is known to be linearly unstable in a certain sense. See [6, 19, 23] for example.

Theorem 1.1 can also be found in [4], p. 98, or [22], p. 111. For the reader’s convenience, we provide a detailed proof following Arnold’s original idea in Appendix A.

By Theorem 1.1, any least eigenstate (x_2 -mode on a short torus, or x_1 -mode on a long torus, or sinusoidal flow of the form (1.22) on a square torus), is nonlinearly stable *up to phase translations and amplitude scalings*. For example, given an x_2 -mode with vorticity $A_0 \sin(\nu_2^{-1}x_2 + \alpha_0)$ on a short torus, if a smooth Euler flow is “close” to this x_2 -mode at initial time, then at any $t > 0$ the evolved flow is “close” to some x_2 -mode with vorticity $A_t \sin(\nu_2^{-1}x_2 + \alpha_t)$. Here “closeness” is measured in terms of L^2 norm of the vorticity. Since A_t and α_t may vary with time, it is not clear whether the x_2 -mode with vorticity $A_0 \sin(\nu_2^{-1}x_2 + \alpha_0)$ is nonlinearly stable.

1.3. Main result. The nonlinear stability of a single sinusoidal flow was listed as an open problem on p. 112 of Marchioro-Pulvirenti’s book [22]. Arnold’s method can not handle this problem since it only involves energy and enstrophy conservations (see Appendix A), which is not enough to distinguish different sinusoidal flows. In [27], Wirosoetisno-Shepherd employed high-order (cubic, quartic and quintic) Casimirs to bound the variation of the amplitudes A, B in the case of a square torus. As a consequence, they obtained the nonlinear stability of a single sinusoidal flow up to phase translations. However, since the bound therein depends on high-order Casimirs of the initial state, rigorous nonlinear stability (even up to phase translations) remains unclear.

Our purpose in this paper is to give an extension of Theorem 1.1 and implement the idea in [27] rigorously. To state our result, for fixed $A \geq 0$, define

$$\mathcal{V}_{A,i} = \left\{ A \sin \left(\frac{x_i}{\nu_i} + \alpha \right) \mid \alpha \in \mathbb{R} \right\}. \quad (1.26)$$

If $\nu_1 = \nu_2 = \nu$, for fixed $A, B \geq 0$, define

$$\mathcal{V}_{A,B} = \left\{ A \sin \left(\frac{x_1}{\nu} + \alpha \right) + B \sin \left(\frac{x_2}{\nu} + \beta \right) \mid \alpha, \beta \in \mathbb{R} \right\}. \quad (1.27)$$

It is easy to see that

$$\mathcal{V}_i = \bigcup_{A>0} \mathcal{V}_{A,i}, \quad i = 1, 2, \quad \mathcal{V} = \bigcup_{A,B \geq 0} \mathcal{V}_{A,B}.$$

Our main result in this paper is the following theorem.

Theorem 1.4. *Let $1 < p < +\infty$ be fixed. Then the following assertions hold:*

- (i) *If $\nu_1 < \nu_2$, then for any $A \geq 0$, $\mathcal{V}_{A,2}$ is nonlinearly stable in L^p norm, i.e., for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any smooth Euler flow on \mathbb{T}^2 with vorticity ω , we have that*

$$\min_{v \in \mathcal{V}_{A,2}} \|\omega(0, \cdot) - v\|_{L^p(\mathbb{T}^2)} < \delta \implies \min_{v \in \mathcal{V}_{A,2}} \|\omega(t, \cdot) - v\|_{L^p(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0. \quad (1.28)$$

- (ii) *If $\nu_1 > \nu_2$, then for any $A \geq 0$, $\mathcal{V}_{A,1}$ is nonlinearly stable in L^p norm, i.e., (1.28) holds with $\mathcal{V}_{A,2}$ replaced by $\mathcal{V}_{A,1}$.*

- (iii) *If $\nu_1 = \nu_2 = \nu$, then for any $A, B \geq 0$, $\mathcal{V}_{A,B}$ is nonlinearly stable in L^p norm, i.e., (1.28) holds with $\mathcal{V}_{A,2}$ replaced by $\mathcal{V}_{A,B}$.*

Remark 1.5. In Theorem 1.4, to avoid some technical (but not essential) difficulties and illustrate the main idea clearly, we assume that the perturbed flows are smooth. However, by checking the proof carefully, Theorem 1.4 actually holds for a large class of less regular perturbations such that (i) the quantities (C1)-(C3) are conserved; (ii) the vorticity is continuous in $L^p(\mathbb{T}^2)$ with respect to the time variable; (iii) a ‘‘follower’’ to the perturbed vorticity as in Section 5 exists.

Remark 1.6. By Lemma 2.4 in Section 2, the nonlinear stabilities in Theorem 1.4 can also be measured in terms of $W^{1,p}$ norm of the normalized velocity, or $W^{2,p}$ norm of the normalized stream function for any $1 < p < +\infty$. For example, Theorem 1.4(i) can be equivalently stated as follows.

- (1) *If $\nu_1 < \nu_2$, then for any $A \geq 0$, it holds that: for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any smooth Euler flow on \mathbb{T}^2 with normalized velocity $\tilde{\mathbf{v}}$, we have that*

$$\min_{\mathbf{u} \in \mathcal{V}_{A,2}} \|\tilde{\mathbf{v}}(0, \cdot) - \mathbf{u}\|_{W^{1,p}(\mathbb{T}^2)} < \delta \implies \min_{\mathbf{u} \in \mathcal{V}_{A,2}} \|\tilde{\mathbf{v}}(t, \cdot) - \mathbf{u}\|_{W^{2,p}(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0,$$

where $\mathcal{V}_{A,2}$ is the set of velocities (also the set of normalized velocities) related to $\mathcal{V}_{A,2}$, given by

$$\mathcal{V}_{A,2} = \left\{ (A\nu_2 \cos(\nu_2^{-1}x_2 + \alpha), 0) \mid \alpha \in \mathbb{R} \right\}.$$

- (2) If $\nu_1 < \nu_2$, then for any $A \geq 0$, it holds that: for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any smooth Euler flow on \mathbb{T}^2 with normalized stream function $\tilde{\psi}$, we have that

$$\min_{\varphi \in \mathcal{S}_{A,2}} \|\tilde{\psi}(0, \cdot) - \varphi\|_{W^{2,p}(\mathbb{T}^2)} < \delta \implies \min_{\varphi \in \mathcal{S}_{A,2}} \|\tilde{\psi}(t, \cdot) - \varphi\|_{W^{2,p}(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0,$$

where $\mathcal{S}_{A,2}$ is the set of stream functions (also the set of normalized stream functions) related to $\mathcal{V}_{A,2}$, given by

$$\mathcal{S}_{A,2} = \{A\nu_2^2 \sin(\nu_2^{-1}x_2 + \alpha) \mid \alpha \in \mathbb{R}\}.$$

By Theorem 1.4, any eigenstate on \mathbb{T}^2 is nonlinearly stable *only up to phase translations*, which noticeably improves Theorem 1.1. Moreover, the stabilities in Theorem 1.4 are measured in terms of L^p norm of the vorticity for any $1 < p < +\infty$, which are also more general than those in Theorem 1.1.

To prove Theorem 1.4, we use a variational approach in combination with a compactness argument, which is very different from the classical energy-Casimir method used by Arnold in [2, 3] and Wirosoetisno-Shepherd in [27]. Our method consists of three ingredients: a suitable variational characterization for the sinusoidal flows under consideration, a compactness argument, and proper use of flow invariants. These three ingredients are also essential in the nonlinear stability analysis of many other stationary Euler flows. See [1, 5, 9, 10, 13–15, 26] for example. The variational characterizations, which states that the sinusoidal flows in Theorem 1.4 are exactly the set of maximizers of the conserved functional E relative to all isovortical flows to them, are the most important step in the whole proof. The advantage of such variational characterizations is that we are able to distinguish the sinusoidal flows with vorticity in $\mathcal{V}_{A,1}$, $\mathcal{V}_{A,2}$ or \mathcal{V} from other least eigenstates.

The nonlinear stability of a single sinusoidal flow still remains open. In our method, we only use energy and vorticity conservations, which are not enough to differentiate one flow from another within the set of flows with vorticity in $\mathcal{V}_{A,1}$, $\mathcal{V}_{A,2}$ or $\mathcal{V}_{A,B}$. Hence to improve Theorem 1.4 further, new flow invariants are needed. This is an interesting further work.

This paper is organized as follows. In Section 2, we give some preliminary materials for later use. In Section 3, we establish variational principles for the sinusoidal flows under consideration. In Section 4, we prove compactness related to the variational principles established in Section 3. In Section 5, we give the proof of Theorem 1.4.

2. PRELIMINARIES

2.1. Definitions, notation and basic facts.

- For $1 \leq p \leq +\infty$, denote by $L^p(\mathbb{T}^2)$ the set of all p -th power integrable (essentially bounded if $p = +\infty$) real-valued functions on \mathbb{T}^2 . The norm of $L^p(\mathbb{T}^2)$ is denoted by $\|\cdot\|_{L^p(\mathbb{T}^2)}$.
- For $1 \leq p \leq +\infty$ and $k \in \mathbb{Z}^+$, where \mathbb{Z}^+ is the set of positive integers, $W^{k,p}(\mathbb{T}^2)$ denotes the set of all real-valued functions whose weak derivatives up to order k are p -th power integrable (essentially bounded if $p = +\infty$). The norm of $W^{k,p}(\mathbb{T}^2)$ is denoted by $\|\cdot\|_{W^{k,p}(\mathbb{T}^2)}$. Note that if we consider $D = (0, 2\pi\nu_1) \times (0, 2\pi\nu_2)$, a

domain of \mathbb{R}^2 , then $W^{k,p}(\mathbb{T}^2)$ can be regarded as a closed subspace of $W^{k,p}(D)$, and $f \in W^{k,p}(\mathbb{T}^2)$ if and only if $f \in W^{k,p}(D)$ and satisfies

$$f(0, x_2) = f(2\pi\nu_1, x_2), \quad \forall 0 \leq x_2 \leq 2\pi\nu_2,$$

$$f(x_1, 0) = f(x_1, 2\pi\nu_2), \quad \forall 0 \leq x_1 \leq 2\pi\nu_1$$

in the sense of traces.

- For $1 \leq p \leq +\infty$ and $k \in \mathbb{Z}^+$, denote

$$\mathring{L}^p(\mathbb{T}^2) = \left\{ f \in L^p(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} f d\mathbf{x} = 0 \right\},$$

$$\mathring{W}^{k,p}(\mathbb{T}^2) = \left\{ f \in W^{k,p}(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} f d\mathbf{x} = 0 \right\}.$$

It is clear that $\mathring{L}^p(\mathbb{T}^2)$ is a closed subspace of $L^p(\mathbb{T}^2)$, and $\mathring{W}^{k,p}(\mathbb{T}^2)$ is a closed subspace of $W^{k,p}(\mathbb{T}^2)$.

- Denote by $L^2(\mathbb{T}^2; \mathbb{C})$ the set of all p -th power integrable complex-valued functions on \mathbb{T}^2 endowed with the following inner product:

$$\langle f, g \rangle = \int_{\mathbb{T}^2} f(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x}, \quad \forall f, g \in L^2(\mathbb{T}^2; \mathbb{C}),$$

where $\overline{g(\mathbf{x})}$ is the complex conjugate of $g(\mathbf{x})$.

- Denote by \mathbb{Z}^2 the set of all points in \mathbb{R}^2 with integer coordinates. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, define

$$\zeta_{\mathbf{k}}(\mathbf{x}) = \frac{1}{\sqrt{4\pi^2\nu_1\nu_2}} e^{i\left(\frac{k_1}{\nu_1}x_1 + \frac{k_2}{\nu_2}x_2\right)}, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{T}^2.$$

Then $\{\zeta_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}$ is an orthonormal basis of $L^2(\mathbb{T}^2; \mathbb{C})$ (see [17], p. 186). For $i, j = 1, 2$, it is easy to check that

$$\partial_{x_i} \zeta_{\mathbf{k}} = i \left(\frac{k_i}{\nu_i} \right) \zeta_{\mathbf{k}}, \quad \partial_{x_i x_j} \zeta_{\mathbf{k}} = - \left(\frac{k_i}{\nu_i} \right) \left(\frac{k_j}{\nu_j} \right) \zeta_{\mathbf{k}}. \quad (2.1)$$

In particular,

$$-\Delta \zeta_{\mathbf{k}} = \left[\left(\frac{k_1}{\nu_1} \right)^2 + \left(\frac{k_2}{\nu_2} \right)^2 \right] \zeta_{\mathbf{k}}. \quad (2.2)$$

- For $f \in L^1(\mathbb{T}^2; \mathbb{C})$ and $\mathbf{k} \in \mathbb{Z}^2$, denote by $\hat{f}_{\mathbf{k}}$ the \mathbf{k} -th Fourier coefficient of f , i.e.,

$$\hat{f}_{\mathbf{k}} = \langle f, \zeta_{\mathbf{k}} \rangle = \int_{\mathbb{T}^2} f(\mathbf{x}) \overline{\zeta_{\mathbf{k}}(\mathbf{x})} d\mathbf{x}.$$

The Fourier series of f is then

$$f \sim \sum_{\mathbf{k} \in \mathbb{Z}^2} \hat{f}_{\mathbf{k}} \zeta_{\mathbf{k}}.$$

- For any $f \in L^1(\mathbb{T}^2; \mathbb{C})$ and $N \in \mathbb{Z}^+$, denote by f_N the N -th *circular* partial sum of the Fourier series of f , i.e.,

$$f_N = \sum_{\mathbf{k} \in \mathbb{Z}^2, |\mathbf{k}|_\infty \leq N} \hat{f}_{\mathbf{k}} \zeta_{\mathbf{k}}, \quad \text{where } |\mathbf{k}|_\infty := \max\{|k_1|, |k_2|\}. \quad (2.3)$$

Note that if f is real-valued, then $\hat{f}_{-\mathbf{k}} = \overline{\hat{f}_{\mathbf{k}}}$ for any $\mathbf{k} \in \mathbb{Z}^2$, thus f_N is also real-valued for any $N \in \mathbb{Z}^+$. Also note that for fixed $1 < p < +\infty$, if $f \in L^p(\mathbb{T}^2; \mathbb{C})$, then $f_N \rightarrow f$ in $L^p(\mathbb{T}^2)$ as $N \rightarrow +\infty$ (see [17], Theorem 4.1.8).

2.2. Poisson equation on a flat torus. In this subsection, we study the following Poisson equation:

$$\begin{cases} -\Delta u = f, & \mathbf{x} \in \mathbb{T}^2, \\ u \in \dot{W}^{2,p}(\mathbb{T}^2), \end{cases} \quad (2.4)$$

where $f \in \dot{L}^p(\mathbb{T}^2)$, $1 < p < +\infty$.

Lemma 2.1. *Let $1 < p < +\infty$. Then for any $f \in \dot{L}^p(\mathbb{T}^2)$, there exists a unique solution u to the Poisson equation (2.4). Moreover, the following estimate holds:*

$$\|u\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f\|_{L^p(\mathbb{T}^2)}, \quad (2.5)$$

where $C > 0$ depends only on ν_1, ν_2 and p .

Proof. First we prove existence. Consider the following approximate equation:

$$\begin{cases} -\Delta u_N = f_N, & \mathbf{x} \in \mathbb{T}^2, \\ u_N \in \dot{W}^{2,p}(\mathbb{T}^2), \end{cases} \quad (2.6)$$

where f_N is N -th circular partial sum of the Fourier series of f , defined by (2.3). Since $f \in \dot{L}^p(\mathbb{T}^2)$, we have $\hat{f}_0 = 0$. Then it is easy to check that (2.6) admits an explicit solution:

$$u_N = \sum_{\mathbf{k} \in \mathbb{Z}^2, 0 < |\mathbf{k}|_\infty \leq N} \frac{\hat{f}_{\mathbf{k}}}{\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2} \zeta_{\mathbf{k}}.$$

Moreover, we have the following uniform estimate for u_N (see [12], Theorem 10):

$$\|\partial_{x_i x_j} u_N\|_{L^p(\mathbb{T}^2)} \leq C \|f_N\|_{L^p(\mathbb{T}^2)}, \quad \forall i, j = 1, 2, \quad (2.7)$$

where $C > 0$ depends only on ν_1, ν_2 and p . Applying the Poincaré inequality (notice that $u_N \in \dot{L}^p(\mathbb{T}^2)$ and $\partial_i u_N \in \dot{L}^p(\mathbb{T}^2)$, $i = 1, 2$), we further have that

$$\|u_N\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f_N\|_{L^p(\mathbb{T}^2)}. \quad (2.8)$$

Similarly, for any $N_1, N_2 \in \mathbb{Z}^+$,

$$\|u_{N_1} - u_{N_2}\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f_{N_1} - f_{N_2}\|_{L^p(\mathbb{T}^2)}. \quad (2.9)$$

From (2.9), taking into account the fact that $f_N \rightarrow f$ in $L^p(\mathbb{T}^2)$ as $N \rightarrow +\infty$, we see that $\{u_N\}$ is a Cauchy sequence in $\dot{W}^{2,p}(\mathbb{T}^2)$, thus u_N converges to some u in $\dot{W}^{2,p}(\mathbb{T}^2)$ as

$N \rightarrow +\infty$. It is clear that $-\Delta u = f$ a.e. $\mathbf{x} \in \mathbb{T}^2$, hence u solves (2.4). Moreover, passing to the limit $N \rightarrow +\infty$ in (2.8) gives

$$\|u\|_{W^{2,p}(\mathbb{T}^2)} \leq C\|f\|_{L^p(\mathbb{T}^2)}.$$

Next we prove uniqueness. Suppose (2.4) has two solutions, say u_1 and u_2 . Then

$$-\Delta(u_1 - u_2) = 0, \quad \mathbf{x} \in \mathbb{T}^2.$$

By integration by parts,

$$\int_{\mathbb{T}^2} |\nabla(u_1 - u_2)|^2 d\mathbf{x} = 0,$$

which implies that $u_1 = u_2 + c$ for some constant c . Taking into account the fact that $u_1, u_2 \in \dot{W}^{2,p}(\mathbb{T}^2)$, we obtain $u_1 \equiv u_2$. \square

By Lemma 2.1, the negative Laplacian on \mathbb{T}^2 has an inverse, denoted by K . The estimate (2.5) indicates that K is a bounded operator from $\dot{L}^p(\mathbb{T}^2)$ to $\dot{W}^{2,p}(\mathbb{T}^2)$.

The following lemma, asserting that K is symmetric and positive definite, is crucial to the proof of Proposition 4.1 in Section 4.

Lemma 2.2. *Let $1 < p < +\infty$ be fixed. Then*

(i) *for any $f, g \in \dot{L}^p(\mathbb{T}^2)$, it holds that*

$$\int_{\mathbb{T}^2} fKgd\mathbf{x} = \int_{\mathbb{T}^2} gKfd\mathbf{x}; \quad (2.10)$$

(ii) *for any $f \in \dot{L}^p(\mathbb{T}^2)$, it holds that*

$$\int_{\mathbb{T}^2} fKfd\mathbf{x} \geq 0, \quad (2.11)$$

and the equality holds if and only if $f \equiv 0$.

Proof. First we prove (i). Denote $u = Kf$, $v = Kg$. By integration by parts,

$$\int_{\mathbb{T}^2} fKgd\mathbf{x} = \int_{\mathbb{T}^2} (-\Delta u)v d\mathbf{x} = \int_{\mathbb{T}^2} \nabla u \cdot \nabla v d\mathbf{x}.$$

Similarly,

$$\int_{\mathbb{T}^2} gKfd\mathbf{x} = \int_{\mathbb{T}^2} (-\Delta v)u d\mathbf{x} = \int_{\mathbb{T}^2} \nabla v \cdot \nabla u d\mathbf{x}.$$

Hence (2.10) holds.

Next we prove (ii). Still denote $u = Kf$. Then integration by parts gives

$$\int_{\mathbb{T}^2} fKfd\mathbf{x} = \int_{\mathbb{T}^2} (-\Delta u)u d\mathbf{x} = \int_{\mathbb{T}^2} |\nabla u|^2 d\mathbf{x}.$$

Hence (2.11) holds. Moreover, (2.11) is an equality if and only if $u \equiv 0$, which is equivalent to $f \equiv 0$. \square

Denote by $\mathcal{V}_1^\perp, \mathcal{V}_2^\perp, \mathcal{V}^\perp$ the orthogonal complements of $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}$ in $\dot{L}^2(\mathbb{T}^2)$, respectively. The following lemma will also be needed in subsequent sections.

Lemma 2.3. *For $i = 1, 2$, it holds that*

$$\int_{\mathbb{T}^2} \nabla K g \cdot \nabla K h d\mathbf{x} = 0, \quad \forall g \in \mathcal{V}_i, h \in \mathcal{V}_i^\perp. \quad (2.12)$$

If additionally $\nu_1 = \nu_2 = \nu$, then

$$\int_{\mathbb{T}^2} \nabla K g \cdot \nabla K h d\mathbf{x} = 0, \quad \forall g \in \mathcal{V}, h \in \mathcal{V}^\perp. \quad (2.13)$$

Proof. Fix $i \in \{1, 2\}$. Observe that

$$-\Delta v = \nu_i^{-2} v, \quad \forall v \in \mathcal{V}_i,$$

which implies that

$$Kv = \nu_i^2 v, \quad \forall v \in \mathcal{V}_i.$$

Hence for $g \in \mathcal{V}_i$ and $h \in \mathcal{V}_i^\perp$, by integration by parts

$$\int_{\mathbb{T}^2} \nabla K g \cdot \nabla K h d\mathbf{x} = \int_{\mathbb{T}^2} (-\Delta K h) K g d\mathbf{x} = \int_{\mathbb{T}^2} h K g d\mathbf{x} = \nu_i^2 \int_{\mathbb{T}^2} g h d\mathbf{x} = 0.$$

Hence (2.12) has been proved. The proof of (2.13) is similar when $\nu_1 = \nu_2 = \nu$. \square

The following lemma is mainly used to illustrate Remark 1.6 in Section 1.

Lemma 2.4. *Consider a smooth Euler flow on \mathbb{T}^2 . Let $\tilde{\mathbf{v}}$, $\tilde{\psi}$ and ω be the normalized velocity, the normalized stream function and the vorticity, respectively. Then for any $1 < p < +\infty$, it holds that*

$$\begin{aligned} \|\tilde{\psi}\|_{W^{2,p}(D)} &\lesssim \|\omega\|_{L^p(D)} \lesssim \|\tilde{\psi}\|_{W^{2,p}(D)}, \\ \|\tilde{\psi}\|_{W^{2,p}(D)} &\lesssim \|\tilde{\mathbf{v}}\|_{W^{1,p}(D)} \lesssim \|\tilde{\psi}\|_{W^{2,p}(D)}. \end{aligned}$$

Here $A \lesssim B$ means $A \leq CB$ for some positive constant C depending only on ν_1, ν_2 and p .

Proof. Recall that $\tilde{\psi}$ and ω satisfy (see Section 1)

$$\begin{cases} -\Delta \tilde{\psi} = \omega, & \mathbf{x} \in \mathbb{T}^2, \\ \int_{\mathbb{T}^2} \tilde{\psi} d\mathbf{x} = 0. \end{cases} \quad (2.14)$$

Then the desired estimates are straightforward consequences of the estimate (2.5). \square

2.3. Energy-entropy inequalities. In this subsection, we deduce several energy-entropy type inequalities for functions in $\dot{L}^2(\mathbb{T}^2)$ based on Fourier series expansion.

Recall that E and Z are defined by (1.13) and (1.6), respectively. It is easy to check that E is well defined in $\dot{L}^p(\mathbb{T}^2)$ for any $1 < p < +\infty$, and Z is well defined in $L^2(\mathbb{T}^2)$.

Lemma 2.5. *Let $f \in \dot{L}^2(\mathbb{T}^2)$. Then the following assertions hold.*

(i) *If $\nu_1 < \nu_2$, then*

$$\begin{aligned} E(f) &= \nu_2^2 Z(f), \quad \forall f \in \mathcal{V}_2, \\ E(f) &\leq \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} Z(f), \quad \forall f \in \mathcal{V}_2^\perp. \end{aligned}$$

(ii) If $\nu_1 > \nu_2$, then

$$E(f) = \nu_1^2 Z(f), \quad \forall f \in \mathcal{V}_1,$$

$$E(f) \leq \max \left\{ \nu_2^2, \frac{\nu_1^2}{4} \right\} Z(f), \quad \forall f \in \mathcal{V}_1^\perp.$$

(iii) If $\nu_1 = \nu_2 = \nu$, then

$$E(f) = \nu^2 Z(f), \quad \forall f \in \mathcal{V},$$

$$E(f) \leq \frac{\nu^2}{4} Z(f), \quad \forall f \in \mathcal{V}^\perp.$$

Proof. First we show that for any $f \in \dot{L}^2(\mathbb{T}^2)$,

$$\|\partial_{x_i} Kf\|_{L^2(\mathbb{T}^2)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^2, \mathbf{k} \neq \mathbf{0}} \frac{\left(\frac{k_i}{\nu_i}\right)^2 |\hat{f}_{\mathbf{k}}|^2}{\left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2\right]^2}, \quad i = 1, 2. \quad (2.15)$$

To prove (2.15), we first show that the Fourier series of $\partial_{x_i} Kf$ has the form:

$$\partial_{x_i} Kf \sim \sum_{\mathbf{k} \in \mathbb{Z}^2} c_{\mathbf{k}} \zeta_{\mathbf{k}}, \quad c_{\mathbf{k}} = \begin{cases} 0 & \text{if } \mathbf{k} = \mathbf{0}, \\ \frac{i\left(\frac{k_i}{\nu_i}\right)}{\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2} \hat{f}_{\mathbf{k}} & \text{if } \mathbf{k} \neq \mathbf{0}. \end{cases} \quad (2.16)$$

In fact, since the integral of $\partial_{x_i} Kf$ on \mathbb{T}^2 is zero, we have $c_{\mathbf{0}} = 0$; for $\mathbf{k} \neq \mathbf{0}$, by integration by parts,

$$\begin{aligned} \bar{c}_{\mathbf{k}} &= \int_{\mathbb{T}^2} (\partial_{x_i} Kf) \zeta_{\mathbf{k}} d\mathbf{x} = - \int_{\mathbb{T}^2} (Kf) \partial_{x_i} \zeta_{\mathbf{k}} d\mathbf{x} \\ &= -i \left(\frac{k_i}{\nu_i}\right) \int_{\mathbb{T}^2} (Kf) \zeta_{\mathbf{k}} d\mathbf{x} \\ &= -i \left(\frac{k_i}{\nu_i}\right) \left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \right]^{-1} \int_{\mathbb{T}^2} (Kf) (-\Delta \zeta_{\mathbf{k}}) d\mathbf{x} \\ &= -i \left(\frac{k_i}{\nu_i}\right) \left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \right]^{-1} \int_{\mathbb{T}^2} (-\Delta Kf) \zeta_{\mathbf{k}} d\mathbf{x} \\ &= -i \left(\frac{k_i}{\nu_i}\right) \left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \right]^{-1} \int_{\mathbb{T}^2} f \zeta_{\mathbf{k}} d\mathbf{x} \\ &= -i \left(\frac{k_i}{\nu_i}\right) \left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \right]^{-1} \hat{f}_{\mathbf{k}}. \end{aligned}$$

Here we used (2.1) and (2.2). Hence (2.16) has been proved. From (2.16), we can apply Parseval's identity (see [17], Proposition 3.2.7) to obtain

$$\|\partial_{x_i} K f\|_{L^2(\mathbb{T}^2)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^2} |c_{\mathbf{k}}|^2 = \sum_{\mathbf{k} \in \mathbb{Z}^2, \mathbf{k} \neq \mathbf{0}} \frac{\left(\frac{k_i}{\nu_i}\right)^2 |\hat{f}_{\mathbf{k}}|^2}{\left[\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2\right]^2},$$

which is exactly (2.15).

As a consequence of (2.15), we obtain

$$E(f) = \frac{1}{2} \|\nabla K f\|_{L^2(\mathbb{T}^2)}^2 = \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^2, \mathbf{k} \neq \mathbf{0}} \frac{|\hat{f}_{\mathbf{k}}|^2}{\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2}, \quad \forall f \in \dot{L}^2(\mathbb{T}^2). \quad (2.17)$$

Besides, by Parseval's identity, the enstrophy can also be expressed in terms of Fourier coefficients:

$$Z(f) = \frac{1}{2} \|f\|_{L^2(\mathbb{T}^2)}^2 = \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^2, \mathbf{k} \neq \mathbf{0}} |\hat{f}_{\mathbf{k}}|^2, \quad \forall f \in \dot{L}^2(\mathbb{T}^2). \quad (2.18)$$

Below we prove the lemma based on (2.17) and (2.18). We only prove (i), since the proofs of (ii) and (iii) are almost identical to that of (i). Notice that $f \in \mathcal{V}_2$ if and only if $\hat{f}_{\mathbf{k}} = 0$ for $\mathbf{k} \neq (0, \pm 1)$, and $f \in \mathcal{V}_2^\perp$ if and only if $\hat{f}_{\mathbf{k}} = 0$ for $\mathbf{k} = (0, \pm 1)$. Hence for $f \in \mathcal{V}_2$,

$$E(f) = \frac{1}{2} \sum_{\mathbf{k}=(0,\pm 1)} \frac{|\hat{f}_{\mathbf{k}}|^2}{\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2} = \frac{1}{2} \nu_2^2 \sum_{\mathbf{k}=(0,\pm 1)} |\hat{f}_{\mathbf{k}}|^2 = \nu_2^2 Z(f).$$

For $f \in \mathcal{V}_2^\perp$,

$$E(f) = \frac{1}{2} \sum_{\mathbf{k} \neq (0,\pm 1), \mathbf{k} \neq \mathbf{0}} \frac{|\hat{f}_{\mathbf{k}}|^2}{\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2}. \quad (2.19)$$

We claim that

$$\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \geq \min\left\{\frac{1}{\nu_1^2}, \frac{4}{\nu_2^2}\right\}, \quad \forall \mathbf{k} \neq (0, \pm 1), \mathbf{k} \neq \mathbf{0}. \quad (2.20)$$

In fact, notice that $\mathbf{k} \neq (0, \pm 1)$, $\mathbf{k} \neq \mathbf{0}$ if and only if

$$|k_1| \geq 1 \quad \text{or} \quad |k_2| \geq 2.$$

If $|k_1| \geq 1$, then

$$\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \geq \frac{1}{\nu_1^2};$$

if $|k_2| \geq 2$, then

$$\left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2 \geq \frac{4}{\nu_2^2}.$$

Hence (2.20) follows. Combining (2.19) and (2.20), we have that

$$\begin{aligned}
E(f) &\leq \frac{1}{2} \sum_{\mathbf{k} \neq (0, \pm 1), \mathbf{k} \neq \mathbf{0}} \frac{|\hat{f}_{\mathbf{k}}|^2}{\min \left\{ \frac{1}{\nu_1^2}, \frac{4}{\nu_2^2} \right\}} \\
&= \frac{1}{2} \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} \sum_{\mathbf{k} \neq (0, \pm 1), \mathbf{k} \neq \mathbf{0}} |\hat{f}_{\mathbf{k}}|^2 \\
&= \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} Z(f).
\end{aligned}$$

□

3. VARIATIONAL CHARACTERIZATIONS

Throughout this section, let $A, B \geq 0$ be fixed. Denote

$$v_{A,i} = A \sin \left(\frac{x_i}{\nu_i} \right), \quad i = 1, 2, \quad (3.1)$$

$$v_{A,B} = A \sin \left(\frac{x_1}{\nu} \right) + B \sin \left(\frac{x_2}{\nu} \right) \quad \text{if } \nu_1 = \nu_2 = \nu. \quad (3.2)$$

Denote by $\mathcal{R}_{A,i}, \mathcal{R}_{A,B}$ the set of rearrangements of $v_{A,i}, v_{A,B}$ on \mathbf{T}^2 , respectively, i.e.,

$$\mathcal{R}_{A,i} = \mathcal{R}(v_{A,i}), \quad i = 1, 2, \quad (3.3)$$

$$\mathcal{R}_{A,B} = \mathcal{R}(v_{A,B}) \quad \text{if } \nu_1 = \nu_2 = \nu. \quad (3.4)$$

It is easy to check that $\mathcal{V}_{A,i} \subset \mathcal{R}_{A,i}$, $i = 1, 2$, and $\mathcal{V}_{A,B} \subset \mathcal{R}_{A,B}$ if $\nu_1 = \nu_2 = \nu$.

Consider the following variational problems:

$$\mathbf{m}_{A,i} = \sup_{v \in \mathcal{R}_{A,i}} E(v), \quad i = 1, 2, \quad (3.5)$$

$$\mathbf{m}_{A,B} = \sup_{v \in \mathcal{R}_{A,B}} E(v) \quad \text{if } \nu_1 = \nu_2 = \nu. \quad (3.6)$$

Our aim in this section is to prove the following variational characterizations for $\mathcal{V}_{A,1}$, $\mathcal{V}_{A,2}$ and $\mathcal{R}_{A,B}$.

Proposition 3.1. *Let $\mathcal{R}_{A,1}, \mathcal{R}_{A,2}, \mathcal{R}_{A,B}$ be defined by (3.3), (3.4), and $\mathbf{m}_{A,1}, \mathbf{m}_{A,2}, \mathbf{m}_{A,B}$ be defined by (3.5), (3.6).*

(i) *If $\nu_1 < \nu_2$, then*

$$\mathcal{V}_{A,2} = \{v \in \mathcal{R}_{A,2} \mid E(v) = \mathbf{m}_{A,2}\}.$$

(ii) *If $\nu_1 > \nu_2$, then*

$$\mathcal{V}_{A,1} = \{v \in \mathcal{R}_{A,1} \mid E(v) = \mathbf{m}_{A,1}\}.$$

(iii) *If $\nu_1 = \nu_2 = \nu$, then*

$$\mathcal{V}_{A,B} = \{v \in \mathcal{R}_{A,B} \mid E(v) = \mathbf{m}_{A,B}\}.$$

Proof. First we prove (i). Let $v_{A,2}$ be given by (3.1). Denote

$$Z_A = Z(v_{A,2}).$$

Then $Z(v) = Z_A$ for any $v \in \mathcal{R}_{A,2}$. Decompose any $v \in \mathcal{R}_{A,2}$ into two components:

$$v = \bar{v} + \tilde{v}, \quad \bar{v} \in \mathcal{V}_2, \quad \tilde{v} \in \mathcal{V}_2^\perp. \quad (3.7)$$

It is clear that

$$Z(\bar{v}) + Z(\tilde{v}) = Z_A. \quad (3.8)$$

Using Lemma 2.3, we have that

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{L^2(\mathbb{T}^2)} |\nabla K v|^2 d\mathbf{x} \\ &= \frac{1}{2} \int_{L^2(\mathbb{T}^2)} |\nabla K \bar{v}|^2 d\mathbf{x} + \int_{L^2(\mathbb{T}^2)} \nabla K \bar{v} \cdot \nabla K \tilde{v} d\mathbf{x} + \frac{1}{2} \int_{L^2(\mathbb{T}^2)} |\nabla K \tilde{v}|^2 d\mathbf{x} \\ &= E(\bar{v}) + E(\tilde{v}). \end{aligned} \quad (3.9)$$

Recalling Lemma 2.5(i), it holds that

$$E(\bar{v}) = \nu_2^2 Z(\bar{v}), \quad E(\tilde{v}) \leq \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} Z(\tilde{v}). \quad (3.10)$$

Combining (3.8)-(3.10), we obtain

$$E(v) \leq \nu_2^2 Z_A, \quad (3.11)$$

and the equality holds if and only if $v \in \mathcal{V}_2$. In other words, we have proved that

$$\mathbf{m}_{A,2} = \nu_2^2 Z_A,$$

and, moreover, for any $v \in \mathcal{R}_{A,2}$, $E(v) = \mathbf{m}_{A,2}$ if and only if $v \in \mathcal{V}_2$. To finish the proof of (i), it is sufficient to show that

$$\mathcal{V}_2 \cap \mathcal{R}_{A,2} = \mathcal{V}_{A,2}.$$

The inclusion $\mathcal{V}_{A,2} \subset \mathcal{V}_2 \cap \mathcal{R}_{A,2}$ is obvious. To prove the inverse inclusion, it is sufficient to show that for any $v \in \mathcal{V}_2 \cap \mathcal{R}_{A,2}$ with the form

$$v = B \sin \left(\frac{x_2}{\nu_2} + \beta \right)$$

for some $B \geq 0$ and $\beta \in \mathbb{R}$, it holds that $B = A$. This is obvious since

$$B = \|v\|_{L^\infty(\mathbb{T}^2)} = \|v_A\|_{L^\infty(\mathbb{T}^2)} = A.$$

The proof of (ii) is almost identical to that of (i), therefore we omit it.

Now we prove (iii). Denote

$$Z_{A,B} = Z(v_{A,B}).$$

Then it is clear that $Z(v) = Z_{A,B}$ for any $v \in \mathcal{R}_{A,B}$. Analogously to (3.7), we decompose any $v \in \mathcal{R}_{A,B}$ into two components:

$$v = \bar{v} + \tilde{v}, \quad \bar{v} \in \mathcal{V}, \quad \tilde{v} \in \mathcal{V}^\perp.$$

Then

$$Z(\bar{v}) + Z(\tilde{v}) = Z_{A,B}. \quad (3.12)$$

As in (3.9), we can prove that

$$E(v) = E(\bar{v}) + E(\tilde{v}). \quad (3.13)$$

Moreover, by Lemma 2.5(iii),

$$E(\bar{v}) = \nu^2 Z(\bar{f}), \quad E(\tilde{v}) \leq \frac{\nu^2}{4} Z(\tilde{v}). \quad (3.14)$$

Therefore we infer from (3.12)-(3.14) that

$$E(v) \leq \nu^2 Z_{A,B}, \quad (3.15)$$

and the equality holds if and only if $v \in \mathcal{V}$. Hence we have proved that

$$\mathbf{m}_{A,B} = \nu^2 Z_{A,B},$$

and, moreover, for any $v \in \mathcal{R}_{A,B}$, v is a maximizer of E relative to $\mathcal{R}_{A,B}$ if and only if $v \in \mathcal{V}$. To finish the proof, it is sufficient to show that

$$\mathcal{V} \cap \mathcal{R}_{A,B} = \mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}.$$

Since it is obvious that $\mathcal{V}_{A,B}, \mathcal{V}_{B,A} \subset \mathcal{R}_{A,B}$, it holds that $\mathcal{V}_{A,B} \cup \mathcal{V}_{B,A} \subset \mathcal{V} \cap \mathcal{R}_{A,B}$. To prove the inverse inclusion, it is sufficient to show that for any $v \in \mathcal{V} \cap \mathcal{R}_{A,B}$ with the form

$$v = C \sin\left(\frac{x_1}{\nu} + \alpha\right) + D \sin\left(\frac{x_2}{\nu} + \beta\right)$$

for some $C, D \geq 0$ and $\alpha, \beta \in \mathbb{R}$, it holds that

$$A = C, B = D \quad \text{or} \quad A = D, B = C. \quad (3.16)$$

To prove (3.16), notice that for any $v \in \mathcal{R}_{A,B}$,

$$\|v\|_{L^\infty(\mathbb{T}^2)} = \|v_{A,B}\|_{L^\infty(\mathbb{T}^2)}, \quad \|v\|_{L^2(\mathbb{T}^2)} = \|v_{A,B}\|_{L^2(\mathbb{T}^2)}, \quad (3.17)$$

which implies that

$$A + B = C + D, \quad A^2 + B^2 = C^2 + D^2. \quad (3.18)$$

From (3.18), we can easily obtain (3.16). In fact, (3.18) can be written as

$$A - C = D - B, \quad (A - C)(A + C) = (D - B)(D + B). \quad (3.19)$$

If $A - C = 0$, then $A = C, B = D$, hence (3.16) holds; if $A - C \neq 0$, then $A + C = D + B$, which together with $A + B = C + D$ gives $A = D, B = C$, hence (3.16) still holds. \square

Remark 3.2. From the above proof, for $i = 1, 2$, $\mathcal{V}_{A,i}$ is in fact the set of maximizers of E relative to

$$\{v \in \dot{L}^p(\mathbb{T}^2) \mid Z(v) = Z_A\}.$$

However, $\mathcal{V}_{A,B}$ does not have such a characterization. To distinguish $\mathcal{V}_{A,B}$ from other sinusoidal flows, it is necessary to study their rearrangements.

4. COMPACTNESS

Throughout this paper, let $A, B \geq 0$, $1 < p < +\infty$ be fixed.

Our purpose in this section is to prove the following proposition, stating that any maximizing sequence for the maximization problem (3.5) or (3.6) is compact in $L^p(\mathbb{T}^2)$.

Proposition 4.1. *Let $\mathcal{R}_{A,1}, \mathcal{R}_{A,2}, \mathcal{R}_{A,B}$ be defined by (3.3), (3.4), and $\mathfrak{m}_{A,1}, \mathfrak{m}_{A,2}, \mathfrak{m}_{A,B}$ be defined by (3.5), (3.6).*

(i) *If $\nu_1 < \nu_2$, then for any sequence $\{v_n\} \subset \mathcal{R}_{A,2}$ satisfying*

$$\lim_{n \rightarrow +\infty} E(v_n) = \mathfrak{m}_{A,2}, \tag{4.1}$$

there exists some subsequence of $\{v_n\}$, denoted by $\{v_{n_j}\}$, and some $\hat{v} \in \mathcal{V}_{A,2}$, such that $v_{n_j} \rightarrow \hat{v}$ in $L^p(\mathbb{T}^2)$ as $j \rightarrow +\infty$.

(ii) *If $\nu_1 > \nu_2$, then for any sequence $\{v_n\} \subset \mathcal{R}_{A,1}$ satisfying*

$$\lim_{n \rightarrow +\infty} E(v_n) = \mathfrak{m}_{A,1}, \tag{4.2}$$

there exists some subsequence of $\{v_n\}$, denoted by $\{v_{n_j}\}$, and some $\hat{v} \in \mathcal{V}_{A,1}$, such that $v_{n_j} \rightarrow \hat{v}$ in $L^p(\mathbb{T}^2)$ as $j \rightarrow +\infty$.

(iii) *If $\nu_1 = \nu_2 = \nu$, then for any sequence $\{v_n\} \subset \mathcal{R}_{A,B}$ satisfying*

$$\lim_{n \rightarrow +\infty} E(v_n) = \mathfrak{m}_{A,B}, \tag{4.3}$$

there exists some subsequence of $\{v_n\}$, denoted by $\{v_{n_j}\}$, and some $\hat{v} \in \mathcal{V}_{A,B}$, such that $v_{n_j} \rightarrow \hat{v}$ in $L^p(\mathbb{T}^2)$ as $j \rightarrow +\infty$.

To prove Proposition 4.1, we need several lemmas.

Lemma 4.2. *For any $1 < p < +\infty$, E is sequentially weakly continuous in $\mathring{L}^p(\mathbb{T}^2)$, i.e., if v_n converges weakly to \hat{v} in $\mathring{L}^p(\mathbb{T}^2)$, then*

$$\lim_{n \rightarrow +\infty} E(v_n) = E(\hat{v}).$$

Proof. Since K is bounded from $\mathring{L}^p(\mathbb{T}^2)$ to $\mathring{W}^{2,p}(\mathbb{T}^2)$, Kv_n converges weakly to $K\hat{v}$ in $W^{2,p}(\mathbb{T}^2)$. Taking into account the fact that the embedding $W^{2,p}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ is compact, we further deduce that Kv_n converges strongly to $K\hat{v}$ in $L^\infty(\mathbb{T}^2)$. Hence the desired result follows immediately. \square

Lemma 4.3 ([7], Theorem 6). *Let $\mathcal{R}(f_0)$ be set of rearrangements of some $f_0 \in L^p(\mathbb{T}^2)$ on \mathbb{T}^2 , and $\overline{\mathcal{R}(f_0)}$ be the weak closure of $\mathcal{R}(f_0)$ in $L^p(\mathbb{T}^2)$. Then $\overline{\mathcal{R}(f_0)}$ is convex, i.e., $\theta f_1 + (1 - \theta)f_2 \in \overline{\mathcal{R}(f_0)}$ whenever $f_1, f_2 \in \mathcal{R}(f_0)$ and $\theta \in [0, 1]$.*

Lemma 4.4 ([7], Theorem 4). *Let $p^* = p/(p - 1)$ be the Hölder conjugate of p . Let $\mathcal{R}(f_0), \mathcal{R}(g_0)$ be sets of rearrangements on \mathbb{T}^2 of some $f_0 \in L^p(\mathbb{T}^2)$ and some $g_0 \in L^q(\mathbb{T}^2)$, respectively. Then for any $\tilde{g} \in \mathcal{R}(g_0)$, there exists $\tilde{v} \in \mathcal{R}(f_0)$, such that*

$$\int_{\mathbb{T}^2} \tilde{f}\tilde{g}dx \geq \int_{\mathbb{T}^2} fgdx, \quad \forall f \in \mathcal{R}(f_0), g \in \mathcal{R}(g_0).$$

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. To prove (i), fix a sequence $\{v_n\} \subset \mathcal{R}_{A,2}$ such that (4.1) holds. Obviously $\{v_n\}$ is bounded in $L^p(\mathbb{T}^2)$. Without loss of generality, we can assume, up to a subsequence, that v_n converges weakly to some $\hat{v} \in \overline{\mathcal{R}_{A,2}}$ in $L^p(\mathbb{T}^2)$. Here $\overline{\mathcal{R}_{A,2}}$ is the weak closure of $\mathcal{R}_{A,2}$ in $L^p(\mathbb{T}^2)$ as in Lemma 4.3. By Lemma 4.2, we have that

$$E(\hat{v}) = \mathfrak{m}_{A,2} \geq E(v), \quad \forall v \in \overline{\mathcal{R}_{A,2}}. \quad (4.4)$$

Here we used the fact that

$$\mathfrak{m}_{A,2} = \sup_{v \in \mathcal{R}_{A,2}} E(v) = \sup_{v \in \overline{\mathcal{R}_{A,2}}} E(v).$$

By Lemma 4.3, $\overline{\mathcal{R}_{A,2}}$ is convex. Hence for any $v \in \mathcal{R}_{A,2}$ and $\theta \in [0, 1]$, we have that $\theta v + (1 - \theta)\hat{v} \in \overline{\mathcal{R}_{A,2}}$. By (4.4), $E(\theta v + (1 - \theta)\hat{v})$ attains its maximum value at $\theta = 0$. Therefore we have that

$$\left. \frac{d}{d\theta} E(\theta v + (1 - \theta)\hat{v}) \right|_{\theta=0^+} \leq 0,$$

which gives

$$\int_{\mathbb{T}^2} v K \hat{v} d\mathbf{x} \leq \int_{\mathbb{T}^2} \hat{v} K \hat{v} d\mathbf{x}. \quad (4.5)$$

Note that (4.5) holds for any $v \in \mathcal{R}_{A,2}$. By Lemma 4.4, there exists some $\tilde{v} \in \mathcal{R}_{A,2}$ such that

$$\int_{\mathbb{T}^2} v K \hat{v} d\mathbf{x} \leq \int_{\mathbb{T}^2} \tilde{v} K \hat{v} d\mathbf{x}, \quad \forall v \in \mathcal{R}_{A,2}. \quad (4.6)$$

By a simple approximation procedure, it is easy to show that (4.6) actually holds for any $v \in \overline{\mathcal{R}_{A,2}}$. In particular,

$$\int_{\mathbb{T}^2} \hat{v} K \hat{v} d\mathbf{x} \leq \int_{\mathbb{T}^2} \tilde{v} K \hat{v} d\mathbf{x}. \quad (4.7)$$

Below we show that $\tilde{v} = \hat{v}$. We compute as follows:

$$\begin{aligned} E(\tilde{v} - \hat{v}) &= E(\tilde{v}) + E(\hat{v}) - \int_{\mathbb{T}^2} \tilde{v} K \hat{v} d\mathbf{x} \\ &\leq E(\tilde{v}) + E(\hat{v}) - \int_{\mathbb{T}^2} \hat{v} K \hat{v} d\mathbf{x} \\ &= \frac{1}{2} E(\tilde{v}) - \frac{1}{2} E(\hat{v}) \\ &\leq 0. \end{aligned} \quad (4.8)$$

Here we used (4.4), (4.7), and the fact that K is symmetric (see Lemma 2.2). Taking into account the fact that K is positive definite (see Lemma 2.2), we infer from (4.8) that $\tilde{v} = \hat{v}$. In particular, $\hat{v} \in \mathcal{R}_{A,2}$. Furthermore, since $E(\hat{v}) = \mathfrak{m}_{A,2}$ (recall (4.4)), we can apply Proposition 3.1(i) to obtain

$$\hat{v} \in \mathcal{V}_{A,2}.$$

To conclude, we have proved that v_n , up to a subsequence, converges weakly to some $\hat{v} \in \mathcal{V}_{A,2}$ in $L^p(\mathbb{T}^2)$ as $n \rightarrow +\infty$. In particular, $\|v_n\|_{L^p(\mathbb{T}^2)} = \|\hat{v}\|_{L^p(\mathbb{T}^2)}$ for all n . By uniform convexity, we further deduce that v_n , up to a subsequence, in fact converges *strongly* to \hat{v} in $L^p(\mathbb{T}^2)$ as $n \rightarrow +\infty$. This completes the proof of (i).

The proofs of (ii)(iii) are almost identical to that of (i), we omit them therefore. □

5. PROOF OF THEOREM 1.4

With the variational characterizations for $\mathcal{V}_{A,1}$, $\mathcal{V}_{A,2}$ and $\mathcal{V}_{A,B}$ established in Section 3, and the compactness proved in Section 4, we are ready to prove Theorem 1.4 in this section.

Throughout this section, let $A, B \geq 0$ and $1 < p < +\infty$ be fixed.

Proof of Theorem 1.4(i). Suppose by contradiction that $\mathcal{V}_{A,2}$ is not nonlinearly stable in the sense of (1.28). Then there exist some $\varepsilon_0 > 0$, a sequence of smooth Euler flows on \mathbb{T}^2 with vorticity $\{\omega^n\}$, and a sequence of times $\{t_n\}$, such that

$$\lim_{n \rightarrow +\infty} \min_{v \in \mathcal{V}_{A,2}} \|\omega_0^n - v\|_{L^p(\mathbb{T}^2)} = 0, \tag{5.1}$$

$$\min_{v \in \mathcal{V}_{A,2}} \|\omega_{t_n}^n - v\|_{L^p(\mathbb{T}^2)} \geq \varepsilon_0, \quad \forall n. \tag{5.2}$$

Here $\omega_t^n := \omega^n(t, \cdot)$.

It is easy to check that $\mathcal{V}_{A,2}$ is compact in $L^p(\mathbb{T}^2)$ (which can also be seen from Proposition 4.1(i)), hence there exist some subsequence of $\{\omega_0^n\}$, still denoted by $\{\omega_0^n\}$, and some $\bar{\omega} \in \mathcal{V}_{A,2}$, such that

$$\lim_{n \rightarrow +\infty} \|\omega_0^n - \bar{\omega}\|_{L^p(\mathbb{T}^2)} = 0. \tag{5.3}$$

Consequently,

$$\lim_{n \rightarrow +\infty} E(\omega_0^n) = E(\bar{\omega}) = \mathfrak{m}_{A,2}. \tag{5.4}$$

By energy conservation, we get from (5.4) that

$$\lim_{n \rightarrow +\infty} E(\omega_{t_n}^n) = \lim_{n \rightarrow +\infty} E(\omega_0^n) = \mathfrak{m}_{A,2}. \tag{5.5}$$

Now we can easily get a contradiction if we only consider perturbed flows with vorticity on $\mathcal{R}_{A,2}$. In fact, if $\{\omega_{t_n}^n\} \subset \mathcal{R}_{A,2}$ for any n , then we can choose $v_n = \omega_{t_n}^n$ in Proposition 4.1(i) (note that (4.1) is satisfied by (5.5)) to deduce that $\{\omega_{t_n}^n\}$, up to a subsequence, converges to some element in $\mathcal{V}_{A,2}$ in $L^p(\mathbb{T}^2)$ as $n \rightarrow +\infty$. This obviously contradicts (5.2).

To deal with the general case, we need to introduce a sequence of “followers” to $\{\omega^n\}$ as in [8, 18]. For fixed n , denote by \mathbf{v}^n the velocity of the Euler flow with vorticity ω^n . Then ω^n satisfies the following nonlinear transport equation (see [21], p. 20):

$$\begin{cases} \partial_t \omega^n + \mathbf{v}^n \cdot \nabla \omega^n = 0, & t > 0, \mathbf{x} \in \mathbb{T}^2, \\ \omega^n(0, \cdot) = \omega_0^n. \end{cases} \tag{5.6}$$

Let ζ^n be the solution of the following linear transport equation:

$$\begin{cases} \partial_t \zeta^n + \mathbf{v}^n \cdot \nabla \zeta^n = 0, & t > 0, \mathbf{x} \in \mathbb{T}^2, \\ \zeta^n(0, \cdot) = \bar{\omega}. \end{cases} \quad (5.7)$$

For simplicity, denote $\zeta_t^n = \zeta^n(t, \cdot)$. Since \mathbf{v}^n is divergence-free, by the Liouville theorem (see [22], p. 48), it holds that

$$\zeta_t^n \in \mathcal{R}(\bar{\omega}) = \mathcal{R}_{A,2}, \quad \forall t \geq 0, \quad (5.8)$$

On the other hand, combining (5.6) and (5.7), we see that $\zeta^n - \omega^n$ satisfies

$$\begin{cases} \partial_t(\zeta^n - \omega^n) + \mathbf{v}^n \cdot \nabla(\zeta^n - \omega^n) = 0, & t > 0, \mathbf{x} \in \mathbb{T}^2, \\ (\zeta^n - \omega^n)(0, \cdot) = \bar{\omega} - \omega_0^n. \end{cases} \quad (5.9)$$

Again, by the Liouville theorem,

$$\zeta_t^n - \omega_t^n \in \mathcal{R}(\bar{\omega} - \omega_0^n), \quad \forall t \geq 0. \quad (5.10)$$

Having introduced the sequence of “followers” $\{\zeta^n\}$, we are ready to deduce a contradiction. By (5.3) and (5.10),

$$\lim_{n \rightarrow +\infty} \|\zeta_{t_n}^n - \omega_{t_n}^n\|_{L^p(\mathbb{T}^2)} = 0, \quad (5.11)$$

which together with (5.5) implies that

$$\lim_{n \rightarrow +\infty} E(\zeta_{t_n}^n) = \mathfrak{m}_{A,2}. \quad (5.12)$$

To conclude, we have found a sequence $\{\zeta_{t_n}^n\}$ such that (5.8) and (5.12) hold. Applying Proposition 4.1(i), we infer that $\zeta_{t_n}^n$, up to a subsequence, converges to some $\tilde{\omega} \in \mathcal{V}_{A,2}$ in $L^p(\mathbb{T}^2)$ as $n \rightarrow +\infty$. Combining (5.11), we deduce that $\omega_{t_n}^n$ converges to $\tilde{\omega}$ in $L^p(\mathbb{T}^2)$ as $n \rightarrow +\infty$, which obviously contradicts (5.2). \square

The proof of Theorem 1.4(ii) can also be proved similarly.

To prove Theorem 1.4(iii), we need the following lemma.

Lemma 5.1. *If $\nu_1 = \nu_2 = \nu$ and $A \neq B$, then*

$$\min_{u \in \mathcal{V}_{A,B}, v \in \mathcal{V}_{B,A}} \|u - v\|_{L^p(\mathbb{T}^2)} > 0.$$

Proof. Observe that

$$\mathcal{V}_{A,B} = \left\{ a \sin\left(\frac{x_1}{\nu}\right) + b \cos\left(\frac{x_1}{\nu}\right) + c \sin\left(\frac{x_2}{\nu}\right) + d \cos\left(\frac{x_2}{\nu}\right) \mid a^2 + b^2 = A^2, c^2 + d^2 = B^2 \right\}.$$

Hence it is easy to see that $\mathcal{V}_{A,B}$ is compact in $L^p(\mathbb{T}^2)$. Similarly, $\mathcal{V}_{B,A}$ is also compact in $L^p(\mathbb{T}^2)$. To finish the proof, it is sufficient to show that $\mathcal{V}_{A,B} \cap \mathcal{V}_{B,A} = \emptyset$.

Fix $u \in \mathcal{V}_{A,B}$, $v \in \mathcal{V}_{B,A}$. Assume that u, v has the form

$$u = A \sin\left(\frac{x_1}{\nu} + \alpha\right) + B \sin\left(\frac{x_2}{\nu} + \beta\right), \quad v = B \sin\left(\frac{x_1}{\nu} + \alpha'\right) + A \sin\left(\frac{x_2}{\nu} + \beta'\right),$$

where $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$. Then u, v can be written as

$$\begin{aligned} u &= A \cos \alpha \sin \left(\frac{x_1}{\nu} \right) + A \sin \alpha \cos \left(\frac{x_1}{\nu} \right) + B \cos \beta \sin \left(\frac{x_2}{\nu} \right) + B \sin \beta \cos \left(\frac{x_2}{\nu} \right), \\ v &= B \cos \alpha' \sin \left(\frac{x_1}{\nu} \right) + B \sin \alpha' \cos \left(\frac{x_1}{\nu} \right) + A \cos \beta' \sin \left(\frac{x_2}{\nu} \right) + A \sin \beta' \cos \left(\frac{x_2}{\nu} \right). \end{aligned}$$

If $u = v$, then we must have

$$A \cos \alpha = B \cos \alpha', \quad A \sin \alpha = B \sin \alpha', \quad B \cos \beta = A \cos \beta', \quad B \sin \beta = A \sin \beta',$$

which implies that $A = B$, a contradiction. □

Proof of Theorem 1.4(iii). Following the proof of Theorem 1.4(i), we can show that $\mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}$ is nonlinearly stable in the sense of (1.28). If $A = B$, then $\mathcal{V}_{A,B} = \mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}$ is nonlinearly stable in the sense of (1.28). If $A \neq B$, then by Lemma 5.1 there is a positive distance between $\mathcal{V}_{A,B}$ and $\mathcal{V}_{B,A}$ in $L^p(\mathbb{T}^2)$, hence by continuity each of them is nonlinearly stable in the sense of (1.28). □

APPENDIX A. PROOF OF THEOREM 1.1

In this appendix, we give the proof of Theorem 1.1 based on the energy-entropy type inequalities established in Lemma 2.5. We only prove Theorem 1.1(i). The other two items can be proved in a similar manner.

For any Euler flow with vorticity ω and velocity mean vector \mathbf{b} , since E and Z are both conserved quantities, we have that

$$\nu_2^2 Z(\omega_t) - E(\omega_t) = \nu_2^2 Z(\omega_0) - E(\omega_0), \quad \forall t \geq 0. \tag{A.1}$$

Decompose ω_t into two components:

$$\omega_t = \bar{\omega}_t + \tilde{\omega}_t, \quad \bar{\omega}_t \in \mathcal{V}_2, \quad \tilde{\omega}_t \in \mathcal{V}_2^\perp.$$

Using Lemma 2.3, we infer from (A.1) that

$$\nu_2^2 Z(\bar{\omega}_t) + \nu_2^2 Z(\tilde{\omega}_t) - E(\bar{\omega}_t) - E(\tilde{\omega}_t) = \nu_2^2 Z(\bar{\omega}_0) + \nu_2^2 Z(\tilde{\omega}_0) - E(\bar{\omega}_0) - E(\tilde{\omega}_0), \quad \forall t \geq 0. \tag{A.2}$$

Applying Lemma 2.5(i), we have that

$$E(\bar{\omega}_t) = \nu_2^2 Z(\bar{\omega}_t), \quad \forall t \geq 0. \tag{A.3}$$

$$E(\tilde{\omega}_t) \leq \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} Z(\tilde{\omega}_t), \quad \forall t \geq 0. \tag{A.4}$$

From (A.2) and (A.3), we have that

$$\nu_2^2 Z(\tilde{\omega}_t) - E(\tilde{\omega}_t) = \nu_2^2 Z(\tilde{\omega}_0) - E(\tilde{\omega}_0), \quad \forall t \geq 0. \tag{A.5}$$

which together with (A.4) gives

$$\begin{aligned} \left(\nu_2^2 - \max \left\{ \nu_1^2, \frac{\nu_2^2}{4} \right\} \right) Z(\tilde{\omega}_t) &\leq \nu_2^2 Z(\tilde{\omega}_0) - E(\tilde{\omega}_0) \\ &\leq \nu_2^2 Z(\tilde{\omega}_0), \quad \forall t \geq 0, \end{aligned} \tag{A.6}$$

or equivalently,

$$Z(\tilde{\omega}_t) \leq C_{\nu_1, \nu_2} Z(\tilde{\omega}_0) \quad \forall t \geq 0, \quad C_{\nu_1, \nu_2} := \left(1 - \max \left\{ \left(\frac{\nu_1}{\nu_2}\right)^2, \frac{1}{4} \right\}\right)^{-1} > 0. \quad (\text{A.7})$$

Taking into account the fact that

$$\min_{v \in \mathcal{V}_2} \|\omega_t - v\|_{L^2(\mathbb{T}^2)} = \|\tilde{\omega}_t\|_{L^2(\mathbb{T}^2)} = \sqrt{2Z(\tilde{\omega}_t)}, \quad \forall t \geq 0,$$

we obtain the desired stability from (A.7) immediately.

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