arXiv:2210.01416v1 [math.CV] 4 Oct 2022

COMPLETE PLURIPOLAR SETS AND REMOVABLE SINGULARITIES OF PLURISUBHARMONIC FUNCTIONS

XIEPING WANG

To Li and Rui'an

Abstract. Inspired by Chen-Wu-Wang (Math. Ann. 362: 305–319, 2015), we prove a Hartogs type extension theorem for plurisubharmonic functions across a compact complete pluripolar set, which is complementary to a classical theorem of Shiffman.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of Ω such that $\Omega\backslash K$ is connected. The famous Hartogs extension theorem states that every holomorphic function on $\Omega\backslash K$ extends holomorphically to the whole domain Ω. An analogue for pluriharmonic functions is also valid, as recently discovered by Chen in [Che17] (see also [Wan22]). When it comes to plurisubharmonic (psh for short) functions the situation, however, is quite different; for instance, every bounded domain in $\mathbb{C}^n$ with smooth boundary is the domain of existence of a psh function (see [BT88]). Thus it is meaningful to prove the following

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact complete pluripolar subset of Ω. Then every psh function on $\Omega\backslash K$ admits a unique psh extension to Ω.*

This somewhat surprising result is complementary to a classical theorem of Shiffman [Shi72] that every psh function on a domain in $\mathbb{C}^n$ extends plurisubharmonically across a closed set of Hausdorff $(2n-2)$-measure zero. It is also worth noting that Theorem 1.1 is of global nature, while Shiffman's theorem is of local nature.

The special case of Theorem 1.1 where Ω is the unit polydisc $\Delta^n \subset \mathbb{C}^n$ is already contained in the beautiful work of Chen-Wu-Wang [CWW15], who also dealt with the more general case of K being a *closed* complete pluripolar subset of Δ^n under certain reasonable conditions. Chen-Wu-Wang proved their result by using an Ohsawa-Takegoshi type extension theorem for a single point in bounded *complete Kähler* domains, which is also one of the main results in [CWW15] and seems to be highly nontrivial due to its connection with an open problem posed by Ohsawa in [Ohs95]. We observe that Theorem 1.1 can be proved by combining this powerful result with an idea of Shiffman.

One may naturally ask whether Theorem 1.1 remains true when $\mathbb{C}^n$ is replaced by a generic Stein manifold of dimension $n \geq 2$. Since it is not clear to us at this moment whether the Ohsawa-Takegoshi type extension theorem by Chen-Wu-Wang applies to this more general case, we instead use other techniques, namely the recently proved Hartogs extension theorem for pluriharmonic functions in [Wan22] and the Skoda-El Mir extension theorem for closed positive currents, to prove the following

Date: October 5, 2022.

2020 *Mathematics Subject Classification.* 32D15, 32D20, 32U30, 32U05.

Key words and phrases. Removable singularities, pluriharmonic functions, complete pluripolar sets.

The author was partially supported by NSFC grant 12001513 and NSF of Anhui Province grant 2008085QA18.

Theorem 1.2. *Let X be a Stein manifold of dimension $n \geq 2$. Suppose Ω is a domain in X such that $H^1(\Omega, \mathcal{O}) = 0$ and $H^2(\Omega, \mathbb{R}) = 0$, and K is a compact complete pluripolar subset of Ω. Then every psh function on $\Omega \backslash K$ admits a unique psh extension to Ω.*

Remark. Since K is holomorphically convex in X, there always exists a Stein neighborhood of K contained in a given domain $\Omega \subset X$ so that the assumption $H^1(\Omega, \mathcal{O}) = 0$ is nonessential for the theorem. Also, it seems that the additional condition $H^2(\Omega, \mathbb{R}) = 0$ is superfluous (and this is the case at least when $X = \mathbb{C}^n$, as shown by Theorem 1.1).

After recalling a fundamental result concerning closed complete pluripolar sets in Stein manifolds in Section 2, we prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

Acknowledgements. The author would like to thank Professor Bo-Yong Chen for kindly explaining his joint work with Wu and Wang [CWW15]. The author would also like to thank Doctors Yong-Xin Gao and Zhi Li for patiently listening to his lectures on the basic parts of pluripotential theory and related topics at the seminar they organized when all three of them were at the Institute of Mathematics, AMSS, Chinese Academy of Sciences from 2017 to 2019.

2. Complete pluripolar sets and their defining functions

We begin by recalling the notion of complete pluripolarity. Let X be a complex manifold and $PSH(X)$ denote the set of all psh functions on X.

Definition. A subset $E \subset X$ is called *complete pluripolar* if for every point $z \in E$ there exists a neighborhood U of z and a function $\varphi \in PSH(U)$ such that

$$E \cap U = \varphi^{-1}(-\infty).$$

The set of all complex subvarieties of X forms a particularly important class of (closed) complete pluripolar sets, but complete pluripolar sets are much more general: for instance, the Cartesian product of finitely many (possibly different) Cantor type sets in the complex plane of logarithmic capacity zero is a compact complete pluripolar set in the corresponding complex Euclidean space (see, e.g., [Ran95]), but far from being complex-analytic; see also [ElM84] for some other nontrivial examples.

In 1990, Colţoiu proved the following important result concerning the existence of a global defining function for a closed complete pluripolar set.

Theorem 2.1 (see [Col90, Corollary 1])**.** *Let X be a Stein manifold and $E \subset X$ be a closed complete pluripolar set. Then there exists a function $\rho \in PSH(X) \cap C^\infty(X \backslash E)$ such that $\rho^{-1}(-\infty) = E$ and $\sqrt{-1}\partial\bar{\partial}\rho > 0$ on $X \backslash E$.*

As we shall see later, this result plays a fundamental role in the proofs of Theorems 1.1 and 1.2.

3. Proof of Theorem 1.1

The idea of the proof is due to Chen-Wu-Wang [CWW15], which in turn was more or less inspired by the celebrated work of Demailly [Dem92]. The key ingredient here is an Ohsawa-Takegoshi type extension theorem for a single point in bounded complete Kähler domains in $\mathbb{C}^n$ (see [CWW15, Theorem 1.3] for details). In order to make use of this theorem, we first need to prove the following result:

Theorem 3.1. *Let X be a Stein manifold and $E \subset X$ be a closed complete pluripolar set. Then $X \backslash E$ carries a complete Kähler metric.*

The important special case of $E \subset X$ being a complex subvariety is well-known and due to Grauert [Gra56].

Proof. The result is an easy consequence of Theorem 2.1. Let φ be a C^∞ strictly psh exhaustion function for X and let $\psi\colon X\setminus E \to \mathbb{R}$ be a C^∞ function such that

$$\psi = -\log(-\rho) \quad \text{on} \;\; U\setminus E,$$

where ρ is a psh function as in Theorem 2.1 and $U := \{\rho < -1\}$. Then one can construct a C^∞ convex, rapidly increasing function χ on $\mathbb{R}$ such that

$$\omega := \sqrt{-1}\partial\bar\partial(\chi\circ\varphi) + \sqrt{-1}\partial\bar\partial\psi \geq \omega_0 \quad \text{on} \;\; X\setminus E$$

for some complete Kähler metric ω_0 on X.

We claim that ω is complete on $X\setminus E$. For this, we may assume without loss of generality that X itself is connected (and so is $X\setminus E$). Suppose $\{z_j\}_{j\geq 1}$ is a bounded sequence in the metric space $(X\setminus E, \omega)$. Then there is a sequence of smooth curves $\{\gamma_j\}_{j\geq1} \subset C^\infty([0,1], X\setminus E)$ with uniformly bounded lengths with respect to ω, joining each z_j to a (fixed) reference point in $X\setminus\overline{U}$. Since $\omega \geq \omega_0$ on $X\setminus E$ and ω_0 is complete on X, we may assume that the sequence $\{z_j\}_{j\geq1}$ itself converges in X by passing to a subsequence if necessary. What now remains is to show that the limit of $\{z_j\}_{j\geq1}$ lies outside E. Suppose the contrary and set

$$t_j := \inf\big\{t\in[0,1]\colon \gamma_j([t,1])\subset U\big\}, \quad j\geq 1.$$

Clearly $0<t_j<1$ and $\gamma_j(t_j)\in\partial U = \{\rho=-1\}$ for all sufficiently large j. Observe also that

$$\omega \geq \sqrt{-1}\partial\bar\partial\big(-\log(-\rho)\big) \geq \sqrt{-1}\partial\log(-\rho)\wedge\bar\partial\log(-\rho) \quad \text{on} \;\; U.$$

It then follows that

$$\begin{aligned}\sqrt{2}\,\mathrm{length}_\omega(\gamma_j) &\geq \int_{t_j}^1 \big|\big(d\log(-\rho)\big)(\gamma'(t))\big|dt \geq \int_{t_j}^1 \big(d\log(-\rho)\big)(\gamma'(t))dt \\ &= \log(-\rho(z_j)) \to \infty \quad \text{as} \;\; j\to\infty,\end{aligned}$$

contradicting the boundedness of $\{\mathrm{length}_\omega(\gamma_j)\}_{j\geq1}$. Therefore the limit of $\{z_j\}_{j\geq1}$ lies outside E, and hence ω is complete on $X\setminus E$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The uniqueness for extension is clear, since two psh functions on Ω which coincide almost everywhere are actually equal everywhere. So it suffices to prove the existence part of the theorem.

We first observe that the problem can be reduced to the case when $\Omega\supset K$ is a bounded pseudoconvex domain. To see this, let ρ be a psh function on $\mathbb{C}^n$, continuous on $\mathbb{C}^n\setminus K$ and satisfying $\rho^{-1}(-\infty) = K$ (see Theorem 2.1). Choose an open set $U\subset\mathbb{C}^n$ such that $K\subset U\subset\subset\Omega$, and set

$$\widetilde{\rho} := \begin{cases} \max\big\{\rho, \inf\limits_{\partial U}\rho\big\} & \text{on} \;\; \mathbb{C}^n\setminus\overline{U}; \\ \rho & \text{on} \;\; \overline{U}. \end{cases}$$

Then $\widetilde\rho$ is a psh function on $\mathbb{C}^n$ with $\widetilde\rho^{-1}(-\infty) = K$. Replacing Ω by any connected component of $\{\widetilde\rho < \inf\limits_{\partial U}\rho\}$, we may assume in what follows that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$.

Let $\varphi\in PSH(\Omega\setminus K)$. To prove the plurisubharmonic extendibility of φ across K, it suffices to show that every point of K admits a small neighborhood on which φ is bounded above. Observe that $\Omega\setminus K$ is a bounded complete Kähler domain, in view of Theorem

3.1. We can therefore invoke [CWW15, Theorem 1.3] to assign to every point $z \in \Omega\backslash K$ a holomorphic function f_z on $\Omega\backslash K$ with the property that $f_z(z) = e^{\varphi(z)/2}$ and

$$\int_{\Omega\backslash K} |f_z|^2 e^{-\varphi} \leq \mathrm{const}_{n,\,\mathrm{diam}\,\Omega}. \tag{3.1}$$

By the Hartogs extension theorem, each such f_z extends holomorphically to Ω. With a slight abuse of notation, we denote the extension still by f_z.

To proceed the proof, we make use of a result of Shiffman (cf. [Har77, Lemma 2.3] and [Dem12, Chapter 3, Lemma 4.7]). Fix an arbitrary point $z_0 \in K$ and recall that being a polar subset of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, K has Hausdorff dimension at most $2n-2$ (see, e.g., [AG01, Theorem 5.9.6]). By suitably selecting affine linear coordinates for $\mathbb{C}^n$, we can find a polydisc $\Delta' \times \Delta'' \subset \mathbb{C}^{n-1} \times \mathbb{C}$ centered at $z_0 =: (z_0', z_0'')$ such that

$$(\Delta' \times \partial\Delta'') \cap K = \emptyset.$$

By shrinking Δ' if necessary, we further arrive at

$$\big(\Delta' \times (\Delta''\backslash(1-\varepsilon)\overline{\Delta''})\big) \cap K = \emptyset$$

for some sufficiently small $\varepsilon > 0$. Now choose R, $r > 0$ such that $1-\varepsilon < r < R < 1$ and ball $B \subset\subset \Delta'$ centered at z_0'. Then the Cauchy estimate and inequality (3.1) imply

$$\begin{aligned} e^{\varphi(z)} = |f_z(z)|^2 &\leq \mathrm{const}_{B,\,R,\,r,\,\varepsilon} \int_{B\times(R\Delta''\backslash r\Delta'')} |f_z|^2 \\ &\leq \mathrm{const}_{B,\,R,\,r,\,\varepsilon} \sup_{B\times(R\Delta''\backslash r\Delta'')} e^{\varphi} \int_{B\times(R\Delta''\backslash r\Delta'')} |f_z|^2 e^{-\varphi} \\ &\leq C \sup_{B\times(R\Delta''\backslash r\Delta'')} e^{\varphi} \end{aligned}$$

for all $z \in \big((1-\varepsilon)(B\times\Delta'')\big)\backslash K$, where $C > 0$ is a constant independent of z. Consequently, φ is bounded above on $(1-\varepsilon)(B \times \Delta'') \ni z_0$ outside K. This completes the proof. □

4. Proof of Theorem 1.2

We start with the following result, which is essentially due to Sibony [Sib85].

Proposition 4.1. *Let X be a Stein manifold of dimension $n \geq 2$ and $K \subset X$ be a compact complete pluripolar set. Then every closed positive (p,p)-current on $X\backslash K$ with $p \leq n-1$ has finite mass near K.*

Proof. As in the proof of Theorem 1.1, we can construct a strongly pseudoconvex neighborhood $\Omega \subset\subset X$ of K with C^∞-boundary. Let u_K denote the relative extremal function of K in Ω, that is

$$u_K = \sup\Big\{u \in PSH(\Omega) \cap C(\Omega)\colon u < 1 \text{ on } \Omega,\ u \leq 0 \text{ on } K\Big\}.$$

Clearly $u_K = 0$ on K. Moreover according to [Sib85, Proposition 1.4], the product $u_K T$ of u_K and every closed positive (p,p)-current T on $\Omega\backslash K$ with $p \leq n-1$ has finite mass near K. (This is true for all compact sets $K \subset \Omega$, regardless of the complete pluripolarity of K.)

It remains to show that u_K is no other than the characteristic function $\chi_{\Omega\backslash K}$ of $\Omega\backslash K$, provided $K \subset \Omega$ is further assumed to be complete pluripolar. For this let ρ be a negative psh function on Ω, continuous on $\Omega\backslash K$ and satisfying $\rho^{-1}(-\infty) = K$, and set

$$\rho_t := \max\big\{\rho/t + 1,\, 0\big\}, \quad t > 0.$$

Then $\{\rho_t\}_{t>0}$ forms a family of candidates for the supremum defining u_K, hence $\rho_t \leq u_K$ for all $t > 0$. Letting $t \to \infty$ yields $u_K = \chi_{\Omega \setminus K}$, as desired. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. As pointed out in the proof of Theorem 1.1, it suffices to prove the existence part of the theorem.

Given a function $\varphi \in PSH(\Omega \setminus K)$, we consider the associated closed positive $(1,1)$-current $T := \sqrt{-1}\partial\bar{\partial}\varphi$ on $\Omega \setminus K$. By virtue of Proposition 4.1 and the Skoda-El Mir extension theorem (see [ElM84, Théorème II.1] or [Sib85, Dem12]), T extends to a closed positive $(1,1)$-current on Ω, which we denote by $\widetilde{T}$. Since $H^1(\Omega, \mathcal{O}) = 0$ and $H^2(\Omega, \mathbb{R}) = 0$, a standard argument shows that $\widetilde{T}$ admits a global potential $\widetilde{\varphi} \in PSH(\Omega)$, i.e., $\sqrt{-1}\partial\bar{\partial}\widetilde{\varphi} = \widetilde{T}$. One can then write

$$\widetilde{\varphi} = \varphi + h \quad \text{on } \Omega \setminus K$$

with h being a pluriharmonic function on $\Omega \setminus K$, in view of Weyl's lemma. On the other hand, the Hartogs extension theorem for pluriharmonic functions (see [Wan22, Theorem 1.1]) implies that h admits a pluriharmonic extension $\widetilde{h}$ to Ω. It now follows that $\widetilde{\varphi} - \widetilde{h}$ is a psh function on Ω that extends φ. □

References

[AG01] D. H. Armitage and S. J. Gardiner, Classical Potential Theory, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2001.

[BT88] E. Bedford and B. A. Taylor, *Smooth plurisubharmonic functions without subextension*, Math. Z. **198** (1988), 331–337.

[Che17] B.-Y. Chen, *Hardy-Sobolev type inequalities and their applications*, arXiv:1712.02044v2.

[CWW15] B.-Y. Chen, J. Wu and X. Wang, *Ohsawa-Takegoshi type theorem and extension of plurisubharmonic functions*, Math. Ann. **362** (2015), 305–319.

[Col90] M. Colţoiu, *Complete locally pluripolar sets*, J. Reine Angew. Math. **412** (1990), 108–112.

[Dem92] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409.

[Dem12] J.-P. Demailly, Complex Analytic and Differential Geometry, available at https://www-fourier.ujf-grenoble.fr/~demailly/books.html.

[ElM84] H. El Mir, *Sur le prolongement des courants positifs fermés*, Acta Math. **153** (1984), 1–45.

[Gra56] H. Grauert, *Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik*, Math. Ann. **131** (1956), 38–75.

[Har77] R. Harvey, *Holomorphic chains and their boundaries*, in Several Complex Variables (Proc. Sympos. Pure Math., Vol. XXX, Part 1, Williams Coll., Williamstown, Mass., 1975), pp. 309–382. Amer. Math. Soc., Providence, R.I., 1977.

[Ohs95] T. Ohsawa, *On the extension of L^2 holomorphic functions III: negligible weights*, Math. Z. **219** (1995), 215–225.

[Ran95] T. Ransford, Potential Theory in the Complex Plane, London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, 1995.

[Shi72] B. Shiffman, *Extension of positive line bundles and meromorphic maps*, Invent. Math. **15** (1972), 332–347.

[Sib85] N. Sibony, *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J. **52** (1985), 157–197.

[Wan22] X. Wang, *Bott-Chern cohomology and the Hartogs extension theorem for pluriharmonic functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), to appear; see also arXiv:2205.02494v2.

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

Email address: xpwang008@ustc.edu.cn