THE ENDOMORPHISM RING OF THE TRIVIAL MODULE IN A LOCALIZED CATEGORY

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ABSTRACT. Suppose that G is a finite group and k is a field of characteristic $p > 0$. Let $\mathcal{M}$ be the thick tensor ideal of finitely generated modules whose support variety is in a fixed subvariety V of the projectivized prime ideal spectrum $\operatorname{Proj} \mathrm{H}^*(G,k)$. Let $\mathcal{C}$ denote the Verdier localization of the stable module category $\mathbf{stmod}(kG)$ at $\mathcal{M}$. We show that if V is a finite collection of closed points and if the p-rank every maximal elementary abelian p-subgroups of G is at least 3, then the endomorphism ring of the trivial module in $\mathcal{C}$ is a local ring whose unique maximal ideal is infinitely generated and nilpotent. In addition, we show an example where the endomorphism ring in $\mathcal{C}$ of a compact object is not finitely presented as a module over the endomorphism ring of the trivial module.

1. INTRODUCTION

Suppose that G is a finite group and that k is a field of characteristic $p > 0$. The stable category $\mathbf{stmod}(kG)$ of finitely generated kG-modules is a tensor triangulated category. A thick tensor ideal in $\mathbf{stmod}(kG)$ is determined by the support variety of its objects. Hence, for any closed subvariety V in $V_G(k) = \operatorname{Proj} \mathrm{H}^*(G,k)$, the full subcategory $\mathcal{M}_V$ of all finitely generated kG-modules M with $V_G(M) \subset V$, is a thick subcategory that is closed under tensor product with any finitely generated kG-module. Moreover, every thick tensor ideal can be defined in this or a similar way. Associated to $\mathcal{M}_V$ is a distinguished triangle

$$\longrightarrow \mathcal{E}_V \longrightarrow k \longrightarrow \mathcal{F}_V \longrightarrow$$

where $\mathcal{E}_V$ and $\mathcal{F}_V$ are idempotent kG-modules that are almost always infinitely generated. In addition, $\mathcal{E}_V \otimes M \cong M$ in the stable category if and only if $V_G(M) \subseteq V$. Tensoring with $\mathcal{F}_V$ is the localizing functor to the Verdier localization $\mathcal{C}_V$ of $\mathbf{stmod}(kG)$ at $\mathcal{M}_V$. Thus the localized category $\mathcal{C}_V$ is embedded in the stable category $\mathbf{StMod}(kG)$ of all kG-modules.

In the localized category $\mathcal{C}_V$, the trivial module k is identified with $\mathcal{F}_V$ and the ring $\operatorname{End}_{\mathcal{C}_V}(k)$ is isomorphic to $\operatorname{End}_{\mathbf{StMod}(kG)}(\mathcal{F}_V)$. For kG-modules M and N, the

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group $\mathrm{Hom}_{\mathcal{C}_V}(M, N)$ is a module over $\mathrm{End}_{\mathcal{C}_V}(k)$. This suggests that modules in the category $\mathcal{C}_V$ can be distinguished by invariants such as the annihilators of their endomorphism rings or cohomology rings. Such is the essence of the support variety theory that classifies the thick tensor ideals in $\mathbf{stmod}(kG)$. In subsequent work [10], we show that there is such a theory that is nontrivial in the case of the colocalized category generated by $\mathcal{E}_V$. However, in the localized category, there are additional complications, as we see in this paper.

In this paper, we complete the task of characterizing $\mathrm{End}_{\mathcal{C}_V}(k)$ in the case that the maximal elementary abelian subgroups of G have sufficiently large p-rank and the variety V is a finite collection of closed points. We prove in such cases that the ring $\mathrm{End}_{\mathcal{C}_V}(k)$ is a local ring whose maximal ideal is infinitely generated and nilpotent.

Our study relies on earlier results in [11] that proves the special case in which V is a single closed point in $V_G(k)$ and G is elementary abelian. More generally, that paper shows that the nonpositive Tate cohomology ring of any finite group H can be realized as the endomorpism ring of the trivial module in the Verdier localization $\mathcal{C}_V$ where V is a single point in the spectrum of the cohomology ring of $G = C \times H$ for C a cyclic group of order p. The proof of our main theorem also requires the fact that nilpotence in cohomology can be detected on restrictions to elementary abelian p-subgroups of a finite group [9]. This theorem does not hold for G a general finite group scheme, and hence the proof of our main theorem does not extend to that realm.

The next section presents an introduction and references to the categories and recalls some theorems on support varieties. In the three sections that follow, we review the main theorem of [11] and extend the result to the case in which the variety V is a finite collection of more than one closed points. In section 5, we prove the main theorem for any finite group whose maximal elementary abelian p-subgroups have p-rank at least three. In section 6, we look at the restriction of the module $\mathcal{F}_V$ from an elementary abelian group to one of its proper subgroup. This result is used in the final section to show by example that even for a compact object M in $\mathbf{stmod}(kG)$, it is possible that $\mathrm{End}_{\mathcal{C}_V}(M)$ is not finitely generated over $\mathrm{End}_{\mathcal{C}_V}(k)$. In the other direction, we show also in Section 7 that if V is the subvariety of all homogeneous prime ideals that contain a single non-nilpotent element of cohomology, then for any compact object M, $\mathrm{End}_{\mathcal{C}_V}(M)$ is finitely generated over $\mathrm{End}_{\mathcal{C}_V}(k)$. In the final section we present an example that shows more of the stucture of the idempotent module $\mathcal{F}_V$ in the case of groups that are not elementary abelian p-groups.

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2. Background

In this section, we review some background. As references, we refer the reader to [12] or [3] for information on the cohomology of finite groups and support varieties in this context. For information on triangulated categories see [14]. A lot of the background material is summarized very well in the paper [2] of Balmer and Favi.

Throughout the paper, we let G be a finite group and k a field of characteristic $p > 0$. For convenience, we assume that k is algebraically closed. Recall that kG is a Hopf algebra so that if M and N are kG-modules, then so is $M \otimes_k N$. In general, we write $\otimes$ for $\otimes_k$.

Let $\mathbf{mod}(kG)$ denote the category of finitely generated kG-modules and $\mathbf{Mod}(kG)$ the category of all kG-module. Let $\mathbf{stmod}(kG)$ be the stable category of finitely generated kG-module modulo projectives. The objects in $\mathbf{stmod}(kG)$ are the same as those in $\mathbf{mod}(kG)$. If M and N are finitely generated kG-modules, then the group of morphisms from M to N in the stable category is the quotient $\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N)$ where $\mathrm{PHom}_{kG}(M,N)$ is the set of homomorphisms that factor through projective modules. The definition of the stable category of all modules $\mathbf{StMod}(kG)$ is similar.

The stable categories $\mathbf{stmod}(kG)$ and $\mathbf{StMod}(kG)$ are tensor triangulated categories. The tensor is the one given by the Hopf algebra structure on kG as mentioned above. Triangles correspond roughly to exact sequences in the module categories. The translation functor for both is Ω^{-1}, so that a triangle looks like

$$A \longrightarrow B \longrightarrow C \longrightarrow \Omega^{-1}(A)$$

where for some projective module P there is an exact sequence $0 \to A \to B \oplus P \to C \to 0$. Here, $\Omega^{-1}(M)$ is the cokernel of an injective hull $M \hookrightarrow I$ for I Injective.

The cohomology ring $\mathrm{H}^*(G,k)$ is a finitely generated, graded-commutative algebra over k. Let $V_G(k) = \mathrm{Proj}(\mathrm{H}^*(G,k)$ be its projectivized prime ideal spectrum, the collection of all homogeneous prime ideals with the Zariski topology. The support variety of a finitely generated kG-module M is the closed subvariety consisting of all homogeneous prime ideals that contain the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $\mathrm{H}^*(G,k)$. The support variety of an infinitely generated kG-module is a subset of $V_G(k)$, not necessarily closed (see [6]).

If H is a subgroup of G, the restriction functor $\mathbf{mod}(kG) \to \mathbf{mod}(kH)$ induces a map on cohomology ring $\mathrm{res}_{G,H} : \mathrm{H}^*(G,k) \to \mathrm{H}^*(H,k)$ and also a map on sectra $\mathrm{res}^*_{G,H} : V_H(k) \to V_G(k)$.

For much of the next few sections, we assume that $G = \langle g_1, \ldots, g_r \rangle$ is an elementary abelian p-group of order p^r. In this case, we set $X_i = g_i - 1 \in kG$ for $i = 1, \ldots, r$. Then $X_i^p = 0$ and $kG \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$. With this structure in mind, we make the following definition.

Definition 2.1. *Suppose that kG is the group algebra of an elementary abelian p-group. A flat subalgebra of kG is the image in kG of a flat map $\alpha : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kG$. We say a flat subalgebra is maximal if $s = r - 1$ where r is the p-rank of G.*

By definition, a map α, as above, is flat if kG is a projective module over the image of the ring $k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p)$. This happens if and only if the images $\alpha(t_1), \dots, \alpha(t_s)$, in $\operatorname{Rad}(kG)/\operatorname{Rad}^2(kG)$ are k-linearly independent. In particular, we have the following.

Lemma 2.2. *Suppose that G is an elementary abelian p-group of p-rank r. Let $\alpha : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kG$ be a flat map. Then there exists another flat map $\beta : k[t_1, \dots, t_{r-s}]/(t_1^p, \dots, t_{r-s}^p) \to kG$ such that kG is the internal tensor product $kG = A \otimes B$, where A and B are the images of α and β, respectively.*

Proof. Choose elements $m_1, \dots, m_{r-s}$ in kG, such that the classes modulo $\operatorname{Rad}^2(kG)$ of $\alpha(t_1), \dots, \alpha(t_s), m_1, \dots, m_{r-s}$ form a basis for $\operatorname{Rad}(kG)/\operatorname{Rad}^2(kG)$. Then let β be defined by $\beta(t_i) = m_i$ for $i = 1, \dots, s$. Then $AB = kG$, by Nakayama's Lemma and a dimension argument. □

If α, as above, is a flat map, then the multiplicative subgroup generated by the images $\alpha(1+t_i)$ is called a shifted subgroup of kG in other papers. It is an elementary abelian p-subgroup of the group of units of kG. In the case that $s = 1$, we have an example of a π-point.

Definition 2.3. [13] *A π-point is a flat map $\alpha_K : K[t]/(t^p) \to KG_K$ where K is an extension of the field k. If G is a finite group scheme that is not elementary abelian, then we assume also that α_K factors by flat maps through a unipotent abelian subgroup scheme of KG_K. Two π-points $\alpha_K : K[t]/(t^p) \to KG_K$ and $\beta_L : L[t]/(t^p) \to LG_L$ are equivalent if for any finitely generated kG-module M, the restriction $\alpha_K^*(K \otimes M)$ is projective if and only if $\beta_L^*(L \otimes M)$ is projective.*

The set of equivalence classes of π-point has a partial order coming from specializations, and that ordering gives the set a topology. With this in mind we have the following, which holds for any finite group scheme G.

Theorem 2.4. [13] *The space of equivalence classes of π-points is homeomorphic to $V_G(k) = \operatorname{Proj} \mathrm{H}^*(G, k)$.*

The point is that if $A = K[t]/(t^p)$, then $\mathrm{H}^*(A, K)/\operatorname{Rad}(\mathrm{H}^*(A, K))$ is a polynomial ring in one variable. So, if $\alpha : A \to KG$ is a π-point, then the kernel of the composition

$$\mathrm{H}^*(G, k) \xrightarrow{\alpha^*} \mathrm{H}^*(A, K) \longrightarrow \mathrm{H}^*(A, K)/\operatorname{Rad}(\mathrm{H}^*(A, K))$$

is a prime ideal. Equivalent π-points determine the same prime ideal.

With the identification given by the theorem, we can define the support variety $\mathcal{V}_G(M)$ of any kG-module M to be the set of all equivalence classes of π-point $\alpha_K : K[t]/(t^p) \to KG_K$ such that the restriction $\alpha_K^*(K \otimes M)$ is not a free KG-module. In the case that M is finitely generated, $\mathcal{V}_G(M) \simeq V_G(M)$ is a closed set.

Remark 2.5. If G is a finite group that is not elementary abelian, then the Quillen Dimension Theorem (see [15] or [12, Theorem 8.4.6]) says that $V_G(k) = \operatorname{Proj} \mathrm{H}^*(G,k) = \cup \operatorname{res}^*_{G,E}(V_E(k))$, where the union is over the elementary abelian p-subgroups E of G. This assures us that every π-point is equivalent to one that factors through the inclusion of the group algebra of some elementary abelian p-subgroup E of G into kG. Or, stated another way, every homogeneous prime ideal in $\mathrm{H}^*(G,k)$ contains the kernel of the restriction $\operatorname{res}_{G,E} : \mathrm{H}^*(G,k) \to \mathrm{H}^*(E,k)$, for some elementary abelian p-subgroup E.

3. Point varieties

A subcategory $\mathcal{M}$ of a triangulated category $\mathcal{C}$ is thick if it is triangulated and closed under taking direct summands. It is a thick tensor ideal if it is thick and if, for any $X \in \mathcal{C}$ and $Y \in \mathcal{M}$, $X \otimes Y$ is in $\mathcal{M}$. For V a closed subset of $V_G(k)$, let $\mathcal{M}_V$ be the thick tensor ideal in $\mathbf{stmod}(kG)$ consisting of all finitely generated kG-modules M with $V_G(M) \subseteq V$. More generally, let $\mathcal{V}$ be a collection of closed subsets $V_G(k)$ that is closed under taking finite unions and specializations (meaning that if $U \subseteq V \in \mathcal{V}$ then $U \in \mathcal{V}$). Then the subcategory $\mathcal{M}_{\mathcal{V}}$ of all finitely generated modules M with $V_G(M) \in \mathcal{V}$ is a thick tensor ideal. Indeed, this is the story.

Theorem 3.1. [7] *If $\mathcal{M}$ is a thick tensor ideal in $\mathbf{stmod}(kG)$, then $\mathcal{M} = \mathcal{M}_{\mathcal{V}}$ for some collection $\mathcal{V}$ of closed subsets of $V_G(k)$ that is closed under finite unions and specializations.*

Corresponding to a thick tensor ideal $\mathcal{M}_{\mathcal{V}}$ in $\mathbf{stmod}(kG)$ is a triangle of idempotent modules in $\mathbf{StMod}(kG)$ having the form

$$\mathcal{S}_{\mathcal{V}} : \qquad \longrightarrow \mathcal{E}_{\mathcal{V}} \xrightarrow{\sigma_{\mathcal{V}}} k \xrightarrow{\tau_{\mathcal{V}}} \mathcal{F}_{\mathcal{V}} \longrightarrow .$$

See [16] for proofs and details. The modules $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{F}_{\mathcal{V}}$ are idempotent in the stable category, meaning that $\mathcal{E}_{\mathcal{V}} \otimes \mathcal{E}_{\mathcal{V}} \cong \mathcal{E}_{\mathcal{V}}$ and $\mathcal{F}_{\mathcal{V}} \otimes \mathcal{F}_{\mathcal{V}} \cong \mathcal{F}_{\mathcal{V}}$ in $\mathbf{StMod}(kG)$, *i. e.* ignoring projective summands. In addition, $\mathcal{E}_{\mathcal{V}} \otimes \mathcal{F}_{\mathcal{V}} \cong 0$ in the stable category. The support variety $\mathcal{V}_G(\mathcal{E}_{\mathcal{V}})$ is the set of all equivalences classes of π-points corresponding to irreducible closed subsets in $\mathcal{V}$, and $\mathcal{V}_G(\mathcal{F}_{\mathcal{V}}) = V_G(k) \setminus \mathcal{V}_G(\mathcal{E}_{\mathcal{V}})$.

For any finitely generated kG module X, the triangle

$$X \otimes \mathcal{S}_{\mathcal{V}} : \qquad \mathcal{E}_{\mathcal{V}}(X) \xrightarrow{\mu_X} X \xrightarrow{\nu_X} \mathcal{F}_{\mathcal{V}}(X) \longrightarrow$$

has a couple of universal properties [16]. Let $\mathcal{M}^{\oplus}$ denote the closure of $\mathcal{M}$ in $\mathbf{StMod}(kG)$ under taking arbitrary direct sums. The map μ_X is universal for maps from objects in $\mathcal{M}_{\mathcal{V}}^{\oplus}$ to X, meaning that if Y is in $\mathcal{M}_{\mathcal{V}}^{\oplus}$, then any map $Y \to X$ factors through μ_X. The map ν_X is universal for maps from X to $\mathcal{M}_{\mathcal{V}}$-local objects, meaning objects Y such that $\underline{\mathrm{Hom}}_{kG}(M, Y) = \{0\}$ for all M in $\mathcal{M}_{\mathcal{V}}$. The universal property says that for an $\mathcal{M}_{\mathcal{V}}$-local module Y, any map $X \to Y$ factors through ν_X.

In the event that V is a closed subset of $V_G(k)$, let $\mathcal{E}_V = \mathcal{E}_{\mathcal{V}}$ and $\mathcal{F}_V = \mathcal{F}_{\mathcal{V}}$ where $\mathcal{V}$ is the collection of all closed subsets of V.

Lemma 3.2. *Suppose that V is a closed subvariety of $V_G(k)$. Suppose that L is a kG-module such that $U \cap V = \emptyset$ for all $U \in \mathcal{V}_G(L)$. Then $\underline{\mathrm{Hom}}_{kG}(\mathcal{E}_V, L) = \{0\}$.*

Proof. The point is that $\mathcal{E}_V$ can be constructed as the direct limit of finitely generated modules having variety equal to V. So L is $\mathcal{M}_V$-local. □

Suppose that $\mathcal{M} = \mathcal{M}_{\mathcal{V}}$ is a thick tensor ideal of $\mathbf{stmod}(kG)$ for an appropriate collection $\mathcal{V}$. The Verdier localization $\mathcal{C} = \mathcal{C}_{\mathcal{V}}$ of $\mathbf{stmod}(kG)$ with respect to $\mathcal{M}$ is the category whose objects are the same as those of $\mathbf{stmod}(kG)$. The collection of morphisms from an object M to an object N is obtained by inverting any morphism with the property that the third object in the triangle of that morphism is in $\mathcal{M}$. Thus, objects in $\mathcal{M}$ are equal to the zero object in $\mathcal{C}$. One of the motivations for this work is that $\underline{\mathrm{End}}(\mathcal{F}_V)$ is isomorphic to the ring of endomorpisms of the trivial module k in the localized category $\mathcal{C}$.

Proposition 3.3. *Suppose that $V = V_1 \cup V_2$ where V_1 and V_2 are closed subvarieties such that $V_1 \cap V_2 = \emptyset$. Then $\mathcal{F}_V$ is the pushout of the diagram*

$$\begin{array}{ccc} k & \xrightarrow{\tau_{V_1}} & \mathcal{F}_{V_1} \\ \Big\downarrow{\scriptstyle \tau_{V_2}} & & \Big\downarrow \\ \mathcal{F}_{V_2} & \longrightarrow & \mathcal{F}_V \end{array}$$

That is, $\mathcal{F}_V \cong (\mathcal{F}_{V_1} \oplus \mathcal{F}_{V_2})/N$ where $N = \{(\tau_{V_1}(a), -\tau_{V_2}(a)) \mid a \in k\}$.

Proof. The thing to note is that $\mathcal{E}_V \cong \mathcal{E}_{V_1} \oplus \mathcal{E}_{V_2}$. That is, because, $V_1 \cap V_2 = \emptyset$, if $M \in \mathcal{M}_V$, then $M \cong M_1 \oplus M_2$ where $M_i \in \mathcal{M}_{V_i}$ for $i = 1, 2$. So in particular, the map $\mathcal{E}_{V_1} \oplus \mathcal{E}_{V_2} \to k$ sending (u, v) to $\sigma_{V_1}(u) + \sigma_{V_2}(v)$ has the desired universal property. The third object in the triangle of the map is the pushout, and it also satisfies the desired universal property. Moreover, we know that $\mathcal{E}_{V_1} \otimes \mathcal{E}_{V_2}$ is projective because the varieties of the two modules are disjoint [6]. So $\mathcal{E}_{V_1} \oplus \mathcal{E}_{V_2}$ is an idempotent module. This is sufficient to prove the proposition. □

Lemma 3.4. *Let G be an elementary abelian group of order p^r. Suppose that H is a subgroup of G or that kH is the image of a flat map $\gamma : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to$*

kG. Let V be a closed subvariety of $V_G(k)$. Then the restriction of the exact triangle $\mathcal{S}_V$ to kH is the triangle

$$\mathcal{S}_{V'}: \qquad \longrightarrow \mathcal{E}_{V'} \xrightarrow{\sigma_{V'}} k \xrightarrow{\tau_{V'}} \mathcal{F}_{V'} \longrightarrow,$$

where $V' = (\mathrm{res}^*_{G,H})^{-1}(V)$, the inverse image of V under the restriction map.

Proof. The proof is a straightforward matter checking that the varieties are correct. □

In the case of G an elementary abelian groups, one of the main theorem in [11] is the following.

Theorem 3.5. *Suppose that G is an elementary abelian p-group having p-rank at least 3. Suppose that V is a subvariety of $V_G(k)$ consisting of a single closed point. Let kH be the image of a flat map $\gamma : k[t_1, \dots, t_{r-1}]/(t_1^p, \dots, t_{r-1}^p) \to kG$ with the property that V is not in $\mathrm{res}^*_{G,H}(V_H(k))$. Suppose that $Z = \alpha(t)$ where $\alpha : k[t]/(t^p) \to kG$ is a π-point whose equivalence class is the point in V. Then, the idempotent module $\mathcal{F}_V$ has a decomposition (as a direct sum of kH-modules)*

$$\mathcal{F}_V = k \oplus P_0^{p-1} \oplus P_1 \oplus P_2^{p-1} \oplus P_3 \oplus \dots$$

where

$$\dots \longrightarrow P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

is a projective kH-resolution of the trivial kH-module. Multiplication by the element Z is zero on the summand k. For $m \in P_{2i-1}$, $i > 0$,

$$Zm \;=\; -(\partial(m), 0, \dots, 0) \in P_{2i-2}^{p-1}.$$

For $m = (m_1, \dots, m_{p-1}) \in P_{2i}^{p-1}$,

$$Zm = \begin{cases} -\varepsilon(m_{p-1}) + (0, m_1, \dots, m_{p-2}) \in k \oplus P_0^{p-1} & \text{if } i = 0, \\ -\partial(m_{p-1}) + (0, m_1, \dots, m_{p-2}) \in P_{2i-1} \oplus P_{2i}^{p-1} & \text{if } i > 0. \end{cases}$$

The map $\tau_V : k \to \mathcal{F}_V$ has image the summand k in the decomposition. Moreover, $\mathrm{Hom}_{kG}(k, \mathcal{F}_V) = \sum_{i \geq 0} H_i$ is a graded ring with

$$H_i \cong \begin{cases} k\tau_V(1) & \text{for } i = 0, \\ \mathrm{Hom}_{kH}(k, P_i) & \text{for } i > 0 \end{cases}$$

A homogeneous element $\theta : k \to \mathcal{F}_V$ lifts to a homomorphism $\hat{\theta} : \mathcal{F}_V \to \mathcal{F}_V$ that is induced by a kH-chain map $\theta_* : (P_*, \varepsilon) \to (P_*, \varepsilon)$ of the augmented projective resolution to itself, that lifts θ.

Proof. Let kC denote the image of α. Because γ is a flat map, the classes modulo $\operatorname{Rad}^2(kG)$ of $\gamma(t_1), \dots, \gamma(t_{r-1})$ span a subspace of $\operatorname{Rad}(kG)/\operatorname{Rad}^2(kG)$ of dimension $r-1$. The fact that V is not in $\operatorname{res}^*_{G,H}(V_H(k))$ implies that the class modulo $\operatorname{Rad}^2(kG)$ of Z is not in that subspace. That is, otherwise there would be a a π-point equivalent to α that factored through γ violating the assumption on the varieties. Thus we have that $kG \cong kC \otimes kH$, is the group algebra of the direct product $C \times H$. Now we apply [11, Theorem 6.2], where V the point $[1, 0, \dots, 0]$ in $V_G(k)$ corresponding to Z. This gives the stated result. That is, for this specific choice of V and generators $Z, \alpha(t_1), \dots, \alpha(t_{r-1})$, the module $\mathcal{F}_V$ has a decomposition as described in [11]. □

We remark that changing the generators of kG, as we have done above, does not preserve the Hopf algebra structure. However, as noted in [11, Remark 7.5], the structure of the idempotent modules does not depend on the coalgebra structure.

Theorem 3.6. *Assume the hypothesis of the previous theorem (Thm. 3.5). Suppose that $X = \alpha(t)$ where $\alpha : k[t]/(t^p) \to kG$ is a π-point not corresponding to the point V. Assume that $\theta : k \to \mathcal{F}_V$ is a homomorphism with the property that $\theta(1) \in X^{p-1}\mathcal{F}_V$. Then, the image of $\hat{\theta} : \mathcal{F}_V \to \mathcal{F}_V$ is contained in $X^{p-1}\mathcal{F}_V$.*

Proof. In the statement of Theorem 3.5, the generators $Y_i = \alpha(t_i)$ of kH can be chosen so that $Y_1 = X$ and $Y_2, \dots, Y_{r-1}$ are any elements so that the classes modulo $\operatorname{Rad}^2(kG)$ of $Z, X, Y_2, \dots, Y_{r-1}$ form a basis for $\operatorname{Rad}(kG)/\operatorname{Rad}^2(kG)$. Thus, $kH \cong kC \otimes kJ$ where $kC \cong k[X]/(X^p)$ is the flat subalgebra of kG generated by X and kJ is the flat subalgebra generated by $Y_2, \dots, Y_{r-1}$. Let (R_*, ε_1) be a minimal projective kC-resolution of k_C and (Q_*, ε_2), minimal projective kJ-resolution of k_J. Then $R_i \cong kC$ for all $i \geq 0$. The minimal kH-resolution of k can be taken to be the tensor product of these two, so that

$$P_n = \sum_{i=0}^{n} R_i \otimes Q_{n-i}$$

In the decomposition of $\mathcal{F}_V$ given in Theorem 3.5, the element $\theta(1) \in X^{p-1}\mathcal{F}_V \subseteq \sum_{n \geq 0} P_n$. Because an assignment of chain maps to elements of $\operatorname{Hom}_{kG}(k, \mathcal{F}_V)$ is additive, it is sufficient to prove the theorem assuming that $\theta(1) \in X^{p-1}P_n$ for some n. Indeed, we may assume that $\theta(1) \in X^{p-1}(R_m \otimes Q_{n-m}) = (X^{p-1}R_m) \otimes Q_{n-m})$ for some m and n. Because $R_m \cong kC$, we have that $\theta(1) = X^{p-1} \otimes u$ for some $u \in Q_{n-m}$.

Let $\varphi : k \to Q_{n-m}$ be given by $\varphi(1) = u$. Then φ lifts to a chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \\
\cdots & \longrightarrow & Q_{n-m+2} & \longrightarrow & Q_{n-m+1} & \longrightarrow & Q_{n-m} & \longrightarrow & \cdots
\end{array}$$

Let $\mu : R_0 \to X^{p-1}R_m$ be given by $\mu(1) = X^{p-1}$. Now define $\theta_i : P_i \to P_{n+i+1}$ as the composition

$$P_i \longrightarrow R_0 \otimes Q_i \longrightarrow R_0/(X^{p-1}R_0) \otimes Q_i \xrightarrow{\mu\otimes\varphi_i} X^{p-1}R_m \otimes Q_{n-m+i+1} \longrightarrow P_{n+i+1}$$

The first map is projection onto the direct summand $R_0 \otimes Q_i$. The second is the natural quotient. Then comes the chain map, and the fourth is the inclusion.

The task to finish the proof amounts to two straightforward exercises which we leave to the reader. The first is to show that $\{\theta_i\}$ is a chain map, and the second is to show that it lifts the map θ. □

Corollary 3.7. *Assume the hypotheses and notation of Theorems 3.5 and 3.6. Let $\mathcal{I}$ be the collection of all $\theta : k \to \mathcal{F}_V$ such that $\theta(1) \in X^{p-1}\mathcal{F}_V$. Then under the correspondence $\underline{\mathrm{Hom}}_{kG}(k, \mathcal{F}_V) \cong \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V)$, $\mathcal{I}$ is the kernel of the restriction map $\underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V) \to \underline{\mathrm{Hom}}_{kH}(\mathcal{F}_V, \mathcal{F}_V)$. In particular, $\mathcal{I}$ is an ideal.*

Proof. The point is, in the notation of the last proof, that $X^{p-1}\mathcal{F}_V \subset P_*$. Thus, $\theta(1)$ and $\hat{\theta}(\mathcal{F}_V)$ are in P_* which is free as a kH-module. Hence, the map θ factors through a kH-projective object and is zero on restriction to kH. □

Remark 3.8. We emphasize that in Theorem 3.5, the choices of the flat map γ and also of the generators for kH are arbitrary except that kH should have rank $r-1$ and the condition on the varieties must be satisfied. Similarly, in Theorem 3.6, any π-point α satisfying the desired conditions can be chosen.

4. Endomorphisms of $\mathcal{F}_V$

Throughout this section, assume that $G = \langle g_1, \dots, g_r \rangle$ is an elementary abelian group of order p^r for $r \geq 3$. We show that if $V \subset V_G(k)$ is a closed subvariety of dimension 0, then the endomorphism ring of the idempotent module $\mathcal{F}_V$ in the stable category has a unique maximal ideal that is nilpotent and has codimension one. We assume the notation of the previous section.

We prove the following result in more generality than is actually needed in the later development.

Proposition 4.1. *Suppose that V_1 and V_2 are disjoint subvarieties of $V_G(k)$. Let kH be the image of a flat map $\gamma : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kG$, for $s \geq 2$. Let $X = \gamma(t_1)$ and let kJ be the flat subalgebra of kG generated by $\gamma(t_2), \dots, \gamma(t_s)$. Assume that we have the following two conditions.*

(1) $V_1 \subseteq \mathrm{res}^*_{G,J}(V_J(k))$.
(2) $V_2 \cap \mathrm{res}^*_{G,H}(V_H(k)) = \emptyset$.

Suppose that $\varphi : k \to \mathcal{F}_2 = \mathcal{F}_{V_2}$ *is a homomorphism such that* $\varphi(1) \in X^{p-1}\mathcal{F}_2$. *Then* φ *extends to a homomorphism* $\psi : \mathcal{F}_1 = \mathcal{F}_{V_1} \to X^{p-1}\mathcal{F}_2$. *That is, we have a commutative diagram*

$$\begin{array}{ccc} k & \xrightarrow{\tau_1} & \mathcal{F}_1 \\ {\scriptstyle\varphi}\big\downarrow & \swarrow{\scriptstyle\psi} & \big\downarrow \\ X^{p-1}\mathcal{F}_2 & \xrightarrow{\iota} & \mathcal{F}_2 \end{array}$$

where ι *is the inclusion.*

Proof. By Condition (2), $(\mathcal{E}_2)_{\downarrow H}$ is free as a kH-module. This implies that $(\mathcal{F}_2)_{\downarrow H} \cong k \oplus P$, where P is a free kH-module. Thus, $\underline{\mathrm{Hom}}_{kH}(k, (\mathcal{F}_2)_{\downarrow H})$ has dimension one, and the fact that $\varphi(1) \in X^{p-1}\mathcal{F}_2$ means that φ factors through a projective kH-module, namely P. This follows because $\varphi(1) \in \mathrm{Rad}(kH)\mathcal{F}_2 \cap \mathrm{Soc}(\mathcal{F}_2) \subset P$.

Note that $X \notin kJ$, and hence the restriction of $\mathcal{E}_1$ to the subalgebra generated by X is a free module. Moreover, $(X^{p-1}\mathcal{F}_2)_{\downarrow H} = X^{p-1}P$ is free as a kJ-module. Consequently, $\mathcal{V}_G(\mathcal{E}_1) \cap \mathcal{V}_G(X^{p-1}\mathcal{F}_2) = \emptyset$, and the composition

$$\mathcal{E}_1 \xrightarrow{\sigma_1} k \xrightarrow{\varphi} X^{p-1}\mathcal{F}_2$$

is the zero map in the stable category by Lemma 3.2. The existence of the map ψ is implied from the distinguished triangle. □

Corollary 4.2. *Suppose that* G *is an elementary abelian* p*-group having rank at least 3. Suppose that* $V_1, V_2 \subset V_G(k)$ *are closed subvarieties each consisting of a single point. Let* $\beta : k[t_1, t_2]/(t_1^p, t_2^p) \to kG$ *be flat map such that the following hold. For notation, let* kH *be the image of* β.

(1) *The class of the* π*-point* $\alpha : k[t]/(t^p) \to kG$ *with* $\alpha(t) = \beta(t_1)$ *is in* V_1.
(2) $V_2 \not\subset \mathrm{res}^*_{G,H}(V_H(k))$.

Let $X = \beta(t_2)$. *Suppose that* $\varphi : k \to \mathcal{F}_2 = \mathcal{F}_{V_2}$ *is a map such that* $\varphi(1) \in X^{p-1}\mathcal{F}_2$. *Then* φ *extends to a map* $\theta : \mathcal{F}_1 \to \mathcal{F}_2$ *such that* $\theta(\mathcal{F}_1) \subseteq X^{p-1}\mathcal{F}_2$.

Proof. Let kJ be the image of α. Then the conditions of Proposition 4.1 are satisfied and the corollary follows. □

We can now prove the main theorem of the section.

Theorem 4.3. *Suppose that* G *is an elementary abelian* p*-group having rank* $r \geq 3$. *Let* $V \subset V_G(k)$ *be a closed subset consisting of a finite number of closed points. Let* kH *be the image of a flat map* $\gamma : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kG$ *such that* $s \geq 2$ *and* $\mathrm{res}^*_{G,H}(V_H(k)) \cap V = \emptyset$. *Let* $\mathcal{I}$ *be the kernel of the restriction* $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V) \to \underline{\mathrm{End}}_{kH}(\mathcal{F}_V)$. *Then* $\mathcal{I}$ *is an ideal of codimension one in* $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$, *and* $\mathcal{I}^2 = \{0\}$. *Thus* $\mathcal{I}$ *is the unique maximal ideal in* $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$.

Proof. We write $V = \cup_{i=1}^{n} V_i$ where each V_i is a closed point in $V_G(k)$. For each i, let $\alpha_i : k[t]/(t^p) \to kG$ be a π-point corresponding to the closed point in V_i. For $i \neq j$ let $kH_{i,j}$ be the image of the flat map $\beta_{i,j} : k[t_1, t_2]/(t_1^p, t_2^p) \to kG$ with $\beta_{i,j}(t_1) = \alpha_i(t)$ and $\beta_{i,j}(t_2) = \alpha_j(t)$. We note that for all i, j, the intersection

$$\mathrm{res}^*_{G,H_{i,j}}(V_{H_{i,j}}(k)) \cap \mathrm{res}^*_{G,H}(V_H(k))$$

either contains only one point or is empty. Let $\alpha : k[t]/(t^p) \to kG$ be a π-point that factors through γ, but is not in $\mathrm{res}^*_{G,H_{i,j}}(V_{H_{i,j}}(k))$ for any pair i, j with $1 \leq i < j \leq n$.

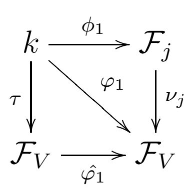
By Proposition 3.3, the idempotent module $\mathcal{F}_V$ is the pushout of the system $\{\tau_i : k \to \mathcal{F}_i\}$, where $\mathcal{F}_i = \mathcal{F}_{V_i}$. For each i, there is a homomorphism $\nu_i : \mathcal{F}_i \to \mathcal{F}$, such that the compositions $\nu_i \mu_i$ coincide. From the conditions on the choice of kH, we see that for each i, the restriction of $\mathcal{F}_i$ to kH has the form $k \oplus P_i$ where P_i is a projective kH-module. In addition, the component isomorphic to k is generated by $\tau_i(1)$. Thus, the restriction $(\mathcal{F}_V)_{\downarrow H} \cong k \oplus \sum P_i$. We see that $\mathcal{I}$ is the subspace of $\underline{\mathrm{Hom}}_{kG}(k, \mathcal{F}_V)$ spanned by all $\varphi : k \to \mathcal{F}_V$ with $\varphi(1) \in X^{p-1}\mathcal{F}_V$.

For any $j = 1, \dots, t$, let $\mathcal{I}_j$ be the set of all $\varphi \in \mathcal{I}$ such that $\varphi(1) \in X^{p-1}P_j$. Thus $\mathcal{I}$ is the direct sum of the subspaces $\mathcal{I}_j$.

Throughout the proof, we make the identification $\underline{\mathrm{Hom}}_{kG}(k, \mathcal{F}_V) \cong \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V)$. If we are given two elements φ_1 and φ_2 in $\underline{\mathrm{Hom}}_{kG}(k, \mathcal{F}_V)$, their product is obtained by first finding lifts $\hat{\varphi}_i : \mathcal{F}_V \to \mathcal{F}_V$, for $i = 1, 2$, taking the composition and composing with the map $\tau : k \to \mathcal{F}_V$. Note that any lift will serve the purpose. Our aim is to show that if $\varphi_1, \varphi_2 \in \mathcal{I}$, then the product is zero. Without loss of generality we may assume that $\varphi_i \in \mathcal{I}_{j_i}$ for some $1 \leq j_i \leq t$.

Letting $j = j_1$, there is a $\phi_1 : k \to X^{p-1}P_j \subset P_j$ such that $\nu_j \phi_1 = \varphi_1 : k \to \mathcal{F}_V$. By Corollary 4.2, for any $i \neq j$ there is an extension $\theta_i : \mathcal{F}_i \to \nu_j(X^{p-1}\mathcal{F}_j)$ extending ϕ_1. Likewise, by Theorem 3.6, there is such an extension also in the case that $i = j$. Thus, for every $i = 1, \dots, t$, there is an extension $\hat{\theta}_i : \mathcal{F}_i \to \mathcal{F}_V$ of φ_1 with the property that $\hat{\theta}_i(\mathcal{F}_i) \subseteq X^{p-1}\mathcal{F}_V$.

The universal property of pushouts, now guarantees that there is a map $\hat{\varphi}_1 : \mathcal{F}_V \to \mathcal{F}_V$ such that, for every i, the diagram



commutes and $\hat{\varphi}_1(\mathcal{F}_V) \subseteq X^{p-1}\mathcal{F}_V$. There is a similar extension $\hat{\varphi}_2 : \mathcal{F}_V \to \mathcal{F}_V$ with the same property.

Now, we see that $\hat{\varphi}_2(\hat{\varphi}_1(\mathcal{F}_V) \subseteq \hat{\varphi}_2(X^{p-1}\mathcal{F}_V) \subseteq X^{2(p-1)}\mathcal{F}_V = \{0\}$. Hence, the product of any two elements in $\mathcal{I}$ is zero. We notice that $\underline{\mathrm{End}}_{kH}(\mathcal{F}_V) \cong k$, implying that $\mathcal{I}$ is a maximal ideal. Because $\mathcal{I}^2 = \{0\}$, any element of $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$ that is not in $\mathcal{I}$ is invertible. □

5. The general case

The aim of this section is to extend the conclusion of Theorem 4.3 to a more general finite group G. The arguments in the proofs depend on the fact (see Remark 2.5) that any prime ideal in $\mathrm{H}^*(G,k)$ contains the kernel of a restriction to an elementary abelian p-subgroup of G. For this reason, the main results of the section do not extend to general finite group schemes. Throughout the section we assume the following.

Hypothesis 5.1. *suppose that G is a finite group whose maximal elementary abelian p-subgroups all have p-rank at least three. Let V be a closed subvariety of $V_G(k)$, that is a union of a finite collection of closed points.*

We make the identification $\underline{\mathrm{Hom}}_{kG}(k, \mathcal{F}_V) \cong \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V) = \underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$, as before. The ideal, that we are interested in, is the following.

Definition 5.2. *Assume that 5.1 holds. Suppose that E is an elementary abelian p-subgroup of G with order p^r for $r \geq 3$. Let kH be the image of a flat map $\gamma : k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kE$, such that $1 \leq s < r$ and*

$$\mathrm{res}^*_{G,H}(V_H(k)) \cap V = \emptyset.$$

Let $\mathcal{I} \subset \underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$ be the kernel of the restriction map $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V) \to \underline{\mathrm{End}}_{kH}(\mathcal{F}_V)$.

Given E, the existence of H follows easily from the geometry. Note that $\mathcal{I}$ is an ideal, because if a kG-homomorphism factors through a kH-projective module, then so does its composition with any other homomorphism.

Proposition 5.3. *Assume that 5.1 holds. The ideal $\mathcal{I}$ does not depend on the choice of E or H, as long as the above conditions are satisfied.*

Proof. The first thing to notice is that the restriction of the module $\mathcal{E}_V$ to kH is projective, and hence, $(\mathcal{F}_V)_{\downarrow H} \cong k \oplus P$ where P is a projective module by Lemma 3.4. Then, the independence of the choice of kH in kE follows from Theorem 4.3.

Now suppose that E_1 and E_2 are elementary abelian p-subgroups of G such that $F = E_1 \cap E_2$ has p-rank at least 2. Let $\mathcal{I}_j$ be the ideal defined by E_j as above for $j = 1, 2$. We claim that $\mathcal{I}_1 = \mathcal{I}_2$. The reason is that there must be a π-point $\alpha : k[t]/(t^p) \to kF \subset kG$ with the property that the equivalence class of α does not correspond to any point of $(\mathrm{res}^*_{G,F})^{-1}(V)$, the inverse image V under the restriction map $\mathrm{res}^*_{G,F} : V_F(k) \to V_G(k)$. Let H be the image of α. Then kH is a flat subalgebra of both kE_1 and kE_2, which satisfies the condition of Definition 5.2. So both $\mathcal{I}_1$ and $\mathcal{I}_2$ are the kernel of the restriction to kH.

Now suppose G is a p-group and that E, E' are any two elementary abelian p-subgroups having p-rank at least 3. By the argument of Alperin (see the bottom of page 8 to top of page 9 of [1]), there is a chain $E = F_1, F_2, \ldots, F_m = E'$ of elementary abelian p-subgroups of G such that $F_i \cap F_{i+1}$ has p-rank at least 2. Thus by an easy induction and the previous paragraph, the ideal $\mathcal{I}$ is independent of the choice of E.

If G is not a p-group we need only notice that if E_1 and E_2 are conjugate elementary abelian p-subgroups, then the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ must be the same. Thus, we may assume that any two elementary abelian p-subgroup are in the same Sylow p-subgroup. In such a case the same proof as above works. □

Theorem 5.4. *Assume that the conditions of* 5.1 *hold. Let $\mathcal{F}_V$ be the idempotent $\mathcal{F}$-module corresponding to V. Then $\underline{\mathrm{End}}(\mathcal{F}_V)$ is a local ring whose unique maximal ideal $\mathcal{I}$ is nilpotent. Moreover, there is a number B depending only on G and p such that the nilpotence degree of $\mathcal{I}$ is at most B.*

Proof. The proof is an easy consequence of the above Proposition, Theorem 4.3 and Theorem 2.5 of [9]. The last mentioned theorem can be interpreted as saying that there is number N, depending only on G and p, such that for any sequence $M_0, \ldots, M_n$ of kG-modules and maps $\theta_i \in \underline{\mathrm{Hom}}_{kG}(M_{i-1}, M_i)$, $1 \leq i \leq n$, such that $n \geq N$ and $\mathrm{res}_{G,E}(\theta_i) = 0$ for every elementary abelian subgroup E of G, the composition $\theta_n \cdots \theta_1 = 0$. In the case that $M_i = \mathcal{F}_V$ for all i and $B = 2N$, choose elements $\theta_i \in \mathcal{I}$. Then, for every i, the restriction of the product $\theta_{2i}\theta_{2i-1}$ to every elementary abelian p-subgroup of G vanishes, by Theorem 4.3. It follows that $\theta_n \cdots \theta_1 = 0$ if $n \geq B$. □

6. Restrctions

The aim of the section is prove a few facts about the restrictions of the endomorphism rings. The first result is known, but perhaps has not been written down.

Lemma 6.1. *Suppose that G is an elementary abelian p-group of order $p^r > 1$. Let $kH \neq kG$ be a flat subalgbra of kG. Then for all $n < 0$, we have that*

$$\mathrm{res}_{G,H} : \widehat{\mathrm{H}}^n(G, k) \to \widehat{\mathrm{H}}^n(H, k)$$

is the zero map.

Proof. Let $\gamma : k[t_1, \ldots, t_s]/(t_1^p, \ldots, t_s^p) \to kG$ be a flat map whose image is kH. Then the classes modulo $\mathrm{Rad}^2(kG)$ of $\gamma(t_1), \ldots, \gamma(t_s)$ are k-linearly independent in $\mathrm{Rad}(kG)/\mathrm{Rad}^2(kG)$. Let $b_{s+1}, \ldots, b_r$ be elements that are chosen so that the classes of $\gamma(t_1), \ldots, \gamma(t_s), b_{s+1}, \ldots, b_r$ form a basis for $\mathrm{Rad}(kG)/\mathrm{Rad}^2(kG)$. Let kJ be the flat subalgebra generated by $b_{s+1}, \ldots, b_r$ so that $kG \cong kH \otimes kJ$ (see Lemma 2.2).

The Tate cohomology group $\widehat{\mathrm{H}}^n(G,k)$ is isomorphic to $\mathrm{Hom}_{kG}(k, P_{n+1})$, where P_* is a minimal projective kG-resolution of k. The restriction is the map $\psi_* : \mathrm{Hom}_{kG}(k, P_{n-1}) \to \mathrm{Hom}_{kG}(k, Q_{n-1})$, where Q_* is a minimal kH-projective resolution of k, and $\psi : P_* \to Q_*$ is a kH-chain map. Because of the decomposition $kG \cong kH \otimes kJ$, we may assume that Q_* is a complex of kG-modules on which kJ acts trivially and that ψ is a kG-chain map. However, P_{n-1} is a free kG-module, implying that any map $\zeta : k \to P_{n-1}$ has its image in $\mathrm{Rad}(kJ)P_{n-1}$. Because kJ acts trivially on Q_{n-1}, the image of ζ is in the kernel of ψ. ☐

Theorem 6.2. *Suppose that G is an elementary abelian p-group of p-rank $r \geq 3$. Let V be a subvariety of $V_G(k)$ that is a union of a finite collection of closed points. Suppose that $kH \neq kG$ is a flat subalgebra of kG. Then the maximal ideal $\mathcal{I} \subseteq \underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$ is the kernel of the restriction map $\mathrm{res}_{G,H} : \underline{\mathrm{End}}_{kG}(\mathcal{F}_V) \to \underline{\mathrm{End}}_{kH}((\mathcal{F}_V)_{\downarrow H})$.*

Proof. We write $V = \cup_{i=1}^n V_i$ where each V_i is a closed point in $V_G(k)$. Recall that by Proposition 3.3, the idempotent module $\mathcal{F}_V$ is the pushout of the system $\{\tau_i : k \to \mathcal{F}_i\}$. Here, $\mathcal{F}_i = \mathcal{F}_{V_i}$ and for each i, there is a homomorphism $\nu_i : \mathcal{F}_i \to \mathcal{F}$, such that the compositions $\nu_i\tau_i$ coincide.

Let $\gamma_i : k[t_1, \dots, t_{r-1}]/(t_1^p, \dots, t_{r-1}^p) \to kG$ with image kJ such that

$$\mathrm{res}^*_{G,J}(V_J(k)) \cap V = \emptyset$$

For each i, we have that the restriction of $\mathcal{F}_i = \mathcal{F}_{V_i}$ to kJ has the form $\mathcal{F}_i \cong k \oplus Q_i$ where Q_i is a projective kJ-module. It follows that

$$\mathcal{F}_{\downarrow J} \;\cong\; k \oplus \nu_1(Q_1) \oplus \cdots \oplus \nu_n(Q_n).$$

If $\zeta : k \to \mathcal{F}_V$ is in $\mathcal{I}$, then $\zeta(1) \in \sum \nu_i(Q_i)$ by Theorem 4.3 and Corollary 3.7 (see also Remark 3.8).

Hence, for the remainder of the proof we fix an element $\zeta \in \mathcal{I}$, and without loss of generality, we may assume that $\zeta(1) \in \nu_i(Q_i)$ for some fixed i. Our object is to show that ζ factors through a kH-projective module. Notice that ζ must factor through $\nu_i : \mathcal{F}_i \to \mathcal{F}_V$. Consequently, it is sufficient to prove the theorem in the case that $\mathcal{F}_V = \mathcal{F}_i$. That is, we may assume that $n = 1$, $V = V_i$.

There are two cases to consider. First assume that $\mathrm{res}^*_{G,H}(V_H(k))$ does not contain the point V. In this case $(\mathcal{E}_V)_{\downarrow H}$ is projective, and hence $(\mathcal{F}_V)_{\downarrow H} \cong k$ in the stable category. In this case, the restriction of $\mathcal{I}$ to kH is zero and we are done.

Next, we assume that $V \subset \mathrm{res}^*_{G,H}(V_H(k))$. There is a π-point $\alpha : k[t]/(t^p) \to kH \subset kG$ whose equivalence class is the one closed point in V. Let kC be the image of α. The flat subalgebra kH has a maximal flat subalgebra kL such that $kH \cong kC \otimes kL$. There is a flat subalgebra kD such that $kG \cong kH \otimes kD$ (see Lemma 2.2). We may assume that the subalgebra kJ has the form $kJ = kL \cdot kD \cong$

$kL \otimes kD$, because V is not contained in $\mathrm{res}^*_{G,J}(V_J(k))$. Therefore, by Theorem 4.3, $(\mathcal{F}_V)_{\downarrow J} \cong k \oplus P$, where P is the sum of the terms (with multiplicities) of a minimal augmented kJ-projective resolution of k such that element $\alpha(1)$ acts as the boundary homomorphism on the augmented complex. Likewise, the restriction of $\mathcal{F}_V$ to kL has the form $(\mathcal{F}_V)_{\downarrow L} \cong k \oplus Q$ is the sum of the terms (with multiplicities) of an augmented minimal kL-projective resolution of k. As in the proof of Lemma 6.1, the restriction map in the stable category is given by a chain map of augmented complexes which can be shown to be a kG-homomorphism. Because $\zeta \in \mathcal{I}$, we have that $\zeta(1) \in P$, and as in the proof of Lemma 6.1, the chain map takes $\zeta(1)$ to zero.

This proves that $\mathcal{I}$ is in the kernel of the restriction to $\underline{\mathrm{End}}_{kH}(\mathcal{F}_V)$. The fact that it is the kernel is a consequence of its maximality. □

7. Finite generation

In this final section we address the issue of the finite generation of the endomorphism rings. Suppose that V is a closed subvariety of $V_G(k)$. Let $\mathcal{M}_V$ be the subcategory of all kG-modules whose support variety is contained in V, and let $\mathcal{C}_V$ be the localization of $\mathbf{StMod}(kG)$ at $\mathcal{M}_V$. Tensoring with $\mathcal{F}_V$ is the localization functor, and for kG-modules M and N, we have that

$$\mathrm{Hom}_{\mathcal{C}_V}(M,N) \cong \underline{\mathrm{Hom}}_{kG}(M \otimes \mathcal{F}_V, N \otimes \mathcal{F}_V)$$

is a module over $\mathrm{End}_{\mathcal{C}_V}(k) \cong \underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$. The question is whether it is a finitely generated module?

The question is most relevant in the case that M and N are finitely generated modules (compact objects in $\mathbf{StMod}(kG)$). Of course, if the support varieties of the finitely generated objects M and N are both disjoint from V then the finite generation is obvious. This is because, in such a case, $\mathcal{E}_V \otimes M$ and $\mathcal{E}_V \otimes N$ are projective modules, and hence $\mathcal{F}_V \otimes M \cong M$ and $\mathcal{F}_V \otimes N \cong N$ in the stable category. Also, if the support variety of either M or N is contained in V, then the finite generation is also clear. In other situations some proof is required. We show that the answer is not unlike the answer to the question of the finite generation of $\mathrm{End}_{\mathcal{C}_V}(k)$.

The proof of the following is straightforward generalization of a well used argument.

Theorem 7.1. *Suppose that $\zeta \in \mathrm{H}^d(G,k)$ is a non-nilpotent element. Let $V = V_G(\zeta)$, the collection of all homogeneous prime ideals that contain ζ. If M and N are finitely generated kG-modules, then $\mathrm{Hom}_{\mathcal{C}_V}(M,N)$ is finitely generated as a module over $\mathrm{End}_{\mathcal{C}_V}(k)$.*

In fact, what we want to show is the following.

Lemma 7.2. *Assume the hypothesis of Theorem 7.1. Then*

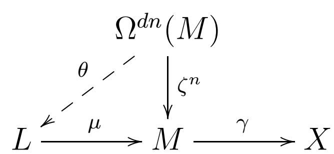
$$\mathrm{Hom}_{\mathcal{C}_V}(M,N) \cong (\mathrm{Ext}^*_{kG}(M,N)[\zeta^{-1}])_0.$$

That is, viewing $\mathrm{Ext}^*_{kG}(M,N)$ *as a module over* $\mathrm{H}^*(G,k)$*, we invert the action of* ζ *and take the zero grading.*

Proof. Suppose that in $\mathcal{C}_V$, we have a morphism

$$\phi = \nu\mu^{-1} : \qquad M \xleftarrow{\ \mu\ } L \xrightarrow{\ \nu\ } N$$

for some kG-module L such that the third object X in the triangle of μ is in $\mathcal{M}_V$. This implies that for n sufficiently large, the element ζ^n annihilates the cohomology of X. Consequently, we have as in the diagram



that the composition $\gamma\zeta^n = 0$. This implies the existence of the map θ, and we have that $\phi = \nu\mu^{-1} = (\nu\theta)\zeta^{-n}$, where $\nu\theta$ represents an element in $\mathrm{Ext}^{nd}_{kG}(M,N)$ and any other representative defines the same element of $\mathrm{Hom}_{\mathcal{C}_V}(M,N)$. Likewise, the class of θ in $\mathrm{Ext}^{nd}_{kG}(M,M)$ is the class of the cocycle $\theta \otimes 1 : \Omega^{nd}(k) \otimes M \to k \otimes M$, and all representatives of this class define the same element of $\mathrm{Ext}^{nd}_{kG}(M,M)$. $\square$

Proof of Theorem 7.1. We now use the fact that $\mathrm{Ext}^*_{kG}(M,N)$ is finitely generated as a module over $\mathrm{H}^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$. Let $\gamma_1,\dots,\gamma_s$ be a set of homogeneous generators. If $A = \sum_{n\geq 0} \mathrm{Ext}^{nd}_{kG}(k,k)$, then by elementary commutative algebra $\mathrm{H}^*(G,k)$ is finitely generated over A. Suppose that $\beta_1,\dots,\beta_r$ is a set of homogeneous generators. Let $B = \sum_{n\geq 0} \mathrm{Ext}^{nd}_{kG}(M,N)$. If $\theta \in B$ is a homogeneous element then $\theta = \sum_{i=1}^s \gamma_i\mu_i$ for $\mu_i \in \mathrm{H}^*(G,k)$. But then, for each i, $\mu_i = \sum_{j=1}^r \beta_j\alpha_{ij}$, for $\alpha_{ij} \in A$. Hence,

$$\theta = \sum_{i,j=1,1}^{s,r} \gamma_i\beta_j\alpha_{ij}.$$

That is, we see that the products $\gamma_i\beta_j$ generate B as a module over A. Note that we need really only consider those whose degree is a multiple of $d = \mathrm{Degree}(\zeta)$. Now we have that $\mathrm{Hom}_{\mathcal{C}_V}(M,N)$ is generated by elements having the form $\gamma_i\beta_j\zeta^{-r}$, where $rd = \mathrm{Degree}(\gamma_i\beta_j)$. $\square$

In the other direction we have the following. The example is far from general, but perhaps the reader can see how other examples can be constructed.

Theorem 7.3. *Suppose that $V \subset V_G(k)$ is a closed subvariety such that there is an elementary abelian p-subgroup E with*

$$\mathrm{res}^*_{G,E}(V_E(k)) \cap V$$

a nonempty finite set of closed points. Assume that $|E| \geq p^3$ and that E has a subgroup F with $|F| = p^2$ and $\mathrm{res}^*_{G,F}(V_F(k)) \cap V \neq \emptyset$. Let $M = k_F^{\uparrow G} = kG \otimes_{kF} k_F$ be the induced module. Then $\mathrm{End}_{\mathcal{C}_V}(M)$ is not finitely generated as a module over $\mathrm{End}_{\mathcal{C}_V}(k)$.

Proof. First we note that, in the category $\mathcal{C}_V$, $M \cong \mathcal{F}_V \otimes M$. By Frobenius Reciprocity,

$$\mathcal{F}_V \otimes M \cong \mathcal{F}_V \otimes k_F^{\uparrow G} \cong (\mathcal{F}_V)_{\downarrow F})^{\uparrow G}$$

By Lemma 3.4, $(\mathcal{F}_V)_{\downarrow F} \cong \mathcal{F}_{V'}$ in the stable category where $V' = (\mathrm{res}^*_{G,F})^{-1}(V)$. From the hypothesis, we know that V' consists of a finite set of closed points and is not equal to $V_F(k)$.

By the Eckmann-Shapiro Lemma, we have the usual adjointness:

$$\begin{aligned} \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V \otimes k_F^{\uparrow G}, \mathcal{F}_V \otimes k_F^{\uparrow G}) &\cong \underline{\mathrm{Hom}}_{kF}((\mathcal{F}_V)_{\downarrow F}, (\mathcal{F}_V \otimes k_F^{\uparrow G})_{\downarrow F}) \\ &\cong \underline{\mathrm{Hom}}_{kF}((\mathcal{F}_{V'}), ((\mathcal{F}_{V'})^{\uparrow G})_{\downarrow F}) \end{aligned}$$

Now notice that $((\mathcal{F}_{V'})^{\uparrow G})_{\downarrow F}$ has a direct summand isomorphic to $\mathcal{F}_{V'}$. That is,

$$(\mathcal{F}_{V'})^{\uparrow G} \cong \sum_{g \in G/F} g \otimes \mathcal{F}_{V'}$$

as k-vector spaces. Here, the sum is over a complete set of left coset representatives of F in G. The subspace $1 \otimes \mathcal{F}_{V'}$ is a kF-submodule and a direct summand. This also follows from the Mackey Theorem. The implication is that $\underline{\mathrm{Hom}}_{kF}((\mathcal{F}_{V'}), ((\mathcal{F}_{V'})^{\uparrow G})_{\downarrow F})$ has a direct sumand isomorphic to $\underline{\mathrm{End}}_{kF}(\mathcal{F}_{V'})$. We have seen in earlier sections of this paper that this has infinite k-dimension.

The action of $\mathrm{End}_{\mathcal{C}_V}(k)$ on $\mathrm{End}_{\mathcal{C}_V}(M)$ is given by

$$\begin{gathered} \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V) \otimes \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V \otimes k_F^{\uparrow G}, \mathcal{F}_V \otimes k_F^{\uparrow G}) \\ \downarrow \\ \underline{\mathrm{Hom}}_{kF}((\mathcal{F}_V)_{\downarrow F}, (\mathcal{F}_V)_{\downarrow F}) \otimes \underline{\mathrm{Hom}}_{kF}((\mathcal{F}_V)_{\downarrow F}, (\mathcal{F}_V \otimes k_F^{\uparrow G})_{\downarrow F}) \\ \downarrow \\ \underline{\mathrm{Hom}}_{kF}((\mathcal{F}_V)_{\downarrow F}, (\mathcal{F}_V \otimes k_F^{\uparrow G})_{\downarrow F}) \end{gathered}$$

where the first arrow is the isomorphism given by the Eckmann-Shapiro Lemma and the second is composition. The Eckmann-Shapiro Lemma is easily seen to hold in the stable category.

The main point of the proof is that in applying the Lemma, the action of $\mathrm{End}_{\mathcal{C}_V}(k)$ factors through the restriction map to kF. However, the restriction map is transitive, and hence must factor through the restriction to kE. By Theorem 6.2, the restriction of the maximal ideal $\mathcal{I}$ in $\mathrm{End}_{\mathcal{C}_V}(k)$ is zero. That is, the image of the restriction of $\mathrm{End}_{\mathcal{C}_V}(k) = \underline{\mathrm{Hom}}_{kG}(\mathcal{F}_V, \mathcal{F}_V)$ to $\underline{\mathrm{Hom}}_{kF}((\mathcal{F}_V)_{\downarrow F}, (\mathcal{F}_V)_{\downarrow F})$ is the identity subring k.

It follows from the above that in order for $\mathrm{End}_{\mathcal{C}_V}(M)$ to be finitely generated over $\mathrm{End}_{\mathcal{C}_V}(k)$, it must be finite dimensional. However, we have already noted that this is not the case. $\square$

8. Examples

We end the paper with a couple of examples and a theorem on the structure of the idempotent modules. For the first example and most of the section suppose that $G = SL_2(p^n)$ for $n > 2$, and let k be an algebraically closed field of characteristic p.

Let a be a generator for the multiplicative group $\mathbb{F}_{p^n}^\times$. The Borel subgroup B of G is generated by the elements

$$t = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \quad \text{and} \quad x_i = \begin{bmatrix} 1 & a^i \\ 0 & 1 \end{bmatrix} \quad \text{for} \quad i = 0, \dots, n-1.$$

Then $S = \langle x_1, \dots, x_n \rangle$ is a Sylow p-subgroup and B is its normalizer in G.

The variety $V_S(k) \cong \mathbb{P}^{n-1}$, projective $(n-1)$-space. The group B acts on S by conjugation and hence also on $V_S(k)$. The action of $T = \langle t \rangle$ is given by the relation

$$\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix} \quad = \quad \begin{bmatrix} 1 & ub^2 \\ 0 & 1 \end{bmatrix}$$

for u in $\mathbb{F}_{p^n}$ and b in $\mathbb{F}_{p^n}^\times$. The thing to notice is that if b^2 is in the prime field $\mathbb{F}_p$, then this element of T operates on S (viewed as an $\mathbb{F}_p$-vector space $S \cong (\mathbb{Z}/(p))^n$) by a scalar matrix with diagonal entries equal to b^2. This implies that the element of T acts trivially on the projective space $V_S(k)$. With this in mind, let

$$d \;=\; \begin{cases} p-1 & \text{if } p = 2 \text{ or } n \text{ is odd,} \\ 2(p-1) & \text{otherwise.} \end{cases}$$

It is easily checked that d is the order of the subgroup of T that acts trivially on $V_G(S)$. Let $m = (p^n - 1)/d$, and let D be the subgroup of B generated by $c = t^m$ and S.

Choose W to be any subvariety of $V_S(k) = V_D(k)$ that consists of a single point whose stabilizer in $T = \langle t \rangle$ is generated by c. Let $V = \mathrm{res}^*_{B,S}(W)$. Then the inverse image of V under the restriction map $\mathrm{res}^*_{B,S}$ is the union of the points in the orbit of W under the action of T. Let $V = V_0, \dots, V_{m-1}$ be the images under V of this action. Let $\mathcal{F}_V$ be the idempotent kD-module corresponding to V.

The induced module $\mathcal{F}_V^{\uparrow B} = kB \otimes_{kD} \mathcal{F}_V$ has the form $\sum_{j=0}^{m-1} \mathcal{F}_{V_j}$ on restriction to D. We may assume that $\mathcal{F}_{V_j} = t^j \otimes \mathcal{F}_V$ in this context. Thus we have maps $t^j \otimes \tau_V : t^j \otimes k \to t^j \otimes \mathcal{F}_V$. That is, T acts on this system and also acts on the pushout that is obtained by identifying the images of the maps $t^j \otimes \tau_V$. Explicitly, let N be the submodule of $k_D^{\uparrow B} \cong kB \otimes_{kD} k$ generated by $1 \otimes 1 - t \otimes 1$. This is a kB-submodule of dimension $m-1$. Let N' be its image in $\mathcal{F}_V^{\uparrow B}$, that is the submodule generated by $1 \otimes \tau_V(1) - t \otimes \tau_V(1)$. Then we have a triangle

$$\longrightarrow k_D^{\uparrow G}/N \longrightarrow \mathcal{F}_V^{\uparrow B}/N' \longrightarrow \mathcal{F}_V^{\uparrow B}/k_D^{\uparrow G} \longrightarrow$$

Next, a check of the varieties can be done at the level of the Sylow p-subgroup S. In particular, the variety of $\mathcal{E}_V$ is $\{V\}$, while that if $\mathcal{F}_V$ is $\mathcal{V}_G(k) \setminus \{V\}$. Thus, by the tensor product theorem $\mathcal{E}_V \otimes \mathcal{F}_V$ is projective, and zero in the stable category. Tensoring with the triangle, we see that both $\mathcal{E}_V$ and $\mathcal{F}_V$ are idempotent modules. It can be seen that the relevant universal properties of the triangle also hold.

The endomorphism ring $\underline{\mathrm{End}}_{kB}(\mathcal{F}_V)$ is the set of all element of $\underline{\mathrm{End}}_{kS}((\mathcal{F}_V)_{\downarrow S})$ that are stable under the action of T. The identity element is certainly T-stable. Let $\mathcal{I}$ be the maximal ideal in $\underline{\mathrm{End}}_{kS}((\mathcal{F}_V)_{\downarrow S})$. Because, by Theorem 4.3, $\mathcal{I}^2 = \{0\}$, any element of $\mathcal{I}$ that is invariant under T is an orbit sum of the T-action. These elements form a maximal ideal of codimension one, and the product of any two elements in this ideal is zero.

The idempotent modules for G can be obtained by using the fact that S is TI-subgroup (trivial intersection). That is, for any element $g \in G$ we have that $S \cap gSg^{-1}$ is S if $g \in N_G(S) = B$ and is $\{1\}$ otherwise. Then by the Mackey Theorem, we have that

$$((\mathcal{F}_V)^{\uparrow G})_{\downarrow B} \cong \mathcal{F}_V \oplus P$$

where P is projective. Consequently, the induced module $(\mathcal{F}_V)^{\uparrow G})$ has a single nonprojective direct summand that is $\mathcal{F}_{V'}$ where $V' = \mathrm{res}^*_{B,G}(V)$. It follows that $\underline{\mathrm{End}}_{kG}(\mathcal{F}_{V'})$ is isomorphic to $\underline{\mathrm{End}}_{kG}(\mathcal{F}_V)$.

All of this has a sweeping generalization that is reminiscent of the work in [4] and [8].

For notation we say that if S is an elementary abelian p-subgroup of a group G, its diagonalizer $D = D_G(S)$ is the subgroup of $N_G(S)$ consisting of all elements whose conjugation action is by a scalar matrix on the $\mathbb{F}_p$-vector space of S when written as an additive group. As in the above example, it is the subgroup of elements of $N_G(S)$ that acts trivially on $V_S(k)$.

Theorem 8.1. *Suppose that S is a normal elementary abelian p-subgroup of G and that $D = D_G(S)$. Let U be a subvariety of $V_S(k)$ consisting of a single point and assume that U is not contained in* $\mathrm{res}^*_{S,R}(V_R(k))$ *for any subgroup R of S. Let $W =$*

$\operatorname{res}^*_{D,S}(U)$. *Let* $V = \operatorname{res}^*_{G,D}(W)$. *Let* N *be the kernel of the natural homomorphism* $\varphi : k_D^{\uparrow G} \to k$ *given by* $g \otimes 1 \mapsto 1$ *for any* $g \in G$. *Then we have a triangle*

$$\longrightarrow k_D^{\uparrow G}/N \xrightarrow{\tau} \mathcal{F}_W^{\uparrow G}/\tau(N) \longrightarrow \Omega^{-1}(\mathcal{E}_W)^{\uparrow G} \longrightarrow$$

where τ *is the map induced on quotients by* $1 \otimes \tau_W : k_D^{\uparrow G} \to \mathcal{F}_W^{\uparrow G}$. *In particular, we have that*

$$\mathcal{F}_V \cong \mathcal{F}_W^{\uparrow G}/\tau(N) \quad and \quad \mathcal{E}_V \cong \mathcal{E}_W^{\uparrow G},$$

and the triangle is the triangle of idempotent modules associated to V.

Proof. This follows by a very similar argument as in the above example. Note that, by an eigenvalue argument, D is precisely the subgroup of G that fixes the point U in $V_S(k)$. The fact that $\mathcal{E}_V$ is induced from a kD-module follows also from Theorem 1.5 of [4], which is proved in even greater generality. □

Remark 8.2. If the group G in the theorem satisfies the Hypothesis 5.1, then Theorem 5.4 assurs us that $\underline{\operatorname{End}}_{kG}(\mathcal{F}_V)$ has a unique maximal ideal $\mathcal{I}$ having codimension one and that $\mathcal{I}$ is nilpotent. Unlike the example we may not assume that $\mathcal{I}^2 = \{0\}$. For an example, let $p = 2$, and $G = H \times S$ where H is a semidihedral group and S has order 2. So if $V = \operatorname{res}^*_{G,S}(V_S(k))$, then by Theorem 7.4 of [11], $\underline{\operatorname{End}}_{kG}(\mathcal{F}_V)$ is the nonpositive Tate cohomology ring of H which by [5] has nonzero products in its maximal ideal. Note that in this particular case $D_G(S) = G$, so that the above theorem says nothing new.

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