

CERTAIN COEFFICIENT PROBLEMS OF \mathcal{S}_e^* AND \mathcal{C}_e

S. SIVAPRASAD KUMAR AND NEHA VERMA

ABSTRACT. In this current study, we consider the classes \mathcal{S}_e^* and \mathcal{C}_e to obtain sharp bounds for the third Hankel determinant for functions within these classes. Additionally, we provide estimates for the sixth and seventh coefficients while establishing the fourth-order Hankel determinant as well.

1. INTRODUCTION

Consider the set of normalized analytic functions, denoted as \mathcal{A} , which are defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. These functions are represented by the expansion:

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \quad (1.1)$$

Within this class, we define a subclass \mathcal{S} , which comprises univalent functions. Also, assume a class of analytic functions defined on the unit disk \mathbb{D} , which possess a positive real part. This class is represented as \mathcal{P} whose elements are of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. We use the notation $h_1 \prec h_2$ to indicate that function h_1 is subordinate to h_2 , which implies the existence of a Schwarz function w with the properties $w(0) = 0$ and $|w(z)| \leq |z|$, such that $h_1(z) = h_2(w(z))$.

The Bieberbach conjecture, as discussed in [3, Page no. 17] has made a substantial contribution to the advancement of geometric function theory and the emergence of coefficient-related challenges. In the wake of this, numerous additional subclasses of \mathcal{S} , encompassing starlike functions denoted as \mathcal{S}^* and convex functions denoted as \mathcal{C} , have been introduced. Notably, in 1992, Ma and Minda [15] introduced the following two classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad (1.2)$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \quad (1.3)$$

which unifies various subclasses of \mathcal{S}^* and \mathcal{C} , respectively. Here φ is an analytic univalent function satisfying the conditions $\operatorname{Re} \varphi(z) > 0$, $\varphi(\mathbb{D})$ symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$.

The notion of Hankel determinants was introduced in [18]. Remarkably, this concept continues to captivate the attention of numerous researchers to this very day. Encompassing a broad spectrum of applications and implications, the q th Hankel determinants $H_q(n)$ of analytic functions belonging to the class \mathcal{A} , as represented in (1.1), have been defined under the premise that a_1 takes the value 1. For $n, q \in \mathbb{N}$, this definition unfolds as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix} \quad (1.4)$$

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Coefficient estimate, Exponential, Starlike, Hankel determinants .

The specific expression for the third-order Hankel determinant, denoted as $H_3(1)$, is obtained by substituting $q = 3$ and $n = 1$ into equation (1.4). This determinant can be precisely defined as:

$$H_3(1) = 2a_2a_3a_4 - a_3^3 - a_4^2 - a_2^2a_5 + a_3a_5. \quad (1.5)$$

Over the time, several authors established sharp bound of second-order Hankel determinants, see [1, 8]. However, the task of computing bounds for third-order Hankel determinants, proves to be considerably more intricate, can be observed from [12, 24, 25]. In the context of the class \mathcal{S}^* , Kwon et al. [12] established the inequality $|H_3(1)| \leq 8/9$, which has recently been best improved to the bound of $4/9$ by Kowalczyk et al. [7]. Furthermore, Lecko et al. [13] successfully derived the bound $|H_3(1)| \leq 1/9$, a result that stands as sharp for functions in $\mathcal{S}^*(1/2)$. For a more comprehensive exploration of Hankel determinants, interested readers can turn to works such as [2, 7, 13, 22].

Below, we enlist specific subclasses of \mathcal{S}^* and \mathcal{C} , resulting from diverse selections of $\varphi(z)$ in Table 1. In a similar manner, Mendiratta et al. [16] introduced and analyzed the classes \mathcal{S}_e^* and \mathcal{C}_e by selecting $\varphi(z) = e^z$ in (1.2) and (1.3), respectively. These classes are defined as follows:

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \right\} \quad \text{and} \quad \mathcal{C}_e = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec e^z \right\}.$$

TABLE 1. List of subclasses of \mathcal{S}^* and \mathcal{C}

$\mathcal{S}^*(\varphi)$	$\mathcal{C}(\varphi)$	$\varphi(z)$	Author(s)	Reference
$\mathcal{S}^*[C, D]$	$\mathcal{C}[C, D]$	$(1 + Cz)/(1 + Dz)$	Janowski	[5]
\mathcal{S}_{SG}^*	\mathcal{C}_{SG}	$2/(1 + e^{-z})$	Goel and Kumar	[4]
\mathcal{S}_ρ^*	\mathcal{C}_ρ	$1 + ze^z$	Kumar and Kamaljeet	[9]
\mathcal{S}_q^*	\mathcal{C}_q	$z + \sqrt{1 + z^2}$	Raina and Sokół	[19]
\mathcal{S}_L^*	\mathcal{C}_L	$\sqrt{1 + z}$	Sokół and Stankiewicz	[21]

Numerous studies have addressed radius problems [16] and investigated implications of first and higher-order differential subordination [17, 23] for the subclasses associated with the exponential function. Zaprawa [25] established bounds for the third Hankel determinants, yielding values of 0.385 and 0.021 for the classes \mathcal{S}_e^* and \mathcal{C}_e , respectively, although the results were not sharp.

In our present investigation, we contribute by establishing sharp bounds for $H_3(1)$ for functions in the classes \mathcal{S}_e^* and \mathcal{C}_e . Additionally, in the upcoming sections, we will provide estimations for the bounds of the sixth and seventh coefficients for the functions belonging to the classes, \mathcal{S}_e^* and \mathcal{C}_e and also evaluate the fourth Hankel determinant.

2. HANKEL DETERMINANTS FOR \mathcal{S}_e^*

2.1. Preliminaries. In this part of the section, we derive the expressions of a_i ($i = 2, 3, \dots, 7$) in terms of Carathéodory coefficients. For this, let $f \in \mathcal{S}_e^*$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = e^{w(z)}. \quad (2.1)$$

Suppose that $p(z) = 1 + p_1z + p_2z^2 + \dots \in \mathcal{P}$ and consider $w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in equation (2.1) and then comparing the coefficients, we obtain the expressions of a_i ($i = 2, 3, \dots, 7$) in terms of p_j ($j = 1, 2, \dots, 5$), given as follows:

$$a_2 = \frac{1}{2}p_1, \quad a_3 = \frac{1}{16} \left(4p_2 + p_1^2 \right), \quad a_4 = \frac{1}{288} \left(-p_1^3 + 12p_1p_2 + 48p_3 \right), \quad (2.2)$$

$$a_5 = \frac{1}{1152} \left(p_1^4 - 12p_1^2p_2 + 24p_1p_3 + 144p_4 \right), \quad (2.3)$$

$$a_6 = \frac{1}{57600} \left(-17p_1^5 + 220p_1^3p_2 - 480p_1p_2^2 - 480p_1^2p_3 - 480p_2p_3 + 720p_1p_4 + 5760p_5 \right), \quad (2.4)$$

and

$$a_7 = \frac{1}{8294400} \left(881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 \right. \\ \left. - 106560p_1p_2p_3 - 57600p_3^2 - 56160p_1^2p_4 - 86400p_2p_4 \right. \\ \left. + 69120p_1p_5 \right). \quad (2.5)$$

The formula for p_i ($i = 2, 3, 4$), which is included in the Lemma 2.1 below, plays a vital role in establishing the sharp bound for Hankel determinants and forms the foundation for our main results.

Lemma 2.1. [11, 14] *Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$2p_2 = p_1^2 + \gamma(4 - p_1^2), \quad (2.6)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta, \quad (2.7)$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta \\ + \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho), \quad (2.8)$$

for some γ , η and ρ such that $|\gamma| \leq 1$, $|\eta| \leq 1$ and $|\rho| \leq 1$.

2.2. Sharp Third Hankel Determinant for \mathcal{S}_e^* . In this subsection, we present the sharp bound for $H_3(1)$ for functions belonging to the class \mathcal{S}_e^* .

Theorem 2.2. *Let $f \in \mathcal{S}_e^*$. Then*

$$|H_3(1)| \leq 1/9. \quad (2.9)$$

This result is sharp.

Proof. Since the class \mathcal{P} is invariant under rotation, the value of p_1 belongs to the interval $[0, 2]$. Let $p := p_1$ and then substitute the values of a_i ($i = 2, 3, 4, 5$) in equation (1.5) from equations (2.2) and (2.3). We get

$$H_3(1) = \frac{1}{331776} \left(-211p^6 + 420p^4p_2 - 1872p^2p_2^2 - 5184p_2^3 + 2544p^3p_3 \right. \\ \left. + 10944pp_2p_3 - 9216p_3^2 - 7776p^2p_4 + 10368p_2p_4 \right).$$

After simplifying the calculations through (2.6)-(2.8), we obtain

$$H_3(1) = \frac{1}{331776} \left(\beta_1(p, \gamma) + \beta_2(p, \gamma)\eta + \beta_3(p, \gamma)\eta^2 + \phi(p, \gamma, \eta)\rho \right),$$

for $\gamma, \eta, \rho \in \mathbb{D}$. Here

$$\begin{aligned} \beta_1(p, \gamma) &:= -13p^6 - 36\gamma^2 p^2(4-p^2)^2 - 360\gamma^3 p^2(4-p^2)^2 + 72\gamma^4 p^2(4-p^2)^2 \\ &\quad + 78\gamma p^4(4-p^2) + 120p^4 \gamma^2(4-p^2) - 324p^4 \gamma^3(4-p^2) \\ &\quad - 1296\gamma^2 p^2(4-p^2), \\ \beta_2(p, \gamma) &:= 24(1-|\gamma|^2)(4-p^2)(17p^3 + 54\gamma p^3 + 30p\gamma(4-p^2) - 12p\gamma^2(4-p^2)), \\ \beta_3(p, \gamma) &:= 144(1-|\gamma|^2)(4-p^2)(-16(4-p^2) - 2|\gamma|^2(4-p^2) + 9p^2\bar{\gamma}), \\ \phi(p, \gamma, \eta) &:= 1296(1-|\gamma|^2)(4-p^2)(1-|\eta|^2)(2(4-p^2)\gamma - p^2). \end{aligned}$$

By choosing $x = |\gamma|$, $y = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_3(1)| \leq \frac{1}{331776} \left(|\beta_1(p, \gamma)| + |\beta_2(p, \gamma)|y + |\beta_3(p, \gamma)|y^2 + |\phi(p, \gamma, \eta)| \right) \leq M(p, x, y),$$

where

$$M(p, x, y) = \frac{1}{331776} \left(m_1(p, x) + m_2(p, x)y + m_3(p, x)y^2 + m_4(p, x)(1-y^2) \right), \quad (2.10)$$

with

$$\begin{aligned} m_1(p, x) &:= 13p^6 + 36x^2 p^2(4-p^2)^2 + 360x^3 p^2(4-p^2)^2 + 72x^4 p^2(4-p^2)^2 \\ &\quad + 78xp^4(4-p^2) + 120p^4 x^2(4-p^2) + 324p^4 x^3(4-p^2) + 1296x^2 p^2(4-p^2), \\ m_2(p, x) &:= 24(1-x^2)(4-p^2)(17p^3 + 54xp^3 + 30px(4-p^2) + 12px^2(4-p^2)), \\ m_3(p, x) &:= 144(1-x^2)(4-p^2)(16(4-p^2) + 2x^2(4-p^2) + 9p^2x), \\ m_4(p, x) &:= 1296(1-x^2)(4-p^2)(2x(4-p^2) + p^2). \end{aligned}$$

In the closed cuboid $U : [0, 2] \times [0, 1] \times [0, 1]$, we now maximise $M(p, x, y)$, by locating the maximum values in the interior of the six faces, on the twelve edges, and in the interior of U .

- (1) We start by taking into account every internal point of U . Assume that $(p, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$. We calculate $\partial M / \partial y$ to identify the points of maxima in the interior of U . We get

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{(4-p^2)(1-x^2)}{13824} \left(24px(5+2x) + p^3(17+24x-12x^2) + 96(8-9x+x^2)y \right. \\ &\quad \left. - 12p^2(25-27x+2x^2)y \right). \end{aligned}$$

Now $\frac{\partial M}{\partial y} = 0$ gives

$$y = y_0 := \frac{p(17p^2 + 120x + 24p^2x + 48x^2 - 12p^2x^2)}{12(-64 + 25p^2 + 72x - 27p^2x - 8x^2 + 2p^2x^2)}.$$

The existence of critical points requires that y_0 belong to $(0, 1)$, which is only possible when

$$\begin{aligned} 300p^2 + 864x + 24p^2x^2 &> 17p^3 + 120px + 24p^3x + 48px^2 - 12p^3x^2 \\ &\quad + 768 + 864x + 24p^2x^2. \end{aligned} \quad (2.11)$$

Now, we find the solution satisfying the inequality (2.11) for the existence of critical points using the hit and trial method. If we assume p tends to 0, then there does not

exist any $x \in (0, 1)$ satisfying the equation (2.11). But, when p tends to 2, the equation (2.11) holds for all $x < 37/54$. We also observe that there does not exist any $p \in (0, 2)$ when $x \in (37/54, 1)$. Similarly, if we assume x tends to 0, then for all $p > 1.68218$, the equation (2.11) holds. After calculations, we observe that there does not exist any $x \in (0, 1)$ when $p \in (0, 1.68218)$. Thus, the domain for the solution of the equation is $(1.68218, 2) \times (0, 37/54)$. Now, we examine that $\frac{\partial M}{\partial y}|_{y=y_0} \neq 0$ in $(1.68218, 2) \times (0, 37/54)$. So, we conclude that the function M has no critical point in $(0, 2) \times (0, 1) \times (0, 1)$.

(2) The interior of each of the cuboid U 's six faces is now being considered.

On $p = 0$, $M(p, x, y)$ turns into

$$s_1(x, y) := \frac{(1-x^2)(8y^2+x^2y^2+9x(1-y^2))}{72}, \quad x, y \in (0, 1). \quad (2.12)$$

Since

$$\frac{\partial s_1}{\partial y} = \frac{(1-x^2)(x-1)(x-8)y}{36} \neq 0, \quad x, y \in (0, 1),$$

indicates that s_1 has no critical points in $(0, 1) \times (0, 1)$.

On $p = 2$, $M(p, x, y)$ reduces to

$$M(2, x, y) := \frac{13}{5184}, \quad x, y \in (0, 1). \quad (2.13)$$

On $x = 0$, $M(p, x, y)$ becomes

$$s_2(p, y) := \frac{13p^6 + (4-p^2)(408p^3y + 2304y^2(4-p^2) + 1296p^2(1-y^2))}{331776} \quad (2.14)$$

with $p \in (0, 2)$ and $y \in (0, 1)$. To determine the points of maxima, we solve $\partial s_2 / \partial p = 0$ and $\partial s_2 / \partial y = 0$. After solving $\partial s_2 / \partial y = 0$, we get

$$y = \frac{17p^3}{12(25p^2 - 64)} (=: y_p). \quad (2.15)$$

In order to have $y_p \in (0, 1)$ for the given range of y , $p_0 := p > \approx 1.68218$ is required. Based on calculations, $\partial s_2 / \partial p = 0$ gives

$$1728p - 864p^3 + 13p^5 + 816p^2y - 340p^4y - 7872py^2 + 2400p^3y^2 = 0. \quad (2.16)$$

After substituting equation (2.15) into equation (2.16), we have

$$21233664p - 27205632p^3 + 11472192p^5 - 1613016p^7 + 2700p^9 = 0. \quad (2.17)$$

A numerical calculation suggests that $p \approx 1.35596 \in (0, 2)$ is the solution of (2.17). So, we conclude that s_2 does not have any critical point in $(0, 2) \times (0, 1)$.

On $x = 1$, $M(p, x, y)$ reforms into

$$s_3(p, y) := M(p, 1, y) = \frac{12672p^2 - 2952p^4 - 41p^6}{331776}, \quad p \in (0, 2). \quad (2.18)$$

While computing $\partial s_3 / \partial p = 0$, $p_0 := p \approx 1.43461$ comes out to be the critical point. Undergoing simple calculations, s_3 achieves its maximum value ≈ 0.0398426 at p_0 .

On $y = 0$, $M(p, x, y)$ can be viewed as

$$s_4(p, x) := \frac{1}{331776} \left(41472x(1 - x^2) + 576p^2(9 - 36x + x^2 + 46x^3 + 2x^4) \right. \\ \left. - 24p^4(54 - 121x - 8x^2 + 174x^3 + 24x^4) \right. \\ \left. + p^6(13 - 78x - 84x^2 + 36x^3 + 72x^4) \right).$$

After undergoing further calculations such as,

$$\frac{\partial s_4}{\partial x} = \frac{1}{331776} \left(-82944x^2 + 41472(1 - x^2) + 576p^2(-36 + 2x + 138x^2 + 8x^3) \right. \\ \left. - 24p^4(-121 - 16x + 522x^2 + 96x^3) + p^6(-78 - 168x \right. \\ \left. + 108x^2 + 288x^3) \right)$$

and

$$\frac{\partial s_4}{\partial p} = \frac{1}{331776} \left(6p^5(13 - 78x - 84x^2 + 36x^3 + 72x^4) - 96p^3(54 - 121x - 8x^2 \right. \\ \left. + 174x^3 + 24x^4) + 1152p(9 - 36x + x^2 + 46x^3 + 2x^4) \right),$$

we observe that no solution in $(0, 2) \times (0, 1)$ exists of the system of equations $\partial s_4/\partial x = 0$ and $\partial s_4/\partial p = 0$.

On $y = 1$, $M(p, x, y)$ reduces to

$$s_5(p, x) := \frac{1}{331776} \left(2304px(5 + 2x - 5x^2 - 2x^3) - 4608(-8 + 7x^2 + x^4) \right. \\ \left. + 576p^2(-32 + 9x + 38x^2 + x^3 + 6x^4) - 24p^5(17 + 24x \right. \\ \left. - 29x^2 - 24x^3 + 12x^4) + 96p^3(17 - 6x - 41x^2 + 6x^3 \right. \\ \left. + 24x^4) - 24p^4(-96 + 41x + 130x^2 + 12x^3 + 36x^4) \right. \\ \left. + p^6(13 - 78x - 84x^2 + 36x^3 + 72x^4) \right).$$

The system of equations $\partial s_5/\partial x = 0$ and $\partial s_5/\partial p = 0$ also do not have any solution in $(0, 2) \times (0, 1)$.

- (3) We next examine the maxima attained by $M(p, x, y)$ on the edges of the cuboid U . From equation (2.14), we have $M(p, 0, 0) = r_1(p) := (5184p^2 - 1296p^4 + 13p^6)/331776$. It is easy to observe that $r_1'(p) = 0$ whenever $p = \delta_0 := 0$ and $p = \delta_1 := 1.4367 \in [0, 2]$ as its points of minima and maxima respectively. Hence,

$$M(p, 0, 0) \leq 0.0159535, \quad p \in [0, 2].$$

Now considering the equation (2.14) at $y = 1$, we get $M(p, 0, 1) = r_2(p) := (36864 - 18432p^2 + 1632p^3 + 2304p^4 - 408p^5 + 13p^6)/331776$. It is easy to observe that $r_2'(p) < 0$ in $[0, 2]$ and hence $p = 0$ serves as the point of maxima. So,

$$M(p, 0, 1) \leq \frac{1}{9}, \quad p \in [0, 2].$$

Through computations, equation (2.14) shows that $M(0, 0, y)$ attains its maxima at $y = 1$. This implies that

$$M(0, 0, y) \leq \frac{1}{9}, \quad y \in [0, 1].$$

Since, the equation (2.18) does not involve x , we have $M(p, 1, 1) = M(p, 1, 0) = r_3(p) := (12672p^2 - 2952p^4 - 41p^6)/331776$. Now, $r_3'(p) = 4224p - 1968p^3 - 41p^5 = 0$ when $p = \delta_2 := 0$ and $p = \delta_3 := 1.43461$ in the interval $[0, 2]$ with δ_2 and δ_3 as points of minima and maxima respectively. Hence

$$M(p, 1, 1) = M(p, 1, 0) \leq 0.0398426, \quad p \in [0, 2].$$

After considering $p = 0$ in (2.18), we get, $M(0, 1, y) = 0$. The equation (2.13) has no variables. So, on the edges, the maximum value of $M(p, x, y)$ is

$$M(2, 1, y) = M(2, 0, y) = M(2, x, 0) = M(2, x, 1) = \frac{13}{5184}, \quad x, y \in [0, 1].$$

Using equation (2.12), we obtain $M(0, x, 1) = r_4(x) := (8 - 7x^2 - x^4)/72$. Upon calculations, we see that $r_4(x)$ is a decreasing function in $[0, 1]$ and attains its maxima at $x = 0$. Hence

$$M(0, x, 1) \leq \frac{1}{9}, \quad x \in [0, 1].$$

Again utilizing the equation (2.12), we get $M(0, x, 0) = r_5(x) := x(1 - x^2)/8$. On further calculations, we get $r_5'(x) = 0$ for $x = \delta_4 := 1/\sqrt{3}$. Also, $r_5(x)$ is an increases in $[0, \delta_4]$ and decreases in $(\delta_4, 1]$. So, it reaches its maximum value at δ_4 . Thus

$$M(0, x, 0) \leq 0.0481125, \quad x \in [0, 1].$$

Given all the cases, the inequality (2.9) holds.

Let the function $f_1(z) \in \mathcal{S}_e^*$, be defined as

$$f_1(z) = z \exp \left(\int_0^z \frac{e^{t^3} - 1}{t} dt \right) = z + \frac{z^4}{3} + \frac{5z^7}{36} + \dots,$$

with $f_1(0) = 0$ and $f_1'(0) = 1$, acts as an extremal function for the bound of $|H_3(1)|$ for $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/3$. ■

2.3. Fourth Hankel Determinant for \mathcal{S}_e^* . In this subsection, we derive the bounds of sixth and seventh coefficients and consequently $H_4(1)$ for functions belonging to the class \mathcal{S}_e^* . We need the following lemma for deriving our results.

Lemma 2.3. [10, 20] Let $p = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$. Then

$$|p_n| \leq 2, \quad n \geq 1,$$

$$|p_{n+k} - \nu p_n p_k| \leq \begin{cases} 2, & 0 \leq \nu \leq 1; \\ 2|2\nu - 1|, & \text{otherwise,} \end{cases}$$

and

$$|p_1^3 - \nu p_3| \leq \begin{cases} 2|\nu - 4|, & \nu \leq 4/3; \\ 2\nu \sqrt{\frac{\nu}{\nu - 1}}, & 4/3 < \nu. \end{cases}$$

We derive the expression of the fourth Hankel determinant when $q = 4$ and $n = 1$ are put into equation (1.4) as follows :

$$H_4(1) = a_7 H_3(1) - a_6 T_1 + a_5 T_2 - a_4 T_3, \quad (2.19)$$

where

$$T_1 := a_6(a_3 - a_2^2) + a_3(a_2 a_5 - a_3 a_4) - a_4(a_5 - a_2 a_4), \quad (2.20)$$

$$T_2 := a_3(a_3 a_5 - a_4^2) - a_5(a_5 - a_2 a_4) + a_6(a_4 - a_2 a_3), \quad (2.21)$$

and

$$T_3 := a_4(a_3 a_5 - a_4^2) - a_5(a_2 a_5 - a_3 a_4) + a_6(a_4 - a_2 a_3). \quad (2.22)$$

Now, using Lemma 2.3, we first determine the bounds of T_1 , T_2 , and T_3 .

By substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (2.20) using (2.2)-(2.4), we obtain

$$\begin{aligned} 5529600T_1 &= 581p_1^7 + 5040p_1^4 p_3 + 25920p_1^2 p_2 p_3 - 7068p_1^5 p_2 + 11040p_1^3 p_4 \\ &\quad - 115200p_3 p_4 + 7920p_1^3 p_2^2 - 69120p_2^2 p_3 + 74880p_1 p_2 p_4 - 25920p_1 p_3^2 \\ &\quad + 57600p_1 p_3^2 + 138240p_2 p_5 - 103680p_1^2 p_5 \end{aligned}$$

or

$$\begin{aligned} 5529600|T_1| &\leq |p_1^4(581p_1^3 + 5040p_3)| + |p_1^2 p_2(25920p_3 - 7068p_1^3)| + |57600p_1 p_3^2| \\ &\quad + |p_2^2(7920p_1^3 - 69120p_3)| + |p_1 p_2(74880p_4 - 25920p_2^2)| \\ &\quad + |p_4(11040p_1^3 - 115200p_3)| + |p_5(138240p_2 - 103680p_1^2)|. \end{aligned}$$

Using Lemma 2.3 and the triangle inequality, we arrive at

$$\begin{aligned} |T_1| &\leq \frac{1848448 + 4976640\sqrt{\frac{15}{1571}} + 1843200\sqrt{\frac{15}{217}} + 442368\sqrt{\frac{30}{17}}}{5529600} \\ &\approx 0.616137. \end{aligned}$$

Now, we calculate the bound of T_2 in the similar way by substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (2.21) from equations (2.2)-(2.4), as follows:

$$\begin{aligned} 22118400T_2 &= 235p_1^8 + 8712p_1^5 p_3 + 37440p_1^3 p_2 p_3 - 1156p_1^6 p_2 - 63360p_1 p_2^2 p_3 \\ &\quad - 14640p_1^4 p_2^2 + 161280p_1 p_3 p_4 - 8400p_1^4 p_4 + 368640p_3 p_5 \\ &\quad - 76800p_1^3 p_5 - 8640p_1^2 p_2^3 + 172800p_2^2 p_4 - 345600p_4^2 - 40320p_1^2 p_3^2 \\ &\quad - 184320p_2 p_3^2 + 178560p_1^2 p_2 p_4 - 184320p_1 p_2 p_5 \end{aligned}$$

or

$$\begin{aligned} 22118400|T_2| &\leq |p_1^5(235p_1^3 + 8712p_3)| + |p_1^3 p_2(37440p_3 - 1156p_1^3)| + |8640p_1^2 p_2^3| \\ &\quad + |p_1 p_2^2(63360p_3 + 14640p_1^3)| + |p_1 p_4(161280p_3 - 8400p_1^3)| \\ &\quad + |p_5(368640p_3 - 76800p_1^3)| + |p_4(172800p_2^2 - 345600p_4)| \\ &\quad + |p_3^2(184320p_2 + 40320p_1^2)| + |p_1 p_2(178560p_1 p_4 - 184320p_5)|. \end{aligned}$$

Lemma 2.3 and the triangle inequality lead us to

$$\begin{aligned} |T_2| &\leq \frac{7821568 + 14376960\sqrt{\frac{65}{9071}} + 2949120\sqrt{\frac{6}{19}} + 737280\sqrt{\frac{42}{13}}}{22118400} \\ &\approx 0.543487. \end{aligned}$$

Next, we determine the bound of T_3 , by replacing the values of a_i 's ($i = 2, 3, \dots, 6$) from equations (2.2)-(2.4) in (2.22), as follows:

$$\begin{aligned}
597196800T_3 &= 6120p_1^8 + 143424p_1^5p_3 - 425p_1^9 - 9000p_1^6p_3 + 9000p_1^7p_2 \\
&\quad + 172800p_1^4p_2p_3 + 302400p_1^3p_3^2 - 2764800p_3^3 + 1036800p_1^3p_2p_4 \\
&\quad + 6220800p_2p_3p_4 - 17280p_1^4p_2^2 + 9953280p_3p_5 - 2073600p_1^3p_5 \\
&\quad + 967680p_1^3p_2p_3 - 64512p_1^6p_2 - 1036800p_1p_2p_3^2 - 32400p_1^5p_2^2 \\
&\quad - 777600p_1^2p_2^2p_3 + 1244160p_1p_3p_4 - 259200p_1^4p_4 - 97200p_1^5p_4 \\
&\quad + 1555200p_1p_2^2p_4 - 4665600p_1p_4^2 - 414720p_1p_2^2p_3 - 172800p_1^3p_3^2 \\
&\quad - 829440p_2p_3^2 - 829440p_1^2p_3^2 + 414720p_1^2p_2^3 - 622080p_1^2p_2p_4 \\
&\quad - 4976640p_1p_2p_5
\end{aligned}$$

or

$$\begin{aligned}
597196800|T_3| &\leq |p_1^5(6120p_1^3 + 143424p_3)| + |p_1^6(425p_1^3 + 9000p_3)| + |17280p_1^4p_2^2| \\
&\quad + |p_1^4p_2(9000p_1^3 + 172800p_3)| + |p_3^2(302400p_1^3 - 2764800p_3)| \\
&\quad + |p_2p_4(1036800p_1^3 + 6220800p_3)| + |p_5(9953280p_3 - 2073600p_1^3)| \\
&\quad + |p_1^3p_2(967680p_3 - 64512p_1^3)| + |1036800p_1p_2p_3^2| + |97200p_1^5p_4| \\
&\quad + |p_1^2p_2^2(32400p_1^3 + 777600p_3)| + |p_1p_4(1244160p_3 - 259200p_1^3)| \\
&\quad + |p_1p_4(1555200p_2^2 - 4665600p_4)| + |p_1^2p_2(414720p_2^2 - 622080p_4)| \\
&\quad + |p_3^2(829440p_2 + 829440p_1^2)| + |172800p_1^3p_3^2| \\
&\quad + |p_1p_2(414720p_2p_3 + 4976640p_5)|.
\end{aligned}$$

By applying Lemma 2.3 and the triangle inequality,

$$\begin{aligned}
|T_3| &\leq \frac{286061056 + 58982400\sqrt{\frac{3}{19}} + 99532800\sqrt{\frac{6}{19}} + 2211840\sqrt{210}}{597196800} \\
&\approx 0.665582.
\end{aligned}$$

Remark 2.4. On the basis of the above calculations, the bounds of T_1 , T_2 and T_3 are 0.616137, 0.543487 and 0.665582 respectively.

To progress further, our next objective is to determine the bounds of the initial coefficients a_i where $i = 2, 3, 4, 5$. These bounds, as derived in [25], are summarized in the following remark.

Remark 2.5. For $f \in \mathcal{S}_e^*$, $|a_2| \leq 1$, $|a_3| \leq 3/4$, $|a_4| \leq 17/36$ and $|a_5| \leq 25/72$. Here the first three bounds are sharp.

Finding coefficient bounds for $n > 5$ becomes notably more challenging. In order to overcome this difficulty, we employ Lemma 2.3 to deduce the bounds for the sixth and seventh coefficients within the class of functions \mathcal{S}_e^* , as demonstrated in the subsequent lemma.

Lemma 2.6. *Let $f \in \mathcal{S}_e^*$. Then $|a_6| \leq 587/1800 \approx 0.326111$ and $|a_7| \leq 1397/4320 \approx 0.32338$.*

Proof. By suitably rearranging the terms given in equation (2.4), we have

$$57600a_6 = 220p_1^3p_2 - 480p_1^2p_3 - 480p_1p_2^2 + 720p_1p_4 - 17p_1^5 - 480p_2p_3 + 5760p_5.$$

Using triangle inequality, it can be viewed as

$$\begin{aligned}
57600|a_6| &\leq |p_1^2(220p_1p_2 - 480p_3)| + |p_1(720p_4 - 480p_2^2)| + |-17p_1^5| \\
&\quad + |5760p_5 - 480p_2p_3|.
\end{aligned} \tag{2.23}$$

Using Lemma 2.3, we arrive at the following inequality:

$$|a_6| \leq \frac{587}{1800} \approx 0.326111.$$

Similarly, considering equation (2.5), we have

$$\begin{aligned} 8294400a_7 &= 881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 - 56160p_1^2p_4 \\ &\quad + 69120p_1p_5 - 106560p_1p_2p_3 - 57600p_3^2 - 86400p_2p_4. \end{aligned}$$

Through the triangle inequality, it can also be seen as

$$\begin{aligned} 8294400|a_7| &\leq |p_1^4(881p_1^2 - 13260p_2)| + |p_2^2(48240p_1^2 - 14400p_2)| \\ &\quad + |p_1(69120p_5 - 106560p_2p_3)| + |p_1^2(29040p_1p_3 - 56160p_4)| \\ &\quad + |57600p_3^2| + |86400p_2p_4|. \end{aligned}$$

Lemma 2.3 implies that $|a_7| \leq 1397/4320 \approx 0.32338$. ■

Theorem 2.7. *Let $f \in \mathcal{S}_e^*$. Then*

$$|H_4(1)| \leq 0.29059.$$

The proof of the above theorem follows by substituting the values obtained from Theorem 2.2, Remark 2.4, Remark 2.5 and Lemma 2.6 in the equation (2.19), therefore, it is skipped here.

3. HANKEL DETERMINANTS FOR \mathcal{C}_e

3.1. Preliminaries. In this segment, we express the expressions of initial coefficients a_i ($i = 2, 3, \dots, 7$) involving Carathéodory coefficients. When $f \in \mathcal{C}_e$, we replace the L.H.S of equation (2.1) by $1 + zf''(z)/f'(z)$ and arrive at the following equation

$$1 + \frac{zf''(z)}{f'(z)} = e^{w(z)}.$$

Proceeding on the similar lines as done for the class \mathcal{S}_e^* , we obtain a_i ($i = 2, 3, \dots, 7$) in terms of p_j ($j = 1, 2, \dots, 5$), then compare the corresponding coefficients as follows:

$$a_2 = \frac{1}{4}p_1, \quad a_3 = \frac{1}{48}\left(p_1^2 + 4p_2\right), \quad a_4 = \frac{1}{1152}\left(-p_1^3 + 12p_1p_2 + 48p_3\right), \quad (3.1)$$

$$a_5 = \frac{1}{5760}\left(p_1^4 - 12p_1^2p_2 + 24p_1p_3 + 144p_4\right), \quad (3.2)$$

$$\begin{aligned} a_6 &= \frac{1}{345600}\left(-17p_1^5 + 220p_1^3p_2 - 480p_1p_2^2 - 480p_1^2p_3 - 480p_2p_3 + 720p_1p_4 \right. \\ &\quad \left. + 5760p_5\right), \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} a_7 &= \frac{1}{58060800}\left(881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 - 106560p_1p_2p_3 \right. \\ &\quad \left. - 57600p_3^2 - 56160p_1^2p_4 - 86400p_2p_4 + 69120p_1p_5\right). \end{aligned} \quad (3.4)$$

3.2. Sharp Third Hankel Determinant for \mathcal{C}_e . In this subsection, we establish the sharp bound of $H_3(1)$ for functions that belong to the class \mathcal{C}_e .

Theorem 3.1. *Let $f \in \mathcal{C}_e$. Then*

$$|H_3(1)| \leq \frac{1}{144}. \quad (3.5)$$

This bound is sharp.

Proof. We follow the same steps which were used to prove Theorem 2.2. The values of $a'_i s(i = 2, 3, 4, 5)$ from equations (3.1) and (3.2) are substituted into equation (1.5). Thus

$$H_3(1) = \frac{1}{6635520} \left(-173p^6 + 552p^4p_2 - 1872p^2p_2^2 - 3840p_2^3 + 2208p^3p_3 \right. \\ \left. + 8064pp_2p_3 - 11520p_3^2 - 6912p^2p_4 + 13824p_2p_4 \right).$$

Using (2.6)-(2.8) for simplification, we arrive at

$$H_3(1) = \frac{1}{6635520} \left(\alpha_1(p, \gamma) + \alpha_2(p, \gamma)\eta + \alpha_3(p, \gamma)\eta^2 + \psi(p, \gamma, \eta)\rho \right),$$

where $\gamma, \eta, \rho \in \mathbb{D}$,

$$\alpha_1(p, \gamma) := -5p^6 - 180\gamma^2p^2(4-p^2)^2 + 1536\gamma^3(4-p^2)^2 - 240\gamma^3p^2(4-p^2)^2 \\ + 144\gamma^4p^2(4-p^2)^2 + 12\gamma p^4(4-p^2) - 120p^4\gamma^2(4-p^2), \\ \alpha_2(p, \gamma) := (1-|\gamma|^2)(4-p^2)(240p^3 - 288p\gamma(4-p^2) - 576p\gamma^2(4-p^2)), \\ \alpha_3(p, \gamma) := (1-|\gamma|^2)(4-p^2)(-2880(4-p^2) - 576|\gamma|^2(4-p^2)), \\ \psi(p, \gamma, \eta) := 3456\gamma(1-|\gamma|^2)(4-p^2)^2(1-|\eta|^2).$$

Since $|\rho| \leq 1$, also for the simplicity of the calculations, assume $x = |\gamma|$ and $y = |\eta|$,

$$|H_3(1)| \leq \frac{1}{6635520} \left(|\alpha_1(p, \gamma)| + |\alpha_2(p, \gamma)|y + |\alpha_3(p, \gamma)|y^2 + |\psi(p, \gamma, \eta)| \right) \leq N(p, x, y),$$

where

$$N(p, x, y) = \frac{1}{6635520} \left(n_1(p, x) + n_2(p, x)y + n_3(p, x)y^2 + n_4(p, x)(1-y^2) \right), \quad (3.6)$$

with

$$n_1(p, x) := 5p^6 + 180x^2p^2(4-p^2)^2 + 1536x^3(4-p^2)^2 + 240x^3p^2(4-p^2)^2 \\ + 144x^4p^2(4-p^2)^2 + 12xp^4(4-p^2) + 120p^4x^2(4-p^2), \\ n_2(p, x) := (1-x^2)(4-p^2)(240p^3 + 288px(4-p^2) + 576px^2(4-p^2)), \\ n_3(p, x) := (1-x^2)(4-p^2)(2880(4-p^2) + 576x^2(4-p^2)), \\ n_4(p, x) := 3456x(1-x^2)(4-p^2)^2.$$

We must maximise $N(p, x, y)$ in the closed cuboid $V : [0, 2] \times [0, 1] \times [0, 1]$. By identifying the maximum values on the twelve edges, the interior of V , and the interiors of the six faces, we can prove this.

- (1) We start by taking into account, every interior point of V . Assume that $(p, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$. We partially differentiate equation (3.6) with respect to y to locate the points of maxima in the interior of V . We obtain

$$\frac{\partial N}{\partial y} = \frac{(1-x^2)(4-p^2)}{138240} \left(24px(1+2x) - p^3(-5+6x+12x^2) + 96(5-6x+x^2)y - 24p^2(5-6x+x^2)y \right).$$

Now $\frac{\partial N}{\partial y} = 0$ gives

$$y = y_1 := \frac{5p^3 + 6px(4-p^2)(1+2x)}{24(4-p^2)(6x-x^2-5)}.$$

Since y_1 must be a member of $(0, 1)$ for critical points to exist, this is only possible if

$$24(20 + (p-24)x + (4+2p-p^2)x^2) + p^3(5-6x-12x^2) < 24p^2(5-6x). \quad (3.7)$$

Now, we find the solutions satisfying the inequality (3.7) for the existence of critical points using the hit and trial method. If we assume p tends to 0 and 2, then no such $x \in (0, 1)$ exists satisfying equation (3.7). Similarly, if we take x tending to 0 and 1, then there does not exist any $p \in (0, 2)$ satisfying equation (3.7). Therefore, we conclude that the function N has no critical point in $(0, 2) \times (0, 1) \times (0, 1)$.

- (2) Now, we study the interior of each of the six faces of the cuboid V .

When $p = 0$, $N(p, x, y)$ becomes

$$c_1(x, y) := \frac{y^2(15 - 12x^2 - 3x^4) + 18x(1 - y^2) - 2x^3(5 - 9y^2)}{2160}, \quad x, y \in (0, 1). \quad (3.8)$$

Since

$$\frac{\partial c_1}{\partial y} = \frac{y(1-x)^2(x+1)(5-x)}{360} \neq 0, \quad x, y \in (0, 1),$$

we note that, in $(0, 1) \times (0, 1)$, c_1 does not have any critical point.

When $p = 2$, $N(p, x, y)$ settles into

$$N(2, x, y) := \frac{1}{20736}, \quad x, y \in (0, 1). \quad (3.9)$$

When $x = 0$, $N(p, x, y)$ turns into

$$c_2(p, y) := \frac{(p^3 + 96y - 24p^2y)^2}{1327104}, \quad p \in (0, 2) \quad \text{and} \quad y \in (0, 1). \quad (3.10)$$

We solve $\partial c_2 / \partial p = 0$ and $\partial c_2 / \partial y = 0$ to locate the points of maxima. On solving $\partial c_2 / \partial y = 0$, we obtain

$$y = -\frac{p^3}{24(4-p^2)} (= y_p).$$

Upon calculations, we observe that such y_p does not belong to $(0, 1)$. Consequently, no such critical point of c_2 exists in $(0, 2) \times (0, 1)$.

When $x = 1$, $N(p, x, y)$ becomes

$$N(p, 1, y) = c_3(p, y) := \frac{24576 - 3264p^2 - 2448p^4 + 437p^6}{6635520}, \quad p \in (0, 2). \quad (3.11)$$

And while computing $\partial c_3/\partial p = 0$, we notice that c_3 has no critical point in $(0, 2)$.
When $y = 0$, $N(p, x, y)$ reduces to

$$c_4(p, x) := \frac{1}{6635520} \left(6144x(9 - 5x^2) + 192p^2x(-144 + 15x + 100x^2 + 12x^3) \right. \\ \left. - 48p^4x(-73 + 20x + 80x^2 + 24x^3) \right. \\ \left. + p^6(5 - 12x + 60x^2 + 240x^3 + 144x^4) \right).$$

Calculations lead to,

$$\frac{\partial c_4}{\partial x} = \frac{1}{6635520} \left(-61440x^2 - 6144(-9 + 5x^2) + 192p^2x(15 + 200x + 36x^2) \right. \\ \left. - 48p^4x(20 + 160x + 72x^2) + 192p^2(-144 + 15x + 100x^2 \right. \\ \left. + 12x^3) - 48p^4(-73 + 20x + 80x^2 + 24x^3) + p^6(-12 \right. \\ \left. + 120x + 720x^2 + 576x^3) \right)$$

and

$$\frac{\partial c_4}{\partial p} = \frac{1}{6635520} \left(384px(-144 + 15x + 100x^2 + 12x^3) - 192p^3x(-73 + 20x \right. \\ \left. + 80x^2 + 24x^3) + 6p^5(5 - 12x + 60x^2 + 240x^3 + 144x^4) \right).$$

No solution exist for the system of equations, $\partial c_4/\partial x = 0$ and $\partial c_4/\partial p = 0$, according to a numerical calculation, in $(0, 2) \times (0, 1)$.

When $y = 1$, $N(p, x, y)$ reduces to

$$c_5(p, x) := \frac{1}{6635520} \left(5p^6 + (4 - p^2)(12p^4x + 120p^4x^2 + 180p^2(4 - p^2)x^2 \right. \\ \left. + 1536(4 - p^2)x^3 + 240p^2(4 - p^2)x^3 + 144p^2(4 - p^2)x^4 \right. \\ \left. + 3456(4 - p^2)x(1 - x^2) + 48(1 - x^2)(p^3(5 - 6x \right. \\ \left. - 12x^2) + 24px(1 + 2x)) \right).$$

The two equations $\partial c_5/\partial x = 0$ and $\partial c_5/\partial p = 0$ also do not assume any solution in $(0, 2) \times (0, 1)$.

- (3) Next, we check the maximum values of $N(p, x, y)$ obtained on the edges of the cuboid V . From equation (3.10), we have $N(p, 0, 0) = t_1(p) := p^6/1327104$. It is easy to observe that $t_1'(p) = 0$ for $p = 0$ in the interval $[0, 2]$. The maximum value of $t_1(p)$ is 0. Now the equation (3.10) reduces to $N(p, 0, 1) = t_2(p) := (96 - 24p^2 + p^3)^2/1327104$ at $y = 1$. Since, $t_2'(p) < 0$ in $[0, 2]$, hence $p = 0$ is the point of maxima. Thus

$$N(p, 0, 1) \leq \frac{1}{144}, \quad p \in [0, 2].$$

Through computations, equation (3.10) shows that $N(0, 0, y)$ attains its maxima at $y = 1$. Hence

$$N(0, 0, y) \leq \frac{1}{144}, \quad y \in [0, 1].$$

Since, the equation (3.11) is free from x , we have $N(p, 1, 1) = N(p, 1, 0) = t_3(p) := (24576 - 3264p^2 - 2448p^4 + 437p^6)/6635520$. Now, we observe that $t_3'(p) < 0$ in $[0, 2]$, consequently, $t_3(p)$ attains its maximum at $p = 0$. Hence

$$N(p, 1, 1) = N(p, 1, 0) \leq 0.0037037, \quad p \in [0, 2].$$

On substituting $p = 0$ in equation (3.11), we get, $N(0, 1, y) = 1/270$. The equation (3.9) does not contain any variable such as p , x and y . Therefore, the maxima of $N(p, x, y)$ on the edges is given by

$$N(2, 1, y) = N(2, 0, y) = N(2, x, 0) = N(2, x, 1) = \frac{1}{20736}, \quad x, y \in [0, 1].$$

Using equation (3.8), we obtain $N(0, x, 1) = t_4(x) := (15 - 12x^2 + 8x^3 - 3x^4)/2160$. Upon calculations, we see that t_4 is a decreasing function in $[0, 1]$ and its maximum value is achieved at $x = 0$. Hence

$$N(0, x, 1) \leq \frac{1}{144}, \quad x \in [0, 1].$$

On again using equation (3.8), we get $N(0, x, 0) = t_5(x) := x(9 - 5x^2)/1080$. On further calculations, we get $t_5'(x) = 0$ for $x = \beta_0 := \sqrt{3/5}$. Also, $t_5(x)$ increases in $[0, \beta_0]$ and decreases in $(\beta_0, 1]$. So, β_0 is the point of maxima. Thus

$$N(0, x, 0) \leq 0.00430331, \quad x \in [0, 1].$$

Because of all the cases discussed above, the inequality (3.5) holds. The function $f_2(z) \in \mathcal{C}_e$, defined as

$$f_2(z) = \int_0^z \left(\exp \left(\int_0^y \frac{e^{t^3} - 1}{t} dt \right) \right) dy = z + \frac{z^4}{12} + \frac{5z^7}{252} + \dots,$$

with $f_2(0) = f_2'(0) - 1 = 0$, plays the role of an extremal function for the bounds of $|H_3(1)|$ having values $a_3 = a_5 = 0$ and $a_4 = 1/12$. ■

3.3. Fourth Hankel Determinant for \mathcal{C}_e . In this part of the section, we derive the bounds of $H_4(1)$ including finding the bounds of sixth and seventh coefficients for functions in the class \mathcal{C}_e . By selecting $q = 4$ and $n = 1$ in the equation (1.4), the expression of $|H_4(1)|$ can be obtained for functions in the class \mathcal{C}_e , which is given as follows:

$$H_4(1) = a_7 H_3(1) - a_6 U_1 + a_5 U_2 - a_4 U_3. \quad (3.12)$$

Here

$$U_1 := a_6(a_3 - a_2^2) + a_3(a_2 a_5 - a_3 a_4) - a_4(a_5 - a_2 a_4), \quad (3.13)$$

$$U_2 := a_3(a_3 a_5 - a_4^2) - a_5(a_5 - a_2 a_4) + a_6(a_4 - a_2 a_3), \quad (3.14)$$

and

$$U_3 := a_4(a_3 a_5 - a_4^2) - a_5(a_2 a_5 - a_3 a_4) + a_6(a_4 - a_2 a_3). \quad (3.15)$$

We start by determining the bounds for U_1 , U_2 , and U_3 .

By substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (3.13) from equations (3.1)-(3.3), we obtain

$$\begin{aligned} 132710400U_1 &= 487p_1^7 - 6304p_1^5 p_2 + 11440p_1^3 p_2^2 - 24960p_1 p_2^3 + 5280p_1^4 p_3 \\ &\quad + 34560p_1 p_3^2 + 19200p_1^2 p_2 p_3 - 53760p_2^2 p_3 + 57600p_1 p_2 p_4 \\ &\quad - 138240p_3 p_4 + 184320p_2 p_5 - 92160p_1^2 p_5 + 8640p_1^3 p_4, \end{aligned}$$

can also be viewed as the following, due to the triangle inequality,

$$\begin{aligned} 132710400|U_1| &\leq |p_1^5(487p_1^2 - 6304p_2)| + |p_1p_2^2(11440p_1^2 - 24960p_2)| \\ &\quad + |p_1p_3(5280p_1^3 + 34560p_3)| + |p_2p_3(19200p_1^2 - 53760p_2)| \\ &\quad + |p_4(57600p_1p_2 - 138240p_3)| + |p_5(184320p_2 - 92160p_1^2)| \\ &\quad + |8640p_1^3p_4|. \end{aligned}$$

Using Lemma 2.3, we arrive at

$$|U_1| \leq \frac{4121}{345600} \approx 0.0119242.$$

We replace the values of a_i 's ($i = 2, 3, \dots, 6$) from equations (3.1)-(3.4) in equation (3.14) and proceed on the same lines to obtain the bound of U_2

$$\begin{aligned} 1592524800U_2 &= 463p_1^8 - 2732p_1^6p_2 - 23472p_1^4p_2^2 - 14400p_1^2p_2^3 + 14592p_1^5p_3 \\ &\quad - 108288p_1^2p_3^2 + 92928p_1^3p_2p_3 - 138240p_1p_2^2p_3 + 1105920p_3p_5 \\ &\quad - 25344p_1^4p_4 + 276480p_2^2p_4 - 995328p_4^2 + 373248p_1^2p_2p_4 \\ &\quad - 276480p_1p_2p_5 + 221184p_1p_3p_4 - 161280p_1^3p_5 - 322560p_2p_3^2, \end{aligned}$$

by implementing the triangle inequality,

$$\begin{aligned} 1592524800|U_2| &\leq |p_1^6(463p_1^2 - 2732p_2)| + |p_1^2p_2^2(-23472p_1^2 - 14400p_2)| \\ &\quad + |p_1^2p_3(14592p_1^3 - 108288p_3)| + |161280p_1^3p_5| \\ &\quad + |p_1^2p_4(373248p_2 - 25344p_1^2)| + |p_4(276480p_2^2 - 995328p_4)| \\ &\quad + |322560p_2p_3^2| + |p_1p_2p_3(92928p_1^2 - 138240p_2)| \\ &\quad + |221184p_1p_3p_4| + |p_5(1105920p_3 - 276480p_1p_2)|. \end{aligned}$$

By applying Lemma 2.3, we have

$$|U_2| \leq \frac{24947200 + 866304\sqrt{\frac{282}{61}}}{1592524800} \approx 0.0168348.$$

Again, substitute the values of a_i 's ($i = 2, 3, \dots, 6$) from equations (3.1)-(3.4) in (3.15) and proceed to calculate the bound of U_3 in the same manner.

$$\begin{aligned} 38220595200U_3 &= 11424p_1^8 - 128256p_1^6p_2 + 10812p_1^7p_2 - 503p_1^9 + 69120p_1^4p_2^2 \\ &\quad + 552960p_1^2p_2^3 - 42192p_1^5p_2^2 - 181440p_1^3p_2^3 + 206208p_1^4p_2p_3 \\ &\quad - 11664p_1^6p_3 + 1889280p_1^3p_2p_3 - 1658880p_1p_2^2p_3 - 2211840p_1^2p_3^2 \\ &\quad - 2211840p_2p_3^2 + 283392p_1^3p_3^2 - 967680p_1p_2p_3^2 + 3317760p_1p_3p_4 \\ &\quad - 483840p_1^4p_4 + 1271808p_1^3p_2p_4 - 117504p_1^5p_4 + 1658880p_1p_2^2p_4 \\ &\quad - 5971968p_1p_4^2 + 6635520p_2p_3p_4 - 331776p_1^2p_3p_4 + 26542080p_3p_5 \\ &\quad - 6635520p_1p_2p_5 + 244224p_1^5p_3 - 794880p_1^2p_2^2p_3 - 2764800p_3^3 \\ &\quad - 829440p_1^2p_2p_4 - 3870720p_1^3p_5, \end{aligned}$$

can be visualized as the following with the help of the triangle inequality,

$$\begin{aligned}
38220595200|U_3| \leq & |p_1^6(11424p_1^2 - 128256p_2)| + |p_1^7(10812p_2 - 503p_1^2)| \\
& + |p_1^2p_2^2(69120p_1^2 + 552960p_2)| + |p_1^3p_2^2(42192p_1^2 + 181440p_2)| \\
& + |p_1^4p_3(206208p_2 - 11664p_1^2)| + |p_1p_2p_3(1889280p_1^2 - 1658880p_2)| \\
& + |p_3^2(2211840p_1^2 + 2211840p_2)| + |p_1p_3^2(283392p_1^2 - 967680p_2)| \\
& + |p_1p_4(3317760p_3 - 483840p_1^3)| + |p_1^3p_4(1271808p_2 - 117504p_1^2)| \\
& + |p_1p_4(1658880p_2^2 - 5971968p_4)| + |p_3p_4(6635520p_2 - 331776p_1^2)| \\
& + |p_5(26542080p_3 - 6635520p_1p_2)| + |244224p_1^5p_3 - 794880p_1^2p_2^2p_3 \\
& - 2764800p_3^3 - 829440p_1^2p_2p_4 - 3870720p_1^3p_5|.
\end{aligned}$$

By applying Lemma 2.3, we get

$$|U_3| \leq \frac{560108544 + 106168320\sqrt{\frac{3}{41}}}{38220595200} \approx 0.015406.$$

Remark 3.2. The bounds of U_1 , U_2 and U_3 , based on the above calculations, are 0.0119242, 0.0168348, and 0.015406 respectively.

The bounds of a_i ($i = 2, 3, 4, 5$) for functions in the class \mathcal{C}_e are obtained in [25], presented below in the following remark:

Remark 3.3. For $f \in \mathcal{C}_e$, $|a_2| \leq 1/2$, $|a_3| \leq 1/4$, $|a_4| \leq 17/144$ and $|a_5| \leq 5/72$. The first three bounds are sharp.

Next, we calculate the bounds of the sixth and seventh coefficient of functions belonging to the class \mathcal{C}_e to establish our main result along the lines of Lemma 2.6.

Lemma 3.4. *Let $f \in \mathcal{C}_e$. Then $|a_6| \leq 587/10800 \approx 0.0543519$ and $|a_7| \leq 0.0343723$.*

Proof. A suitable rearrangement of the terms given in equation (3.3) provides us

$$345600a_6 = 5760p_5 - 480p_2p_3 + 720p_1p_4 - 480p_1p_2^2 - 17p_1^5 + 220p_1^3p_2 - 480p_1^2p_3.$$

Further, through the triangle inequality, it can be viewed as

$$\begin{aligned}
345600|a_6| \leq & |5760p_5 - 480p_2p_3| + |p_1(720p_4 - 480p_2^2)| + |17p_1^5| \\
& + |p_1^2(220p_1p_2 - 480p_3)|.
\end{aligned}$$

Using Lemma 2.3, we arrive at

$$|a_6| \leq \frac{587}{10800} \approx 0.0543519.$$

Similarly, considering equation (3.4), we have

$$\begin{aligned}
58060800a_7 = & 881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 106560p_1p_2p_3 + 29040p_1^3p_3 \\
& - 57600p_3^2 + 69120p_1p_5 - 56160p_1^2p_4 - 86400p_2p_4 - 14400p_2^3.
\end{aligned}$$

It can also be seen as with the aid of the triangle inequality,

$$\begin{aligned}
58060800|a_7| \leq & |p_1^4(881p_1^2 - 13260p_2)| + |p_1p_2(48240p_1p_2 - 106560p_3)| \\
& + |p_3(29040p_1^3 - 57600p_3)| + |p_1(69120p_5 - 56160p_1p_4)| \\
& + |p_2(86400p_4 + 14400p_2^2)|.
\end{aligned} \tag{3.16}$$

Lemma 2.3 takes us at

$$|a_7| \leq \frac{2014080 + 921600\sqrt{\frac{15}{119}}}{58060800} \approx 0.0403246.$$

■

We obtain the following result by omitting the proof as it directly follows from Theorem 3.1, Remark 3.2, Remark 3.3, Lemma 3.4 and equation (3.12).

Theorem 3.5. *Let $f \in \mathcal{C}_e$. Then*

$$|H_4(1)| \leq 0.00101775.$$

Acknowledgment. Neha is thankful to the Department of Applied Mathematics, Delhi Technological University, New Delhi-110042 for providing Research Fellowship.

REFERENCES

- [1] N. M. Alarifi, R. M. Ali and V. Ravichandran, On the second Hankel determinant for the k th-root transform of analytic functions, *Filomat* **31** (2017), no. 2, 227–245
- [2] S. Banga and S. S. Kumar, The sharp bounds of the second and third Hankel determinants for the class \mathcal{SL}^* , *Math. Slovaca* **70** (2020), 849–862.
- [3] A. W. Goodman, *Univalent functions*. Vol. I, Mariner Publishing Co., Inc., Tampa, FL, 1983
- [4] P. Goel and S. S. Kumar, Certain class of starlike functions associated with modified sigmoid function, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), no. 1, 957–991.
- [5] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* **23** (1970/71), 159–177
- [6] B. Kowalczyk and A. Lecko, The sharp bound of the third Hankel determinant for functions of bounded turning, *Bol. Soc. Mat. Mex. (3)* **27** (2021), no. 3, Paper No. 69, 13 pp.
- [7] B. Kowalczyk, A. Lecko, and D. K. Thomas, The sharp bound of the third Hankel determinant for starlike functions, *Forum Mathematicum*. De Gruyter, (2022)
- [8] D. V. Krishna and T. RamReddy, Second Hankel determinant for the class of Bazilevic functions, *Stud. Univ. Babeş-Bolyai Math.* **60** (2015), no. 3, 413–420
- [9] S. S. Kumar and G. Kamaljeet, A cardioid domain and starlike functions, *Anal. Math. Phys.* **11** (2021), no. 2, Paper No. 54, 34 pp.
- [10] V. Kumar, N. E. Cho, V. Ravichandran and H. M. Srivastava, Sharp coefficient bounds for starlike functions associated with the Bell numbers, *Math. Slovaca* **69** (2019), no. 5, 1053–1064
- [11] O. S. Kwon, A. Lecko and Y. J. Sim, On the fourth coefficient of functions in the Carathéodory class, *Comput. Methods Funct. Theory* **18** (2018), no. 2, 307–314
- [12] O. S. Kwon, A. Lecko and Y. J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, *Bull. Malays. Math. Sci. Soc.* **42** (2019), no. 2, 767–780
- [13] A. Lecko, Y. J. Sim and B. Śmiarowska, The sharp bound of the Hankel determinant of the third kind for starlike functions of order $1/2$, *Complex Anal. Oper. Theory* **13** (2019), no. 5, 2231–2238
- [14] R. J. Libera and E. J. Złotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* **85** (1982), no. 2, 225–230
- [15] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA
- [16] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, *Bull. Malays. Math. Sci. Soc.* **38** (2015), no. 1, 365–386
- [17] A. Naz, S. Nagpal and V. Ravichandran, Starlikeness associated with the exponential function, *Turkish J. Math.* **43** (2019), no. 3, 1353–1371
- [18] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. London Math. Soc.*, **41** (1966), 111–122
- [19] R. K. Raina and J. Sokół, Some properties related to a certain class of starlike functions, *C. R. Math. Acad. Sci. Paris*, **353** (2015), no. 11, 973–978.
- [20] V. Ravichandran and S. Verma, Bound for the fifth coefficient of certain starlike functions, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 6, 505–510

- [21] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* No. 19 (1996), 101–105
- [22] N. Verma, S. S. Kumar, A Conjecture on $H_3(1)$ for certain Starlike Functions, *Math. Slovaca.* **73** (2023), no. 5, 1–10
- [23] N. Verma and S. S. Kumar, Higher Order Differential Subordination for \mathcal{S}_e^* , arXiv e-prints, pp.arXiv:2306.11215(2023).
- [24] P. Zaprawa, Third Hankel determinants for subclasses of univalent functions, *Mediterr. J. Math.* **14** (2017), no. 1, Paper No. 19, 10 pp.
- [25] P. Zaprawa, Hankel determinants for univalent functions related to the exponential function, *Symm.* **11** (2019), no. 3, Paper No. 10, 1211 pp.

DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI-110042, INDIA
Email address: `spkumar@dce.ac.in`

DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI-110042, INDIA
Email address: `nehaverma1480@gmail.com`