ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC TENSORS

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ABSTRACT. In this article we characterize the range of the attenuated and non-attenuated X-ray transform of compactly supported symmetric tensor fields in the Euclidean plane. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

We consider here the problem of the range characterization of (non)-attenuated X-ray transform of a real valued symmetric m-tensors in a strictly convex bounded domain in the Euclidean plane. As the X-ray and Radon transform [38] for planar functions (0-tensors) differ merely by the way lines are parameterized, the $m = 0$ case is the classical Radon transform [38], for which the range characterization has been long established independently by Gelfand and Graev [13], Helgason [14], and Ludwig [22]. Models in the presence of attenuation have also been considered in the homogeneous case [21, 2], and in the non-homogeneous case in the breakthrough works [3, 32, 33], and subsequently [28, 6, 5, 17, 25]. The references here are by no means exhaustive.

The interest in the range characterization problem in the 0-tensors case stems out from their applications to data enhancement in medical imaging methods such as Single Photon Emission Computed Tomography or Positron Emission Computed Tomography [27, 12]. The X-ray transform of 1-tensors (Doppler transform [29, 46]) appears in the investigation of velocity distribution in a flow [7], in ultrasound tomography [47, 44], and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid [30, 31]. The X-ray transform of second order tensors arises as the linearization of the boundary rigidity problem [46]. The case of tensor fields of rank four describes the perturbation of travel times of compressional waves propagating in slightly anisotropic elastic media [46, Chapters 6,7]. Thus, due to the various applications the range characterization problem has been a continuing subject of research.

Unlike the scalar case, the X-ray transform of tensor fields has a non-zero kernel, and the null-space becomes larger as the order of the tensor field increases. For tensors of order $m \geq 1$, it is easy to check that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors, and it is possible to reconstruct uniquely (without additional information of moment ray transforms [46]) only the solenoidal part of a tensor field. The non-injectivity of the X-ray transform makes the range characterization problem even more interesting.

For the attenuating media in planar domains, interesting enough, the 1-tensor field can be recovered in the regions of positive absorption as shown in [18, 5, 48, 40], without using some additional data information [45, 9, 23]. It is due to a surprising fact that the two-dimensional attenuated Doppler transform with positive attenuation is injective while the non-attenuated Doppler transform is not.

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The systematic study of tensor tomography in non-Euclidean spaces originated in [46]. On simple Riemannian surfaces, the range characterization of the geodesic X-ray of compactly supported 0 and 1-tensors has been established in terms of the scattering relation in [37], and the results were extended in [4, 11, 20] to symmetric tensors of arbitrary order. Explicit inversion approaches in the Euclidean case have been proposed in [17, 10, 24]. In the attenuating media, tensor tomography was solved for the cases $m = 0, 1$ in [43]. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher can be found in [35], with a range characterization in [36, 25, 4].

The original characterization in [13, 14, 22] was extended to arbitrary symmetric m-tensors in [34]; see [10] for a partial survey on the tensor tomography in the Euclidean plane. The connection between the Euclidean version of the characterization in [37] and the characterization in [13, 14, 22] was established in [24]. Recently, in [41] the connection between the range characterization result in [39] and the original range characterization in [13, 14, 22] has been established.

In here we build on the results in [39, 40, 42], and extends them to symmetric tensor fields of any arbitrary order. In particular, the range characterization therein are given in terms of the Bukhgeim-Hilbert transform [39] (the Hilbert-like transform associated with A-analytic maps in the sense of Bukhgeim [8]). The characterization in here can be viewed as an explicit description of the scattering relation in [35, 36] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible m-tensors yielding identical X-ray data; see (43) and (69) for the non-attenuated case and (94) and (122) for the attenuated case.

This article is organized as follows: All the details establishing notations and basic properties of symmetric tensor fields needed here are in Section 2. In Section 3 we briefly recall existing results on A-analytic maps that are used in the proofs. In Section 4 and Section 5, we provide range characterization of symmetric tensor field $\mathbf{f}$ of even order, respectively, odd order in the non-attenuated case. In Section 6 and Section 7, we provide range characterization of symmetric tensor field $\mathbf{f}$ of even order, respectively, odd order in the attenuated case.

2. Preliminaries

Given an integer $m \geq 0$, let $\mathbf{T}^m(\mathbb{R}^2)$ denote the space of all real-valued covariant tensor fields of rank m:

$$\mathbf{f}(x^1, x^2) = f_{i_1 \cdots i_m}(x^1, x^2) dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_m}, \quad i_1, \cdots, i_m \in \{1, 2\}, \tag{1}$$

where $\otimes$ is the tensor product, $f_{i_1 \cdots i_m}$ are the components of tensor field $\mathbf{f}$ in the Cartesian basis (x^1, x^2), and where by repeating superscripts and subscripts in a monomial a summation from 1 to 2 is meant.

We denote by $\mathbf{S}^m(\mathbb{R}^2)$ the space of symmetric covariant tensor fields of rank m on $\mathbb{R}^2$. Let $\sigma : \mathbf{T}^m(\mathbb{R}^2) \to \mathbf{S}^m(\mathbb{R}^2)$ be the canonical projection (symmetrization) defined by $(\sigma \mathbf{f})_{i_1 \cdots i_m} = \dfrac{1}{m!} \displaystyle\sum_{\pi \in \Pi_m} f_{i_{\pi(1)} \cdots i_{\pi(m)}}$, where the summation is over the group Π_m of all permutations of the set $\{1, \cdots, m\}$.

A planar covariant symmetric tensor field of rank m has $m + 1$ independent component, which we denote by

$$\tilde{f}_k := f_{\underbrace{1 \cdots 1}_{m-k} \underbrace{2 \cdots 2}_{k}}, \quad (k = 0, \cdots, m), \tag{2}$$

in connection with this, a symmetric tensor $\mathbf{f} = (f_{i_1\cdots i_m},\ i_1,\cdots,i_m = 1,2)$ of rank m will be given by a pseudovector of size $m+1$

$$\mathbf{f} = (\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{m-1}, \tilde{f}_m).$$

We identify the plane $\mathbb{R}^2$ by the complex plane $\mathbb{C}$, $z^1 \equiv z = x^1 + \imath x^2, z^2 \equiv \bar{z} = x^1 - \imath x^2$. We consider the Cauchy-Riemann operators

$$\frac{\partial}{\partial z^1} \equiv \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x^1} - \imath\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial z^2} \equiv \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x^1} + \imath\frac{\partial}{\partial x^2}\right), \tag{3}$$

and the inverse relation by $\dfrac{\partial}{\partial x^1} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}, \quad \dfrac{\partial}{\partial x^2} = \imath\dfrac{\partial}{\partial z} - \imath\dfrac{\partial}{\partial \bar{z}}$.

Let $\mathbf{f} = (f_{i_1\cdots i_m}(x^1,x^2),\ i_1,\cdots,i_m = 1,2)$ be real valued symmetric m-tensor field in Cartesian coordinates (x^1,x^2), then in complex coordinates (z^1,z^2) it will have new components $(F_{i_1\cdots i_m}(z,\bar{z}))$, which are formally expressed by the covariant tensor law:

$$\begin{aligned} F_{i_1\cdots i_m}(z,\bar{z}) &= \frac{\partial x^{s_1}}{\partial z^{i_1}}\cdots\frac{\partial x^{s_m}}{\partial z^{i_m}} f_{s_1\cdots s_m}(x^1,x^2), \quad \text{and} \\ f_{i_1\cdots i_m}(x^1,x^2) &= \frac{\partial z^{s_1}}{\partial x^{i_1}}\cdots\frac{\partial z^{s_m}}{\partial x^{i_m}} F_{s_1\cdots s_m}(z,\bar{z}), \end{aligned} \tag{4}$$

where the Jacobian matrix has the form

$$J := \begin{pmatrix} \frac{\partial x^1}{\partial z^1} & \frac{\partial x^1}{\partial z^2} \\ \frac{\partial x^2}{\partial z^1} & \frac{\partial x^2}{\partial z^2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\imath & \imath \end{pmatrix}, \quad \text{and} \quad J^{-1} = \begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \frac{\partial z^1}{\partial x^2} \\ \frac{\partial z^2}{\partial x^1} & \frac{\partial z^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 1 & \imath \\ 1 & -\imath \end{pmatrix}.$$

Adopting the notation in [17], we shall write the transformations (4) as

$$\begin{aligned} \mathbf{f} = \{f_{i_1\cdots i_m}(x^1,x^2)\} &\quad\rightarrowtail\quad \mathbf{F} = \{F_{i_1\cdots i_m}(z,\bar{z})\}, \quad \text{and} \\ \mathbf{F} = \{F_{i_1\cdots i_m}(z,\bar{z})\} &\quad\rightarrowtail\quad \mathbf{f} = \{f_{i_1\cdots i_m}(x^1,x^2)\}. \end{aligned} \tag{5}$$

A symmetric tensor $\mathbf{F}$ of rank m, obtained from the real symmetric tensor $\mathbf{f}$ by passing to complex variables, we also define a pseudovector $(F_0, F_1, \cdots, F_{m-1}, F_m)$ with components

$$F_k = F_{\underbrace{1\cdots1}_{m-k}\underbrace{2\cdots2}_{k}}, \quad k = 0,\cdots,m, \tag{6}$$

and subject to the conditions

$$F_k = \overline{F}_{m-k}, \quad k = 0,\cdots,m. \tag{7}$$

Taking into account the tensor law (4), we obtain formulas relating the components of pseudovectors in (2) and pseudovectors in (6):

$$F_k = \frac{(-\imath)^{m-k}}{2^m}\sum_{q=0}^{m-k}\sum_{p=0}^{k}\binom{m-k}{q}\binom{k}{p}\imath^{k-p+q}\tilde{f}_{p+q}, \quad k = 0,1,\cdots,m, \tag{8}$$

$$\tilde{f}_k = \imath^k\sum_{q=0}^{m-k}\sum_{p=0}^{k}\binom{m-k}{q}\binom{k}{p}(-1)^{k-p}F_{p+q}, \quad k = 0,1,\cdots,m. \tag{9}$$

In Cartesian coordinates covariant and contravariant components are the same, and thus contravariant components of the tensor field $\mathbf{f}$ coincide with its corresponding covariant components, $f_{i_1\cdots i_m} = f^{i_1\cdots i_m}$. The dot product on $\mathbf{S}^m(\mathbb{R}^2)$ induced by the Euclidean metric is defined by

$$\langle \mathbf{f}, \mathbf{h}\rangle := f_{i_1\cdots i_m}h^{i_1\cdots i_m}. \tag{10}$$

Note that if $\mathbf{f}_1 \rightarrowtail \mathbf{F}_1$ and $\mathbf{f}_2 \rightarrowtail \mathbf{F}_2$, then the pointwise inner product of tensors is invariant:

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \langle \mathbf{F}_1, \mathbf{F}_2 \rangle. \tag{11}$$

For $\boldsymbol{\theta} = (\theta^1, \theta^2) = (\cos\theta, \sin\theta) \in \mathbb{S}^1$, we denote by $\boldsymbol{\theta}^m$ the tensor product $\boldsymbol{\theta}^m := \underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m}$ and $\boldsymbol{\theta}^m$ will be an m-contravariant tensor in Cartesian coordinates. According to the tensor law for contravariant components its representation in complex coordinates will look like

$$\boldsymbol{\theta} \rightarrowtail \Theta, \qquad \Theta^k = \frac{\partial z^k}{\partial x^s}\theta^s, \qquad \Theta = (\Theta^1, \Theta^2) = (e^{\imath\theta}, e^{-\imath\theta}),$$

and $\Theta^m := \underbrace{\Theta \otimes \Theta \otimes \cdots \otimes \Theta}_{m}$ be an m-contravariant tensor, and we also have $\boldsymbol{\theta}^m \rightarrowtail \Theta^m$. Using (11), we get

$$\begin{aligned} \langle \mathbf{f}, \boldsymbol{\theta}^m \rangle = \langle \mathbf{F}, \Theta^m \rangle &= \sum_{k=0}^{m} \binom{m}{k} F_k \, e^{\imath\theta(m-k)} e^{-\imath\theta k} = \sum_{k=0}^{m} \binom{m}{k} F_k e^{\imath(m-2k)\theta} \\ &= \begin{cases} \displaystyle\sum_{k=0}^{q} f_{-2k} e^{\imath(2k)\theta} + \sum_{k=1}^{q} f_{2k} e^{-\imath(2k)\theta}, & (\text{if } m = 2q,\ q \geq 0), \\ \displaystyle\sum_{k=0}^{q} f_{-(2k+1)} e^{\imath(2k+1)\theta} + f_{2k+1} e^{-\imath(2k+1)\theta}, & (\text{if } m = 2q+1,\ q \geq 0), \end{cases} \end{aligned} \tag{12}$$

where

$$f_{-2k} = \binom{2q}{q-k} F_{q-k}, \qquad 0 \leq k \leq q,\ q \geq 0, \quad \left(q = \frac{m}{2}, m \text{ even}\right), \tag{13}$$

$$f_{-(2k+1)} = \binom{2q+1}{q-k} F_{q-k}, \qquad 0 \leq k \leq q,\ q \geq 0, \quad \left(q = \frac{m-1}{2}, m \text{ odd}\right), \tag{14}$$

and $f_n = \overline{f_{-n}}$ and $F_n = \overline{F}_{m-n}$, for $0 \leq n \leq m$.

Let $\mathbf{f}$ be a real valued symmetric m-tensor, with integrable components of compact support in $\mathbb{R}^2$, and $a \in L^1(\mathbb{R}^2)$ a real valued function. The attenuated X-ray transform of $\mathbf{f}$ is given by

$$X_a\mathbf{f}(x, \boldsymbol{\theta}) := \int_{-\infty}^{\infty} \langle \mathbf{f}(x + t\boldsymbol{\theta}), \boldsymbol{\theta}^m \rangle \exp\left\{-\int_t^{\infty} a(x + s\boldsymbol{\theta}) ds\right\} dt, \tag{15}$$

where $x \in \mathbb{R}^2$, $\boldsymbol{\theta} \in \mathbb{S}^1$, and $\langle \cdot, \cdot \rangle$ is the inner product in (10). For the non attenuated case $(a \equiv 0)$, we use the notation $X\mathbf{f}$.

In here, we consider the tensor field $\mathbf{f}$ be defined on a strongly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary Γ; further regularity and the order of vanishing will be specified in the theorems. In the statements below we use the notations in [46]:

$$C^{\mu}(\mathbf{S}^m; \Omega) = \{\mathbf{f} = (f_{i_1 \cdots i_m}) \in \mathbf{S}^m(\Omega) : f_{i_1 \cdots i_m} \in C^{\mu}(\Omega)\}$$

$0 < \mu < 1$, for the space of real valued, symmetric tensor fields of order m with locally Hölder continuous components. Similarly, $L^1(\mathbf{S}^m; \Omega)$ denotes the tensor fields of order m with integrable components.

For any $(x, \boldsymbol{\theta}) \in \overline{\Omega} \times \mathbb{S}^1$, let $\tau(x, \boldsymbol{\theta})$ be length of the chord passing through x in the direction of $\boldsymbol{\theta}$. Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$\Gamma_{\pm} := \{(x,\boldsymbol{\theta}) \in \Gamma \times \mathbb{S}^1 : \pm\boldsymbol{\theta}\cdot\nu(x) > 0\}, \tag{16}$$

and the variety

$$\Gamma_0 := \{(x,\boldsymbol{\theta}) \in \Gamma \times \mathbb{S}^1 : \boldsymbol{\theta}\cdot\nu(x) = 0\}, \tag{17}$$

where $\nu(x)$ denotes outer normal.

The a-attenuated X-ray transform of $\mathbf{f}$ is realized as a function on $\varGamma_+$ by

$$X_a\mathbf{f}(x,\boldsymbol{\theta}) = \int_{-\tau(x,\boldsymbol{\theta})}^{0} \langle \mathbf{f}(x+t\boldsymbol{\theta}), \boldsymbol{\theta}^m\rangle \, e^{-\int_t^0 a(x+s\boldsymbol{\theta})ds}\, dt,\ (x,\boldsymbol{\theta}) \in \varGamma_+. \tag{18}$$

We approach the range characterization via the well-known connection with the transport model as follows: The boundary value problem

$$\boldsymbol{\theta}\cdot\nabla u(x,\boldsymbol{\theta}) + a(x)u(x,\boldsymbol{\theta}) = \langle \mathbf{f}(x), \boldsymbol{\theta}^m\rangle, \quad (x,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1, \tag{19a}$$

$$u|_{\varGamma_-} = 0, \tag{19b}$$

has a unique solution in $\Omega\times\mathbb{S}^1$ and

$$u|_{\varGamma_+}(x,\boldsymbol{\theta}) = X_a\mathbf{f}(x,\boldsymbol{\theta}), \quad (x,\boldsymbol{\theta}) \in \varGamma_+. \tag{20}$$

The range characterization is given in terms of the trace

$$g := u|_{\varGamma\times\mathbb{S}^1} = \begin{cases} X_a\mathbf{f}, & \text{on } \varGamma_+, \\ 0, & \text{on } \varGamma_-\cup\varGamma_0. \end{cases} \tag{21}$$

We note that from (12), the expression $\langle \mathbf{f}, \boldsymbol{\theta}^m\rangle$ in the transport equation (19a) is represented in the Fourier decomposition in $\boldsymbol{\theta}$ as in terms of the following Fourier modes:

$$\langle \mathbf{f}, \boldsymbol{\theta}^m\rangle = \begin{cases} f_0 + f_{\pm 2}e^{\mp 2\imath\theta} + f_{\pm 4}e^{\mp 4\imath\theta} + \cdots + f_{\pm m}e^{\mp m\imath\theta} & (m \text{ even}), \\ f_{\pm 1}e^{\mp\imath\theta} + f_{\pm 3}e^{\mp 3\imath\theta} + \cdots + f_{\pm m}e^{\mp m\imath\theta} & (m \text{ odd}). \end{cases}$$

3. Ingredients from A-analytic theory

In this section we briefly introduce the properties of A-analytic maps needed later.
For $0<\mu<1$, $p=1,2$, we consider the Banach spaces:

$$\begin{aligned} l^{1,p}_{\infty}(\varGamma) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle \ : \|\mathbf{g}\|_{l^{1,p}_{\infty}(\varGamma)} := \sup_{\xi\in\varGamma}\sum_{j=0}^{\infty}\langle j\rangle^p |g_{-j}(\xi)| < \infty \right\}, \\ C^{\mu}(\varGamma; l_1) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle : \sup_{\xi\in\varGamma}\|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta\in\varGamma \\ \xi\neq\eta}} \frac{\|\mathbf{g}(\xi)-\mathbf{g}(\eta)\|_{l_1}}{|\xi-\eta|^{\mu}} < \infty \right\}, \\ Y_{\mu}(\varGamma) &:= \left\{ \mathbf{g} : \mathbf{g}\in l^{1,2}_{\infty}(\varGamma) \text{ and } \sup_{\substack{\xi,\eta\in\varGamma \\ \xi\neq\eta}} \sum_{j=0}^{\infty}\langle j\rangle \frac{|g_{-j}(\xi)-g_{-j}(\eta)|}{|\xi-\eta|^{\mu}} < \infty \right\}, \end{aligned} \tag{22}$$

where $l_{\infty}(,l_1)$ is the space of bounded (, respectively summable) sequences, and for brevity, we use the notation $\langle j\rangle = (1+|j|^2)^{1/2}$. Similarly, we consider $C^{\mu}(\overline{\Omega}; l_1)$, and $C^{\mu}(\overline{\Omega}; l_{\infty})$.

A sequence valued map $\Omega \ni z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), ...\rangle$ in $C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ is called L^k*-analytic* (in the sense of Bukhgeim), $k = 1, 2$, if

$$\overline{\partial}\mathbf{v}(z) + L^k \partial \mathbf{v}(z) = 0, \quad z \in \Omega, \tag{23}$$

where L is the left shift operator $L\langle v_0, v_{-1}, v_{-2}, \cdots\rangle = \langle v_{-1}, v_{-2}, \cdots\rangle$, and $L^2 = L \circ L$.

Bukhgeim's original theory in [8] shows that solutions of (23), satisfy a Cauchy-like integral formula,

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_\Gamma](z), \quad z \in \Omega, \tag{24}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{v}|_\Gamma$. We use the formula in [12], where $\mathcal{B}$ is defined component-wise for $n \geq 0$ by

$$(\mathcal{B}\mathbf{g})_{-n}(z) := \frac{1}{2\pi\imath}\int_\Gamma \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\imath}\int_\Gamma \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j, \ z \in \Omega. \tag{25}$$

The following regularity result in [39, Proposition 4.1] is needed.

Proposition 3.1. [39, Proposition 4.1] *Let $\mu > 1/2$ and $\mathbf{g} = \langle g_0, g_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of non-positive Fourier modes of g.*
(i) If $g \in C^\mu(\Gamma; C^{1,\mu}(\mathbb{S}^1))$, then $\mathbf{g} \in l^{1,1}_\infty(\Gamma) \cap C^\mu(\Gamma; l_1)$.
(ii) If $g \in C^\mu(\Gamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$, then $\mathbf{g} \in Y_\mu(\Gamma)$.

Similar to the analytic maps, the traces of L-analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [39]. More precisely, the Bukhgeim-Hilbert transform $\mathcal{H}$ acting on $\mathbf{g}$,

$$\Gamma \ni z \mapsto (\mathcal{H}\mathbf{g})(z) = \langle (\mathcal{H}\mathbf{g})_0(z), (\mathcal{H}\mathbf{g})_{-1}(z), (\mathcal{H}\mathbf{g})_{-2}(z), ...\rangle \tag{26}$$

is defined component-wise for $n \geq 0$ by

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi}\int_\Gamma \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi}\int_\Gamma \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j, \ z \in \Gamma, \tag{27}$$

and we refer to [39] for its mapping properties.

Note that the Bukhgeim-Cauchy integral formula in (25) above is restated in terms of L-analytic maps as opposed to L^2-analytic as in [39]. The only change is the index relabeling. In particular, the index g_{-n-j} will change to g_{-n-2j} therein to account for L^2-analytic. Moreover, the same index relabelling in the Bukhgeim-Hilbert transform formula (27) is made to account for the difference between L-analytic and L^2-analytic.

The following result recalls the necessary and sufficient conditions for a sufficiently regular map to be the boundary value of an L^k-analytic function, $k = 1, 2$.

Theorem 3.1. *Let $0 < \mu < 1$, and $k = 1, 2$. Let $\mathcal{B}$ be the Bukhgeim-Cauchy operator in* (25)*.*
Let $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle \in Y_\mu(\Gamma)$ for $\mu > 1/2$ be defined on the boundary Γ, and let $\mathcal{H}$ be the Bukhgeim-Hilbert transform acting on $\mathbf{g}$ as in (27)*.*
(i) If $\mathbf{g}$ is the boundary value of an L^k-analytic function, then $\mathcal{H}\mathbf{g} \in C^\mu(\Gamma; l_1)$ and satisfies

$$(I + \imath\mathcal{H})\mathbf{g} = \mathbf{0}. \tag{28}$$

(ii) If $\mathbf{g}$ *satisfies* (28)*, then there exists an* L^k*-analytic function* $\mathbf{v} := \mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$*, such that*

$$\mathbf{v}|_{\varGamma} = \mathbf{g}. \tag{29}$$

For the proof of Theorem 3.1 we refer to [39, Theorem 3.2, Corollary 4.1, and Proposition 4.2] and [40, Proposition 2.3].

Another ingredient, in addition to L^2-analytic maps, consists in the one-to-one relation between solutions $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ...\rangle$ satisfying

$$\overline{\partial} u_{-n}(z) + \partial u_{-n-2}(z) + a(z) u_{-n-1}(z) = 0, \quad z \in \Omega,\ n \geq 0, \tag{30}$$

and the L^2-analytic map $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ satisfying

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad z \in \Omega,\ n \geq 0; \tag{31}$$

via a special function h, see [42, Lemma 4.2] for details. The function h is defined as

$$h(z, \boldsymbol{\theta}) := Da(z, \boldsymbol{\theta}) - \frac{1}{2}\left(I - \imath H\right) Ra(z \cdot \boldsymbol{\theta}^{\perp}, \boldsymbol{\theta}^{\perp}), \tag{32}$$

where $\boldsymbol{\theta}^{\perp}$ is the counter-clockwise rotation of $\boldsymbol{\theta}$ by $\pi/2$, $Ra(s, \boldsymbol{\theta}^{\perp}) = \displaystyle\int_{-\infty}^{\infty} a\left(s\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta}\right) dt$ is the Radon transform in $\mathbb{R}^2$ of the attenuation a, $Da(z, \boldsymbol{\theta}) = \displaystyle\int_0^{\infty} a(z + t\boldsymbol{\theta}) dt$ is the divergent beam transform of the attenuation a, and $Hh(s) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(t)}{s-t} dt$ is the classical Hilbert transform [26], taken in the first variable and evaluated at $s = z \cdot \boldsymbol{\theta}^{\perp}$. The function h appeared first in [27] and enjoys the crucial property of having vanishing negative Fourier modes yielding the expansions

$$e^{-h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \alpha_k(z) e^{\imath k\theta}, \quad e^{h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \beta_k(z) e^{\imath k\theta}, \quad (z, \boldsymbol{\theta}) \in \overline{\Omega} \times \mathbb{S}^1. \tag{33}$$

Using the Fourier coefficients of $e^{\pm h}$, define the integrating operators $e^{\pm G}\mathbf{u}$ component-wise for each $n \leq 0$, by

$$(e^{-G}\mathbf{u})_n = (\boldsymbol{\alpha} * \mathbf{u})_n = \sum_{k=0}^{\infty} \alpha_k u_{n-k}, \quad \text{and} \quad (e^{G}\mathbf{u})_n = (\boldsymbol{\beta} * \mathbf{u})_n = \sum_{k=0}^{\infty} \beta_k u_{n-k}, \tag{34}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is given by

$$\overline{\Omega} \ni z \mapsto \boldsymbol{\alpha}(z) := \langle \alpha_0(z), \alpha_1(z), \alpha_2(z), ..., \rangle, \quad \overline{\Omega} \ni z \mapsto \boldsymbol{\beta}(z) := \langle \beta_0(z), \beta_1(z), \beta_2(z), ..., \rangle.$$

Note that $e^{\pm G}$ can also be written in terms of left translation operator as

$$e^{-G}\mathbf{u} = \sum_{k=0}^{\infty} \alpha_k L^k \mathbf{u}, \quad \text{and} \quad e^{G}\mathbf{u} = \sum_{k=0}^{\infty} \beta_k L^k \mathbf{u}, \tag{35}$$

where L^k is the k-th composition of left translation operator. It is important to note that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$. We refer [42, Lemma 4.1] for the properties of h, and we restate the following result [39, Proposition 5.2] to incorporate the operators $e^{\pm G}$ notation used in here.

Proposition 3.2. [39, Proposition 5.2] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$. Then $\boldsymbol{\alpha}, \partial\boldsymbol{\alpha}, \boldsymbol{\beta}, \partial\boldsymbol{\beta} \in l^{1,1}_{\infty}(\overline{\Omega})$, and the operators*

$$
\begin{aligned}
&(i)\, e^{\pm G} : C^{\mu}(\overline{\Omega}; l_{\infty}) \to C^{\mu}(\overline{\Omega}; l_{\infty});\\
&(ii)\, e^{\pm G} : C^{\mu}(\overline{\Omega}; l_{1}) \to C^{\mu}(\overline{\Omega}; l_{1});\\
&(iii)\, e^{\pm G} : Y_{\mu}(\varGamma) \to Y_{\mu}(\varGamma).
\end{aligned} \tag{36}
$$

Lemma 3.1. [40, Lemma 4.2] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$, and $e^{\pm G}$ be operators as defined in* (34).

(i) If $\mathbf{u} \in C^{1}(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = 0$, then $\mathbf{v} = e^{-G}\mathbf{u} \in C^{1}(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = 0$.

(ii) Conversely, if $\mathbf{v} \in C^{1}(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = 0$, then $\mathbf{u} = e^{G}\mathbf{v} \in C^{1}(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = 0$.

4. Even order m-tensor - non-attenuated case

We establish necessary and sufficient conditions for a sufficiently smooth function on $\varGamma \times \mathbb{S}^1$ to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of even order $m = 2q,\ q \geq 0$. In this non-attenuated case, the transport equation (19a) becomes

$$
\boldsymbol{\theta} \cdot \nabla u(x, \boldsymbol{\theta}) = \sum_{k=-q}^{q} f_{2k}(x) e^{-\imath(2k)\theta}, \quad x \in \Omega, \tag{37}
$$

where f_{2k} defined in (13), and $f_{2k} = \overline{f_{-2k}}$, for $-q \leq k \leq q$, $q \geq 0$. Note that f_0 is real-valued while other modes are complex conjugates.

For $z = x_1 + \imath x_2 \in \Omega$, the advection operator $\boldsymbol{\theta} \cdot \nabla$ in complex notation becomes $e^{-\imath\theta}\overline{\partial} + e^{\imath\theta}\partial$, where $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$, and $\overline{\partial}, \partial$ are the Cauchy-Riemann operators in (3).

If $\sum_{n\in\mathbb{Z}} u_n(z) e^{\imath n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (37), then, provided some sufficient decay (to be specified later) of u_n to allow regrouping, the equation (37) reduces to the system:

$$
\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = f_{2n}(z), \qquad 0 \leq n \leq q,\ q \geq 0, \tag{38}
$$

$$
\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq q+1,\ q \geq 0, \tag{39}
$$

$$
\overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \qquad n \geq 0. \tag{40}
$$

Recall that the trace $u|_{\varGamma\times\mathbb{S}^1} := g$ as in (21), with $g = X\mathbf{f}$ on $\varGamma_+$ and $g = 0$ on $\varGamma_- \cup \varGamma_0$.

The range characterization is given in terms of the Fourier modes of g in the angular variables:

$$
g(\zeta, \boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{\imath n\theta}, \quad \zeta \in \varGamma.
$$

Since the trace g is also real valued, its Fourier modes will satisfy $g_{-n} = \overline{g}_n$, for $n \geq 0$.
From the non-positive Fourier modes, we built the sequences

$$
\mathbf{g}^{\text{even}} := \langle g_0, g_{-2}, g_{-4}, \ldots \rangle, \quad \text{and} \quad \mathbf{g}^{\text{odd}} := \langle g_{-1}, g_{-3}, g_{-5}, \ldots \rangle. \tag{41}
$$

From the negative odd modes starting from mode $(2q+1)$, we built the sequence

$$
L^q \mathbf{g}^{\text{odd}} := \langle g_{-(2q+1)}, g_{-(2q+3)}, g_{-(2q+5)}, \ldots \rangle, \quad q \geq 0, \tag{42}
$$

where L^q is the q-th composition of left translation operator.

We characterize next the non-attenuated X-ray data g in terms of the Bukhgeim-Hilbert Transform $\mathcal{H}$ in (27). We will construct the solution u of the transport equation (37), whose trace matches the boundary data g, and also construct the right hand side of the (37). The construction of solution u is in terms of its Fourier modes in the angular variable. We first construct the non-positive Fourier modes and then the positive Fourier modes are constructed by conjugation. For even $m = 2q$, $q \geq 1$, apart from q many Fourier modes $u_{-1}, u_{-3}, \cdots u_{-(2q-1)}$, all non-positive Fourier modes are defined by Bukhgeim-Cauchy integral formula (25) using boundary data. Other than having the traces $u_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1$, on the boundary, the q many Fourier modes $u_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1$, are unconstrained. They are chosen arbitrarily from the class Ψ_g^{even} of functions of cardinality $q = \frac{m}{2}$ with prescribed trace on the boundary $\varGamma$ defined as

$$
\begin{aligned}
\Psi_g^{\text{even}} := \Big\{ &\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \left(C^{1,\mu}(\overline{\Omega};\mathbb{C})\right)^q, 2\mu > 1 : \\
&\psi_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1 \Big\}. \qquad (43)
\end{aligned}
$$

Remark 4.1. *In the 0-tensor case ($m = 0$), there is no class, and the characterization of the X-ray data g is in terms of the Fourier modes* $\mathbf{g}$.

Theorem 4.1 (Range characterization for even order tensors)**.** *(i) Let $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$, $\mu > 1/2$, be a real-valued symmetric tensor field of even order $m = 2q$, $q \geq 0$, and*

$$g = X\mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_- \cup \varGamma_0.$$

Then $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$ satisfy

$$[I + \imath\mathcal{H}]\mathbf{g}^{\text{even}} = \mathbf{0}, \qquad (44)$$

$$[I + \imath\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}^{\text{odd}} = \mathbf{0}, \qquad (45)$$

where $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}}$ are sequences in (41)*, and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in* (27)*.*

(ii) Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. For $q = 0$, if the corresponding sequences $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$ satisfies (44) *and* (45)*, then there is a unique real valued symmetric 0-tensor $\mathbf{f}$ such that $g|_{\varGamma_+} = X\mathbf{f}$. Moreover, for $q \geq 1$, if $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$ satisfies* (44) *and* (45)*, and for each element $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$, then there is a unique real valued symmetric m-tensors $\mathbf{f}_{\Psi} \in C^{\mu}(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X\mathbf{f}_{\Psi}$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$ are compactly supported inside Ω, then for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\mu}(\varGamma \times \mathbb{S}^1)$. Moreover, g is the trace on $\varGamma \times \mathbb{S}^1$ of a solution $u \in C^{1,\mu}(\overline{\Omega} \times \mathbb{S}^1)$ of the transport equation (37). By [39, Proposition 4.1] $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$.

If u solves (37) then its Fourier modes satisfy (38), (39), and (40). Since the negative even Fourier modes u_{2n} for $n \leq 0$, satisfies the system (40), then the sequence valued map

$$\Omega \ni z \mapsto \mathbf{u}^{\text{even}}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \cdots \rangle$$

is L-analytic in Ω and the necessity part in Theorem 3.1 yields the condition (44).

The equation (39) for negative odd Fourier modes starting from negative $2q + 1$ mode, yield that the sequence valued map

$$z \mapsto \langle u_{-(2q+1)}, u_{-(2q+3)}, u_{-(2q+5)}, \ldots \rangle$$

is L-analytic in Ω and the necessity part in Theorem 3.1 gives the condition (45).

(ii) **Sufficiency:** Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. Since g is real valued, its Fourier modes in the angular variable occurs in conjugates

$$g_{-n}(\zeta) = \overline{g}_n(\zeta), \quad \text{for } n \geq 0, \ \zeta \in \varGamma. \tag{46}$$

Let the corresponding sequences $\mathbf{g}^{\text{even}}$ satisfying (44) and $\mathbf{g}^{\text{odd}}$ satisfying (45). By Proposition (3.1), $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$.

Let $m = 2q$, $q \geq 0$, be an even integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\varGamma \times \mathbb{S}^1} = g$ and u solves (37) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of even modes u_{2n} for $n \in \mathbb{Z}$.

Apply the Bukhgeim-Cauchy Integral operator (25) to construct the negative even Fourier modes:

$$\langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), ...\rangle := \mathcal{B}\mathbf{g}^{\text{even}}(z), \quad z \in \Omega. \tag{47}$$

By Theorem 3.1, the sequence valued map

$$z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), ...\rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1),$$

is L-analytic in Ω, thus the equations

$$\overline{\partial} u_{-2n} + \partial u_{-2n-2} = 0, \tag{48}$$

are satisfied for all $n \geq 0$. Moreover, the hypothesis (44) and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\varGamma$ and $u_{-2n}|_{\varGamma} = g_{-2n}$, for all $n \geq 0$.

Construct the positive even Fourier modes by conjugation: $u_{2n} := \overline{u_{-2n}}$, for all $n \geq 1$.

By conjugating (48) we note that the positive even Fourier modes also satisfy

$$\overline{\partial} u_{2n+2} + \partial u_{2n} = 0, \quad n \geq 0.$$

Moreover, by reality of g in (46) they extend continuously to $\varGamma$ and

$$u_{2n}|_{\varGamma} = \overline{u_{-2n}}|_{\varGamma} = \overline{g_{-2n}} = g_{2n}, \quad n \geq 1.$$

Thus, as a summary from above equations, we have shown that the even modes u_{2n} satisfy

$$\overline{\partial} u_{2n} + \partial u_{2n-2} = 0, \quad \text{and} \quad u_{2n}\big|_{\varGamma} = g_{2n}, \quad \text{for all } n \in \mathbb{Z}. \tag{49}$$

Step 2: The construction of odd modes u_{2n-1} for $|n| \geq q$, $q \geq 0$.

Apply the Bukhgeim-Cauchy Integral operator (25) to construct the other odd negative modes:

$$\langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), \cdots \rangle := \mathcal{B}L^q \mathbf{g}^{\text{odd}}(z), \quad z \in \Omega. \tag{50}$$

By Theorem 3.1, the sequence valued map

$$z \mapsto \langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), u_{-(2q+5)}(z), ..., \rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1),$$

is L-analytic in Ω, thus the equations

$$\overline{\partial} u_{-(2n+1)} + \partial u_{-(2n+3)} = 0, \tag{51}$$

are satisfied for all $n \geq q$, $q \geq 0$. Moreover, the hypothesis (45) and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\varGamma$ and

$$u_{-(2n+1)}|_{\varGamma} = g_{-(2n+1)}, \quad \forall n \geq q, \ q \geq 0. \tag{52}$$

Construct the positive odd Fourier modes by conjugation: $u_{2n+1} := \overline{u}_{-(2n+1)}$, for all $n \geq q$, $q \geq 0$.

By conjugating (51) we note that the positive odd Fourier modes also satisfy

$$\overline{\partial} u_{2n+3} + \partial u_{2n+1} = 0, \quad \forall n \geq q, \ q \geq 0. \tag{53}$$

Moreover, by (46) they extend continuously to $\varGamma$ and

$$u_{2n+1}|_{\varGamma} = \overline{u}_{-(2n+1)}|_{\varGamma} = \overline{g}_{-(2n+1)} = g_{2n+1}, \quad n \geq q,\ q \geq 0. \tag{54}$$

Step 3: The construction of the tensor field f in the $q = 0$ case. In the case of the 0-tensor, $\mathbf{f} = f_0$, and f_0 is uniquely determined from the odd Fourier mode u_{-1} in (50), by

$$f_0 := 2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}, \quad \text{(for } q = 0 \text{ case)}. \tag{55}$$

We consider next the case $q \geq 1$ of tensors of order 2 or higher. In this case the construction of the tensor field $\mathbf{f}_\Psi$ is in terms of the Fourier mode $u_{-(2q+1)}$ in (50) and the class Ψ_g^{even} in (43).

Step 4: The construction of odd modes $u_{\pm(2n-1)}$, for $1 \leq n \leq q,\ q \geq 1$.

Recall the non-uniqueness class Ψ_g^{even} in (43).

For $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$ arbitrary, define the modes $u_{\pm 1}, u_{\pm 3}, ..., u_{\pm(2q-1)}$ in Ω by

$$u_{-(2n-1)} := \psi_{-(2n-1)} \text{ and } u_{2n-1} := \overline{\psi}_{-(2n-1)}, \quad 1 \leq n \leq q,\ q \geq 1. \tag{56}$$

By the definition of the class (43), and the reality of g in (46), we have

$$u_{-(2n-1)}|_{\varGamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\varGamma} = \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq q,\ q \geq 1. \tag{57}$$

Step 5: The construction of the tensor field $\mathbf{f}_\Psi$ whose X-ray data is g.

The components of the m-tensor $\mathbf{f}_\Psi$ are defined via the one-to-one correspondence between the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$ and the functions $\{f_{2n} :\ -q \leq n \leq q\}$ as follows.

For $q \geq 1$, we define f_{2q} by using $\psi_{-(2q-1)}$ from the non-uniqueness class (43), and Fourier mode $u_{-(2q+1)}$ from the Bukhgeim-Cauchy formula (50). Then, define $\{f_{2n} :\ 0 \leq n \leq q-1\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-2n} :\ 1 \leq n \leq q\}$ by conjugation.

$$\begin{aligned}
&f_{2q} := \overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}, && q \geq 1,\\
&f_{2n} := \overline{\partial}\psi_{-(2n-1)} + \partial \psi_{-(2n+1)}, && 1 \leq n \leq q-1,\ q \geq 2,\\
&f_0 := 2\,\mathbb{R}\mathrm{e}\,\partial\psi_{-1}, && q \geq 1, \quad \text{and}\\
&f_{-2n} := \overline{f_{2n}}, && 1 \leq n \leq q,\ q \geq 1,
\end{aligned} \tag{58}$$

By construction, $f_{2n} \in C^\mu(\Omega)$, for $-q \leq n \leq q$, as $\psi_{-1}, \cdots, \psi_{-2q+1} \in C^{1,\mu}(\Omega)$. We use these Fourier modes $f_0, f_{\pm 2}, f_{\pm 4}, \cdots, f_{\pm 2q}$ for $q \geq 1$, and equations (13), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$, and thus the m-tensor field $\mathbf{f}_\Psi \in C^\mu(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\varGamma_+} = X\mathbf{f}_\Psi$ for $q \geq 1$, with $\mathbf{f}_\Psi$ being constructed as in (58), we define the real valued function u via its Fourier modes for $q \geq 1$,

$$u(z, \boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} u_{2n} e^{\imath 2n\theta} + \sum_{|n| \geq q} u_{2n+1} e^{\imath(2n+1)\theta} + \sum_{n=1}^{q} \psi_{-(2n-1)} e^{-\imath(2n-1)\theta} + \sum_{n=1}^{q} \overline{\psi}_{-(2n-1)} e^{\imath(2n-1)\theta}. \tag{59}$$

Since $g \in C^\mu\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$, we use Proposition 3.1 (ii) and [39, Proposition 4.1 (iii)] to conclude that u defined in (59) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^\mu(\overline{\Omega} \times \mathbb{S}^1)$. Using (49), (52), (54), (57), and definition of $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$ for $q \geq 1$, the trace $u(\cdot, \boldsymbol{\theta})$ in (59) extends to the boundary,

$$u(\cdot, \boldsymbol{\theta})|_{\varGamma} = g(\cdot, \boldsymbol{\theta}).$$

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, then the term by term differentiation in (59) is now justified, and u satisfy (37):

$$\begin{aligned}\boldsymbol{\theta} \cdot \nabla u &= \overline{\partial}\, \overline{\psi_{-1}} + \partial \psi_{-1} + \sum_{n=1}^{q-1} (\overline{\partial} \psi_{-(2n-1)} + \partial \psi_{-(2n+1)}) e^{-\imath(2n)\theta} + \sum_{n=1}^{q-1} (\overline{\partial}\, \overline{\psi}_{-(2n+1)} + \partial \overline{\psi}_{-(2n-1)}) e^{\imath(2n)\theta} \\ &\quad + e^{-\imath(2q)\theta} (\overline{\partial} \psi_{-(2q-1)} + \partial u_{-(2q+1)}) + e^{\imath(2q)\theta} (\partial \overline{\psi}_{-(2q-1)} + \overline{\partial}\, \overline{u}_{-(2q+1)}) \\ &= \sum_{n=-q}^{q} f_{2n}(z) e^{-\imath(2n)\theta} = \langle \mathbf{f}, \boldsymbol{\theta}^{2q} \rangle,\end{aligned}$$

where the cancellation uses equations (49), (51), (53), (56), and the second equality uses the definition of f_{2k}'s in (58).

□

5. Odd order m-tensor - non-attenuated case

In this section we establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times \mathbb{S}^1$ to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of odd order $m = 2q+1,\ q \geq 0$.

In the non-attenuated odd m-tensor case, the transport equation (19a) becomes

$$\boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) = \sum_{n=0}^{q} \left(f_{2n+1}(z) e^{-\imath(2n+1)\theta} + f_{-(2n+1)}(z) e^{\imath(2n+1)\theta} \right), \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbb{S}^1, \tag{60}$$

where f_{2n+1} defined in (14), and $f_{2n+1} = \overline{f_{-2n-1}}$, for $0 \leq n \leq q,\ q \geq 0$.

If $\displaystyle\sum_{n\in\mathbb{Z}} u_n(z) e^{\imath n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (60), then, by identifying the Fourier modes of the same order, the equation (60) reduces to the system:

$$\overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) = f_{2n+1}(z), \qquad 0 \leq n \leq q,\ q \geq 0, \tag{61}$$

$$\overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \qquad n \geq q+1,\ q \geq 0, \tag{62}$$

$$\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq 0. \tag{63}$$

In the odd m-tensor case, the even and odd Fourier modes of u plays a different role, unlike the even m-tensor case in the previous section. To emphasize this difference we separate the non-positive even modes $\mathbf{u}^{\text{even}} := \langle u_0, u_{-2}, u_{-4} ... \rangle$, and negative odd modes $\mathbf{u}^{\text{odd}} := \langle u_{-1}, u_{-3}, ... \rangle$, and note that if $\langle u_0(z), u_{-1}(z), u_{-2}(z), ... \rangle$ is L^2-analytic, then $\mathbf{u}^{\text{even}}, \mathbf{u}^{\text{odd}}$ are L-analytic.

Let us consider the sequence $\{\mathbf{u}^{2k-1}\}_{k\geq 1} \subset C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ given by

$$\mathbf{u}^{2k-1} := \langle u_{2k-1}, u_{2k-3}, ..., u_1, u_{-1}, u_{-3}, u_{-5}, ... \rangle, \quad k \geq 1, \tag{64}$$

obtained by augmenting the sequence of negative odd indices $\langle u_{-1}, u_{-3}, u_{-5}, ... \rangle$ by k many terms in the order $\underbrace{u_{2k-1}, u_{2k-3}, ..., u_1}_{k}$.

One of the ingredients in our characterization of the odd m-tensor is the following simple property of L-analytic maps, shown in [39, Lemma 2.6].

Lemma 5.1. [39, Lemma 2.6] *Let $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ be the sequence of L-analytic maps defined in* (64). *Assume that*

$$u_{2k-1}|_{\Gamma} = \overline{u_{-(2k-1)}}|_{\Gamma}, \quad \forall k \geq 1. \tag{65}$$

Then, for each $k \geq 1$*,*

$$u_{2k-1}(z) = \overline{u_{-(2k-1)}(z)}, \quad z \in \Omega. \tag{66}$$

The range characterization of data g will be given in terms of its Fourier modes:

$$g(\zeta, \theta) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{\imath n\varphi}, \quad \zeta \in \varGamma.$$

Since the trace g is also real valued, its Fourier modes will satisfy $g_{-n} = \overline{g}_n$, for $n \geq 0$. From the non-positive even modes, we build the sequence

$$\mathbf{g}^{\text{even}} := \langle g_0, g_{-2}, g_{-4}, g_{-6}, ...\rangle. \tag{67}$$

For each $k \geq 1$, we use the odd modes $\{g_{-1}, g_{-3}, g_{-5}, ...\}$ to build the sequence

$$\mathbf{g}^{2k-1} := \langle g_{2k-1}, g_{2k-3}, ..., g_1, g_{-1}, g_{-3}, g_{-5}, ...\rangle \tag{68}$$

by augmenting the negative odd indices by k-many terms in the order $\underbrace{g_{2k-1}, g_{2k-3}, ..., g_1}_{k}$.

Similar to the non-attenuated even m-tensor case before, we will construct the solution u of the transport equation (60), whose trace matches the boundary data g, and also construct the right hand side of the (60). The construction of solution u is in terms of its Fourier modes in the angular variable. Except for non-positive modes $u_0, u_{-2}, \cdots, u_{-2q}$, all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (25) using boundary data. Other than having the traces $u_{-2j}\big|_{\varGamma} = g_{-2j},\ 0 \leq j \leq q,\ q \geq 0$, on the boundary, the $q+1$ many Fourier modes $u_{-2j},\ 0 \leq j \leq q,\ q \geq 0$, are unconstrained. They are chosen arbitrarily from the class of functions

$$\begin{aligned}\Psi_g^{\text{odd}} := \Big\{&(\psi_0, \psi_{-2}, \cdots, \psi_{-2q}) \in C^{1,\mu}(\overline{\Omega}; \mathbb{R}) \times \left(C^{1,\mu}(\overline{\Omega}; \mathbb{C})\right)^q : 2\mu > 1 : \\ &\psi_{-2j}\big|_{\varGamma} = g_{-2j},\ 0 \leq j \leq q,\ q \geq 0\Big\}.\end{aligned} \tag{69}$$

Remark 5.1. *In the 1-tensor case* ($m = 1$)*, only Fourier mode* u_0 *be an* arbitrary *function in* $C^1(\Omega) \cap C(\overline{\Omega})$ *with* $u_0|_{\varGamma} = g_0$*. The arbitrariness of* u_0 *characterizes the non-uniqueness (up to the gradient field of a function which vanishes at the boundary) in the reconstruction of a vector field from its Doppler data.*

Theorem 5.1 (Range characterization for odd tensors.)**.** *Let* $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m; \Omega)$*,* $\mu > 1/2$*, be a real-valued symmetric tensor field of odd order* $m = 2q+1$*,* $q \geq 0$*, and*

$$g = X\mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_- \cup \varGamma_0.$$

Then $\mathbf{g}^{\text{even}}, \mathbf{g}^{2k-1} \in l^{1,1}_{\infty}(\Gamma) \cap C^\mu(\Gamma; l_1)$ *for* $k \geq 1$*, and satisfy*

$$[I + \imath\mathcal{H}] L^{\frac{m+1}{2}} \mathbf{g}^{\text{even}} = \mathbf{0}, \tag{70}$$

$$[I + \imath\mathcal{H}] \mathbf{g}^{2k-1} = \mathbf{0}, \quad \forall k \geq 1, \tag{71}$$

where $\mathbf{g}^{\text{even}}$ *is the sequence in* (41)*,* $\mathbf{g}^{2k-1}$ *for* $k \geq 1$ *is the sequence in* (68)*, and* $\mathcal{H}$ *is the Bukhgeim-Hilbert operator in* (27)*.*

(ii) Let $g \in C^\mu(\varGamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ *be real valued with* $g|_{\varGamma_- \cup \varGamma_0} = 0$*. If the corresponding sequence* $\mathbf{g}^{\text{even}} \in Y_\mu(\varGamma)$ *satisfies* (70)*,* $\mathbf{g}^{2k-1} \in Y_\mu(\varGamma)$ *for* $k \geq 1$*, satisfies* (71)*, and for each element* $(\psi_0, \psi_{-2}, \cdots, \psi_{-2q}) \in \Psi_g^{\text{odd}}$*, then there is a unique real valued symmetric* m*-tensor* $\mathbf{f}_\Psi \in C^\mu(\mathbf{S}^m; \Omega)$ *such that* $g|_{\varGamma_+} = X\mathbf{f}_\psi$*.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$, $X\mathbf{f} \in C^{1,\mu}(\varGamma_+)$, and, thus, the solution u to the transport equation (60) is in $C^{1,\mu}(\overline{\Omega}\times\mathbb{S}^1)$. Moreover, its trace $g = u|_{\varGamma\times\mathbb{S}^1} \in C^{1,\mu}(\varGamma\times\mathbb{S}^1)$. By [39, Proposition 4.1] $\mathbf{g}^{\text{even}}, \mathbf{g}^{2k-1} \in l^{1,1}_{\infty}(\Gamma)\cap C^{\mu}(\Gamma;l_1)$ for all $k\geq 1$.

If u solves (60) then its Fourier modes satisfy (61), (62), and (63). Since the negative even Fourier modes u_{-2n} for $n\geq \frac{m+1}{2}$, satisfies the system (62), then the sequence valued map

$$\Omega \ni z \mapsto \langle u_{-(m+1)}(z), u_{-(m+3)}(z), u_{-(m+5)}(z), \cdots\rangle$$

is L-analytic in Ω and the necessity part in Theorem 3.1 yields the condition (70).

The system (63) yield that the sequence valued map

$$\Omega \ni z \mapsto \mathbf{u}^1(z) := \langle u_1(z), u_{-1}(z), u_{-3}(z)\cdots\rangle$$

is L-analytic in Ω with the trace satisfying $u_{2k-1}|_{\varGamma} = g_{2k-1}$, for all $k\leq 1$.
By Theorem 3.1 necessity part, the sequence $\mathbf{g}^1 = \langle g_1, g_{-1}, g_{-3}, ...\rangle$ must satisfy

$$[I + \imath\mathcal{H}]\mathbf{g}^1 = \mathbf{0}.$$

Recall that u is real valued so that its Fourier modes occur in conjugates $u_n = \overline{u_{-n}}$ for all $n\geq 0$. Consider now the equation (63) for $n = 1$ and take its conjugate to yield

$$\overline{\partial}u_3 + \partial u_1 = 0. \tag{72}$$

Equation (72) together with (63) yield that the sequence valued map

$$\Omega \ni z \mapsto \mathbf{u}^3(z) := \langle u_3(z), u_1(z), u_{-1}(z), u_{-3}(z)\cdots\rangle$$

is L-analytic in Ω with the trace satisfying $u_{2k-1}|_{\varGamma} = g_{2k-1}$ for all $k\leq 2$.
By the necessity part in Theorem 3.1, it must be that $\mathbf{g}^3 = \langle g_3, g_1, g_{-1}, g_{-3}, ...\rangle$ satisfies

$$[I + \imath\mathcal{H}]\mathbf{g}^3 = \mathbf{0}.$$

Inductively, the argument above holds for any odd index $2k-1$ to yield that the sequence

$$\Omega \ni z \mapsto \mathbf{u}^{2k-1}(z) := \langle u_{2k-1}(z), u_{2k-3}(z), ..., u_1(z), u_{-1}(z), u_{-3}(z)\cdots\rangle$$

is L-analytic in Ω. Then, again by the necessity part in Theorem 3.1, its trace $\mathbf{u}^{2k-1}|_{\varGamma} = \mathbf{g}^{2k-1}$ must satisfy the condition (71):

$$[I + \imath\mathcal{H}]\mathbf{g}^{2k-1} = \mathbf{0}, \quad \text{for all } k\geq 1.$$

(ii) **Sufficiency:** Let $g\in C^{\mu}(\varGamma;C^{1,\mu}(\mathbb{S}^1))\cap C(\varGamma;C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0} = 0$. Since g is real valued, its Fourier modes in the angular variable occurs in conjugates

$$g_{-n}(\zeta) = \overline{g}_n(\zeta), \quad \text{for } n\geq 0,\ \zeta\in\varGamma. \tag{73}$$

Let the corresponding sequences $\mathbf{g}^{\text{even}}$ satisfying (44) and $\mathbf{g}^{\text{odd}}$ satisfying (45). By Proposition (3.1), $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$.

Let $m = 2q+1,\ q\geq 0$, be an odd integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u\in C^1(\Omega\times\mathbb{S}^1)\cap C(\overline{\Omega}\times\mathbb{S}^1)$ such that $u|_{\varGamma\times\mathbb{S}^1} = g$ and u solves (60) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of even modes u_{2n} for $|n|\geq 2q+1,\ q\geq 0$.

Apply the Bukhgeim-Cauchy integral formula (25) to construct the negative even Fourier modes:

$$\langle u_{-2(q+1)}, u_{-2(q+2)}, u_{-2(q+3)}, ...,\rangle := \mathcal{B}L^{q+1}\mathbf{g}^{\text{even}}. \tag{74}$$

By Theorem 3.1, the sequence valued map

$$\Omega \ni z \mapsto \langle u_{-2(q+1)}(z), u_{-2(q+2)}(z), u_{-2(q+3)}(z), ...\rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1),$$

is L-analytic in Ω, thus the equations

$$\overline{\partial} u_{-2n} + \partial u_{-(2n+2)} = 0, \tag{75}$$

are satisfied for all $n \geq q+1$, $q \geq 0$. Moreover, the hypothesis (70) and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\varGamma$ and

$$u_{-2n}|_{\varGamma} = g_{-2n}, \quad n \geq q+1, \, q \geq 0. \tag{76}$$

Construct the positive even Fourier modes by conjugation: $u_{2n} := \overline{u_{-2n}}$, for all $n \geq q+1$, $q \geq 0$.

By conjugating (75) we note that the positive even Fourier modes also satisfy

$$\overline{\partial} u_{2n+2} + \partial u_{2n} = 0, \quad n \geq q+1, \, q \geq 0. \tag{77}$$

Moreover, by reality of g in (73), they extend continuously to $\varGamma$ and

$$u_{2n}|_{\varGamma} = \overline{u_{-2n}}|_{\varGamma} = \overline{g_{-2n}} = g_{2n}, \quad n \geq q+1, \, q \geq 0. \tag{78}$$

Step 2: The construction of even modes u_{2n}, for $|n| \leq 2q, \ q \geq 0$.

Recall the non-uniqueness class Ψ_g^{odd} in (69).

For $(\psi_0, \psi_{-2}, \cdots, \psi_{-2q}) \in \Psi_g^{\text{odd}}$ arbitrary, define the modes $u_0, u_{\pm 2}, u_{\pm 4}, ..., u_{\pm 2q}$ in Ω by

$$u_{-2n} := \psi_{-2n}, \quad \text{and} \quad u_{2n} := \overline{\psi_{-2n}}, \quad 0 \leq n \leq q. \tag{79}$$

By the definition of the class (69), and reality of g in (73), we have

$$u_{2n}|_{\varGamma} = \overline{g_{-2n}} = g_{2n}, \quad 0 \leq n \leq q. \tag{80}$$

Step 3: The construction of negative modes u_{2n-1} for $n \in \mathbb{Z}$.

Use the Bukhgeim-Cauchy Integral formula (25) to construct the negative odd Fourier modes:

$$\langle u_{-1}(z), u_{-3}(z), u_{-5}(z), ...\rangle := \mathcal{B}\mathbf{g}^{\text{odd}}(z), \quad z \in \Omega. \tag{81}$$

By Theorem 3.1, the sequence valued map

$$\Omega \ni z \mapsto \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), ...\rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1),$$

is L-analytic in Ω, thus the equations

$$\overline{\partial} u_{-2n-1} + \partial u_{-2n-3} = 0, \tag{82}$$

are satisfied for all $n \geq 0$.

Note that $L\mathbf{g}^1 = \mathbf{g}^{\text{odd}}$. By hypothesis (71), $[I + \imath\mathcal{H}]\mathbf{g}^1 = \mathbf{0}$. Since $\mathcal{H}$ commutes with the left translation L, then

$$\mathbf{0} = L[I + \imath\mathcal{H}]\mathbf{g}^1 = [I + \imath\mathcal{H}]L\mathbf{g}^1 = [I + \imath\mathcal{H}]\mathbf{g}^{\text{odd}}.$$

By applying Theorem 3.1 sufficiency part, we have that each u_{2n-1} extends continuously to Γ:

$$u_{-2n-1}|_{\varGamma} = g_{-2n-1}, \quad n \geq 1.$$

If we were to define the positive odd index modes by conjugating the negative ones (as we did for the non-attenuated even tensor case) it would not be clear why the equation (63) for $n = 0$:

$$\overline{\partial} u_1 + \partial u_{-1} = 0,$$

should hold. To solve this problem we will define the positive odd modes by using the Bukhgeim-Cauchy integral formula (25) inductively.

Let $\mathbf{u}^1 = \langle u_1, u_{-1}^1, u_{-3}^1, \cdots \rangle$ be the L-analytic map defined by

$$\mathbf{u}^1 := \mathcal{B}\mathbf{g}^1. \tag{83}$$

The hypothesis (71) for $k = 1$,

$$[I + \imath\mathcal{H}]\mathbf{g}^1 = \mathbf{0},$$

allows us to apply the sufficiency part of Theorem 3.1 to yield that $\mathbf{u}^1$ extends continuously to $\varGamma$ and has trace $\mathbf{g}^1$ on $\varGamma$. However, $L\mathbf{u}^1 = \mathbf{u}^{\text{odd}}$ is also L-analytic with the same trace $\mathbf{g}^{\text{odd}}$ as $\mathbf{u}^{\text{odd}}$. By the uniqueness of L-analytic maps with the given trace we must have the equality

$$\langle u_{-1}^1, u_{-3}^1, \cdots \rangle = \langle u_{-1}, u_{-3}, \cdots \rangle.$$

In other words the formula (83) constructs only one new function u_1 and recovers the previously defined negative odd functions $u_{-1}, u_{-3}, \ldots$. In particular $\mathbf{u}^1 = \langle u_1, u_{-1}, u_{-3}, \cdots \rangle$ is L-analytic, and the equation $\overline{\partial} u_1 + \partial u_{-1} = 0$ holds in Ω. We stress here that, at this stage, we do not know that u_1 is the complex conjugate of u_{-1}.

Inductively, for $k \geq 1$, the formula

$$\mathbf{u}^{2k-1} = \langle u_{2k-1}, u_{2k-3}^{2k-1}, \ldots, u_1^{2k-1}, u_{-1}^{2k-1}, \cdots \rangle := \mathcal{B}\mathbf{g}^{2k-1} \tag{84}$$

defines a sequence $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ of L-analytic maps with $\mathbf{u}^{2k-1}|_{\varGamma} = \mathbf{g}^{2k-1}$. By the uniqueness of L-analytic maps with the given trace, a similar reasoning as above shows

$$L\mathbf{u}^{2k-1} = \mathbf{u}^{2k-3}, \quad \forall k \geq 2.$$

In particular for all $k \geq 1$, the sequence

$$\mathbf{u}^{2k-1} = \langle u_{2k-1}, u_{2k-3}, \ldots, u_1, u_{-1}, \cdots \rangle$$

is L-analytic. Note that the sequence $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ constructed above satisfies the hypotheses of the Lemma 5.1, and therefore for each $k \geq 1$,

$$u_{2k-1}(z) = \overline{u}_{-(2k-1)}(z), \quad z \in \Omega. \tag{85}$$

We stress here that the identities (85) need the hypothesis (71) for all $k \geq 1$, cannot be inferred directly from the Bukhgeim-Cauchy integral formula (25) for finitely many k's.

We have shown that

$$\overline{\partial} u_{2n-1} + \partial u_{2n-3} = 0, \quad \text{and} \quad u_{2n-1}|_{\Gamma} = g_{2n-1}, \quad \forall n \in \mathbb{Z}. \tag{86}$$

Step 4: The construction of the tensor field $\mathbf{f}_\psi$ whose X-ray data is g.

The components of the m-tensor $\mathbf{f}_\Psi$ are defined via the one-to-one correspondence between the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$ and the functions $\{f_{\pm(2n+1)} : \ 0 \leq n \leq q\}$ as follows.

For $q \geq 0$, we define f_{2q+1} by using ψ_{-2q} from the non-uniqueness class in (69), and Fourier mode $u_{-(2q+2)}$ from the Bukhgeim-Cauchy formula (74). Then, define $\{f_{2n+1} : \ 0 \leq n \leq q-1\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-(2n+1)} : \ 0 \leq n \leq q\}$ by conjugation.

$$\begin{aligned} f_{2q+1} &:= \overline{\partial}\psi_{-2q} + \partial u_{-(2q+2)}, && q \geq 0, \\ f_{2n+1} &:= \overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}, && 0 \leq n \leq q-1, \ q \geq 1, \quad \text{and} \\ f_{-(2n+1)} &:= \overline{f_{2n+1}}, && 0 \leq n \leq q, \ q \geq 0, \end{aligned} \tag{87}$$

By construction, $f_{\pm(2n+1)} \in C^{\mu}(\Omega)$, for $0 \leq n \leq q$, as $\psi_0, \psi_{-2}, \cdots, \psi_{-2q} \in C^{1,\mu}(\Omega)$. We use these Fourier modes $f_{\pm 1}, f_{\pm 3}, \cdots, f_{\pm m}$ for $m = 2q+1,\ q \geq 0$, and equations (14), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$, and thus the m-tensor field $\mathbf{f}_\Psi \in C^{\mu}(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\Gamma_+} = X\mathbf{f}_\Psi$ with $\mathbf{f}_\Psi$ being constructed from pseudovectors via Fourier modes as in (87) from class Ψ_g^{odd}, we define the real valued function u via its Fourier modes

$$u(z,\boldsymbol{\theta}) := \sum_{n=-\infty}^{\infty} u_{2n-1}(z) e^{\imath(2n-1)\theta} + \sum_{|n|\geq q+1} u_{2n}(z) e^{\imath 2n\theta} + \sum_{n=0}^{q} \psi_{-2n}(z) e^{-\imath 2n\theta} + \sum_{n=0}^{q} \overline{\psi}_{-2n}(z) e^{\imath 2n\theta}. \tag{88}$$

Since $g \in C^{\mu}\left(\Gamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$, we use Proposition 3.1 (ii) and [39, Proposition 4.1 (iii)] to conclude that u defined in (88) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$.

Using (76), (78), (80), (86), and element $(\psi_0, \psi_{-2}, \cdots, \psi_{-2q}) \in \Psi_g^{\text{odd}}$, the $u(\cdot, \boldsymbol{\theta})$ in (88) extends to the boundary

$$u(\cdot, \boldsymbol{\theta})|_{\Gamma} = g(\cdot, \boldsymbol{\theta}),$$

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, then the term by term differentiation in (88) is now justified, satisfying the transport equation (60):

$$\begin{aligned} \boldsymbol{\theta} \cdot \nabla u &= 2\,\mathbb{R}\mathrm{e}\left\{ (\overline{\partial}\psi_{-2q} + \partial u_{-(2q+2)}) e^{\imath(2q+1)\theta} \right\} + 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{n=0}^{q-1} (\overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}) e^{\imath(2n+1)\theta} \right\} \\ &= \sum_{n=0}^{q} \left(f_{2n+1} e^{-\imath(2n+1)\theta} + f_{-(2n+1)} e^{\imath(2n+1)\theta} \right) = \langle \mathbf{f}, \boldsymbol{\theta}^{2q+1} \rangle, \end{aligned}$$

where the cancellation uses equations (75), (77), (86), and the second equality uses the definition of f_{2k+1}'s in (87).

□

6. EVEN ORDER m-TENSOR - ATTENUATED CASE

Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$, with $\min_{\overline{\Omega}} a > 0$. We now establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times \mathbb{S}^1$ to be the attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of even order $m = 2q,\ q \geq 0$. In this case $a \neq 0$, the transport equation (19a) becomes

$$\boldsymbol{\theta} \cdot \nabla u(x,\boldsymbol{\theta}) + a(x) u(x,\boldsymbol{\theta}) = \sum_{k=0}^{q} f_{-2k} e^{\imath(2k)\theta} + \sum_{k=1}^{q} f_{2k} e^{-\imath(2k)\theta}, \tag{89}$$

where f_{2k} defined in (13), and $f_{2k} = \overline{f_{-2k}}$, for $-q \leq k \leq q,\ q \geq 0$.

If $\sum_{n\in\mathbb{Z}} u_n(z) e^{\imath n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (89), then by identifying the Fourier coefficients of the same order, equation (89) reduces to the system:

$$\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) + a u_{-2n}(z) = f_{2n}(z), \qquad 0 \leq n \leq q,\ q \geq 0, \tag{90}$$

$$\overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) + a u_{-2n-1}(z) = 0, \qquad 0 \leq n \leq q-1,\ q \geq 1, \tag{91}$$

$$\overline{\partial} u_{-n}(z) + \partial u_{-(n+2)}(z) + a u_{-(n+1)}(z) = 0, \qquad n \geq 2q,\ q \geq 0. \tag{92}$$

Recall that the trace $u|_{\varGamma\times\mathbb{S}^1} := g$ as in (21), with $g = X_a\mathbf{f}$ on $\varGamma_+$ and $g = 0$ on $\varGamma_- \cup \varGamma_0$. We expand the attenuated X-ray data g in terms of its Fourier modes in the angular variables:

$$g(\zeta, \boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{\imath n\theta}, \quad \zeta \in \varGamma.$$

Since the trace g is also real valued, its Fourier modes will satisfy $g_{-n} = \overline{g}_n$, for $n \geq 0$. From the negative modes, we built the sequence $\mathbf{g} := \langle g_0, g_{-1}, g_{-2}, g_{-3}, ...\rangle$. From the special function h defined in (32) and the data g, we built the sequence

$$\mathbf{g}_h := e^{-G}\mathbf{g} := \langle \gamma_0, \gamma_{-1}, \gamma_{-2}, ...\rangle,$$

where $e^{\pm G}$ as defined in (34). From the negative even, respectively, negative odd Fourier modes, we built the sequences

$$\mathbf{g}_h^{\text{even}} = \langle \gamma_0, \gamma_{-2}, \gamma_{-4}, ...\rangle, \quad \text{and} \quad \mathbf{g}_h^{\text{odd}} = \langle \gamma_{-1}, \gamma_{-3}, \gamma_{-5}, ...\rangle. \tag{93}$$

Next we characterize the attenuated X-ray data g in terms of its Fourier modes $\underbrace{g_0, g_{-1}, g_{-2}, \cdots g_{-(m-1)}}_{m}$, and the Fourier modes

$$L^m\mathbf{g}_h := L^m e^{-G}\mathbf{g} := \langle \gamma_{-m}, \gamma_{-(m+1)}, \gamma_{-(m+2)}, ...\rangle.$$

Similar to the non-attenuated case as before, we construct simultaneously the right hand side of the transport equation (89) together with the solution u via its Fourier modes. For $m = 2q$, $q \geq 1$, apart from modes $\underbrace{u_0, u_{-1}, u_{-2}, \cdots u_{-(2q-1)}}_{2q}$, all Fourier modes are constructed uniquely from the data $L^{2q}\mathbf{g}_h$. The modes $u_0, u_{-2}, u_{-4}, \cdots u_{-(2q-2)}$ will be chosen arbitrarily from the class $\Psi_{a,g}^{\text{even}}$ of cardinality $q = \frac{m}{2}$ with prescribed trace and gradient on the boundary $\varGamma$ defined as

$$\begin{aligned}
\Psi_{a,g}^{\text{even}} := \Big\{ \big(\psi_0, \psi_{-2}, \cdots, \psi_{-2(q-1)}\big) &\in C^2(\overline{\Omega}; \mathbb{R}) \times \big(C^2(\overline{\Omega}; \mathbb{C})\big)^q : \\
&\psi_{-2j}\big|_{\varGamma} = g_{-2j}, \quad 0 \leq j \leq q-1,\ q \geq 1, \\
&\overline{\partial}\psi_{-2(q-1)}\big|_{\varGamma} = -\partial(e^G\mathcal{B}e^{-G}\mathbf{g})_{-2q}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-(2q-1)}, \quad q \geq 1, \\
&\overline{\partial}\psi_{-2j}\big|_{\varGamma} = -\partial\psi_{-(2j+2)}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-(2j+1)}, \quad 0 \leq j \leq q-2,\ q \geq 2 \Big\}
\end{aligned} \tag{94}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (25), and the operators $e^{\pm G}$ as defined in (34).

Remark 6.1. *In the 2-tensor case ($m = 2$), apart from zeroth mode u_0 and negative one mode u_{-1}, all Fourier modes are constructed uniquely from the data $L^2\mathbf{g}_h$. The mode u_0 will be chosen arbitrarily from the class $\Psi_{a,g}^{m=2}$. We rewrite the above class $\Psi_{a,g}^{\text{even}}$ explicitly for $m = 2$, as*

$$\Psi_{a,g}^{m=2} := \left\{ \psi_0 \in C^2(\overline{\Omega}; \mathbb{R}) : \psi_0\big|_{\varGamma} = g_0, \quad \overline{\partial}\psi_0\big|_{\varGamma} = -\partial(e^G\mathcal{B}e^{-G}\mathbf{g})_{-2}\big|_{\varGamma} - a|_{\varGamma}\, g_{-1} \right\}. \tag{95}$$

In the 0-tensor case ($m = 0$), there is no class, and the characterization of the attenuated X-ray data g is in terms of the Fourier modes $\mathbf{g}_h := e^{-G}\mathbf{g}$.

Next, we characterize the range for even $m = 2q$, $q \geq 0$, in the attenuated case.

Theorem 6.1 (Range characterization for even order tensors). *Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$ with $\min_{\overline{\Omega}} a > 0$. (i) Let $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$, be a real-valued symmetric tensor field of even order $m = 2q$, $q \geq 0$, and $g = X_a \mathbf{f}$ on $\varGamma_+$ and $g = 0$ on $\varGamma_- \cup \varGamma_0$. Then $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$ satisfy*

$$[I + \imath\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}_h^{\text{even}} = \mathbf{0}, \quad [I + \imath\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}_h^{\text{odd}} = \mathbf{0}. \tag{96}$$

where $\mathbf{g}_h^{even}, \mathbf{g}_h^{\text{odd}}$ are sequences in (93)*, and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in* (27)*.*

(ii) Let $g \in C^\mu(\varGamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0} = 0$. For $q = 0$, if the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$ satisfies (96)*, then there is a unique real valued symmetric* 0*-tensor* $\mathbf{f}$ *such that $g|_{\varGamma_+} = X_a\mathbf{f}$. Moreover, for $q \geq 1$, if $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$ satisfies* (96)*, and for each element $\left(\psi_0, \psi_{-2}, \cdots, \psi_{-2(q-1)}\right) \in \Psi_{a,g}^{\text{even}}$, then there is a unique real valued symmetric m-tensor $\mathbf{f}_\Psi \in C(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X_a\mathbf{f}_\Psi$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$ are compactly supported inside Ω, then for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\mu}(\varGamma \times \mathbb{S}^1)$. Moreover, g is the trace on $\varGamma \times \mathbb{S}^1$ of a solution $u \in C^{1,\mu}(\overline{\Omega} \times \mathbb{S}^1)$ of the transport equation (89). By Proposition 3.1(i) and Proposition 3.2, $\mathbf{g}_h = e^{-G}\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$.

If u solves (89) then its Fourier modes satisfies (90), (91) and (92). In particular, the sequence valued map $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, \cdots \rangle$, satisfies $\overline{\partial} L^m\mathbf{u} + L^2\partial L^m\mathbf{u} + aL^{m+1}\mathbf{u} = 0$.

Let $\mathbf{v} := e^{-G}L^m\mathbf{u}$, then by Lemma 3.1, and the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$, the sequence $\mathbf{v} = L^m e^{-G}\mathbf{u}$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = 0$, i.e $\mathbf{v}$ is L^2 analytic. Thus, the negative even subsequence $\langle v_0, v_{-2}, \cdots \rangle$, and negative odd subsequence $\langle v_{-1}, v_{-3}, \cdots \rangle$, respectively, are L analytic, with traces $L^{\frac{m}{2}}\mathbf{g}_h^{\text{even}}$, respectively, $L^{\frac{m}{2}}\mathbf{g}_h^{\text{odd}}$. The necessity part in Theorem 3.1 yields (96):

$$[I + \imath\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}_h^{\text{even}} = \mathbf{0}, \quad [I + \imath\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}_h^{\text{odd}} = \mathbf{0}.$$

This proves part (i) of the theorem.

(ii) **Sufficiency:** Let $g \in C^\mu(\varGamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0} = 0$. Let the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ as in (93) satisfying (96). By Proposition 3.1(ii) and Proposition 3.2(iii), we have $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$.

Let $m = 2q$, $q \geq 0$, be an even integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\varGamma\times\mathbb{S}^1} = g$ and u solves (89) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of modes u_{-n} for $|n| \geq 2q,\ q \geq 0$.

Use the Bukhgeim-Cauchy Integral formula (25) to define the L-analytic maps

$$\mathbf{v}^{\text{even}}(z) = \langle v_0(z), v_{-2}(z), v_{-4}(z), ... \rangle := \mathcal{B}L^q\mathbf{g}_h^{\text{even}}(z), \quad z \in \Omega,$$
$$\mathbf{v}^{\text{odd}}(z) = \langle v_{-1}(z), v_{-3}(z), v_{-5}(z), ... \rangle := \mathcal{B}L^q\mathbf{g}_h^{\text{odd}}(z), \quad z \in \Omega.$$

By intertwining the above L-analytic maps, define also the L^2-analytic map

$$\mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), v_{-3}(z), ... \rangle, \quad z \in \Omega.$$

By Theorem 3.1 (ii),

$$\mathbf{v}, \mathbf{v}^{\text{even}}, \mathbf{v}^{\text{odd}} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{97}$$

Moreover, since $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ satisfy the hypothesis (96), by Theorem 3.1 sufficiency part, we have

$$\mathbf{v}^{\text{even}}|_{\varGamma} = L^q \mathbf{g}_h^{\text{even}} \quad \text{and} \quad \mathbf{v}^{\text{odd}}|_{\varGamma} = L^q \mathbf{g}_h^{\text{odd}}.$$

In particular, $\mathbf{v}$ is L^2-analytic map with trace:

$$\mathbf{v}|_{\varGamma} = L^{2q}\mathbf{g}_h = L^{2q} e^{-G}\mathbf{g}, \tag{98}$$

where $\mathbf{g}_h$ is formed by intertwining $\mathbf{g}_h^{\text{even}}$ and $\mathbf{g}_h^{\text{odd}}$.

Define the sequence valued map

$$\Omega \ni z \mapsto L^{2q}\mathbf{u}(z) = \langle u_{-2q}(z), u_{-2q-1}(z), u_{-2q-2}(z), \cdots \rangle := e^{G}\mathbf{v}(z), \tag{99}$$

where the operator e^G as defined in (34). Since convolution preserves l_1, by Proposition 3.2,

$$L^{2q}\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1). \tag{100}$$

Moreover, since $\mathbf{v} \in C^2(\Omega; l_\infty)$ as in (97), we also conclude from convolution that $L^{2q}\mathbf{u} \in C^2(\Omega; l_\infty)$.

As $\mathbf{v}$ is L^2 analytic, by Lemma 3.1, $L^{2q}\mathbf{u}$ satisfies

$$\overline{\partial} L^{2q}\mathbf{u} + L^2 \partial L^{2q}\mathbf{u} + aL^{2q+1}\mathbf{u} = 0,$$

which in component form is written as:

$$\overline{\partial} u_{-n} + \partial u_{-n-2} + a u_{-n-1} = 0, \quad n \geq 2q,\ q \geq 0. \tag{101}$$

The trace satisfy

$$L^{2q}\mathbf{u}|_{\varGamma} = e^{G}\mathbf{v}|_{\varGamma} = e^{G} L^{2q} e^{-G}\mathbf{g} = L^{2q}\mathbf{g}, \tag{102}$$

where the second equality follows from (98) and in the last equality we use the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$.

Construct the positive Fourier modes by conjugation: $u_n := \overline{u_{-n}}$, for all $n \geq 2q,\ q \geq 0$. Moreover using (102), the traces $u_n|_{\varGamma}$ for each $n \geq 2q,\ q \geq 0$, satisfy

$$u_n|_{\varGamma} = \overline{u_{-n}}|_{\varGamma} = \overline{g_{-n}} = g_n, \quad n \geq 2q,\ q \geq 0. \tag{103}$$

By conjugating (101) we note that the positive Fourier modes also satisfy

$$\overline{\partial} u_{n+2} + \partial u_n + a u_{n+1} = 0, \quad n \geq 2q,\ q \geq 0. \tag{104}$$

Step2: The construction of the tensor field f in the $q = 0$ case.

In the case of the 0-tensor, $\mathbf{f} = f_0$, and f_0 is uniquely determined from the odd Fourier mode u_{-1}, and the zeroth mode u_0 in (99), by

$$\mathbf{f} := 2\,\mathbb{R}\mathrm{e}\, \partial u_{-1} + a u_0, \quad (\text{for } q = 0 \text{ case}). \tag{105}$$

We consider next the case $m = 2q, q \geq 1$ of tensors of order 2 or higher. In this case the construction of the tensor field $\mathbf{f}_\psi$ is in terms of the mode u_{-2q} in (99) and the class $\Psi_{a,g}^{\text{even}}$ in (94).

Step 3: The construction of modes u_n for $|n| \leq 2q - 1$ $q \geq 1$.

Recall that $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$ with $\min\limits_{\overline{\Omega}} a > 0$, and the non-uniqueness class $\Psi_{a,g}^{\text{even}}$ in (94).

For $\left(\psi_0, \psi_{-2}, \cdots, \psi_{-2(q-1)}\right) \in \Psi_{a,g}^{\text{even}}$ arbitrary, define the modes $u_0, u_{\pm 2}, ..., u_{\pm(2(q-1))}$ in Ω by

$$u_{-2j} := \psi_{-2j}, \quad \text{and} \quad u_{2j} := \overline{\psi_{-2j}}, \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{106}$$

Using the mode u_{-2q} from (99) and $\psi_{-2(q-1)}$, define the modes $u_{\pm(2q-1)}$ by

$$u_{-(2q-1)} := -\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}, \quad \text{and} \quad u_{2q-1} := \overline{u}_{-(2q-1)}, \quad \text{for all } q \geq 1. \tag{107}$$

As $\psi_0 \in C^2(\overline{\Omega};\mathbb{R})$ and $\psi_{-(2j+2)} \in C^2(\overline{\Omega};\mathbb{C})$, for $0 \leq j \leq q-2,\ q \geq 2$, define modes
(108)
$$u_{-(2j+1)} := -\frac{\overline{\partial}\psi_{-2j} + \partial\psi_{-(2j+2)}}{a}, \text{ and } u_{2j+1} := \overline{u}_{-(2j+1)}, \text{ for all } 0 \leq j \leq q-2,\ q \geq 2.$$

By the construction in (106), (107), and (108):

$$\begin{aligned} &u_{-2j} \in C^2(\Omega; l_\infty), && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \\ &u_{-(2j+1)} \in C^1(\Omega; l_\infty), && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \quad \text{and} \\ &\overline{\partial}u_{-2j} + \partial u_{-(2j+2)} + a u_{-(2j+1)} = 0, && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \end{aligned} \tag{109}$$

are satisfied. Moreover, by conjugating the last equation in (109) yields

$$\partial u_{2j} + \overline{\partial}u_{(2j+2)} + a u_{(2j+1)} = 0, \quad \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{110}$$

By the definition of the class (94), and reality of g, we have the trace of u_{-2j} in (106) satisfies

$$u_{-2j}|_\Gamma = g_{-2j}, \quad \text{and} \quad u_{2j}|_\Gamma = \overline{g_{-2j}} = g_{2j}, \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{111}$$

We check next that the trace of $u_{-(2j+1)}$ is $g_{-(2j+1)}$ for $0 \leq j \leq q-2,\ q \geq 2$:

$$u_{-(2j+1)}\big|_\Gamma = -\frac{\overline{\partial}\psi_{-2j} + \partial\psi_{-(2j+2)}}{a}\bigg|_\Gamma = g_{-(2j+1)}, \tag{112}$$

where the last equality uses the condition in class (94). Similar calculation to (112) for mode $u_{-(2q-1)}$ give the trace

$$u_{-(2q-1)}\big|_\Gamma = -\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\bigg|_\Gamma = g_{-(2q-1)}. \tag{113}$$

Thus, from (111) - (113), we have the traces:

$$u_n\big|_\Gamma = g_n, \quad \forall |n| \leq 2q-1. \tag{114}$$

Step 4: The construction of the tensor field $\mathbf{f}_\Psi$ whose attenuated X-ray data is g.
The components of the m-tensor $\mathbf{f}_\Psi$ are defined via the one-to-one correspondence between the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$ and the functions $\{f_{2n}:\ -q \leq n \leq q\}$ as follows.
We define first f_{2q} by using $\psi_{-2(q-1)}$ from the non-uniqueness class, and Fourier modes $u_{-2q}, u_{-(2q+1)} \in C^2(\Omega; l_\infty)$ from (99). Then, next define f_{2q-2} by using $\psi_{-2(q-1)}, \psi_{-2(q-2)}$ from the non-uniqueness class, and Fourier mode u_{-2q} from (99). Then, define $\{f_{2n}:\ 0 \leq n \leq q-2\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-2n}:\ 1 \leq n \leq q\}$ by conjugation.

(115)

$$\begin{aligned}
f_{2q} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) + \partial u_{-(2q+1)} + a u_{-2q}, \quad q \geq 1,\\
f_{2q-2} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(q-2)} + \partial \psi_{-2(q-1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) + a\psi_{-2(q-1)}, \quad q \geq 2,\\
f_{2n} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(n-1)} + \partial \psi_{-2n}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-2n} + \partial \psi_{-2(n+1)}}{a}\right) + a\psi_{-2n}, \quad 1 \leq n \leq q-2, \quad q \geq 3,\\
f_0 &:= \begin{cases} -2\,\mathbb{R}\mathrm{e}\,\partial\left(\dfrac{\overline{\partial}\psi_0 + \partial u_{-2}}{a}\right) + a\psi_0, & q = 1,\\ -2\,\mathbb{R}\mathrm{e}\,\partial\left(\dfrac{\overline{\partial}\psi_0 + \partial \psi_{-2}}{a}\right) + a\psi_0, & q \geq 2,\end{cases}\\
f_{-2n} &:= \overline{f_{2n}}, \quad 0 \leq n \leq q,\ q \geq 1,
\end{aligned}$$

By construction, $f_{2n} \in C(\Omega)$, for $0 \leq n \leq q,\ q \geq 1$, as $\psi_{-2n} \in C^2(\Omega; l_\infty)$, for $0 \leq n \leq q-1$, from (94). Note that f_{2n} satisfy (90). We use these Fourier modes $\langle f_0, f_{\pm 2}, f_{\pm 4}, \cdots, f_{\pm m}\rangle$ and equations (13), (7) and (9) to construct pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m\rangle$, and thus m-tensor field $\mathbf{f}_\Psi \in C(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\varGamma_+} = X_a \mathbf{f}_\Psi$ with $\mathbf{f}_\Psi$ being constructed from pseudovectors via Fourier modes as in (115) from class $\Psi_{a,g}^{\text{even}}$, we define the real valued function u via its Fourier modes

(116)

$$\begin{aligned}
u(z,\boldsymbol{\theta}) := &\sum_{|n|\geq 2q} u_n(z) e^{\imath n\theta} + 2\,\mathbb{R}\mathrm{e}\left(-\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) e^{-\imath(2q-1)\theta}\\
&+ 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=0}^{q-1} \psi_{-2n}(z) e^{-\imath(2n)\theta}\right\} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=0}^{q-2}\left(-\frac{\overline{\partial}\psi_{-2j} + \partial\psi_{-(2j+2)}}{a}\right) e^{-\imath(2n+1)\theta}\right\}
\end{aligned}$$

and check that it has the trace g on $\varGamma$ and satisfies the transport equation (89).

Since $g \in C^\mu(\varGamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$, we use Proposition 3.1 (ii) and [39, Proposition 4.1 (iii)] to conclude that u defined in (116) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^\mu(\overline{\Omega} \times \mathbb{S}^1)$. In particular $u(\cdot, \boldsymbol{\theta})$ for $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ extends to the boundary and its trace satisfies

$$u(\cdot,\boldsymbol{\theta})|_{\varGamma} = \sum_{|n|\geq 2q} u_n\big|_{\varGamma} e^{\imath n\theta} + \sum_{|n|\leq 2q-1} u_n\big|_{\varGamma} e^{\imath n\theta} = \sum_{|n|\geq 2q} g_n e^{\imath n\theta} + \sum_{|n|\leq 2q-1} g_n e^{\imath n\theta} = g(\cdot,\boldsymbol{\theta}),$$

where in the second equality above we use (98), (103) and (114).

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^\mu(\overline{\Omega} \times \mathbb{S}^1)$, then using (101), (104), (107), (109), (110), and the definition of f_{2n} for $-q \leq n \leq q,\ q \geq 1$ in (115), the real valued u defined in (116) satisfies the transport equation (89):

$$\boldsymbol{\theta}\cdot\nabla u + au = \langle \mathbf{f}_\Psi, \boldsymbol{\theta}^{2q}\rangle, \quad q \geq 1.$$

□

7. ODD ORDER m-TENSOR - ATTENUATED CASE

In this section, we establish necessary and sufficient conditions for a sufficiently smooth function on $\varGamma \times \mathbb{S}^1$ to be the attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of odd order $m = 2q+1,\ q \geq 0$.

In this case $a \neq 0$, the transport equation becomes

$$\boldsymbol{\theta} \cdot \nabla u(x, \boldsymbol{\theta}) + a(x) u(x, \boldsymbol{\theta}) = \sum_{n=0}^{q} \left(f_{2n+1}(x) e^{-\imath(2n+1)\theta} + f_{-(2n+1)}(x) e^{\imath(2n+1)\theta} \right), \quad x \in \Omega, \tag{117}$$

where $\overline{f}_{2n+1} = f_{-(2n+1)},\ 0 \leq n \leq q,\ q \geq 0$.

If $\displaystyle\sum_{n \in \mathbb{Z}} u_n(z) e^{\imath n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (117), then by identifying the Fourier coefficients of the same order, the equation (117) reduces to the system:

$$\overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) + a u_{-(2n+1)}(z) = f_{2n+1}(z), \qquad 0 \leq n \leq q,\ q \geq 0, \tag{118}$$

$$\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) + a u_{-2n}(z) = 0, \qquad 0 \leq n \leq q,\ q \geq 0, \tag{119}$$

$$\overline{\partial} u_{-n}(z) + \partial u_{-(n+2)}(z) + a u_{-(n+1)}(z) = 0, \qquad n \geq 2q+1,\ q \geq 0, \tag{120}$$

Recall that the trace $u|_{\varGamma \times \mathbb{S}^1} := g$ as in (21), with $g = X_a \mathbf{f}$ on $\varGamma_+$ and $g = 0$ on $\varGamma_- \cup \varGamma_0$. We expand the attenuated X-ray data g in terms of its Fourier modes in the angular variables: $g(\zeta, \boldsymbol{\theta}) = \displaystyle\sum_{n=-\infty}^{\infty} g_n(\zeta) e^{\imath n\theta}$, for $\zeta \in \varGamma$. From the non-positive modes of g, we built the sequences $\mathbf{g} := \langle g_0, g_{-1}, g_{-2}, ... \rangle$, and $\mathbf{g}_h := e^{-G}\mathbf{g} := \langle \gamma_0, \gamma_{-1}, \gamma_{-2}, ... \rangle$, where $e^{\pm G}$ as defined in (34). From the non-positive even, respectively, negative odd Fourier modes, we built the sequences

$$\mathbf{g}_h^{\text{even}} = \langle \gamma_0, \gamma_{-2}, \gamma_{-4}, ... \rangle, \quad \text{and} \quad \mathbf{g}_h^{\text{odd}} = \langle \gamma_{-1}, \gamma_{-3}, \gamma_{-5}, ... \rangle. \tag{121}$$

Next we characterize the attenuated X-ray data g in terms of its m many modes $g_0, g_{-1}, \cdots g_{-(m-1)}$, and the Fourier modes $L^m \mathbf{g}_h := L^m e^{-G}\mathbf{g} := \langle \gamma_{-m}, \gamma_{-(m+1)}, \gamma_{-(m+2)}, ... \rangle$.
As before we construct simultaneously the right hand side of the transport equation (117) together with the solution u. Construction of u is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. For $m = 2q+1$ (odd integer), $q \geq 1$, the modes will be chosen arbitrarily from the class $\Psi_{a,g}^{\text{odd}}$ of cardinality $q = \frac{m-1}{2}$ with prescribed trace and gradient on the boundary $\varGamma$ defined as

$$\begin{aligned} \Psi_{a,g}^{\text{odd}} := \Big\{ & \left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)} \right) \in \left(C^2(\overline{\Omega}; \mathbb{C}) \right)^q : \\ & \psi_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1, \\ & \overline{\partial} \psi_{-(2q-1)}\big|_{\varGamma} = -\partial (e^{G} \mathcal{B} e^{-G} \mathbf{g})_{-(2q+1)}\big|_{\varGamma} - a\big|_{\varGamma} g_{-2q}, \quad q \geq 1, \\ & \overline{\partial} \psi_{-(2j-1)}\big|_{\varGamma} = -\partial \psi_{-(2j+1)}\big|_{\varGamma} - a\big|_{\varGamma} g_{-2j}, \quad 1 \leq j \leq q-1, \quad q \geq 2, \\ & 2 \left(\mathbb{R}\mathrm{e}\, \partial \psi_{-1}\big|_{\varGamma} \right) = -a|_{\varGamma}\, g_0, \Big\} \end{aligned} \tag{122}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (25), and the operators $e^{\pm G}$ as defined in (34).

Remark 7.1. *In the 1-tensor case ($q=0$), there is no class, and the characterization of the attenuated X-ray data g is in terms of its zero-th mode $g_0 = \oint g(\cdot,\theta)d\theta$ and negative Fourier modes of $\mathbf{g}_h := e^{-G}\mathbf{g}$.*

Theorem 7.1 (Range characterization for odd order tensors)**.** *Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$ with $\min_{\overline{\Omega}} a > 0$. and $m = 2q+1$, $q \geq 0$. (i) Let $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$ be a real-valued symmetric m-tensor field of odd order and*

$$g = X_a \mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_- \cup \varGamma_0.$$

Then $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$ satisfy

$$[I + \imath\mathcal{H}]L^{\frac{m+1}{2}}\mathbf{g}_h^{\text{even}} = \mathbf{0}, \quad [I + \imath\mathcal{H}]L^{\frac{m-1}{2}}\mathbf{g}_h^{\text{odd}} = \mathbf{0}, \quad \text{for} \quad q \geq 0, \tag{123}$$

where $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ are sequences in (121)*. Additionally, in $q=0$ case, for each $\zeta \in \varGamma$, the zero-th Fourier mode g_0 of g satisfy*

$$g_0(\zeta) = \lim_{\Omega \ni z \to \zeta \in \varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\partial(e^G \mathcal{B}\mathbf{g}_h)_{-1}(z)}{a(z)}, \quad \text{for} \quad q = 0, \tag{124}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (25)*, and the operators $e^{\pm G}$ as defined in* (34)*.*

(ii) Let $g \in C^\mu(\varGamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. For $q=0$, if the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$ satisfies (123)*, and g_0 satisfies* (124)*, then there exists a unique real valued vector field (1-tensor) $\mathbf{f} \in C(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X_a\mathbf{f}$. Moreover, for $q \geq 1$, if $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$ satisfies* (123)*, and for each element $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_{a,g}^{\text{odd}}$, then there is a unique real valued symmetric m-tensor $\mathbf{f}_\Psi \in C(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X_a\mathbf{f}_\Psi$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$, $X_a\mathbf{f} \in C^{1,\mu}(\varGamma_+)$, and, thus, the solution u to the transport equation (117) is in $C^{1,\mu}(\overline{\Omega} \times \mathbb{S}^1)$. Moreover, its trace $g = u|_{\varGamma \times \mathbb{S}^1} \in C^{1,\mu}(\varGamma \times \mathbb{S}^1)$. By Proposition 3.1(i) and Proposition 3.2, $\mathbf{g}_h = e^{-G}\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$.

If u solves (117) then its Fourier modes satisfies (118), (119) and (120). In particular, the sequence valued map $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, \ldots\rangle$ satisfy $\overline{\partial}L^m\mathbf{u} + L^2\partial L^m\mathbf{u} + aL^{m+1}\mathbf{u} = \mathbf{0}$.

Let $\mathbf{v} := e^{-G}L^m\mathbf{u}$, then by Lemma 3.1, and the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$, the sequence $\mathbf{v} = L^m e^{-G}\mathbf{u}$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$, i.e $\mathbf{v}$ is L^2 analytic. The non-positive even subsequence $\langle v_0, v_{-2}, \cdots\rangle$, and negative odd subsequence $\langle v_{-1}, v_{-3}, \cdots\rangle$, respectively, are L analytic, with traces $L^{\frac{m+1}{2}}\mathbf{g}_h^{\text{even}}$, respectively, $L^{\frac{m-1}{2}}\mathbf{g}_h^{\text{odd}}$. The necessity part in Theorem 3.1 yields (123):

$$[I + \imath\mathcal{H}]L^{\frac{m+1}{2}}\mathbf{g}_h^{\text{even}} = \mathbf{0}, \quad [I + \imath\mathcal{H}]L^{\frac{m-1}{2}}\mathbf{g}_h^{\text{odd}} = \mathbf{0}, \quad \text{for} \quad m = 2q+1,\ q \geq 0.$$

Additionally, in the $q=0$ case, the Fourier modes u_0, u_{-1}, u_1 of u solve (119) for $n=0$. Since $a > 0$ in Ω, we have

$$u_0(z) = \frac{-2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{a(z)}, \quad z \in \Omega. \tag{125}$$

Since the left hand side of (125) is continuous all the way to the boundary, so is the right hand side. Moreover, the limit below exists and in the $q=0$ case, we have

$$g_0(z_0) = \lim_{\Omega \ni z \to z_0 \in \varGamma} u_0(z) = \lim_{\Omega \ni z \to z_0 \in \varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{a(z)},$$

thus (124) holds. This proves part (i) of the theorem.

(ii) **Sufficiency:** Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. Let the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ as in (121) satisfying (123). By Proposition 3.1(ii) and Proposition 3.2(iii), $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_\mu(\varGamma)$.

Let $m = 2q+1,\ q \geq 0$, be an odd integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\varGamma \times \mathbb{S}^1} = g$ and u solves (117) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of modes u_n for $|n| \geq 2q+1,\ q \geq 0$.

Use the Bukhgeim-Cauchy Integral formula (25) to define the L-analytic maps

$$\mathbf{v}^{even}(z) = \langle v_0(z), v_{-2}(z), v_{-4}(z), ...\rangle := \mathcal{B}L^{q+1}\mathbf{g}_h^{\text{even}}(z), \quad z \in \Omega,$$
$$\mathbf{v}^{odd}(z) = \langle v_{-1}(z), v_{-3}(z), v_{-5}(z), ...\rangle := \mathcal{B}L^{q}\mathbf{g}_h^{\text{odd}}(z), \quad z \in \Omega.$$

By intertwining let also define L^2-analytic map

$$\mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), v_{-3}(z), ...\rangle, \quad z \in \Omega.$$

By Theorem 3.1 (ii),

$$\mathbf{v}^{\text{even}}, \mathbf{v}^{\text{odd}}, \mathbf{v} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{126}$$

Moreover, since $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ satisfy the hypothesis (96), by Theorem 3.1 sufficiency part, we have

$$\mathbf{v}^{\text{even}}|_{\varGamma} = L^{q+1}\mathbf{g}_h^{\text{even}} \quad \text{and} \quad \mathbf{v}^{\text{odd}}|_{\varGamma} = L^{q}\mathbf{g}_h^{\text{odd}}, \quad q \geq 0.$$

In particular, $\mathbf{v}$ is L^2-analytic with trace:

$$\mathbf{v}|_{\varGamma} = L^{2q+1}\mathbf{g}_h = L^{2q+1}e^{-G}\mathbf{g}, \quad q \geq 0, \tag{127}$$

where $\mathbf{g}_h$ is formed by intertwining $\mathbf{g}_h^{\text{even}}$ and $\mathbf{g}_h^{\text{odd}}$.

For $q \geq 0$, define the sequence valued map

$$\Omega \ni z \mapsto L^{2q+1}\mathbf{u}(z) = \langle u_{-(2q+1)}(z), u_{-(2q+2)}(z), u_{-(2q+3)}(z), \cdots \rangle := e^{G}\mathbf{v}(z). \tag{128}$$

By Proposition 3.2, $L^{2q+1}\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$. Moreover, since $\mathbf{v} \in C^2(\Omega; l_\infty)$ as in (126), we also conclude from convolution that $L^{2q+1}\mathbf{u} \in C^2(\Omega; l_\infty)$. Thus,

$$L^{2q+1}\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{129}$$

As $\mathbf{v}$ is L^2 analytic, by Lemma 3.1, $L^{2q+1}\mathbf{u}$ satisfies $\overline{\partial}L^{2q+1}\mathbf{u} + L^2\partial L^{2q+1}\mathbf{u} + aL^{2q+2}\mathbf{u} = 0$, for $q \geq 0$, which in component form is written as:

$$\overline{\partial}u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad n \geq 2q+1,\ q \geq 0. \tag{130}$$

The trace satisfy

$$L^{2q+1}\mathbf{u}|_{\varGamma} = e^{G}\mathbf{v}|_{\varGamma} = e^{G}L^{2q+1}e^{-G}\mathbf{g} = L^{2q+1}\mathbf{g}, \quad q \geq 0, \tag{131}$$

where the second equality follows from (127) and in the last equality we use $[e^{\pm G}, L] = 0$.

Construct the positive Fourier modes by conjugation: $u_n := \overline{u_{-n}}$, for all $n \geq 2q+1,\ q \geq 0$. Moreover using (131), and the reality of g, the traces $u_n|_{\varGamma}$ satisfy

$$u_n|_{\varGamma} = \overline{u_{-n}}|_{\varGamma} = \overline{g_{-n}} = g_n, \quad n \geq 2q+1,\ q \geq 0. \tag{132}$$

By conjugating (130), and from (131) and (132), we thus have the Fourier modes satisfy

$$\overline{\partial}u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad \text{and} \quad u_n\big|_{\varGamma} = g_n, \quad \forall |n| \geq 2q+1,\ q \geq 0. \tag{133}$$

Step 2: The construction of 1-tensor ($q = 0$ case).

Since $a > 0$ in Ω, we can define u_0 (in $q = 0$ case) by using the Fourier mode u_{-1} from (128):

$$u_0(z) := -\frac{2\,\mathbb{R}\mathrm{e}\, \partial u_{-1}(z)}{a(z)}, \quad z \in \Omega, \quad (\text{for } q = 0 \text{ case}). \tag{134}$$

Note that u_0 satisfy (133) for $n = -1$. In particular $\overline{\partial} u_1 + \partial u_{-1} + a u_0 = 0$ holds.
From (124), u_0 defined above extends continuously to the boundary $\varGamma$ and

$$u_0\big|_{\varGamma} = g_0, \quad (\text{for } q = 0 \text{ case}).$$

Moreover, since $u_{-1} \in C^2(\Omega)$ as shown in (129) and $a \in C^2(\Omega)$ we get $u_0 \in C^1(\Omega)$.

Using the Fourier modes u_{-1}, u_{-2} from (128) and u_0 as in (134), we next define the real valued vector field $\mathbf{f} \in C(\Omega; \mathbb{R}^2)$ (for $q = 0$ case) by

$$\mathbf{f} = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1 \rangle, \quad \text{where} \quad f_1 := \overline{\partial} u_0 + \partial u_{-2} + a u_{-1}. \tag{135}$$

We consider next the case $q \geq 1$ of tensors of order 3 or higher. In this case the construction of the tensor field $\mathbf{f}_\Psi$ is in terms of the Fourier modes $u_{-(2q+1)}, u_{-(2q+2)}$ in (128) and the class $\Psi_{a,g}^{\text{odd}}$ as in (122).

Step 3: The construction of modes u_n for $|n| \leq 2q,\ q \geq 1$.

Recall the non-uniqueness class $\Psi_{a,g}^{\text{odd}}$ as in (122).

For $\big(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\big) \in \Psi_{a,g}^{\text{odd}}$ arbitrary, firstly define the odd modes

$$u_{-(2n-1)} := \psi_{-(2n-1)}, \quad \text{and} \quad u_{2n-1} := \overline{\psi}_{-(2n-1)}, \quad 1 \leq n \leq q,\ q \geq 1. \tag{136}$$

Secondly, by using $\psi_{-1}, \psi_{-(2q-1)}$ and the mode $u_{-(2q+1)}$ from (128), we define the modes

$$u_0 := -\frac{2\,\mathbb{R}\mathrm{e}\, \partial \psi_{-1}}{a}, \tag{137}$$

$$u_{-2q} := -\frac{\overline{\partial} \psi_{-(2q-1)} + \partial u_{-(2q+1)}}{a}, \quad \text{and} \quad u_{2q} := \overline{u_{-2q}} \quad \text{for} \quad q \geq 1. \tag{138}$$

Lastly, by using $\psi_{-(2n-1)} \in C^2(\overline{\Omega}; \mathbb{C})$, for $1 \leq n \leq q-1,\ q \geq 2$, we define the even modes

$$\begin{aligned} &u_{-2n} := -\frac{\overline{\partial} \psi_{-(2n-1)} + \partial \psi_{-(2n+1)}}{a}, \quad 1 \leq n \leq q-1,\ q \geq 2, \quad \text{and} \\ &u_{2n} := \overline{u}_{-2n}, \quad 1 \leq n \leq q-1,\ q \geq 2. \end{aligned} \tag{139}$$

By the construction in (137), (138), and (139), we have

$$\begin{aligned} &u_{-(2n-1)} \in C^2(\Omega; l_\infty), \quad \text{for} \quad 1 \leq n \leq q,\ q \geq 1, \\ &u_{-2n} \in C^1(\Omega; l_\infty), \quad \text{for} \quad 0 \leq n \leq q,\ q \geq 1, \quad \text{and} \\ &\overline{\partial} u_{-(2n-1)} + \partial u_{-(2n+1)} + a u_{-2n} = 0, \quad \text{for} \quad 0 \leq n \leq q,\ q \geq 1, \end{aligned} \tag{140}$$

is satisfied. Moreover, by conjugating the last equation in (140), we have the Fourier modes satisfy

$$\overline{\partial} u_{-(2n-1)} + \partial u_{-(2n+1)} + a u_{-2n} = 0, \quad \text{for} \quad |n| \leq q,\ q \geq 1. \tag{141}$$

By the class (122), and reality of g, we have the trace of $u_{-(2n-1)}$ in (136) satisfy

$$u_{-(2n-1)}|_{\varGamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\varGamma} = \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq q,\ q \geq 1. \tag{142}$$

We check next that the trace of u_{-2n} is g_{-2n} for $1 \leq n \leq q-1,\ q \geq 2$:

$$u_{-2n}\big|_{\varGamma} = -\frac{\overline{\partial} \psi_{-(2n-1)} + \partial \psi_{-(2n+1)}}{a}\Bigg|_{\varGamma} = g_{-2n}, \tag{143}$$

where the last equality uses the condition in class (122). Similar calculation to (143) for mode u_0 in (137), and mode u_{-2q} in (138), give the trace

$$u_0\big|_{\Gamma} = g_0, \quad \text{and} \quad u_{-2q}\big|_{\Gamma} = g_{-2q}, \quad q \geq 1. \tag{144}$$

Thus, from (142), (143) and (144), we have the traces:

$$u_n\big|_{\Gamma} = g_n, \quad \forall |n| \leq 2q,\ q \geq 1. \tag{145}$$

Step 4: The construction of the tensor field $\mathbf{f}_\Psi$ whose attenuated X-ray data is g.

The components of the m-tensor $\mathbf{f}_\Psi$ are defined via the one-to-one correspondence between the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m\rangle$ and the functions $\{f_{\pm(2n+1)} :\ 0 \leq n \leq q\}$ as follows.

We first define f_{2q+1} by using $\psi_{-(2q-1)}$ from the non-uniqueness class, and the Fourier modes $u_{-(2q+1)}, u_{-(2q+2)}$ in (128). Next, define f_{2q-1} by using $\psi_{-(2q-1)}, \psi_{-(2q-3)}$ from the non-uniqueness class, and Fourier mode $u_{-(2q+1)}$ in (128). Then, define $\{f_{2n+1} :\ 0 \leq n \leq q-2\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-(2n+1)} :\ 0 \leq n \leq q\}$ by conjugation.

(146)

$$f_{2q+1} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}}{a}\right) + \partial u_{-(2q+2)} + a u_{-(2q+1)}, \quad q \geq 1,$$

$$f_{2q-1} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2q-3)} + \partial \psi_{-(2q-1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}}{a}\right) + a\psi_{-(2q-1)}, \quad q \geq 2,$$

$$f_{2n+1} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2n-1)} + \partial \psi_{-(2n+1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-(2n+1)} + \partial \psi_{-(2n+3)}}{a}\right) + a\psi_{-(2n+1)},\ 1 \leq n \leq q-2,$$

$$f_1 := \begin{cases} -2\overline{\partial}\left(\frac{\mathbb{R}\mathrm{e}\,\partial\psi_{-1}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-1} + \partial u_{-3}}{a}\right) + a\psi_{-1}, & q = 1,\\ -2\overline{\partial}\left(\frac{\mathbb{R}\mathrm{e}\,\partial\psi_{-1}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-1} + \partial \psi_{-3}}{a}\right) + a\psi_{-1}, & q \geq 2,\end{cases}$$

$$f_{-(2n+1)} := \overline{f_{2n+1}}, \quad 0 \leq n \leq q,\ q \geq 1,$$

By construction, $f_{2n+1} \in C(\Omega)$ for $0 \leq n \leq q,\ q \geq 1$, as $u_{-(2q+1)} \in C^2(\Omega; l_\infty)$ from (129), and $\psi_{-(2n-1)} \in C^2(\Omega; l_\infty)$, for $1 \leq n \leq q-1,\ q \geq 1$, from (122). We use these $m+1$ Fourier modes $\langle f_{\pm 1}, f_{\pm 3}, \cdots, f_{\pm m}\rangle$, and equations (14), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m\rangle$, and thus the m-tensor field $\mathbf{f}_\Psi \in C(\mathbf{S}^m; \Omega)$.

Define the real valued function u via its Fourier modes

$$\begin{aligned} u(z, \boldsymbol{\theta}) := &\sum_{|n| \geq 2q+1} u_n(z) e^{\imath n\theta} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{q} \psi_{-(2n-1)}(z) e^{-\imath(2n-1)\theta}\right\} + \frac{-2\,\mathbb{R}\mathrm{e}\,\partial\psi_{-1}(z)}{a} \\ &+ 2\,\mathbb{R}\mathrm{e}\left(-\frac{\overline{\partial}\psi_{-(2q-1)}(z) + \partial u_{-(2q+1)}(z)}{a}\right) e^{-\imath(2q)\theta} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{q-1} u_{-2n} e^{-\imath(2n\theta)}\right\}. \end{aligned} \tag{147}$$

Using (133) and (145), and definition of $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_{a,g}^{\mathrm{odd}}$ for $q \geq 1$, the trace $u(\cdot, \boldsymbol{\theta})$ in (147) extends to the boundary, and its trace satisfy $u(\cdot, \boldsymbol{\theta})|_{\Gamma} = g(\cdot, \boldsymbol{\theta})$.

Moreover, by using (133), (141) and the definition of f_{2n-1} for $|n| \leq q,\ q \geq 1$ in (146), the real valued u defined in (147) satisfies the transport equation (117):

$$\boldsymbol{\theta} \cdot \nabla u + a u = \langle \mathbf{f}_\Psi, \boldsymbol{\theta}^{2q+1}\rangle, \quad q \geq 1.$$

□

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