Extremal behavior of stationary marked point processes

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Abstract

We consider stationary configurations of points in Euclidean space which are marked by positive random variables called scores. The scores are allowed to depend on the relative positions of other points and outside sources of randomness. Such models have been thoroughly studied in stochastic geometry, e.g. in the context of random tessellations or random geometric graphs.

It turns out that in a neighbourhood of a point with an extreme score one can often rescale positions and scores of nearby points to obtain a limiting point process, which we call the tail configuration. Under some assumptions on dependence between scores, this local limit determines the global asymptotics for extreme scores within increasing windows in \mathbb{R}^d . The main result establishes the convergence of rescaled positions and clusters of high scores to a Poisson cluster process, quantifying the idea of the Poisson clumping heuristic by D. Aldous (in the point process setting).

We apply our results to models based on (marked) Poisson processes where the scores depend on the distance to the kth nearest neighbor and where scores are allowed to propagate through a random network of points depending on their locations.

Keywords: cluster; extreme score; marked point process; moving maxima; nearest neighbor; Poisson convergence

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1 Introduction

In studies of extreme, or rare, features in a point process configuration, Poisson limits and extreme value distributions naturally appear, see, for instance, Owada [19], Otto [18] or Bobrowski et al. [7]. For geometric structures without a significant local dependence

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Poisson limits are not unexpected. In this case, high scores essentially arrive as isolated points which makes analysis relatively simple. In more complicated cases, one can try to remove clumps of large scores in the language of Aldous [1]. However, if one simply substitutes a clump (or a cluster) of large scores by a single score, one looses important information about the geometric properties of the limiting extremal objects. Here we derive and present a mathematically rigorous solution to this problem.

In particular, we present techniques suitable to analyze extremes of marked point processes in Euclidean space. Intuitively, these processes can be viewed as a sequence of scores (i.e., random values) recorded at random locations. We concentrate on points whose scores are high with the aim to understand the appearance of other such points nearby, meaning that we allow extreme scores arising in clusters. Such clustering phenomenon has been well described by Aldous [1], see Chenavier and Robert [8] for a recent analysis of extremes in random tessellations.

Our key concept is that of the tail configuration, which is closely related to the tail process introduced in [5] and since then widely used in time series [11, 13, 15]. For stationary time series this process appears after conditioning upon a high score at zero. When adapting this idea to spatial setting, one needs to work with Palm versions of marked point processes, which necessarily has a point located at the origin, and then condition upon the fact that the score at 0 is high. Passing to the limit involves scaling down the scores, but often also requires scaling of the locations.

The resulting tail configuration provides a local description of the clustering phenomenon related to high scores. Looking at high scores globally in a big domain shows that they build islands (clusters of points) in space. Our main result Theorem 4.2 provides a limit theorem showing convergence of such global cluster process to a marked Poisson process in the space, whose marks are point processes themselves considered equivalent up to translation. In this way we factor out positions of the clusters and explicitly describe the limiting distribution and the extremal index which provides the information about the mean cluster size. One can compare our result with its classical counterpart for discrete time stationary sequences as presented in [4, Theorem 3.6] or [15, Theorem 6.1.4].

Although our theory applies much more generally, we illustrate our results on two examples with scores obtained from a background Poisson point process. In the first one, the scores are simply reciprocals to the distance between a point and its kth nearest neighbor. As a special case of this example, for k = 1 one describes the limiting structure of the process of points with large inradii in the Poisson–Voronoi tessellation, studied in [8] in dimension 2. In our second example, the points are initially marked by i.i.d. random values, but the actual score at a point t say, depends also on the (weighted) values at the points in a possibly random neighborhood of t. The example can be seen as a generalization of the moving maxima model from time series analysis. But here, we are particularly interested to see how large values propagate in such a random network.

The paper is organized as follows. Section 2.1 sets up our basic definitions. Section 3 introduces the tail process and its spectral counterpart. Its central result is a certain invariance property of the tail configuration (Theorem 3.4), which is related to the time-change formula from time series, see [5]; cf. also [22] and [16] for a discussion on the

connection to standard Palm theory. Section 4 contains the main result which provides a Poisson approximation of extremal clusters. The construction involves the standard idea of splitting the space into rectangular blocks. Three main assumptions consist of conditions on block sizes and dependence within and between extremal blocks, see Section 4.4. In Section 4.5 we discuss the representation of the extremal index and the distribution of the typical cluster. The key concept here is the idea of an anchoring function, which is motivated by a similar concept from random fields indexed over \mathbb{Z}^d , see [3]. The proof of the main result is postponed to Section 6. Section 5.1 treats in detail the case of scores derived from the neighboring structure of a stationary Poisson process in Euclidean space. In Section 5.2 we deal with the moving maxima model. In both examples the main steps consist of determining the tail configuration and then checking the appropriate dependence conditions.

2 Random counting measures and marked point processes

2.1 Basic definitions

Consider the space $\mathbb{E} := \mathbb{R}^d \times (0, \infty)$ with the standard product topology and the Borel σ -algebra $\mathcal{B}(\mathbb{E})$. A point in \mathbb{E} is written as (t, s), where $t \in \mathbb{R}^d$ is said to be the *position* and s > 0 is the mark or *score* (at t).

A Borel measure μ on \mathbb{E} is said to be a *counting measure* if it takes values $\{0, 1, 2, \ldots\} \cup \{\infty\}$ on $\mathcal{B}(\mathbb{E})$. Denote by μ' the projection of μ on \mathbb{R}^d , that is, $\mu'(B) = \mu(B \times (0, \infty))$ for each Borel B in \mathbb{R}^d . We call a counting measure simple if its projection on \mathbb{R}^d is simple, that is, $\mu(\{t\} \times (0, \infty)) \leq 1$ for all $t \in \mathbb{R}^d$. We write $(t, s) \in \mu$ and $\mu(t) = s$ if $\mu(\{(t, s)\}) = 1$; for convenience, if $\mu(\{t\} \times (0, \infty)) = 0$ we will sometimes write $\mu(t) = 0$. Each simple counting measure μ is uniquely represented by the set of its atoms. To emphasize this, we write

$$\mu = \sum_{(t,s)\in\mu} \delta_{(t,s)},$$

or equivalently, $\mu = \{(t_i, s_i) : i = 1, ..., k\}$, where k is the total number of atoms in μ (which may be infinite). For a Borel function $f : \mathbb{E} \to \mathbb{R}$, denote

$$\mu(f) \equiv \int f d\mu := \sum_{(t,s) \in \mu} f(t,s).$$

In the following, it is essential to single out families of sets where counting measures take finite values. Introduce subfamilies $\mathcal{B}_{11}, \mathcal{B}_{10}, \mathcal{B}_{01} \subseteq \mathcal{B}(\mathbb{E})$:

- i) $A \in \mathcal{B}_{11}$ if $A \subseteq B \times (\varepsilon, \infty)$ for some $B \subseteq \mathbb{R}^d$ bounded and $\varepsilon > 0$;
- ii) $A \in \mathcal{B}_{10}$ if $A \subseteq B \times (0, \infty)$ for some $B \subseteq \mathbb{R}^d$ bounded;
- iii) $A \in \mathcal{B}_{01}$ if $A \subseteq \mathbb{R}^d \times (\varepsilon, \infty)$ for some $\varepsilon > 0$.

For consistency, we sometimes write $\mathcal{B}_{00} := \mathcal{B}(\mathbb{E})$. These families provide examples of a *boundedness* or *bornology* on \mathbb{E} , see [2], and clearly satisfy $\mathcal{B}_{11} = \mathcal{B}_{10} \cap \mathcal{B}_{01}$.

Let \mathcal{N}_{ij} denote the family of simple counting measures with finite values on \mathcal{B}_{ij} , $i, j \in \{0, 1\}$. For convenience, in the sequel denote $\mathcal{N} := \mathcal{N}_{11}$. Note that $\mathcal{N}_{00} \subset \mathcal{N}_{01} \cap \mathcal{N}_{10}$ and $\mathcal{N}_{01} \cup \mathcal{N}_{10} \subset \mathcal{N}_{11} = \mathcal{N}$. The families \mathcal{N}_{ij} are equipped with the vague topology determined by the choice of the boundedness, see [2].

Definition 2.1. Counting measures $(\mu_n)_{n\in\mathbb{N}}$ from \mathcal{N}_{ij} with $i, j \in \{0, 1\}$ are said to converge to $\mu \in \mathcal{N}_{ij}$ as $n \to \infty$ in \mathcal{B}_{ij} or \mathcal{B}_{ij} -vaguely (notation $\mu_n \xrightarrow{v} \mu$) if $\mu_n(f) \to \mu(f)$ as $n \to \infty$ for all continuous bounded functions $f : \mathbb{E} \to \mathbb{R}$ whose support is in some $B \in \mathcal{B}_{ij}$.

The notion of \mathcal{B}_{ij} -vague convergence on \mathcal{N}_{ij} can be seen as convergence with respect to the smallest topology on \mathcal{N}_{ij} which makes the mappings $\mu \mapsto \mu(f)$ continuous for all continuous bounded functions $f : \mathbb{E} \to \mathbb{R}$ whose support is in some $B \in \mathcal{B}_{ij}$; call this topology the \mathcal{B}_{ij} -vague topology. Since the extension of this topology to the larger space of all Borel measures on \mathbb{E} which are finite on elements of \mathcal{B}_{ij} , is known to be Polish (see [14, Theorem 4.2] and [2, Theorem 3.1]), the \mathcal{B}_{ij} -vague topology on \mathcal{N}_{ij} is separable and metrizable.

We have used the phrase \mathcal{B}_{ij} -vague instead of simply vague since one can consider \mathcal{N}_{ij} with respect to the (weaker) $\mathcal{B}_{i'j'}$ -vague topology whenever $\mathcal{N}_{ij} \subseteq \mathcal{N}_{i'j'}$ which is equivalent to $\mathcal{B}_{i'j'} \subseteq \mathcal{B}_{ij}$.

Define the *shift operators* φ_z , $z \in \mathbb{R}^d$, on \mathbb{E} by letting

$$\varphi_z(t,s) := (t-z,s),$$

and let

$$\varphi_z \mu := \sum_{(t,s)\in\mu} \delta_{\varphi_z(t,s)} = \sum_{(t,s)\in\mu} \delta_{(t-z,s)}$$

Thus, if μ at z has score s, then s becomes the score of $\varphi_{z}\mu$ at 0, so that the shift applies only to positions, leaving the scores unchanged. Since the families \mathcal{B}_{ij} are invariant under shifts, the families \mathcal{N}_{ij} are also invariant. Observe that the mapping $(z,\mu) \mapsto \varphi_{z}\mu$ from $\mathbb{R}^{d} \times \mathcal{N}_{ij}$ to \mathcal{N}_{ij} is continuous if \mathcal{N}_{ij} is equipped with the vague topology generated by any $\mathcal{B}_{i'j'} \subseteq \mathcal{B}_{ij}$.

The sets \mathcal{N}_{ij} , i, j = 0, 1, are equipped with the Borel σ -algebra generated by the maps $B \mapsto \mu(B)$ for all $B \in \mathcal{B}(\mathbb{E})$; this coincides with the Borel σ -algebra generated by \mathcal{B}_{ij} -vaguely open sets. A random counting measure is a random element X in \mathcal{N} . It is called stationary if $\varphi_z X$ and X coincide in distribution for all $z \in \mathbb{R}^d$.

If a random counting measure X takes values from a smaller family \mathcal{N}_{10} , then X is called a marked point process on \mathbb{R}^d (with marks in $(0, \infty)$). Then, for all bounded $A \subset \mathbb{R}^d$, we have $X(A \times (0, \infty)) < \infty$ a.s., that is, the number of $t \in A$ such that $(t, s) \in X$ for some s > 0, is almost surely finite. We assume throughout that this number has a finite mean. If X is stationary, the expected value of $X(A \times (0, \infty))$ is proportional to the Lebesgue measure of $A \in \mathcal{B}(\mathbb{R}^d)$. The coefficient of proportionality λ is said to be the intensity of X. Later on we usually assume that $\lambda = 1$. Each stationary marked point process X on \mathbb{R}^d of finite intensity admits its *Palm* version \tilde{X} , which is a marked point process on \mathbb{R}^d , satisfying

$$\mathbf{E}\left[\sum_{(t,s)\in X} h(t,\varphi_t X)\right] = \lambda \int_{\mathbb{R}^d} \mathbf{E}\left[h(t,\tilde{X})\right] \mathrm{d}t$$
(2.1)

for all measurable $h : \mathbb{R}^d \times \mathcal{N}_{10} \to \mathbb{R}_+$. The Palm version \tilde{X} has the following invariance property

$$\mathbf{E}\left[\sum_{(t,s)\in\tilde{X}}h(-t,\varphi_t\tilde{X})\right] = \mathbf{E}\left[\sum_{(t,s)\in\tilde{X}}h(t,\tilde{X})\right]$$
(2.2)

for all measurable $h : \mathbb{R}^d \times \mathcal{N}_{10} \to \mathbb{R}_+$, see [10, Theorem 13.2.VIII]. Note that \tilde{X} almost surely contains the point $(0,\xi)$; the random variable $\xi := \tilde{X}(0)$ is said to be the score of the Palm version at the origin.

Let \xrightarrow{w} denote weak convergence of probability measures and \xrightarrow{d} the corresponding convergence in distribution. Distributional convergence of random counting measures in \mathcal{N} is understood with respect to a particular version of the vague topology, and so relies on the choice of the corresponding boundedness. It is well known that $X_n \xrightarrow{d} X$ in \mathcal{B}_{ij} if and only if the Laplace functionals of X_n

$$L_f(X_n) := \mathbf{E} \, \exp\left\{-\sum_{(t,s)\in X_n} f(t,s)\right\}$$

converge to $L_f(X)$ as $n \to \infty$ for all continuous $f : \mathbb{E} \to [0, \infty)$ with support in \mathcal{B}_{ij} , see [14, Theorem 4.11].

2.2 A general construction of scores

In our examples we deal with marked point processes X derived from a marked Poisson point process using the following general construction. Let P be an independently marked stationary Poisson process in \mathbb{R}^d , where \mathbb{R}^d is the space of locations and the marks take values from $(0, \infty)$. Note that trivial amendments make it possible to consider the marks taking values in a general Polish space, and allow this construction to be applied to a general marked point processes. The intensity measure of P is the product of the Lebesgue measure on \mathbb{R}^d (possibly, scaled by a constant) and a probability measure m on $(0, \infty)$.

Consider a measurable function $\psi : \mathbb{R}^d \times \mathcal{N}_{10} \to (0, \infty)$ such that $\psi(t-z, \varphi_z \mu) = \psi(t, \mu)$ for all $z \in \mathbb{R}^d$. In the following ψ is called a *scoring function*. For $\mu := \sum \delta_{(t,z)} \in \mathcal{N}_{10}$ denote

$$\Psi(\mu) := \sum_{(t,z)\in\mu} \delta_{(t,\psi(t,\mu))} \,.$$

This defines a mapping from \mathcal{N}_{10} to \mathcal{N}_{10} . While this mapping does not change locations of points, it equips each point $t \in \mu'$ with a new score $s = \psi(t, \mu)$. The same construction can be clearly applied to a Poisson process in \mathbb{R}^d without marks, which fits in the above framework by letting all the marks equal to 0 say. The shift-invariance property of ψ implies that

$$\Psi(\varphi_z \mu) = \varphi_z(\Psi(\mu)). \tag{2.3}$$

By the Poisson assumption, the Palm version of P is given by

$$P = P + \delta_{(0,\zeta)},$$

where ζ has distribution m, and is independent of P.

Lemma 2.2. The Palm version of $\Psi(P)$ is given by $\Psi(P)$.

Proof. The Palm version \tilde{P} satisfies

$$\mathbf{E}\Big[\sum_{(t,u)\in P} h(t,\varphi_t P)\Big] = \lambda \int_{\mathbb{R}^d} \mathbf{E}[h(t,\tilde{P})] \mathrm{d}t \,.$$

Therefore, (2.3) yields that

$$\mathbf{E}\Big[\sum_{(t,u)\in P} h(t,\varphi_t\Psi(P))\Big] = \mathbf{E}\Big[\sum_{(t,u)\in P} h(t,\Psi\varphi_t(P))\Big] = \lambda \int_{\mathbb{R}^d} \mathbf{E}[h(t,\Psi(\tilde{P}))] dt.$$

By the definition of Palm measure, the left-hand side is

$$\lambda \int_{\mathbb{R}^d} \mathbf{E} \left[h(t, \widetilde{\Psi(P)}) \right] \mathrm{d}t \,,$$

so that the Palm version of $\Psi(P)$ is indeed $\Psi(\tilde{P})$.

3 Tail configuration

Define a family of scaling operators $T_{v,u}: \mathbb{E} \to \mathbb{E}, u, v > 0$, by

$$T_{v,u}(t,s) := (t/v, s/u), \quad (t,s) \in \mathbb{E}.$$
 (3.1)

For every $\mu \in \mathcal{N}$, define its scaled version $T_{v,u}\mu$ by letting $(T_{v,u}\mu)(A) := \mu(T_{v^{-1},u^{-1}}A)$ for all Borel A. Equivalently, $T_{v,u}\mu = \sum \delta_{t/v,s/u}$ if $\mu = \sum \delta_{t,s}$.

In the following we mostly work with counting measures scaled by $T_{r(u),u}$ for u > 0, where a function $r: (0, \infty) \to (0, \infty)$, is fixed and regularly varying at infinity, i.e.,

$$r(u) = l(u)u^{\beta}, \quad u > 0,$$
 (3.2)

for some $\beta \in \mathbb{R}$ and a slowly varying function l. We will refer to r and β as the scaling function and scaling index, respectively. Note that $\beta = 0$ and $r(u) \equiv 1$ are allowed.

Definition 3.1. Fix a function r which is regularly varying at infinity. Let $X \in \mathcal{N}_{10}$ be a stationary marked point process on \mathbb{E} with Palm version \tilde{X} and the score at the origin being ξ . If there exists a random counting measure $Y \in \mathcal{N}$ such that $Y(\{0\} \times (1, \infty)) = 1$ a.s. and

$$\mathbf{P}(T_{r(u),u}\tilde{X} \in \cdot \mid \xi > u) \xrightarrow{w} \mathbf{P}(Y \in \cdot) \quad \text{as } u \to \infty$$
(3.3)

with respect to the \mathcal{B}_{11} -vague topology, then Y is called the *tail configuration* of X (with respect to the scaling function r).

Note that the tail configuration Y is assumed to be simple and it necessarily contains the point $(0,\eta)$ with $\mathbf{P}(\eta > 1) = 1$. We call the random variable η the *tail score* at the origin. While X and \tilde{X} are marked point processes and thus belong to \mathcal{N}_{10} , the tail configuration Y in general takes values in \mathcal{N} , which is a larger family.

For c > 0, denote by B_c the open Euclidean ball of radius c centered at 0. It is convenient to denote

$$\mathcal{A}_y := \big\{ \mu \in \mathcal{N} : (0, s) \in \mu \text{ for some } s > y \big\}.$$
(3.4)

Observe that the tail process Y almost surely belongs to \mathcal{A}_1 .

Proposition 3.2. If (3.3) holds then the score at the origin ξ has a regularly varying tail, that is,

$$\lim_{u \to \infty} \frac{\mathbf{P}(\xi > uy)}{\mathbf{P}(\xi > u)} = y^{-\alpha}, \quad y > 0,$$
(3.5)

for some $\alpha > 0$, and the tail score at the origin η is Pareto(α)-distributed, that is,

$$\mathbf{P}(\eta > y) = y^{-\alpha}, \quad y \ge 1.$$
(3.6)

Proof. Observe that for a sequence of counting measures μ_n in \mathcal{A}_1 and a counting measure $\mu \in \mathcal{A}_1$ such that $\mu_n \xrightarrow{v} \mu$ (in \mathcal{B}_{11}), one can always find a bounded set $B \in \mathcal{B}_{11}$ of the form $B = B_{\varepsilon} \times (1 + \varepsilon, \infty)$ such that $\mu(B) = 1$ and $\mu(\partial B) = 0$. It is known that the convergence of simple counting measures can be reformulated as convergence of their support points with a suitable enumeration, e.g. see [2, Proposition 2.8], hence $\mu_n(0) \to \mu(0)$, i.e., the score at the origin is a continuous function on $\mathcal{N} \cap \mathcal{A}_1$. By a continuous mapping argument, $\xi/u = T_{r(u),u}\tilde{X}(0)$, conditionally on $\xi > u$, converges in distribution to $\eta = Y(0)$ as $u \to \infty$. More precisely,

$$\frac{\mathbf{P}(\xi > uy, \xi > u)}{\mathbf{P}(\xi > u)} \to \mathbf{P}(\eta > y) \quad \text{as} \ u \to \infty$$

for all y > 0 which are continuity points for η . Standard arguments now yield that (3.5) holds for some $\alpha > 0$, and (3.6) follows immediately.

The constant $\alpha > 0$ from (3.6) will be called the *tail index* of X, ξ and η . Observe that α is in general unrelated to the scaling index β of the scaling function r. Further, the point process

$$\Theta := T_{\eta^{\beta},\eta}Y \tag{3.7}$$

in \mathcal{N} is called the *spectral tail configuration* of X. By definition, Θ almost surely contains the point (0, 1).

Proposition 3.3. The spectral tail configuration is independent of η , and satisfies

$$\mathbf{P}(T_{r(\xi),\xi}\tilde{X} \in \cdot \mid \xi > u) \xrightarrow{w} \mathbf{P}(\Theta \in \cdot) \quad as \ u \to \infty.$$
(3.8)

Proof. Recall the set \mathcal{A}_y from (3.4) and note that \mathcal{A}_1 consists of all $\mu \in \mathcal{N}$ such that $\mu(\{0\} \times (1, \infty)) \geq 1$. Consider the family of mappings H_u , u > 0, defined by $H_u(\mu) := (s, r(su)/r(u), \mu)$, where $s = \mu(0)$ is the score at the origin of $\mu \in \mathcal{A}_1$.

Let $\mu_u \xrightarrow{v} \mu$ as $u \to \infty$ (in \mathcal{B}_{11}) for some μ_u and μ from \mathcal{A}_1 . Denote $s_u := \mu_n(0)$ and $s := \mu(0)$. Then $s_u \to s$ as $u \to \infty$. Since r is regularly varying with index β , the convergence $r(yu)/r(u) \to y^\beta$ as $u \to \infty$ holds locally uniformly in y on $(0, \infty)$, see e.g. [24, Proposition 2.4], so that $r(s_u u)/r(u) \to s^\beta$ as $u \to \infty$. Therefore, $H_u(\mu_u) \to (s, s^\beta, \mu)$ as $u \to \infty$. The extended continuous mapping theorem applied to (3.3) (see [6, Theorem 5.5]) yields that

$$\mathbf{P}((\xi/u, r(\xi)/r(u), T_{r(u), u}\tilde{X}) \in \cdot \mid \xi > u) \xrightarrow{w} \mathbf{P}((\eta, \eta^{\beta}, Y) \in \cdot)$$

on $(1,\infty) \times (0,\infty) \times \mathcal{N}$. Another application of the continuous mapping theorem yields

$$\mathbf{P}((\xi/u, T_{r(\xi),\xi}\tilde{X}) \in \cdot \mid \xi > u) \xrightarrow{w} \mathbf{P}((\eta, \Theta) \in \cdot)$$

on $(1, \infty) \times \mathcal{N}$. This yields (3.8) and, together with (3.5),

$$\mathbf{P}(\eta > y, \Theta \in B) = \lim_{u \to \infty} \mathbf{P}(\xi > uy, T_{r(\xi),\xi} \tilde{X} \in B \mid \xi > u)$$
$$= \lim_{u \to \infty} \frac{\mathbf{P}(\xi > uy)}{\mathbf{P}(\xi > u)} \mathbf{P}(T_{r(\xi),\xi} \tilde{X} \in B \mid \xi > uy)$$
$$= y^{-\alpha} \mathbf{P}(\Theta \in B) = \mathbf{P}(\eta > y) \mathbf{P}(\Theta \in B)$$

for all $y \ge 1$ and all Borel subsets $B \subseteq \mathcal{N}$ such that $\mathbf{P}(\Theta \in \partial B) = 0$. This implies that η and Θ are independent, since the class of all such B's (denoted by \mathcal{S}) is closed under finite intersections and generates the Borel σ -algebra on \mathcal{N} . The latter fact follows, since the vague topology on \mathcal{N} is separable and metrizable, so one can represent every open subset of \mathcal{N} as a countable union of open balls which are elements of \mathcal{S} .

To conclude this section, we show that the invariance property (2.2) of the Palm distribution \tilde{X} induces a similar property of the tail configuration Y which, as in [22, Section 2], could be called *exceedance-stationarity*, cf. also [16].

Theorem 3.4. For every measurable $h : \mathbb{R}^d \times \mathcal{N} \to [0, \infty)$,

$$\mathbf{E}\left[\sum_{(t,s)\in Y} h(t,Y)\mathbf{1}_{\{s>1\}}\right] = \mathbf{E}\left[\sum_{(t,s)\in Y} h(-t,\varphi_t Y)\mathbf{1}_{\{s>1\}}\right].$$
(3.9)

Proof. Since $T_{r(u),u}$ scales the scores with u^{-1} , a score in $T_{r(u),u}\tilde{X}$ exceeding 1 corresponds to a score in \tilde{X} exceeding u. Thus, by (2.2),

$$\begin{split} \mathbf{E} \Big[\sum_{(t,s)\in T_{r(u),u}\tilde{X}} h(t,T_{r(u),u}\tilde{X}) \mathbf{1}_{\{s>1\}} \mathbf{1}_{\{\xi>u\}} \Big] \\ &= \mathbf{E} \left[\sum_{(t,s)\in\tilde{X}} h(t/r(u),T_{r(u),u}\tilde{X}) \mathbf{1}_{\{\tilde{X}(t)>u\}} \mathbf{1}_{\{(T_{r(u),u}\tilde{X})(0)>1\}} \right] \\ &= \mathbf{E} \left[\sum_{(t,s)\in\tilde{X}} h(-t/r(u),T_{r(u),u}\varphi_{t}\tilde{X}) \mathbf{1}_{\{(\varphi_{t}\tilde{X})(-t)>u\}} \mathbf{1}_{\{(T_{r(u),u}\varphi_{t}\tilde{X})(0)>1\}} \right] \\ &= \mathbf{E} \left[\sum_{(t,s)\in\tilde{X}} h(-t/r(u),\varphi_{t/r(u)}T_{r(u),u}\tilde{X}) \mathbf{1}_{\{\tilde{X}(0)>u\}} \mathbf{1}_{\{\tilde{X}(t/r(u))>u\}} \right] \\ &= \mathbf{E} \left[\sum_{(t,s)\in T_{r(u),u}\tilde{X}} h(-t,\varphi_{t}T_{r(u),u}\tilde{X}) \mathbf{1}_{\{\xi>u\}} \mathbf{1}_{\{s>1\}} \right]. \end{split}$$

We aim to show that both sides (if normalized by $\mathbf{P}(\xi > u)$) converge to the corresponding sides of (3.9). However, a direct application of (3.3) is not possible since the involved functionals are not necessarily bounded.

By an approximation argument, it suffices to consider a bounded continuous function $h : \mathbb{R}^d \times \mathcal{N} \to [0, \infty)$ such that $h(t, \mu) = 0$ for all t outside a ball B_c . Choose a > 2c such that $Y(\partial B_a \times (0, \infty)) = 0$ a.s. Such an a exists, since the projection Y' of Y onto \mathbb{R}^d is the projection Θ' of Θ scaled by η^{β} . Thus, we can write $Y' = \eta^{\beta} \Theta'$ and so

$$\mathbf{P}(Y' \cap \partial B_a \neq \emptyset) = \mathbf{P}(\eta^{\beta} \Theta'(\partial B_a) > 0) = 0.$$

This is easily seen, since the set of u such that $u\Theta'$ intersects ∂B_a is a countable set and η has an absolutely continuous distribution. By the same argument it is possible to choose a such that the probability that ||t - t'|| = a for some $t, t' \in Y'$ is also zero.

Fix $k \in \mathbb{N}$ and consider the maps $H_1, H_2 : \mathcal{A}_0 \to \mathbb{R}_+$ given by

$$H_1(\mu) := \sum_{(t,s)\in\mu} h(t,\mu) \mathbf{1}_{\{\mu(B_a \times (1,\infty)) \le k\}} \mathbf{1}_{\{s>1\}}, H_2(\mu) := \sum_{(t,s)\in\mu} h(-t,\varphi_t\mu) \mathbf{1}_{\{(\varphi_t\mu)(B_a \times (1,\infty)) \le k\}} \mathbf{1}_{\{s>1\}}.$$

Both maps are bounded by $k \sup h$. We need to show that Y belongs to the set of their discontinuity points with probability zero.

Since $Y \in \mathcal{N}$, we have $Y(B_c \times (y, \infty)) < \infty$ with probability one for all y > 0. By (3.7),

$$Y(B_c \times \{y\}) = \Theta(B_{c\eta^{-\beta}} \times \{y\eta^{-1}\}).$$

Since η has a non-atomic distribution, it is easy to see that $Y(B_c \times \{y\}) = 0$ with probability one. Given the choice of a, the tail configuration Y is a discontinuity point of H_1 with probability zero. For the functional H_2 , note that μ is a discontinuity point of $\mathbf{1}\{(\varphi_t\mu)(B_a \times (1,\infty)) \leq k\}$ only if ||t - t'|| = a for some $(t, s), (t', s') \in \mu$. By the choice of a, the tail configuration Yis almost surely not a point of discontinuity.

Since $\mathbf{E}[H_1(T_{r(u),u}\tilde{X}) | \xi > u] = \mathbf{E}[H_2(T_{r(u),u}\tilde{X}) | \xi > u]$ for all u > 0 (argue exactly as in the beginning of the proof), applying (3.3) we obtain that $\mathbf{E}H_1(Y) = \mathbf{E}H_2(Y)$. Letting $k \to \infty$ yields (3.9) for all continuous bounded functions h which vanish for $t \notin B_c$.

Remark 3.5. Exceedance-stationarity property (3.9) and the polar decomposition from Proposition 3.3 yield

$$\mathbf{E}\left[\sum_{(y,s)\in\Theta}h(-y/s^{\beta},T_{r(s),s}\varphi_{y}\Theta)\right] = \mathbf{E}\left[\sum_{(y,s)\in\Theta}h(y,\Theta)s^{\alpha}\right]$$

for every measurable $h : \mathbb{R}^d \times \mathcal{N} \to [0, \infty)$, see [22, Remark 2.11]. This property of the spectral tail configuration can be seen as the analogue of the *time-change formula* known to characterize the class of all spectral tail processes (and thus tail processes) of regularly varying time series, see [11, 13].

4 Poisson approximation for extremal clusters

In what follows assume that X is a stationary marked point process on \mathbb{R}^d of unit intensity with marks (scores) in $(0, \infty)$, which admits a tail configuration Y in the sense of Definition 3.1. The main goal of this section is to describe the limiting behavior of scores of X in $[0, \tau]^d$ which exceed a suitably chosen high threshold, as τ and the threshold size tend to infinity.

4.1 Extremal blocks

Let $(a_{\tau})_{\tau>0}$ be a family of positive real numbers chosen such that

$$\lim_{\tau \to \infty} \mathbf{E} \left[\sum_{(t,s) \in X, t \in [0,\tau]^d} \mathbf{1}_{\{s > a_\tau\}} \right] = \lim_{\tau \to \infty} \tau^d \mathbf{P}(\xi > a_\tau) = 1, \qquad (4.1)$$

where the first equality follows from the refined Campbell's theorem (2.1). By (3.5),

$$\lim_{\tau \to \infty} \tau^d \mathbf{P}(\xi > a_\tau \varepsilon) = \varepsilon^{-\alpha}, \quad \varepsilon > 0.$$
(4.2)

Let $(b_{\tau})_{\tau>0}$ be a family of positive real numbers such that $b_{\tau}/\tau \to 0$ as $\tau \to \infty$. Divide the hypercube $[0, \tau]^d$ into blocks of side length b_{τ} defined as

$$J_{\tau,\boldsymbol{i}} := \bigotimes_{j=1}^{d} [(i_j - 1)b_{\tau}, i_j b_{\tau}] \subset \mathbb{R}^d$$

$$(4.3)$$

for $\mathbf{i} = (i_1, \ldots, i_d) \in I_\tau := \{1, \ldots, k_\tau\}^d$, where $k_\tau := \lfloor \tau/b_\tau \rfloor$.¹ For every $\mathbf{i} \in I_\tau$, define

$$X_{\tau,\boldsymbol{i}} := X_{J_{\tau,\boldsymbol{i}}},\tag{4.4}$$

which is the restriction of X to $J_{\tau,i} \times (0, \infty)$.

For fixed τ and ε , think of *clusters* of extremal scores of X as blocks $X_{\tau,i}$ which contain at least one score exceeding $a_{\tau}\varepsilon$. For every $i \in I_{\tau}$, by (4.2) and since $b_{\tau}/\tau \to 0$,

$$\mathbf{P}(X_{\tau,i} \text{ is an extremal cluster}) = \mathbf{P}(\max_{(t,s)\in X_{\tau,i}} s > a_{\tau}\varepsilon)$$
$$\leq b_{\tau}^{d}\mathbf{P}(\xi > a_{\tau}\varepsilon) \to 0 \quad \text{as } \tau \to \infty,$$

where we used refined Campbell's theorem (2.1) in the second step.

4.2 Space for extremal blocks

Recall that X is a random element of the space \mathcal{N}_{10} so that the blocks $X_{\tau,i}$ can be considered as elements of \mathbb{N}_{01} which consists of simple counting measures on \mathbb{E} with finite values on $\mathbb{R}^d \times (\varepsilon, \infty)$ for all $\varepsilon > 0$. Recall further that $\mathcal{N}_{01} \subset \mathcal{N} = \mathcal{N}_{11}$ and that, on $\mathcal{N}_{01}, \mathcal{B}_{01}$ topology is stronger than the \mathcal{B}_{11} -topology. We now define a metric **m**, generating the \mathcal{B}_{01} -vague topology on \mathcal{N}_{01} .

Let $\mu, \nu \in \mathcal{N}_{01}$ be such that $\mu(\mathbb{E}), \nu(\mathbb{E}) < \infty$ (i.e., $\mu, \nu \in \mathcal{N}_{00}$). If $\mu(\mathbb{E}) \neq \nu(\mathbb{E})$ set $\mathsf{m}_0(\mu, \nu) = 1$, and if $\mu(\mathbb{E}) = \nu(\mathbb{E}) = k \in \mathbb{N}_0$ and $\mu = \sum_{i=1}^k \delta_{(t_i, s_i)}, \nu = \sum_{i=1}^k \delta_{(t'_i, s'_i)}$, define

$$\mathbf{m}_{0}(\mu,\nu) = \min_{\Pi} \max_{i=1,\dots,k} \left[(|t_{i} - t'_{\Pi(i)}| \vee |s_{i} - s'_{\Pi(i)}|) \wedge 1 \right] \,,$$

where the minimum is taken over all permutations Π of $\{1, \ldots, k\}$. Note that \mathbf{m}_0 is a metric generating the weak (that is, \mathcal{B}_{00} -vague) topology on \mathcal{N}_{00} , see [25, Proposition 2.3]. While the authors of [25] assume that the ground space is compact, an easy argument justifies the claim for the space $\mathbb{E} = \mathbb{R}^d \times (0, \infty)$. Observe also that \mathbf{m}_0 is by construction bounded by 1 and shift-invariant, that is, $\mathbf{m}_0(\varphi_y\mu,\varphi_y\nu) = \mathbf{m}_0(\mu,\nu)$ for all $y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{N}_{00}$.

For general $\mu, \nu \in \mathcal{N}_{01}$, set

$$\mathbf{m}(\mu,\nu) := \int_0^\infty \mathbf{m}_0(\mu^{1/u},\nu^{1/u})e^{-u}\mathrm{d}u\,, \tag{4.5}$$

where $\mu^{1/u}$ (and similarly $\nu^{1/u}$) is the restriction of μ on $\mathbb{R}^d \times (1/u, \infty)$. One can show that this is indeed a metric on \mathcal{N}_{01} which is bounded by 1, is shift-invariant, and that it generates the \mathcal{B}_{01} -vague topology. The latter claim follows since μ_n converges to μ in \mathcal{N}_{01} with respect to this topology if and only if $\mu_n^{1/u}$ converges weakly to $\mu^{1/u}$ for Lebesgue almost all $u \in (0, \infty)$, see [9, Proposition A2.6.II] and [17].

¹Technically speaking, we are dividing the hypercube $[0, k_{\tau}b_{\tau}]^d$. However, in applications this edge effect is easily seen to be negligible.

We will actually work with a *quotient space* of \mathcal{N}_{01} . For $\mu, \nu \in \mathcal{N}_{01}$, set $\mu \sim \nu$ if $\varphi_y \mu = \nu$ for some $y \in \mathbb{R}^d$, and denote by $\tilde{\mathcal{N}}_{01}$ the quotient space of shift-equivalent counting measures in \mathcal{N}_{01} . Denote by

$$[\mu] = \{\nu \in \mathcal{N}_{01} : \mu \sim \nu\} = \{\varphi_y \mu : y \in \mathbb{R}^d\}$$

the equivalence class of $\mu \in \mathcal{N}_{01}$. Define

$$\tilde{\mathsf{m}}([\mu],[\nu]) := \inf_{y,z \in \mathbb{R}^d} \mathsf{m}(\varphi_y \mu, \varphi_z \nu), \quad \mu, \nu \in \mathcal{N}_{01}.$$

Lemma 4.1. Function $\tilde{\mathbf{m}}$ is a metric on $\tilde{\mathcal{N}}_{01}$, and $(\tilde{\mathcal{N}}_{01}, \tilde{\mathbf{m}})$ is a separable metric space. Moreover, $\tilde{\mathbf{m}}([\mu_n], [\mu]) \to 0$ as $n \to \infty$ (denoted by $[\mu_n] \xrightarrow{v} [\mu]$) for $\mu_n, \mu \in \mathcal{N}_{01}$ if and only if there exist $y_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that $\varphi_{y_n}\mu_n \xrightarrow{v} \mu$ in \mathcal{N}_{01} .

Proof. Since m is shift-invariant,

$$\tilde{\mathsf{m}}([\mu], [\nu]) = \inf_{z \in \mathbb{R}^d} \mathsf{m}(\mu, \varphi_z \nu) = \inf_{z \in \mathbb{R}^d} \mathsf{m}(\varphi_z \mu, \nu)$$
(4.6)

for all $\mu, \nu \in \mathcal{N}_{01}$. It now follows from [21, Lemma 2.5.1] that \tilde{m} is a pseudo-metric on $\tilde{\mathcal{N}}_{01}$, and that $(\tilde{\mathcal{N}}_{01}, \tilde{m})$ is separable since $(\mathcal{N}_{01}, \mathbf{m})$ is. To show that \tilde{m} is actually a metric, assume that $\mathbf{m}(\varphi_{z_n}\mu,\nu) \to 0$ for a sequence (z_n) and conclude that (z_n) is bounded, so that it admits a convergent subsequence; it is crucial that μ and ν have finitely many points in $\mathbb{R}^d \times (\varepsilon, \infty)$ for every ε . The second claim follows directly from (4.6).

Next, recall the scaling operator $T_{r(u),u}$ defined at (3.1). Observe that

$$T_{r(u),u}\varphi_y\mu=\varphi_{y/r(u)}T_{r(u),u}\mu$$

for all $u > 0, y \in \mathbb{R}^d$ and $\mu \in \mathcal{N}_{01}$. Since $[T_{r(u),u}\mu] = [T_{r(u),u}\nu]$ whenever $\mu \sim \nu$, the scaling operator $T_{r(u),u}[\mu] := [T_{r(u),u}\mu]$ is well defined on $\tilde{\mathcal{N}}_{01}$. For $\mu \in \mathcal{N}_{01}$, denote by

$$M(\mu) := \max_{(t,s)\in\mu} s$$

the maximal score of μ , which is necessarily finite. Observe that the maximal score is shift-invariant and thus it is well defined on $\tilde{\mathcal{N}}_{01}$.

Finally, for every $\mu \in \mathcal{N}_{01}$ such that $M(\mu) > 0$ (that is, μ is not the null measure), define

$$A^{\rm fm}(\mu) = \min\{t : (t,s) \in \mu, s = M(\mu)\}, \qquad (4.7)$$

where the minimum is taken with respect to the lexicographic order on \mathbb{R}^d . Thus, $A^{\text{fm}}(\mu)$ is the position where the maximal score of μ is attained; if there is a tie, the first position is chosen. This is well defined since μ has at most finitely many scores exceeding any $\varepsilon > 0$.

4.3 Main result

For each $\tau > 0$ consider the point process

$$N_{\tau} = \left\{ \left(\boldsymbol{i} b_{\tau} / \tau \,, \, T_{r(a_{\tau}), a_{\tau}}[X_{\tau, \boldsymbol{i}}] \right) : \boldsymbol{i} \in I_{\tau} \right\}$$

$$(4.8)$$

of (rescaled) blocks together with their "positions"; observe that ib_{τ} is one of the corners of $J_{\tau,i}$ defined at (4.3). While the definition of N_{τ} also depends on the choice of the block size length b_{τ} , for simplicity we do not include it in the notation. The exact choice of the position component is immaterial: subsequent results hold if it is arbitrarily chosen inside the block using any deterministic or randomized procedure.

Let $\tilde{\mathcal{N}}_{01}^*$ be equal to the space $\tilde{\mathcal{N}}_{01}$, but without the null measure. Furthermore, let N be the space of all counting measures on $[0,1]^d \times \tilde{\mathcal{N}}_{01}^*$ which are finite on all Borel sets $B \subseteq [0,1]^d \times \tilde{\mathcal{N}}_{01}^*$ such that, for some $\varepsilon = \varepsilon(B) > 0$, $M(\mu) > \varepsilon$ for all $(t,\mu) \in B$. Equip N with the vague topology generated by the same family of sets.

Theorem 4.2. Let X be a stationary marked point process on \mathbb{E} which admits a tail configuration Y in the sense of Definition 3.1 with tail index $\alpha > 0$ and a scaling function r of scaling index $\beta \in \mathbb{R}$. Let Θ be the corresponding spectral tail configuration defined in (3.7). Finally, fix a family of block side lengths $(b_{\tau})_{\tau>0}$.

If Assumptions 4.3, 4.4 and 4.5 below hold, then $\mathbf{P}(Y \in \mathcal{N}_{01}) = \mathbf{P}(\Theta \in \mathcal{N}_{01}) = 1$ and

$$N_{\tau} \xrightarrow{d} N := \sum_{i=1}^{\infty} \delta_{(U_i, T_{\Gamma_i^{\beta/\alpha}, \Gamma_i^{1/\alpha}[Q_i])}} \quad as \ \tau \to \infty$$

$$(4.9)$$

in N, where $\{(U_i, \Gamma_i, Q_i), i \geq 1\}$ are points of the Poisson process on $[0, 1]^d \times \mathbb{R}_+ \times \mathcal{N}_{01}$ with the intensity measure being the product of the Lebesgue measure on $[0, 1]^d$, the Lebesgue measure on \mathbb{R}_+ scaled by

$$\vartheta := \mathbf{P}(A^{\text{fm}}(\Theta) = 0) \in (0, 1], \qquad (4.10)$$

and the probability distribution of a random element Q in \mathcal{N}_{01} given by

$$\mathbf{P}(Q \in \cdot) = \mathbf{P}(\Theta \in \cdot \mid A^{\text{fm}}(\Theta) = 0).$$
(4.11)

The proof of Theorem 4.2 is presented in Section 6. The mark component in limiting point process N from (4.9) can be viewed as the scaling transformation of the equivalence classes $[Q_i]$ by multiplying the scores with $y_i := \Gamma_i^{-1/\alpha}$ and positions with y_i^{β} . Note that $\{y_i, i \ge 1\}$ form a Poisson process on $(0, \infty)$ with intensity $\vartheta \alpha y^{\alpha-1} dy$. Furthermore, note that the point process Q with distribution (4.11) necessarily satisfies M(Q) = 1 almost surely.

If $X_{\tau,(j)}$ and $i_{(j)}b_{\tau}$, $j = 1, 2, \ldots, k_{\tau}^d$, denote the original blocks and their positions, relabelled so that

$$M(X_{\tau,(1)}) \ge M(X_{\tau,(2)}) \ge \cdots \ge M(X_{\tau,(k_{\tau}^d)}),$$

the continuous mapping theorem applied to (4.9) yields the convergence

$$\left(\boldsymbol{i}_{(j)}b_{\tau}/\tau, T_{r(a_{\tau}), a_{\tau}}[X_{\tau, (j)}]\right)_{j=1, \dots, k} \xrightarrow{d} \left(U_{i}, T_{\Gamma_{i}^{\beta/\alpha}, \Gamma_{i}^{1/\alpha}}[Q_{i}]\right)_{i=1, \dots, k} \quad \text{as } \tau \to \infty$$

in the space $([0,1]^d \times \tilde{\mathcal{N}}_{01})^k$ for every fixed $k \ge 1$. In particular, for k = 1 this (modulo some edge effects which are easily shown to be negligible) implies that

$$\lim_{\tau \to \infty} \mathbf{P} \left(a_{\tau}^{-1} \max_{(t,s) \in X, t \in [0,\tau]^d} s \le y \right) = \mathbf{P} (\Gamma_1^{-1/\alpha} \le y) = e^{-\vartheta y^{-\alpha}}, \quad y > 0.$$
(4.12)

Thus, the limiting distribution of the rescaled maximal score of X in $[0, \tau]^d$ is the nonstandard Fréchet distribution. Since the point process of locations in X is assumed to be a unit rate stationary process and since the marginal score satisfies (4.2), the value of ϑ deserves to be called the *extremal index* of X.

The second ingredient of the limiting point process in (4.9) is the distribution of Q which can be seen as the asymptotic distribution of a normalized *typical* cluster of exceedances of X. The tail configuration Y can be seen as a cluster of exceedances which contains the uniformly chosen exceedance of X and thus not typical since, compared to the typical cluster, its distribution is biased towards clusters with more exceedances. In fact, the relationship between the tail configuration and the typical cluster of exceedances of X is similar to the relationship between a stationary point process on \mathbb{R}^d and its Palm version, and this relationship is discussed in detail in [22] for random fields over \mathbb{Z}^d .

4.4 Assumptions of Theorem 4.2

Fix a family of positive real numbers $(b_{\tau})_{\tau>0}$ which represent the block side lengths.

Assumption 4.3 (on scaling). The family $(b_{\tau})_{\tau>0}$ satisfies

$$\lim_{\tau \to \infty} \frac{r(a_{\tau})}{b_{\tau}} = \lim_{\tau \to \infty} \frac{b_{\tau}}{\tau} = 0.$$
(4.13)

By regular variation of the scaling function r, (4.13) yields that $r(a_{\tau}\varepsilon)/b_{\tau} \to 0$ as $\tau \to \infty$ for all $\varepsilon > 0$ as well. If the scaling function r is a constant (as it always was for the case of random fields over \mathbb{Z}^d), then necessarily $b_{\tau} \to \infty$. However, if $r(a_{\tau}) \to 0$, one can take $(b_{\tau})_{\tau}$ to be a constant or even such that $b_{\tau} \to 0$.

Recall that, for $B \in \mathcal{B}(\mathbb{R}^d)$, we denote by μ_B the restriction of $\mu \in \mathcal{N}$ to $B \times (0, \infty)$. Furthermore, recall that \tilde{X} denotes a Palm version of X and $\xi = \tilde{X}(0)$.

Assumption 4.4 (on dependence within a block/ anticlustering). For all $\varepsilon, \delta, c > 0$,

$$\lim_{u \to \infty} \limsup_{\tau \to \infty} \mathbf{P} \left(M(\tilde{X}_{C_{\tau,u}}) > a_{\tau} \delta \mid \xi > a_{\tau} \varepsilon \right) = 0, \qquad (4.14)$$

where

$$C_{\tau,u} := B_{b_{\tau}c} \setminus B_{r(a_{\tau}\varepsilon)u}, \quad \tau, u > 0.$$

Our next and final assumption essentially requires that extremal blocks asymptotically behave as if they were independent. To state it, we introduce some additional notation. Let \mathcal{F} be the family of all shift-invariant measurable functions $f : \mathcal{N}_{01} \to [0, \infty)$ such that f(0) = 0, and for which there exists some $\delta > 0$ such that, for all $\mu \in \mathcal{N}_{01}$,

$$f(\mu) = f(\mu^{\delta}), \tag{4.15}$$

where μ^{δ} denotes the restriction of μ to $\mathbb{R}^d \times (\delta, \infty)$, that is, the value of f depends only on scores of μ which are larger than δ .

For a family of positive real numbers $(l_{\tau})_{\tau}$, for every $\tau > 0$ and $\mathbf{i} = (i_1, \ldots, i_d) \in I_{\tau}$, cut off the edges of $J_{\tau, \mathbf{i}}$ by l_{τ} , that is, consider

$$\hat{J}_{\tau,i} := \bigotimes_{j=1}^{d} [(i_j - 1)b_{\tau} + l_{\tau}, i_j b_{\tau} - l_{\tau}], \qquad (4.16)$$

and define the corresponding trimmed block

$$\widehat{X}_{\tau,i} := X_{\widehat{J}_{\tau,i}}$$

Assumption 4.5 (on dependence between extremal blocks). There exists a family $(l_{\tau})_{\tau>0}$, satisfying

$$\lim_{\tau \to \infty} \frac{l_{\tau}}{b_{\tau}} = 0$$

and such that

$$\mathbf{E}\Big[\prod_{i\in I_{\tau}}\exp\left\{-f_{\tau,i}(T_{r(a_{\tau}),a_{\tau}}\widehat{X}_{\tau,i})\right\}\Big] - \prod_{i\in I_{\tau}}\mathbf{E}\Big[\exp\left\{-f_{\tau,i}(T_{r(a_{\tau}),a_{\tau}}\widehat{X}_{\tau,i})\right\}\Big] \to 0 \quad (4.17)$$

as $\tau \to \infty$ for any family of functions $f_{\tau,i} \in \mathcal{F}, \tau > 0, i \in I_{\tau}$, which satisfy (4.15) for the same $\delta > 0$.

Informally speaking, (4.17) holds if extremal scores are only locally dependent. The crucial issue in (4.17) is that the value of $f_{\tau,i}(T_{r(a_{\tau}),a_{\tau}}\hat{X}_{\tau,i})$ depends only on points $(t,s) \in \hat{X}_{\tau,i}$ with $s > a_{\tau}\delta$, and that for any $(t,s) \in \hat{X}_{\tau,i}$ and $(t',s') \in \hat{X}_{\tau,i'}$ for $i \neq i'$, one has $|t-t'| \geq l_{\tau}$.

4.5 Alternative representations of ϑ and Q

Assume that Y is the tail configuration of a stationary marked point process X on \mathbb{E} , with tail index $\alpha > 0$ and a scaling function r of scaling index β . In this subsection we give some Palm-like properties of the tail configuration under the assumption that $Y \in \mathcal{N}_{01}$ a.s., which e.g. holds under the anticlustering condition (4.14), see Proposition 6.1. All of the following results are based on the exceedance-stationarity property (3.9) of the tail configuration. For any $\mu \in \mathcal{N}_{01}$, denote

$$e(\mu) := \{ t \in \mathbb{R}^d : (t, s) \in \mu \text{ and } s > 1 \}.$$

By definition, $0 \in e(Y)$ a.s. A function $A : \mathcal{N}_{01} \to \mathbb{R}^d$ will be called an *anchoring function* if, for all $\mu \in \mathcal{N}_{01}$ such that $e(\mu) \neq \emptyset$,

- (i) $A(\mu) \in e(\mu);$
- (ii) $A(\varphi_z \mu) = A(\mu) z$ for all $z \in \mathbb{R}^d$ (shift-equivariance).

A typical example of an anchoring function is the first maximum anchor A^{fm} from (4.7). Another one is the first exceedance anchor

$$A^{\rm fe}(\mu) := \min e(\mu) \,, \tag{4.18}$$

where the minimum is taken with respect to the lexicographic order.

Lemma 4.6. If $Y \in \mathcal{N}_{01}$ a.s., then

$$\vartheta_A = \mathbf{P}(A(Y) = 0) > 0$$

for any anchoring function A.

Proof. We adapt the proof of [3, Lemma 3.4]. Using (3.9) and the shift-equivariance property of A,

$$1 = \mathbf{E} \left[\sum_{(t,s)\in Y} \mathbf{1}_{\{A(Y)=t,s>1\}} \right] = \mathbf{E} \left[\sum_{(t,s)\in Y} \mathbf{1}_{\{A(\varphi_t Y)=-t,s>1\}} \right]$$
$$= \mathbf{E} \left[\sum_{(t,s)\in Y} \mathbf{1}_{\{A(Y)=0,s>1\}} \right] = \mathbf{E} \left[\mathbf{1}_{\{A(Y)=0\}} \sum_{(t,s)\in Y} \mathbf{1}_{\{s>1\}} \right].$$

Note that $\mathbf{P}(Y \in \mathcal{N}_{01}) = 1$ implies $\sum_{(t,s)\in Y} \mathbf{1}_{\{s>1\}} < \infty$ almost surely. Thus, $\mathbf{P}(A(Y) = 0) > 0$, since otherwise the last expression above would vanish.

Proposition 4.7. If $\mathbf{P}(Y \in \mathcal{N}_{01}) = 1$, then ϑ_A and the distribution $\mathbf{P}([Y] \in \cdot | A(Y) = 0)$ on $\tilde{\mathcal{N}}_{01}$ do not depend on the choice of the anchoring function A.

Proof. This result parallels [3, Lemma 3.5] and it can proved in the same manner by using (3.9) instead of [3, Property (3.8)]. We omit the details.

The above results yield alternative representations for ϑ in (4.10). Indeed, recall the spectral tail configuration $\Theta = T_{\eta^{\beta},\eta}Y$ (where $\eta = Y(0)$) and observe that $A^{\text{fm}}(\Theta) = 0$ if and only if $A^{\text{fm}}(Y) = 0$. Thus $\vartheta = \vartheta_{A^{\text{fm}}} = \mathbf{P}(A^{\text{fm}}(\Theta) = 0)$ which is further equal to $\vartheta_A = \mathbf{P}(A(Y) = 0)$ for an arbitrary anchoring function A. Since $Y = T_{\eta^{-\beta},\eta^{-1}}\Theta$ with Θ independent of the Pareto(α) random variable η ,

$$\mathbf{P}(Y \in \cdot \mid A^{\mathrm{fm}}(Y) = 0) = \int_{1}^{\infty} \mathbf{P}(T_{u^{-\beta}, u^{-1}}Q \in \cdot) \alpha u^{-\alpha - 1} \mathrm{d}u$$
(4.19)

on \mathcal{N}_{01} , where Q has distribution (4.11) on \mathcal{N}_{01} . Using this fact, one can, as in [22, Proposition 3.9], prove that Q from (4.11) satisfies

$$\mathbf{P}([Q] \in \cdot) = \vartheta^{-1} \mathbf{E} \left[\mathbf{1} \left\{ T_{M(\Theta)^{\beta}, M(\Theta)}[\Theta] \in \cdot \right\} \frac{M(\Theta)^{\alpha}}{\sum_{(t,s) \in \Theta} s^{\alpha}} \right]$$
(4.20)

on $\tilde{\mathcal{N}}_{01}$. In particular,

$$\vartheta = \mathbf{E}\left[\frac{M(\Theta)^{\alpha}}{\sum_{(t,s)\in\Theta} s^{\alpha}}\right],\tag{4.21}$$

and $(Q_i)_{i\geq 1}$ in (4.9) can be chosen such that their common distribution on \mathcal{N}_{01} satisfies

$$\mathbf{P}(Q \in \cdot) = \vartheta^{-1} \mathbf{E} \left[\mathbf{1} \left\{ T_{M(\Theta)^{\beta}, M(\Theta)} \Theta \in \cdot \right\} \frac{M(\Theta)^{\alpha}}{\sum_{(t,s) \in \Theta} s^{\alpha}} \right].$$
(4.22)

5 Examples: tail configurations, extremal indices and typical clusters

5.1 Small distance to the kth nearest neighbor

Fix a $k \in \mathbb{N}$. For a set $I \subset \mathbb{R}^d$ which has at most a finite number of points in any bounded region and $t \in \mathbb{R}^d$, let $\rho_k(t, I)$ denote the distance from t to its kth nearest neighbor in $I \setminus \{t\}$. Note that $\rho_k(t, I) < a$ if and only if $I \setminus \{t\}$ has at least k points in the open ball of radius a centered at t.

Let P be a homogeneous unit intensity Poisson process on \mathbb{R}^d , and let X be the marked point process obtained by attaching the score $\rho_k(t, P)^{-1}$ to each of $t \in P$, so that the score is the reciprocal to the distance from $t \in P$ to its kth nearest neighbor. Thus, one can write $X = \Psi(P)$ with the scoring function $\psi(s, \mu) = \rho_k(s, \mu)^{-1}$, see Section 2.2. By Lemma 2.2, the Palm version \tilde{X} is obtained by the same procedure applied to $P + \delta_0$, so that the points in \tilde{X} are located at all points t from $P + \delta_0$ and the score at t is given by $s = \rho_k(t, P + \delta_0)^{-1}$. In particular, the score of the Palm version at the origin is

$$\xi = \rho_k (0, P + \delta_0)^{-1}.$$

It is straightforward to see that

$$\mathbf{P}(\xi > u) = \mathbf{P}(P(B_{1/u}) \ge k) \sim \mathbf{P}(P(B_{1/u}) = k) = e^{-C_d u^{-d}} \frac{(C_d u^{-d})^k}{k!} \quad \text{as} \ u \to \infty,$$

where C_d is the volume of the unit ball in \mathbb{R}^d . Thus,

$$\mathbf{P}(\xi > u) \sim \frac{C_d^k}{k!} u^{-dk} \quad \text{as } u \to \infty \,, \tag{5.1}$$

i.e., ξ has a regularly varying tail with tail index $\alpha = dk$.

Tail configuration

Proposition 5.1. For every $k \in \mathbb{N}$, the tail configuration of X exists with the normalizing function $r(u) = u^{-1}$ (i.e., $\beta = -1$) and is given by

$$Y := \Psi(\mathcal{Y}) = \{ (U_i, \rho_k(U_i, \mathcal{Y})^{-1}), i = 0, \dots, k \},\$$

where $\mathcal{Y} := \{U_0, ..., U_k\}, U_0 = 0, and U_1, ..., U_k are$ *i.i.d.*uniform on B₁.

Remark 5.2. In this case the tail configuration Y is an element of $\mathcal{N}_{10} \subset \mathcal{N}$, and as the proof below shows, the convergence in (3.3) is valid even in the stronger \mathcal{B}_{10} -vague topology.

Proof of Proposition 5.1. Recall that $X = \Psi(P)$ for the scoring function $\psi(s, \mu) = \rho_k(s, \mu)^{-1}$, and let $\tilde{P} = \delta_0 + P$ be a Palm version of P. Observe that the conditional distribution of \tilde{X} given $\xi > u$ coincides with the conditional distribution of $\Psi(\tilde{P})$ given $P(B_{1/u}) \ge k$. Considering Laplace functionals, it is easy to see that for any nonnegative f,

$$L_f(T_{r(u),u}\tilde{X} \mid \xi > u) = L_f(T_{r(u),u}\Psi(\tilde{P}) \mid P(B_{1/u}) \ge k) \sim L_f(T_{r(u),u}\Psi(\tilde{P}) \mid P(B_{1/u}) = k)$$

as $u \to \infty$.

Furthermore, \tilde{P} conditionally on $P(B_{1/u}) = k$ has the distribution of

$$P^{(u)} := \delta_0 + \sum_{i=1}^k \delta_{U_i/u} + P_{B_{1/u}^c}$$

where U_1, \ldots, U_k on the right-hand side are uniformly distributed in B_1 and independent of P, and $P_{B_{1/u}^c}$ is the restriction of P to the set $B_{1/u}^c$. Since r(u) = 1/u and $u\rho_k(t, A) = \rho_k(ut, uA)$,

$$T_{r(u),u}\Psi(P^{(u)}) = \Psi(uP^{(u)}),$$

where

$$uP^{(u)} = \sum_{i=0}^{k} \delta_{U_i} + Z^{(u)},$$

and $Z^{(u)}$ is uP restricted to B_1^c . Thus, for every nonnegative f,

$$L_f(T_{r(u),u}\tilde{X} \mid \xi > u) \sim L_f(\Psi(\mathcal{Y} + Z^{(u)})) \quad \text{as } u \to \infty.$$

Take now $f : \mathbb{E} \to \mathbb{R}_+$ whose support is contained in the set $B_a \times (0, \infty) \in \mathcal{B}_{10}$ for some a > 0. We can and will assume that $a \ge 2$. Observe that

$$\mathbf{P}(Z^{(u)}(B_a) \ge 1) \le \mathbf{E}[Z^{(u)}(B_a)] = \mathbf{E}[P(B_{a/u} \setminus B_{1/u})] \le \frac{1}{u^d} C_d a^d \to 0,$$

so that $\mathbf{P}(Z^{(u)}(B_a) = 0) \to 1$ as $u \to \infty$. Since f(t,s) = 0 for $t \notin B_a$, on the event $\{Z^{(u)}(B_a) = 0\}$ one has that

$$\int f d\Psi(\mathcal{Y} + Z^{(u)}) = \sum_{i=0}^{k} f(U_i, \rho_k^{-1}(U_i, \mathcal{Y} + Z^{(u)})) = \sum_{i=0}^{k} f(U_i, \rho_k^{-1}(U_i, \mathcal{Y})) = \int f d\Psi(\mathcal{Y}),$$

where the second equality follows since $a \ge 2$, and so the kth nearest neighbor of each U_i is necessarily in \mathcal{Y} . Thus, for any such f,

$$L_f(T_{r(u),u}\tilde{X} \mid \xi > u) \sim L_f(\Psi(\mathcal{Y} + Z^{(u)})) \to L_f(\Psi(\mathcal{Y})) = L_f(Y) \text{ as } u \to \infty.$$

In particular, this holds for any nonnegative, continuous and bounded function f on \mathbb{E} whose support is in \mathcal{B}_{11} since $\mathcal{B}_{11} \subset \mathcal{B}_{10}$, and thus (3.3) holds.

By Proposition 3.2, the tail score at the origin η is Pareto(dk)-distributed. Indeed,

$$\eta^{-1} = \rho_k(0, \mathcal{Y}) = \max_{1 \le i \le k} \|U_i\| =: U^*.$$

By Proposition 3.3, η is independent of the spectral tail configuration, which is (since $\beta = -1$) given by

$$\Theta := T_{\eta^{-1},\eta}Y = \left\{ \left(U_i/U^*, \, \rho_k(U_i/U^*, \mathcal{Y}/U^*)^{-1} \right), i = 0, \dots, k \right\}.$$

Moreover, it is easy to check that

$$\Theta \stackrel{d}{=} \Psi(\mathcal{Y}^*) = \sum_{i=0}^{k-1} \delta_{(U_i, \rho_k(U_i, \mathcal{Y}^*)^{-1})} + \delta_{(U'_k, \rho_k(U'_k, \mathcal{Y}^*)^{-1})}, \qquad (5.2)$$

where $\mathcal{Y}^* := \{0, U_1, \ldots, U_{k-1}, U'_k\}$ with U'_k uniformly distributed on ∂B_1 and independent of U_1, \ldots, U_{k-1} , which are i.i.d. and uniformly distributed on B_1 .

Point process convergence

First, (5.1) implies that the sequence of thresholds (a_{τ}) in (4.1) can be chosen as

$$a_{\tau} := \tau^{1/k} \cdot \sqrt{\pi} (k!)^{-1/dk} \Gamma(d/2+1)^{-\frac{1}{d}}, \quad \tau > 0.$$
(5.3)

Scaling scores with a_{τ}^{-1} is (up to a transformation) equivalent to scaling distances $\rho_k(t, P)$ with a_{τ} .

Let A^{fm} be defined at (4.7). Then the extremal index (depending on k and dimension) is given by

$$\vartheta_{k,d} := \mathbf{P}(A^{\mathrm{fm}}(\Theta) = 0) \,,$$

and (4.11) implies that Q has the conditional distribution of Θ given that $A^{\text{fm}}(\Theta) = 0$. Observe that $A^{\text{fm}}(\Theta) = 0$ if and only if \mathcal{Y}^* is not contained in $B_1(U_i)$ for all $i = 1, \ldots, k-1$, and U'_k is lexicographically larger than 0 if \mathcal{Y}^* is a subset of the closure of $B_1(U'_k)$.

If k = 1, then $\mathcal{Y}^* = \{0, U_1'\}$, so that $A^{\text{fm}}(\Theta) = 0$ if and only if U_1' is lexicographically larger than 0. Thus, $\vartheta_{1,d} = 1/2$ and

$$Q \stackrel{d}{=} \Psi(\{0, U_1''\}) = \delta_{(0,1)} + \delta_{(U_1'',1)}$$

in all dimensions, where U_1'' is uniform on $\partial B_1 \cap \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \ge 0\}$. The value $\vartheta_{1,d} = 1/2$ is intuitively obvious since, asymptotically, large values always come in pairs (with exactly the same score). In dimension d = 2, $\vartheta_{1,2} = 1/2$ was obtained in [8, Section 4.2] when analyzing the extremal properties of the inradius of a Poisson–Voronoi tessellation.

If k = 2, $A^{\text{fm}}(\Theta) = 0$ if and only if $U_1 \notin B_1(U_2')$. Thus, in all dimensions,

$$Q \stackrel{d}{=} \Psi(\{0, U_1', U_2'\}),$$

where U'_1 is, conditionally on U'_2 , uniform on $B_1 \setminus B_1(U'_2)$. Furthermore, due to rotational invariance,

$$\vartheta_{2,d} = 1 - \frac{\operatorname{Leb}(B_1 \cap B_1(e_1))}{\operatorname{Leb}(B_1)} = 1 - \frac{2\Gamma(1+d/2)}{\sqrt{\pi}\Gamma((d+1)/2)} \int_0^{\pi/3} \sin^d u \, \mathrm{d}u,$$

where e_1 is the first basis vector of \mathbb{R}^d and $\text{Leb}(\cdot)$ is the Lebesgue measure. In particular, $\vartheta_{2,1} = 1/2$,

$$\vartheta_{2,2} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \approx 0.609,$$

and $\vartheta_{2,3} = 33/48$. Since $\Gamma(x + \alpha) \sim \Gamma(x) x^{\alpha}$ as $x \to \infty,^2$

$$1 - \vartheta_{2,d} \sim \sqrt{\frac{2d}{\pi}} \cdot 2\frac{(\sqrt{3}/2)^{d+1}}{d+1} \sim \sqrt{\frac{6}{\pi}} \cdot \frac{(\sqrt{3}/2)^d}{\sqrt{d}}, \text{ as } d \to \infty,$$

i.e., $1 - \vartheta_{2,d}$ goes to zero exponentially fast as the dimension grows.

For d = 1, we have already seen that $\vartheta_{1,1} = \vartheta_{2,1} = 1/2$. Interestingly, one can check that $\vartheta_{k,1} = 1/2$ for all $k \in \mathbb{N}$. Indeed, one can assume that $U'_k = 1$ without loss of generality, and then the maximal score is attained at zero if the unit ball around any of $j \in \{0, \ldots, k-2\}$ points that fall in (0, 1) does not cover k - j points uniformly distributed in (-1, 0). This probability can be calculated explicitly, and then the result follows by noticing that j is binomially distributed.

The exact calculations of $\vartheta_{k,d}$ become quite involved for $k \ge 3$ and $d \ge 2$.

Proposition 5.3. For arbitrary $k, d \in \mathbb{N}$, define X as in the beginning of Section 5.1 and $(a_{\tau})_{\tau}$ as in (5.3). For any $(b_{\tau})_{\tau}$ such that $\tau^{-1/k}/b_{\tau} \to 0$ and $b_{\tau}/\tau \to 0$ as $\tau \to \infty$, the assumptions of Theorem 4.2 hold (with $\alpha = dk$ and $r(u) = u^{-1}$). Therefore, the convergence of the extremal blocks in (4.9) holds as well.

²One can obtain the asymptotics for the intregral $I_d = \int_0^{\pi/3} \sin^d u \, du$ by showing that

$$\lim_{d\to\infty}\frac{d+1}{\sin^d(\pi/3)}I_d = \tan(\pi/3)\,.$$

Proof. Assumption 4.3 holds, since $r(u) = u^{-1}$ and a_{τ} has the order $\tau^{1/k}$. For each $\tau > 0$ and $i \in I_{\tau}$ define the block of indices $J_{\tau,i}$ by (4.3) and $X_{\tau,i}$ by (4.4). The key step in checking other assumptions of Theorem 4.2 is that for every $t \in P$, the condition X(t) > y(i.e., $\rho_k(t, P) < y^{-1}$) is equivalent to $P(B_{y^{-1}}(t)) \ge k + 1$. Thus, given that $X(t) > a_{\tau}\varepsilon$, X(t) depends only on the points of P in $B_{a_{\tau}^{-1}\varepsilon^{-1}}(t)$, where $a_{\tau}^{-1} \to 0$ as $\tau \to \infty$.

Fix $\varepsilon, \delta, c > 0$ and recall the notation from (4.14). The event $\{\xi > a_{\tau}\varepsilon\} = \{\rho_k(0, \tilde{P}) < (a_{\tau}\varepsilon)^{-1}\}$ depends only on \tilde{P} (that is, P) restricted to $B_{(a_{\tau}\varepsilon)^{-1}}$. Since $C_{\tau,u} \subset B_{(a_{\tau}\varepsilon)^{-1}u}^c$, as soon as $u > 1 + \delta^{-1}/\varepsilon^{-1}$ the event $\{M(\tilde{X}_{C_{\tau,u}}) > a_{\tau}\delta\} = \{\min_{t\in\tilde{P}\cap C_{\tau,u}}\rho_k(t,\tilde{P}) < (a_{\tau}\delta)^{-1}\}$ is determined only by \tilde{P} restricted to $B_{(a_{\tau}\varepsilon)^{-1}}^c$ (on this set one has that $P = \tilde{P}$ and, consequently, $X = \tilde{X}$). Since P is a Poisson process, this implies that for all such u,

$$\begin{aligned} \mathbf{P}\big(M(\tilde{X}_{C_{\tau,u}}) > a_{\tau}\delta \mid \xi > a_{\tau}\varepsilon\big) &= \mathbf{P}\big(M(\tilde{X}_{C_{\tau,u}}) > a_{\tau}\delta) = \mathbf{P}\big(M(X_{C_{\tau,u}}) > a_{\tau}\delta\big) \\ &\leq \mathbf{P}(M(X_{B_{b_{\tau}c}}) > a_{\tau}\delta) \leq \mathbf{E}\left[\sum_{t \in P'} \mathbf{1}\{\psi(t, P) > a_{\tau}\delta, t \in B_{b_{\tau}c}\}\right] \\ &= \mathbf{E}\left[\sum_{t \in P'} \mathbf{1}\{\psi(0, \varphi_t P) > a_{\tau}\delta, t \in B_{b_{\tau}c}\}\right] \\ &= \operatorname{const}(cb_{\tau})^d \mathbf{P}(\psi(0, \tilde{P}) > a_{\tau}\delta) \\ &= \operatorname{const}\frac{b_{\tau}^d}{\tau^d}\tau^d \mathbf{P}(\xi > a_{\tau}\delta) \to \operatorname{const} \cdot 0 \cdot \delta^{-\alpha} = 0 \quad \text{as } \tau \to \infty. \end{aligned}$$

In the sixth step we used the refined Campbell's theorem (2.1), and in the penultimate step we used (4.2) and the fact that b_{τ} is chosen such that $b_{\tau}/\tau \to 0$. Thus, (4.14) holds, i.e., Assumption 4.4 is satisfied.

Now consider Assumption 4.5. Take $(l_{\tau})_{\tau}$ such that $l_{\tau}/b_{\tau} \to 0$ and $a_{\tau}^{-1}/l_{\tau} \to 0$ as $\tau \to \infty$; since $a_{\tau}^{-1}/b_{\tau} = r(a_{\tau})/b_{\tau} \to 0$, one can, e.g., take $l_{\tau} = \sqrt{b_{\tau}/a_{\tau}^{-1}}$. Let $f_{\tau,i}, \tau > 0$, $i \in I_{\tau}$, be an arbitrary family of shift-invariant measurable functions from \mathcal{N}_{01} to $[0,\infty)$ such that for some $\delta > 0$ and for all $\tau > 0$, $i \in I_{\tau}$ and $\mu \in \mathcal{N}_{01}$,

$$f_{\tau, \mathbf{i}}(\mu) = \begin{cases} 0 & \text{if } M(\mu) \le \delta, \\ f_{\tau, \mathbf{i}}(\mu^{\delta}) & \text{otherwise}, \end{cases}$$

where μ^{δ} denotes the restriction of μ to $\mathbb{R}^d \times (\delta, \infty)$.

Similarly as above, for $t \in P$, the random variable $X(t)\mathbf{1}\{X(t) > a_{\tau}\delta\}$ depends only on P restricted to $B_{(a_{\tau}\delta)^{-1}}(t)$. Moreover, if $(a_{\tau}\delta)^{-1} < l_{\tau}$, then

$$\bigcup_{t\in \hat{J}_{\tau,i}} B_{(a_{\tau}\delta)^{-1}}(t) \subseteq J_{\tau,i}$$

for all $i \in I_{\tau}$; recall that $\hat{J}_{\tau,i}$ in (4.16) are obtained from the original blocks $J_{\tau,i}$ by trimming the edges by l_{τ} . Thus, since $\hat{X}_{\tau,i} := X_{\hat{J}_{\tau,i}}$, the value of $f_{\tau,i}(T_{r(a_{\tau}),a_{\tau}}\hat{X}_{\tau,i})$ depends only on P restricted to $J_{\tau,i}$, for all $i \in I_{\tau}$. For such τ , since P is a Poisson process and $J_{\tau,i}$'s are disjoint, the left-hand side of (4.17) vanishes, hence, Assumption 4.5 holds. A simple consequence of Proposition 5.3 concerns the behavior of the minimal distance to the k-th nearest neighbour in a Poisson configuration of points on an increasing hypercube $[0, \tau]^d$. For all $\tau > 0$, denote $m_{\rho,\tau} = \min\{\rho_k(t, P) : t \in P \cap [0, \tau]^d\}$. Then, for a_{τ} as in (5.3), (4.12) implies

$$\mathbf{P}(a_{\tau}m_{\rho,\tau} \ge v) \to e^{-\vartheta_{k,d}v^{dk}}$$
 as $n \to \infty$

for any v > 0, that is, the scaled minimum distance to the kth nearest neighbour in a Poisson configuration in $[0, \tau]^d$ converges to a (scaled) Weibull distribution.

5.2 Moving maxima model

Let ζ be a positive random variable whose tail is regularly varying of index $\alpha > 0$, and denote the distribution of ζ by m. Assume that $P \in \mathcal{N}_{10}$ is a unit rate independently marked stationary Poisson process on \mathbb{R}^d with marks in $(0, \infty)$, whose intensity measure is the product of the Lebesgue measure on \mathbb{R}^d and a probability measure m, so that ζ is the typical mark. In the sequel, write $P = \sum \delta_{(t,\zeta_t)}$ where $(\zeta_t)_t, t \in \mathbb{R}^d$, are i.i.d. with distribution m.

Let \mathcal{N}_g be the space of all simple locally finite point measures on \mathbb{R}^d equipped with the usual σ -algebra. Consider a measurable function $\Phi : \mathbb{R}^d \times \mathcal{N}_g \to \mathcal{N}_g$ such that, for each $\mu' \in \mathcal{N}_g$, and each $t \in \mu'$, $\Phi(t, \mu')$ is a finite subset of μ' which contains t. It is useful to interpret $\Phi(t, \mu')$ as a *neighborhood* of t in μ' , and all points $x \in \Phi(t, \mu')$ as *neighbors* of t. Furthermore, assume that Φ is shift-equivariant in the sense that $\Phi(t-x, \varphi_x \mu') = \Phi(t, \mu') - x$ for all $\mu' \in \mathcal{N}_g$, $t \in \mu'$ and $x \in \mathbb{R}^d$. Finally, denote by $N(t, \mu')$ the cardinality of $\Phi(t, \mu')$. It can been easily seen that the arguments below also work if Φ depends on some external sources of randomness, but stays independent of the marks (ζ_t) .

Consider a (deterministic) weight function $w : \mathbb{R}^d \to [0, \infty)$ such that w(0) > 0, and define the scoring function $\psi : \mathbb{R}^d \times \mathcal{N}_{10} \to [0, \infty)$ by

$$\psi(t,\mu) := \max_{x \in \Phi(t,\mu')} w(x-t)\mu(x), \quad t \in \mu',$$

where μ' is μ projected to \mathbb{R}^d . Without loss of generality, in the sequel assume that w(0) = 1. If the weight function is identically one, the score at t is the maximum of the scores of its neighbors.

Since ψ is shift-invariant, $X := \Psi(P)$ is a stationary marked point process on \mathbb{R}^d . Its Palm version is given by $\tilde{X} := \Psi(\tilde{P})$, where $\tilde{P} = P + \delta_{(0,\zeta_0)}$, with ζ_0 having distribution mand being independent of P. For notational convenience, denote $\tilde{\Phi}(t) = \Phi(t, \tilde{P}')$, and let $\tilde{N}(t) = N(t, \tilde{P}')$ be the cardinality of $\tilde{\Phi}(t)$ for all $t \in \tilde{P}'$.

Tail configuration

Theorem 5.4. Assume that w is a bounded function and that $\mathbf{E}[N(0)] < \infty$. Then the following statements hold.

(i) The score at the origin ξ satisfies

$$\lim_{u \to \infty} \frac{\mathbf{P}(\xi > u)}{\mathbf{P}(\zeta > u)} \to \mathbf{E}\left[\sum_{t \in \tilde{\Phi}(0)} w(t)^{\alpha}\right] =: \kappa \in (0, \infty).$$
(5.4)

In particular, ξ is regularly varying with tail index α .

(ii) The tail configuration of X exists for a constant scaling function $r \equiv 1$ (thus, with no scaling of the positions) and the distribution of its spectral part $\Theta \in \mathcal{N}$ is given by

$$\mathbf{E}[h(\Theta)] = \kappa^{-1} \mathbf{E} \left[\sum_{x \in \widetilde{\Phi}(0)} h\left(\left\{ \left(t, \frac{w(x-t)}{w(x)}\right) : t \in \widetilde{P}', \ x \in \widetilde{\Phi}(t) \right\} \right) w(x)^{\alpha} \right]$$
(5.5)

for all measurable $h : \mathcal{N} \to \mathbb{R}_+$, where the summand corresponding to x is understood to be 0 if w(x) = 0.

Remark 5.5. For a point measure μ on $\mathbb{R}^d \times [0, \infty)$ whose restriction to $\mathbb{E} = \mathbb{R}^d \times (0, \infty)$ is an element of \mathcal{N} , for any function h on \mathcal{N} we define $h(\mu)$ to be equal to the value of h on this restriction. In other words, we simply neglect the points of the form (t, 0). This is relevant for (5.5) since the weighting function w can in general attain the value 0.

Remark 5.6. Observe that the distribution of Θ , in addition to w, α and Φ , depends only on the distribution of $\tilde{P}' = P' + \delta_0$. It can be obtained as follows. First, let P^* be distributed as \tilde{P}' but from the tilted distribution:

$$\mathbf{P}(P^* \in \cdot) := \kappa^{-1} \mathbf{E} \left[\mathbf{1}_{\{\tilde{P}' \in \cdot\}} \sum_{x \in \tilde{\Phi}(0)} w(x)^{\alpha} \right] , \qquad (5.6)$$

and denote $\Phi^*(t) := \Phi(t, P^*)$ for all $t \in \mathbb{R}^d$. Conditionally on P^* , let V be a $\Phi^*(0)$ -valued random element such that V equals $x \in \Phi^*(0)$ with probability proportional to $w(x)^{\alpha}$. Finally, let

$$\Theta := \left\{ \left(t, \frac{w(V-t)}{w(V)}\right) : t \in P^*, \ V \in \Phi^*(t) \right\}$$

restricted to \mathbb{E} .

Example 5.7. Here we use the notation of Remark 5.6.

- (i) Assume that w(t) = 0 for all $t \neq 0$ (and w(0) = 1), so that $\psi(t, P) = \zeta_t$. Then $P^* \stackrel{d}{=} \tilde{P}'$ and V = 0 almost surely, so $\Theta = \{(0, 1)\}$, i.e., the extreme scores of X appear in isolation.
- (ii) Let now $w(t) \equiv 1$, so that

$$\psi(t, P) = \max_{x \in \Phi(t, P')} \zeta_x$$

and assume that $\mathbf{E}\tilde{N}(0) < \infty$. Then (5.6) implies that P^* is, compared to \tilde{P}' , biased towards configurations in which the origin has more neighbors. Furthermore, $\Theta = \{(t, 1) : t \in P^*, V \in \Phi^*(t)\}$, where V is uniform on $\Phi^*(0)$. Note that necessarily $(0, 1) \in \Theta$. Observe also that (5.4) implies that

$$\lim_{u \to \infty} \frac{\mathbf{P}(\max_{x \in \tilde{\Phi}(0)} \zeta_x > u)}{\mathbf{P}(\zeta > u)} = \mathbf{E}[\tilde{N}(0)].$$
(5.7)

Proof of Theorem 5.4. Let $f: \mathbb{E} \to [0,\infty)$ be an arbitrary continuous function such that

$$f(t,s) = 0$$
 whenever $t \notin B_a$ or $s \le \varepsilon$, (5.8)

for some $a, \varepsilon > 0$. We extend f to a continuous function on $\mathbb{R}^d \times [0, \infty)$ by letting f(t, 0) = 0for all t. For notational convenience set $h(\mu) := e^{-\mu(f)}$ for all $\mu \in \mathcal{N}_{11}$. The result would follow if we show that

$$\lim_{u \to \infty} \frac{\mathbf{E}[h(T_{1,u}X)\mathbf{1}_{\{\tilde{X}(0) > u\}}]}{\mathbf{P}(\zeta > u)} = \kappa \mathbf{E}[h(Y)], \qquad (5.9)$$

where $Y := T_{1,\eta^{-1}}\Theta$ for Θ from (5.5) and η is $Pareto(\alpha)$ distributed and independent of Θ .

First, write $\tilde{P} = \{(t_i, \zeta_i) : i \geq 1\}$, where $\{t_1, \ldots, t_{\tilde{N}(0)}\} = \tilde{\Phi}(0)$. Given \tilde{P}' , the score $\psi(t, \tilde{P})$ depends only on ζ_x for $x \in \tilde{\Phi}(t)$. By (5.8),

$$h(T_{1,u}\tilde{X}) = \exp\left\{-\sum_{(t,s)\in\tilde{X}} f(t,s/u)\right\} = \exp\left\{-\sum_{(t,s)\in\tilde{X}_{B_a}} f(t,s/u)\right\}$$
$$= \exp\left\{-\sum_{i=1}^{K_a} f(t_i,\psi(t_i,\tilde{P})/u)\right\} =: g(\tilde{P}',\zeta_1/u,\zeta_2/u,\ldots,\zeta_{K_a}/u), \quad (5.10)$$

where $K_a = K_a(\tilde{P}')$ is the smallest nonnegative integer such that $\{t_1, \ldots, t_{K_a}\}$ contains $\tilde{\Phi}(t)$ for all $t \in \tilde{P}'_{B_a}$. Observe that $K_a \geq 1$. Since \tilde{P}' and $(\zeta_1, \zeta_2, \ldots)$ are independent, conditioning on \tilde{P}' yields that

$$\mathbf{E}\Big[h(T_{1,u}\tilde{X})\mathbf{1}_{\{\tilde{X}(0)>u\}}\Big] = \mathbf{E}\Big[g_u(\tilde{P}')\Big], \qquad (5.11)$$

where, for all μ' from the support of \tilde{P}' ,

$$g_{u}(\mu) = \mathbf{E} \Big[g\big(\mu', \zeta_{1}/u, \dots, \zeta_{K_{a}(\mu')}/u\big) \mathbf{1} \big\{ \max_{i \le N(0,\mu')} w(t_{i})\zeta_{i} > u \big\} \Big],$$
(5.12)

and $(t_i)_{i\geq 1}$ are deterministic and depend only on μ' . Observe that for every fixed μ (write $k := K_a(\mu')$ and $n := N(0, \mu')$, so $k \geq n$), the function under the expectation

$$(y_1,\ldots,y_k)\mapsto g(\mu,y_1,\ldots,y_k)\mathbf{1}\{\max_{i\leq n}w(t_i)y_i>1\}$$

is bounded (since h is bounded) and continuous except on the set

$$\left\{(y_1,\ldots,y_k):\max_{i\leq n}w(t_i)y_i=1\right\}$$

(since f and hence h is continuous). Furthermore, this function has support bounded away from the origin in \mathbb{R}^k , since it vanishes whenever

$$\max\{y_1, y_2, \dots, y_n\} \le 1/\max_{i \le n} w(t_i).$$

Since $(\zeta_i)_i$ are i.i.d. regularly varying with index α , the vector $(\zeta_1, \ldots, \zeta_k)$ is multivariate regularly varying in $\mathbb{R}^k_+ \setminus \{0\}$ with the same index, see [24, p. 192]. In particular,

$$\lim_{u \to \infty} \frac{g_u(\mu')}{\mathbf{P}(\zeta > u)} = \int_{\mathbb{R}^k_+ \setminus \{0\}} g(\mu', y_1, \dots, y_k) \mathbf{1}\{\max_{i \le n} w(t_i) y_i > 1\} \, \mathrm{d}\nu(y_1, \dots, y_k) \quad (5.13)$$

for a certain measure ν on $\mathbb{R}^k_+ \setminus \{0\}$ concentrated on the axes. More precisely, by [24, p. 192], if η is Pareto(α) distributed, the right-hand side of (5.13) equals

$$\sum_{i=1}^{n} \int_{0}^{\infty} g(\mu', \underbrace{0, \dots, 0}_{i-1}, y, 0, \dots, 0) \mathbf{1}_{\{w(t_i)y>1\}} \alpha y^{-\alpha-1} dy$$

= $\sum_{i=1}^{n} \mathbf{E} \Big[g(\mu', \underbrace{0, \dots, 0}_{i-1}, \eta/w(t_i), 0, \dots, 0) \Big] w(t_i)^{\alpha},$

where the *i*th summand on the right-hand side is set to be 0 if $w(t_i) = 0$. Recalling that g was defined at (5.10), this equals

$$\mathbf{E}\Big[\sum_{i=1}^{n}\exp\left\{-\sum_{j\leq k:\,t_i\in\Phi(t_j,\mu')}f\left(t_j,\frac{w(t_i-t_j)}{w(t_i)}\eta\right)-\sum_{j\leq k:\,t_i\notin\Phi(t_j,\mu')}f(t_j,0)\right\}w(t_i)^{\alpha}\Big].$$

Recall that $\{t_1, \ldots, t_n\} = \Phi(0, \mu')$ and that $n \leq k$. Since f(t, 0) = 0 for all $t \in \mathbb{R}^d$ and since for $j > k = K_a(\mu')$, one has $t_j \notin B_a$ and $f(t_j, y) = 0$ regardless of y, the expression above (and therefore the right-hand side of (5.13)) actually equals

$$\mathbf{E}\left[\sum_{x\in\Phi(0,\mu')}\exp\left\{-\sum_{t\in\mu':x\in\Phi(t,\mu')}f\left(t,\frac{w(x-t)}{w(x)}\eta\right)\right\}w(x)^{\alpha}\right].$$

Going back to (5.11), (5.13) yields

$$\lim_{u \to \infty} \frac{\mathbf{E}[h(T_{1,u}\tilde{X})\mathbf{1}_{\{\tilde{X}(0)>u\}}]}{\mathbf{P}(\zeta > u)} = \lim_{u \to \infty} \mathbf{E}\Big[\frac{g_u(\tilde{P}')}{\mathbf{P}(\zeta > u)}\Big] = \mathbf{E}\Big[\lim_{u \to \infty} \frac{g_u(\tilde{P}')}{\mathbf{P}(\zeta > u)}\Big]$$
$$= \mathbf{E}\Big[\sum_{x \in \Phi(0,\tilde{P}')} \exp\left\{-\sum_{t \in \tilde{P}': x \in \tilde{\Phi}(t)} f\left(t, \frac{w(x-t)}{w(x)}\eta\right)\right\} w(x)^{\alpha}\Big], \quad (5.14)$$

where η and \tilde{P}' are independent, which is precisely (5.9). It remains to justify the interchange of the limit and expectation in (5.14).

Since g is bounded by 1 and $w_* := \sup_{t \in \mathbb{R}^d} w(t) < \infty$, for each μ' and u > 0, we have $g_u(\mu') \leq N(0, \mu') \mathbf{P}(\zeta > u/w_*)$. The regular variation property of ζ yields that

$$0 \le \frac{g_u(P')}{\mathbf{P}(\zeta > u)} \le \tilde{N}(0) \frac{\mathbf{P}(\zeta > u/w_*)}{\mathbf{P}(\zeta > u)} \to \tilde{N}(0) w_*^{\alpha}$$

almost surely as $u \to \infty$. Moreover, since $\mathbf{E}[\tilde{N}(0)w_*^{\alpha}] = \mathbf{E}[\tilde{N}(0)]w_*^{\alpha}$ is finite by assumption, Pratt's extension of the dominated convergence theorem (see [23, Theorem 1]) justifies the interchange

$$\lim_{u \to \infty} \mathbf{E} \left[\frac{g_u(\dot{P}')}{\mathbf{P}(\zeta > u)} \right] = \mathbf{E} \left[\lim_{u \to \infty} \frac{g_u(\dot{P}')}{\mathbf{P}(\zeta > u)} \right],$$

and this finishes the proof.

Point process convergence

In the following assume that the assumptions of Theorem 5.4 hold so that X admits a tail configuration Y whose spectral configuration Θ is given by (5.5); recall the convention explained in Remark 5.5. In order to determine the ingredients of the limiting point process in Theorem 4.2, the following result is crucial; our approach used here is similar to the one taken by [22, Section 4] for random fields over \mathbb{Z}^d .

Lemma 5.8. Denote by W the random element in \mathcal{N} equal to

$$\left\{ (t, w(-t)) : t \in \tilde{P}', 0 \in \tilde{\Phi}(t) \right\}$$
(5.15)

restricted to \mathbb{E} . Then the spectral tail configuration Θ from (5.5) satisfies

$$\mathbf{E}[h(\Theta)] = \kappa^{-1} \mathbf{E}\left[\sum_{(t,s)\in W} h(T_{1,s}\varphi_t W) s^{\alpha}\right]$$
(5.16)

for all $h: \mathcal{N} \to [0, \infty)$. In particular,

$$\kappa = \mathbf{E}\left[\sum_{(t,s)\in W} s^{\alpha}\right].$$
(5.17)

Proof. Observe that

where we used the shift-equivariance of Φ to obtain the second equality. The pointstationarity of \tilde{P}' (see (2.2)) yields that

$$\begin{split} \kappa \mathbf{E}[h(\Theta)] &= \mathbf{E}\Big[\sum_{x \in \tilde{P}'} g(x, \tilde{P}')\Big] \\ &= \mathbf{E}\Big[\sum_{x \in \tilde{P}'} h\Big(\big\{\big(t - x, \frac{w(-t)}{w(-x)}\big) : t \in \tilde{P}', \ 0 \in \Phi(t, \tilde{P}')\big\}\Big) \mathbf{1}_{\{0 \in \Phi(x, \tilde{P}')\}} w(-x)^{\alpha}\Big] \\ &= \mathbf{E}\Big[\sum_{x \in \tilde{P}'} h\Big(T_{1, w(-x)} \varphi_x W\Big) \mathbf{1}_{\{0 \in \tilde{\Phi}(x)\}} w(-x)^{\alpha}\Big], \end{split}$$

which is precisely what we wanted to prove. Expression (5.17) follows by taking $h \equiv 1$. By setting $w \equiv 1$, the two expressions for κ in (5.4) and (5.17) imply that

$$\mathbf{E}\left[\sum_{t\in\tilde{P}'}\mathbf{1}_{\{0\in\tilde{\Phi}(t)\}}\right] = \mathbf{E}[\tilde{N}(0)] < \infty \,.$$

This implies that W, and therefore Θ , almost surely has finitely many points in $\mathbb{E} = \mathbb{R}^d \times (0, \infty)$. In particular, we can regard W and Θ as elements of \mathcal{N}_{01} , i.e. $\mathbf{P}(W \in \mathcal{N}_{01}) = \mathbf{P}(\Theta \in \mathcal{N}_{01}) = 1$; the same holds for the tail configuration as well.

We now turn our attention to the process Q defined in (4.22). Recall that $M(\mu)$ denotes the maximal score of $\mu \in \mathcal{N}_{01}$.

Proposition 5.9. The distribution of Q from (4.22) in $\tilde{\mathcal{N}}_{01}$ is given by

$$\mathbf{P}([Q] \in \cdot) = \frac{1}{\mathbf{E}[M(W)^{\alpha}]} \mathbf{E} \Big[\mathbf{1}_{\{T_{1,M(W)}[W] \in \cdot\}} M(W)^{\alpha} \Big].$$
(5.18)

Moreover,

$$\vartheta = \frac{\mathbf{E}\left[\max_{(t,s)\in W} s^{\alpha}\right]}{\mathbf{E}\left[\sum_{(t,s)\in W} s^{\alpha}\right]}.$$
(5.19)

Proof. Using (4.20) and since $\beta = 0$, for an arbitrary shift-invariant and bounded $h : \mathcal{N}_{01} \to [0, \infty)$, we have

$$\vartheta \mathbf{E}[h(Q)] = \mathbf{E}\Big[h(T_{1,M(\Theta)}\Theta) \; \frac{M(\Theta)^{\alpha}}{\sum_{(t,s)\in\Theta} s^{\alpha}}\Big] =: \mathbf{E}[\tilde{h}(\Theta)].$$

Observe that \tilde{h} is shift-invariant and homogenous in the sense that $\tilde{h}(T_{1,y}\mu) = \tilde{h}(\mu)$ for all $\mu \in \mathcal{N}_{01}, y > 0$. By (5.16),

$$\begin{split} \vartheta \, \mathbf{E}[h(Q)] &= \kappa^{-1} \mathbf{E} \left[\sum_{(t,s) \in W} \tilde{h}(T_{1,s} \varphi_t W) s^{\alpha} \right] \\ &= \kappa^{-1} \mathbf{E} \left[\sum_{(t,s) \in W} \tilde{h}(W) s^{\alpha} \right] \\ &= \kappa^{-1} \mathbf{E} \left[\sum_{(t,s) \in W} h(T_{1,M(W)} W) \frac{M(W)^{\alpha}}{\sum_{(x,y) \in W} y^{\alpha}} s^{\alpha} \right] \\ &= \kappa^{-1} \mathbf{E} \left[h(T_{1,M(W)} W) M(W)^{\alpha} \frac{\sum_{(t,s) \in W} s^{\alpha}}{\sum_{(x,y) \in W} y^{\alpha}} \right] \\ &= \kappa^{-1} \mathbf{E} \left[h(T_{1,M(W)} W) M(W)^{\alpha} \right]. \end{split}$$

Due to (5.17), taking $h \equiv 1$ yields (5.19), while (5.18) follows since h was arbitrary.

We now give sufficient conditions under which the assumptions of Theorem 4.2 are satisfied. First, take a family $(a_{\tau})_{\tau}$ such that (4.1) holds, and fix an arbitrary family $(b_{\tau})_{\tau}$ such that $b_{\tau} \to \infty$ and $b_{\tau}/\tau \to 0$ as $\tau \to \infty$; since r is a constant function, this is equivalent to choosing (b_{τ}) such that Assumption 4.3 holds.

Assumption 5.10. Let $\tilde{\mathcal{N}}_g$ be the space of all simple locally finite point measures μ' on \mathbb{R}^d such that $0 \in \mu'$. Assume that there exists a measurable function $R : \tilde{\mathcal{N}}_g \to [0, \infty]$ such that $R(\tilde{P}') < \infty$ a.s. and for all $\mu' \in \tilde{\mathcal{N}}_g$,

- (i) for all $t \in \mu'$ such that $t \notin B_{R(\mu')}$, $\Phi(0,\mu') \cap \Phi(t,\mu') = \emptyset$ and $\Phi(t,\mu') = \Phi(t,\mu' \setminus \{0\});$
- (ii) $\Phi(0,\mu') = \Phi(0,\nu')$ for all $\nu' \in \mathcal{N}_g$ such that μ' and ν' coincide on $B_{R(\mu')}$, i.e. $\Phi(0,\mu')$ is unaffected by changing points in μ' outside of $B_{R(\mu')}$.

Note that (ii) above necessarily implies that $\Phi(0, \mu') = \Phi(0, \mu' \cap B_{R(\mu')}) \subseteq B_{R(\mu')}$, for all $\mu' \in \tilde{\mathcal{N}}_g$.

Example 5.11. (a) Assume that for some $r_0 > 0$, with probability 1,

$$\Phi(t, \tilde{P}') = \Phi(t, \tilde{P}' \cap B(t, r_0)), \text{ for all } t \in \tilde{P}'.$$

In this case Assumption 5.10 is clearly satisfied if we take $R = 2r_0$.

(b) If $\Phi(t, \mu')$ is the set containing t and the k nearest neighbors of t in $\mu' \setminus \{t\}$ (with respect to the Euclidean distance), R satisfying Assumption 5.10 can be constructed as in the proof of [20, Lemma 6.1], see also [12, p. 104]. Observe that in this case taking $R(\mu')$ to be the distance to the k-th nearest neighbor of 0 in $\mu' \setminus \{0\}$ is not sufficient for property (i), and this property is crucial to ensure that the anticlustering condition holds.

Proposition 5.12. If Assumption 5.10 holds, the anticlustering condition (4.14) also holds.

Proof. Denote $\tilde{R} := R(\tilde{P}')$ and $w_* := \max_{x \in \mathbb{R}^d} w(x)$. Recall that $w_* \in [1, \infty)$ since w(0) = 1 and w is a bounded function. For notational convenience, assume that $w_* = 1$; the proof below is easily extended to the general case. Then, for every $t \in \tilde{P}'$,

$$\psi(t, P) \le \max_{x \in \Phi(t, \tilde{P}')} \zeta_x.$$

Due to (5.4), for arbitrary $\varepsilon, \delta, c > 0$, to show (4.14) it suffices to prove that

$$\lim_{u \to \infty} \limsup_{\tau \to \infty} \frac{\mathbf{P}(A_{\tau,u})}{\mathbf{P}(\zeta > a_{\tau}\varepsilon)} = 0, \qquad (5.20)$$

where

$$A_{\tau,u} := \left\{ \max_{t \in \tilde{P}' \cap C_{\tau,u}, x \in \tilde{\Phi}(t)} \zeta_x > a_\tau \delta, \ \max_{x \in \tilde{\Phi}(0)} \zeta_x > a_\tau \varepsilon \right\}.$$

Fix a u > 0. If $\tilde{R} < u$, since $C_{\tau,u} \subseteq B_u^c$, Assumption 5.10(i) implies that the families $\{\zeta_x : x \in \tilde{\Phi}(t) \text{ for some } t \in \tilde{P}' \cap C_{\tau,u}\}$ and $\{\zeta_x : x \in \tilde{\Phi}(0)\}$ consist of completely different sets of the ζ_x 's, hence are independent (given \tilde{P}'). Moreover, if $\tilde{R} < u$, since $\tilde{P}' = P' \cup \{0\}$, Assumption 5.10(i) also implies that

$$\max_{t\in\tilde{P}'\cap C_{\tau,u},x\in\tilde{\Phi}(t)}\zeta_x=\max_{t\in P'\cap C_{\tau,u},x\in\Phi(t,P')}\zeta_x$$

Therefore,

$$\mathbf{P}(A_{\tau,u}, \tilde{R} < u \mid \tilde{P}') = \mathbf{1}_{\{\tilde{R} < u\}} \mathbf{P}(\max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t, P')} \zeta_x > a_\tau \delta \mid \tilde{P}') \mathbf{P}(\max_{x \in \tilde{\Phi}(0)} \zeta_x > a_\tau \varepsilon \mid \tilde{P}')$$

$$\leq \mathbf{P}(\max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t, P')} \zeta_x > a_\tau \delta \mid \tilde{P}') \tilde{N}(0) \mathbf{P}(\zeta > a_\tau \varepsilon),$$

where the inequality relies on the fact that \tilde{P}' is independently marked. Furthermore,

$$\mathbf{P}(A_{\tau,u}, \tilde{R} \ge u \mid \tilde{P}') \le \mathbf{1}_{\{\tilde{R} \ge u\}} \mathbf{P}(\max_{x \in \tilde{\Phi}(0)} \zeta_x > a_\tau \varepsilon \mid \tilde{P}') \le \mathbf{1}_{\{\tilde{R} \ge u\}} \tilde{N}(0) \mathbf{P}(\zeta > a_\tau \varepsilon).$$

Conditioning on \tilde{P}' yields that

$$\frac{\mathbf{P}(A_{\tau,u})}{\mathbf{P}(\zeta > a_{\tau}\varepsilon)} \le \mathbf{E} \left[\mathbf{1} \left\{ \max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t,P')} \zeta_x > a_{\tau}\delta \right\} \tilde{N}(0) \right] + \mathbf{E} \left[\mathbf{1}_{\{\tilde{R} \ge u\}} \tilde{N}(0) \right]$$

for all $u, \tau > 0$. The second term on the right-hand side does not depend on τ and vanishes as $u \to \infty$ by the dominated convergence theorem since $\mathbf{P}(\tilde{R} < \infty) = 1$ and $\mathbf{E}\tilde{N}(0) < \infty$. For the first term, observe that for each u > 0 and $n \in \mathbb{N}$,

$$\limsup_{\tau \to \infty} \mathbf{E} \left[\mathbf{1} \{ \max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t,P')} \zeta_x > a_\tau \delta \} \tilde{N}(0) \right]$$

$$\leq n \limsup_{\tau \to \infty} \mathbf{P}(\max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t,P')} \zeta_x > a_\tau \delta) + \mathbf{E}[\tilde{N}(0) \mathbf{1}_{\{\tilde{N}(0) > n\}}]$$

Since $\lim_{n\to\infty} \mathbf{E}[\tilde{N}(0)\mathbf{1}_{\{\tilde{N}(0)>n\}}] = 0$ due to $\mathbf{E}\tilde{N}(0) < \infty$, to show (5.20) it suffices to prove that

$$\lim_{u \to \infty} \limsup_{\tau \to \infty} \mathbf{P}(\max_{t \in P' \cap C_{\tau,u}, x \in \Phi(t, P')} \zeta_x > a_\tau \delta) = 0.$$
(5.21)

This holds since a_{τ} is chosen so that (4.1) holds, while for all u, $|C_{\tau,u}|/\tau^d \leq |B_{b_{\tau}c}|/\tau^d = \text{const} \cdot b_{\tau}^d/\tau^d \to 0$ as $\tau \to \infty$ by the choice of $(b_{\tau})_{\tau}$. Indeed, the refined Campbell's formula (2.1) gives that

$$\mathbf{P}(\max_{t\in P'\cap C_{\tau,u},x\in\Phi(t,P')}\zeta_x > a_\tau\delta) \le |B_{b_\tau c}|\mathbf{P}(\max_{x\in\tilde{\Phi}(0)}\zeta_x > a_\tau\delta)$$
$$= \frac{|B_{b_\tau c}|}{\tau^d} \frac{\mathbf{P}(\max_{x\in\tilde{\Phi}(0)}\zeta_x > a_\tau\delta)}{\mathbf{P}(\xi > a_\tau\delta)} n^d \mathbf{P}(\xi > a_\tau\delta),$$

for all $u, \tau > 0$. Due to (4.2), (5.4) and (5.7), $b_{\tau}/\tau \to 0$ implies that (5.21) holds.

Proposition 5.13. If Assumption 5.10 holds, Assumption 4.5 is satisfied for any $(l_{\tau})_{\tau}$ such that $l_{\tau} \to \infty$ and $l_{\tau}/b_{\tau} \to 0$ as $\tau \to \infty$.

Proof. Recall that for a fixed family $(l_{\tau})_{\tau}$, and each $\tau > 0$ and $i \in I_{\tau}$, the block of indices $J_{\tau,i}$ is defined by (4.3), its trimmed version $\hat{J}_{\tau,i}$ by (4.16), and that $\hat{X}_{\tau,i} := X_{\hat{J}_{\tau,i}}$. Let now

 $f_{\tau,i} \in \mathcal{F}, \tau > 0, i \in I_{\tau}$, be an arbitrary family of functions which satisfy (4.15) for a same $\delta > 0$. For notational convenience, write

$$W_{\tau,i} = \exp\left\{-f_{\tau,i}(T_{1,a_{\tau}}\widehat{X}_{\tau,i})\right\}$$

for all $\tau > 0$, $i \in I_{\tau}$. To confirm Assumption 4.5, we need to show that

$$\lim_{\tau \to \infty} \left| \mathbf{E} \left[\prod_{i \in I_{\tau}} W_{\tau, i} \right] - \prod_{i \in I_{\tau}} \mathbf{E} \left[W_{\tau, i} \right] \right| = 0.$$
(5.22)

We first extend the definition of the radius R from Assumption 5.10 by setting $R(t, \mu') := R(\varphi_t \mu')$, for all $\mu' \in \tilde{\mathcal{N}}_g$, $t \in \mu'$. Using shift-equivariance of Φ , Campbell's formula and the assumption $\mathbf{P}(R(\tilde{P}') < \infty) = 1$, it is not difficult to show that almost surely, for all $t \in P'$, $\Phi(t, P') = \Phi(t, \nu')$ for all ν' which coincide with P' on $B_{R(t,\mu')}(t)$.

Thus, for every $t \in P'$, R(t, P') < u implies

$$X(t) = \psi(t, P) = \max_{x \in \Phi(t, P')} w(x - t)\zeta_x = \psi(t, P_{B_u(t)}).$$

Furthermore, the value of $W_{\tau,i}$ depends only on those $t \in P' \cap \hat{J}_{\tau,i}$ with score $\psi(t, P) > a_{\tau}\delta$. In particular, if

$$\max_{t \in \hat{J}_{\tau,i} \cap P'} R(t,P') \mathbf{1}_{\{\psi(t,P) > a_{\tau}\delta\}} < l_{\tau} ,$$

the random variable $W_{\tau,i}$ depends only on P restricted to $J_{\tau,i}$.

Construct now k_{τ}^{d} i.i.d. Poisson processes $P_{(i)}, i \in I_{\tau}$, with common distribution equal to the distribution of P and such that for each $i \in I_{\tau}$, restrictions of $P_{(i)}$ and P on the block $J_{\tau,i}$ coincide. Furthermore, let for each $i \in I_{\tau}, W_{\tau,i}^{*}$ be constructed from $P_{(i)}$ in the same way as $W_{\tau,i}$ is constructed from P. In particular, since $W_{\tau,i}^{*}$'s are independent,

$$\left|\mathbf{E}\left[\prod_{i\in I_{\tau}}W_{\tau,i}\right]-\prod_{i\in I_{\tau}}\mathbf{E}\left[W_{\tau,i}\right]\right|=\left|\mathbf{E}\left[\prod_{i\in I_{\tau}}W_{\tau,i}-\prod_{i\in I_{\tau}}W_{\tau,i}^{*}\right]\right|,$$

and moreover, one has $W_{\tau,i} = W^*_{\tau,i}$ whenever

$$\max_{t \in \hat{J}_{\tau,i} \cap P'} R(t,P') \mathbf{1}_{\{\psi(t,P) > a_{\tau}\delta\}} < l_{\tau} \text{ and } \max_{t \in \hat{J}_{\tau,i} \cap P'} R(t,P'_{(i)}) \mathbf{1}_{\{\psi(t,P_{(i)}) > a_{\tau}\delta\}} < l_{\tau}.$$

Thus, since $\cup_{i \in I_{\tau}} \hat{J}_{\tau,i} \subseteq [0, \tau]^d$ and $0 \leq W_{\tau,i} \leq 1$,

$$\begin{aligned} \left| \mathbf{E} \left[\prod_{\boldsymbol{i} \in I_{\tau}} W_{\tau, \boldsymbol{i}} - \prod_{\boldsymbol{i} \in I_{\tau}} W_{\tau, \boldsymbol{i}}^* \right] \right| &\leq 2 \mathbf{P} \left(\max_{\boldsymbol{t} \in [0, \tau]^d \cap P'} R(\boldsymbol{t}, P') \mathbf{1}_{\{\psi(\boldsymbol{t}, P) > a_{\tau}\delta\}} \geq l_{\tau} \right) \\ &\leq 2 \mathbf{E} \left[\sum_{\boldsymbol{t} \in P' \cap [0, \tau]^d} \mathbf{1} \{ R(\boldsymbol{t}, P') \geq l_{\tau}, \psi(\boldsymbol{t}, P) > a_{\tau}\delta \} \right] \end{aligned}$$

Using shift-invariance and the refined Campbell's theorem (2.1) we obtain

$$\begin{aligned} \left| \mathbf{E} \left[\prod_{\boldsymbol{i} \in I_{\tau}} W_{\tau, \boldsymbol{i}} \right] - \prod_{\boldsymbol{i} \in I_{\tau}} \mathbf{E} \left[W_{\tau, \boldsymbol{i}} \right] \right| &\leq 2 \mathbf{E} \left[\sum_{t \in P' \cap [0, \tau]^d} \mathbf{1} \{ R(0, \varphi_t P') \geq l_{\tau}, \psi(0, \varphi_t P) > a_{\tau} \delta \} \right] \\ &= 2 \tau^d \mathbf{P} (R(0, \tilde{P}') \geq l_{\tau}, \psi(0, \tilde{P}) > a_{\tau} \delta) \,, \end{aligned}$$

for all $\tau > 0$. Since

$$\xi = \psi(0, P) \le w_* \max_{x \in \widetilde{\Phi}(0)} \zeta_x$$

for $w_* = \max_{t \in \mathbb{R}^d} w(t) \in [1, \infty)$ and \tilde{P}' is independent of the ζ 's, conditioning on \tilde{P}' yields that

$$2\tau^{d}\mathbf{P}(R(0,\tilde{P}') \ge l_{\tau},\psi(0,\tilde{P}) > a_{\tau}\delta) \le 2\tau^{d}\mathbf{E}[\mathbf{1}_{\{R(0,\tilde{P}')\ge l_{\tau}\}}\tilde{N}(0)]\mathbf{P}(\zeta > a_{\tau}\delta/w_{*}).$$

Due to (5.4) and (4.2), $\tau^d \mathbf{P}(\zeta > a_\tau \delta / w_*)$ converges to a positive constant, so (5.22) follows by dominated convergence since $l_\tau \to \infty$, $\mathbf{P}(R(0, \tilde{P}') < \infty) = 1$ and $\mathbf{E}\tilde{N}(0) < \infty$.

The above arguments lead to the following conclusion.

Proposition 5.14. Let X be defined as in the beginning of Section 5.2. Assume w is a bounded function, that $\mathbf{E}[\tilde{N}(0)] < \infty$, and that there exists an R satisfying Assumption 5.10. Let $(a_{\tau})_{\tau}$ be as in (4.1). Then any family $(b_{\tau})_{\tau}$, such that $b_{\tau} \to \infty$ and $b_{\tau}/\tau \to 0$ as $\tau \to \infty$, satisfies all of the assumptions of Theorem 4.2 (with α being equal to the tail index of ζ and $r \equiv 1$) and, therefore, the convergence of the extremal blocks in (4.9) holds.

6 Proof of Theorem 4.2

Recall that X is a stationary marked point process on \mathbb{E} which admits a tail configuration Y with tail index $\alpha > 0$ and a scaling function r of scaling index $\beta \in \mathbb{R}$. Moreover, assume that $(a_{\tau})_{\tau>0}$ satisfies (4.1) and fix a family of block side lengths $(b_{\tau})_{\tau>0}$.

For notational convenience, for all $\tau > 0$ and y > 0 denote in the sequel

$$X_{\tau} := X_{[0,b_{\tau}]^d}$$
, and $T_y^{\tau} := T_{r(a_{\tau}y),a_{\tau}y}$.

We first extend (3.3) to convergence in the space \mathcal{N}_{01} with the \mathcal{B}_{01} -vague topology.

Proposition 6.1. Assume that $(b_{\tau})_{\tau>0}$ satisfies Assumptions 4.3 and 4.4. Then

$$\mathbf{P}(Y \in \mathcal{N}_{01}) = 1. \tag{6.1}$$

Furthermore, for all y > 0

$$\mathbf{P}(T_y^{\tau}(\tilde{X}_{D_{\tau}}) \in \cdot \mid \xi > a_{\tau}y) \xrightarrow{w} \mathbf{P}(Y \in \cdot) \quad as \ \tau \to \infty$$
(6.2)

on \mathcal{N}_{01} , where $(D_{\tau})_{\tau>0}$ is any family of subsets of \mathbb{R}^d such that

$$B_{c_1b_\tau} \subseteq D_\tau \subseteq B_{c_2b_\tau}, \quad \tau > 0, \tag{6.3}$$

for some constants $0 < c_1 < c_2$.

Proof. Recall that $\mathbf{P}(Y \in \mathcal{N}_{11}) = 1$, i.e., Y a.s. has finitely many points in $B \times (\delta, \infty)$ for all bounded $B \subset \mathbb{R}^d$ and all $\delta > 0$. To prove (6.1), we need to show that $Y(\mathbb{R}^d \times (\delta, \infty)) < \infty$ a.s. for all $\delta > 0$. Fix $u, \delta > 0$ and observe that

$$\mathbf{P}\Big(\sup_{|t|\geq u} Y(t) \geq \delta\Big) = \lim_{u' \to \infty} \mathbf{P}\Big(\max_{u' \geq |t|\geq u} Y(t) > \delta\Big).$$

By definition (3.3) of Y, for all but at most countably many u' > u,

$$\mathbf{P}\Big(\max_{u' \ge |t| > u} Y(t) \ge \delta\Big) = \lim_{\tau \to \infty} \mathbf{P}\Big(\max_{u' \ge |t| / r(a_\tau) \ge u} \tilde{X}(t) > a_\tau \delta \mid \tilde{X}(0) > a_\tau\Big).$$

Since $b_{\tau}/r(a_{\tau}) \to \infty$ by Assumption 4.3, the two previous relations imply that

$$\mathbf{P}\Big(\sup_{|t|\geq u} Y(t) \geq \delta\Big) \leq \limsup_{\tau \to \infty} \mathbf{P}\Big(\max_{b_{\tau}/r(a_{\tau})\geq |t|/r(a_{\tau})\geq u} \tilde{X}(t) > a_{\tau}\delta \mid \tilde{X}(0) \geq a_{\tau}\Big).$$

Assumption 4.4 (with $c = \varepsilon = 1$) yields

$$\lim_{u \to \infty} \mathbf{P}\Big(\sup_{|t| \ge u} Y(t) \ge \delta\Big) = 0.$$

Thus,

$$\mathbf{P}\Big(\bigcup_{u>0}\bigcap_{|t|>u}\{Y(t)<\delta\}\Big)=1\,,\tag{6.4}$$

and since $\mathbf{P}(Y \in \mathcal{N}_{11}) = 1$, one has that $Y(\mathbb{R}^d \times [\delta, \infty)) < \infty$ a.s. Since δ was arbitrary, this proves (6.1).

We now turn to (6.2). Fix a y > 0. For all but at most countably many u > 0, the definition of the tail configuration implies that

$$\mathbf{P}\left((T_y^{\tau}\tilde{X})_{B_u} \in \cdot \mid \tilde{X}(0) > a_{\tau}y\right) \xrightarrow{w} \mathbf{P}(Y_{B_u} \in \cdot) \quad \text{as } \tau \to \infty$$
(6.5)

in \mathcal{N}_{11} with the \mathcal{B}_{11} -vague topology. For measures in \mathcal{N}_{11} whose support is in $B_u \times (0, \infty)$, \mathcal{B}_{11} -vague topology is actually equivalent to the (in general stronger) \mathcal{B}_{01} -vague topology. Thus, (6.5) holds on \mathcal{N}_{01} with respect to the \mathcal{B}_{01} -vague topology as well. Furthermore, it is easy to see that, due to (6.4), $Y_{B_u} \to Y$ almost surely in \mathcal{N}_{11} as $u \to \infty$. In particular,

$$\mathbf{P}(Y_{B_u} \in \cdot) \xrightarrow{w} \mathbf{P}(Y \in \cdot) \quad \text{as } u \to \infty$$
(6.6)

on \mathcal{N}_{11} with respect to the \mathcal{B}_{01} -vague topology. By the classical result on weak convergence of probability measures (see [6, Theorem 4.2]), to prove (6.2) it suffices to show that

$$\lim_{u \to \infty} \limsup_{\tau \to \infty} \mathbf{E} \Big[\mathsf{m}((T_y^{\tau} \tilde{X})_{B_u}, T_y^{\tau}(\tilde{X}_{D_{\tau}})) \mid \tilde{X}(0) > a_{\tau} y \Big] = 0, \qquad (6.7)$$

where m is the metric from (4.5) (which generates the \mathcal{B}_{01} -vague topology on \mathcal{N}_{01}). For this, observe that

$$(T_y^{\tau}\tilde{X})_{B_u} = T_y^{\tau}(\tilde{X}_{B_{r(a\tau y)u}}).$$

Let $u, \varepsilon > 0$ be arbitrary. Due to the first inclusion in (6.3) and since $b_{\tau}/r(a_{\tau}y) \to \infty$ by Assumption 4.3, $B_{r(a_{\tau}y)u} \subseteq D_{\tau}$ for all sufficiently large τ . Observe that if

$$\max_{t \in D_{\tau} \setminus B_{r(a_{\tau}y)u}} \tilde{X}(t) \le a_{\tau} y \varepsilon, \qquad (6.8)$$

then $T_y^{\tau}(\tilde{X}_{B_{r(a_{\tau}y)u}})$ and $T_y^{\tau}(\tilde{X}_{D_{\tau}})$ coincide when restricted to $\mathbb{R}^d \times (1/r, \infty)$ with $r < 1/\varepsilon$. Since \mathbf{m}_0 is bounded by 1,

$$\mathsf{m}(T_y^{\tau}(\tilde{X}_{B_{r(a\tau y)u}}), T_y^{\tau}(\tilde{X}_{D_{\tau}})) \le \int_{1/\varepsilon}^{\infty} e^{-r} \mathrm{d}r = e^{-1/\varepsilon} \,. \tag{6.9}$$

If (6.8) does not hold, we use the fact that m is bounded by 1. Thus,

$$\begin{split} \limsup_{\tau \to \infty} \mathbf{E} \Big[\mathsf{m}(T_y^{\tau}(\tilde{X}_{B_{r(a_{\tau}y)u}}), T_y^{\tau}(\tilde{X}_{D_{\tau}})) \mid \tilde{X}(0) > a_{\tau}y \Big] \\ &\leq e^{-1/\varepsilon} + \limsup_{\tau \to \infty} \mathbf{P} \Big(\max_{t \in D_{\tau} \setminus B_{r(a_{\tau}y)u}} \tilde{X}(t) > a_{\tau}y\varepsilon \mid \tilde{X}(0) > a_{\tau}y \Big) \\ &\leq e^{-1/\varepsilon} + \limsup_{\tau \to \infty} \mathbf{P} \Big(\max_{t \in B_{b_{\tau}c_2} \setminus B_{r(a_{\tau}y)u}} \tilde{X}(t) > a_{\tau}y\varepsilon \mid \tilde{X}(0) > a_{\tau}y \Big) \,, \end{split}$$

where the last inequality holds, since $D_{\tau} \subseteq B_{b_{\tau}c_2}$. If now one lets $u \to \infty$ and then $\varepsilon \to 0$, Assumption 4.4 implies (6.7), and this proves (6.2).

For all $\mu \in \mathcal{N}_{01}$ and y > 0, let

$$A_y(\mu) := \min\{t \in \mathbb{R}^d : (t,s) \in \mu, s > y\},\$$

where the minimum is taken with respect to the lexicographic order; if $M(\mu) \leq y$, set $A_y(\mu) := 0$. Note that A_y is well defined, since for every $\mu \in \mathcal{N}_{01}$ and y > 0 there are at most finitely many $(t, s) \in \mu$ with s > y. Observe also that A_y is equivariant under translations, that is, if $M(\mu) > y$,

$$A_y(\varphi_z \mu) = A_y(\mu) - z, \quad z \in \mathbb{R}^d.$$
(6.10)

In particular, A_1 is precisely the first exceedance anchoring function A^{fe} from (4.18). As shown in Lemma 4.6, $\mathbf{P}(A_1(Y) = 0)$ is positive whenever $\mathbf{P}(Y \in \mathcal{N}_{01}) = 1$, which holds, for instance, under Assumptions 4.3 and 4.4. Recall also that $X_{\tau} := X_{[0,b_{\tau}]^d}$.

Proposition 6.2. Assume that Assumptions 4.3 and 4.4 hold. Then for every y > 0,

$$\frac{\mathbf{P}(M(X_{\tau}) > a_{\tau}y)}{b_{\tau}^{d}\mathbf{P}(\xi > a_{\tau}y)} \to \mathbf{P}(A_{1}(Y) = 0) \quad as \ \tau \to \infty,$$
(6.11)

and

$$\mathbf{P}\Big([T_y^{\tau}(X_{\tau})] \in \cdot \mid M(X_{\tau}) > a_{\tau}y\Big) \xrightarrow{w} \mathbf{P}\Big([Y] \in \cdot \mid A_1(Y) = 0\Big) \quad on \ \tilde{\mathcal{N}}_{01}.$$
(6.12)

Proof. Consider a function $h : \mathcal{N}_{01} \to [0, \infty)$, which is bounded continuous and shift-invariant. First, observe that we can decompose

$$h(T_y^{\tau}(X_{\tau}))\mathbf{1}\{M(X_{\tau}) > a_{\tau}y\} = \sum_{(t,s)\in X} h(T_y^{\tau}(X_{\tau}))\mathbf{1}\{t \in [0, b_{\tau}]^d, A_{a_{\tau}y}(X_{\tau}) = t, X(t) > a_{\tau}y\}.$$

Since h is shift-invariant and since T_y^{τ} scales the positions with $r(a_{\tau}y)^{-1}$,

$$h(T_y^{\tau}(X_{\tau})) = h(\varphi_{t/r(a_{\tau}y)}T_y^{\tau}(X_{\tau})) = h(T_y^{\tau}(\varphi_t X_{\tau})) = h(T_y^{\tau}((\varphi_t X)_{[0,b_{\tau}]^d - t})).$$

By the definition of $A_{a_{\tau}y}$,

$$\{A_{a_{\tau}y}(X_{\tau})=t\}=\{A_{a_{\tau}y}(\varphi_{t}X_{\tau})=0\}=\{A_{a_{\tau}y}((\varphi_{t}X)_{[0,b_{\tau}]^{d}-t})=0\}.$$

Next, $\{X(t) > a_{\tau}y\} = \{\varphi_t X(0) > a_{\tau}y\}$. All of the above implies that

$$\mathbf{E}\Big[h(T_y^{\tau}(X_{\tau}))\mathbf{1}_{\{M(X_{\tau})>a_{\tau}y\}}\Big] = \mathbf{E}\Big[\sum_{(t,s)\in X} g(t,\varphi_t X)\Big]$$

for an obvious choice of a function $g : \mathbb{R}^d \times \mathcal{N}_{01} \to [0, \infty)$. Refined Campbell's theorem (2.1) yields that

$$\begin{split} \mathbf{E} \Big[h(T_y^{\tau}(X_{\tau})) \mathbf{1}_{\{M(X_{\tau}) > a_{\tau}y\}} \Big] &= \int_{\mathbb{R}^d} \mathbf{E} [g(t, \tilde{X})] \mathrm{d}t \\ &= \int_{[0, b_{\tau}]^d} \mathbf{E} \Big[h(T_y^{\tau}(\tilde{X}_{[0, b_{\tau}]^d - t})) \mathbf{1} \Big\{ A_{a_{\tau}y}(\tilde{X}_{[0, b_{\tau}]^d - t}) = 0, \tilde{X}(0) > a_{\tau}y \Big\} \Big] \mathrm{d}t \\ &= b_{\tau}^d \int_{[0, 1]^d} \mathbf{E} \Big[h(T_y^{\tau}(\tilde{X}_{[0, b_{\tau}]^d - b_{\tau}s})) \mathbf{1} \Big\{ A_1(T_y^{\tau}(\tilde{X}_{[0, b_{\tau}]^d - b_{\tau}s})) = 0, \tilde{X}(0) > a_{\tau}y \Big\} \Big] \mathrm{d}s \,, \end{split}$$

where for the last equality we used the substitution $s = t/b_{\tau}$ and the fact that $A_{a_{\tau}y}(\mu) = 0$ if and only if $A_1(T_y^{\tau}\mu) = 0$. In particular (recall that $\xi = \tilde{X}(0)$),

$$\frac{\mathbf{E}\Big[h(T_y^{\tau}(X_{\tau}))\mathbf{1}_{\{M(X_{\tau})>a_{\tau}y\}}\Big]}{b_{\tau}^{d}\mathbf{P}(\xi>a_{\tau}y)} = \int_{[0,1]^{d}} \mathbf{E}\Big[f(T_y^{\tau}(\tilde{X}_{[0,b_{\tau}]^{d}-b_{\tau}s})) \mid \tilde{X}(0)>a_{\tau}y\Big] \,\mathrm{d}s\,,\qquad(6.13)$$

where $f(\mu) := h(\mu) \mathbf{1}_{\{A_1(\mu)=0\}}$. Since for every fixed $s \in (0,1)^d$, sets $D_{\tau} := [0, b_{\tau}]^d - b_{\tau}s$, $\tau > 0$, satisfy (6.3), (6.2) implies that

$$\lim_{\tau \to \infty} \mathbf{E} \left[f(T_y^{\tau}(\tilde{X}_{[0,b_{\tau}]^d - b_{\tau}s})) \mid \tilde{X}(0) > a_{\tau}y \right] = \mathbf{E} \left[f(Y) \right] = \mathbf{E} \left[h(Y) \mathbf{1}_{\{A_1(Y)=0\}} \right].$$
(6.14)

Observe here that f is not continuous on the whole \mathcal{N}_{01} , but since the probability that $(t, 1) \in Y$ for some $t \in \mathbb{R}^d$ is zero (due to Propositions 3.2 and 3.3), it is continuous on the support of Y – this justifies the use of (6.2). By the dominated convergence theorem and since the limit in (6.14) does not depend on s, (6.13) yields that

$$\lim_{\tau \to \infty} \frac{\mathbf{E} \left[h(T_y^{\tau}(X_{\tau})) \mathbf{1}_{\{M(X_{\tau}) > a_{\tau}y\}} \right]}{b_{\tau}^d \mathbf{P}(\xi > a_{\tau}y)} = \mathbf{E} \left[h(Y) \mathbf{1}_{\{A_1(Y) = 0\}} \right].$$
(6.15)

Convergence in (6.11) follows from (6.15) with $h \equiv 1$. To prove (6.12), we notice that

$$\lim_{\tau \to \infty} \mathbf{E} \Big[h(T_y^{\tau}(X_{\tau})) \mid M(X_{\tau}) > a_{\tau} y \Big] = \mathbf{E} \big[h(Y) \mid A_1(Y) = 0 \big]$$

follows directly from (6.11) and (6.15).

Let M^* be the space of all Borel measures on $\tilde{\mathcal{N}}_{01}^*$ which are finite on all sets from the family

$$\mathcal{B}^* = \left\{ B \subseteq \tilde{\mathcal{N}}_{01}^* : \exists \varepsilon > 0, \forall [\mu] \in B, M([\mu]) > \varepsilon \right\}.$$

Equip M^* with the vague topology generated by \mathcal{B}^* . Furthermore, let M be the space of all Borel measures on $[0,1]^d \times \tilde{\mathcal{N}}_{01}^*$ taking finite values on $[0,1]^d \times B$ for all $B \in \mathcal{B}^*$, and equip it with the corresponding vague topology. Observe that the intensity measure

$$\mathbf{E}[N_{\tau}(\cdot)] = \sum_{i \in I_{\tau}} \mathbf{P}((ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) \in \cdot)$$

of the point process N_{τ} from (4.8), is an element of M for all $\tau > 0$. Recall that $N \subset M$ defined right before Theorem 4.2 is the subset of all *counting* measures.

Proposition 6.3 (Intensity convergence). Assume that Assumptions 4.3 and 4.4 hold. Then

$$k_{\tau}^{d} \mathbf{P} \Big([T_{1}^{\tau} X_{\tau}] \in \cdot \Big) \xrightarrow{v} \nu \quad in \, \boldsymbol{M}^{*} \text{ as } \tau \to \infty , \qquad (6.16)$$

where

$$\nu(\cdot) = \vartheta \int_0^\infty \mathbf{P}([T_{u^{-\beta}, u^{-1}}Q] \in \cdot) \alpha u^{-\alpha - 1} \mathrm{d}u, \qquad (6.17)$$

and $Q \in \mathcal{N}_{01}$ has distribution (4.11). In particular,

$$\mathbf{E}[N_{\tau}(\,\cdot\,)] \xrightarrow{v} \operatorname{Leb} \times \nu(\cdot) \quad as \ \tau \to \infty, \tag{6.18}$$

in N, where Leb is the Lebesgue measure on \mathbb{R}^d .

Proof. Let $h : \mathbf{M}^* \to [0, \infty)$ be a bounded and continuous function such that for some $\varepsilon > 0, h([\mu]) = 0$ whenever $M([\mu]) \le \varepsilon$. Then

$$k_{\tau}^{d} \mathbf{E} \left[h([T_{1}^{\tau} X_{\tau}]) \right] = k_{\tau}^{d} \mathbf{E} \left[h([T_{1}^{\tau} X_{\tau}]) \mathbf{1}_{\{M(X_{\tau}) > a_{\tau}\varepsilon\}} \right].$$
(6.19)

Since $k_{\tau} \sim \tau/b_{\tau}$ as $\tau \to \infty$, (4.1) implies that $k_{\tau} \sim (b_{\tau}^d \mathbf{P}(\xi > a_{\tau}))^{-1}$ as $\tau \to \infty$. Thus, as $\tau \to \infty$,

$$k_{\tau}^{d} \mathbf{E} \Big[h([T_{1}^{\tau} X_{\tau}]) \mathbf{1}_{\{M(X_{\tau}) > a_{\tau}\varepsilon\}} \Big] \\ \sim \mathbf{E} \Big[h([T_{1}^{\tau} X_{\tau}]) \mid M(X_{\tau}) > a_{\tau}\varepsilon \Big] \; \frac{\mathbf{P}(M(X_{\tau}) > a_{\tau}\varepsilon)}{b_{\tau}^{d} \mathbf{P}(\xi > a_{\tau}\varepsilon)} \; \frac{\mathbf{P}(\xi > a_{\tau}\varepsilon)}{\mathbf{P}(\xi > a_{\tau})} \cdot \quad (6.20)$$

By (6.11), the second term on the right-hand side of (6.20) converges to $\mathbf{P}(A_1(Y) = 0)$, which in turn by Proposition 4.7 equals $\vartheta = \mathbf{P}(A(Y) = 0)$. By regular variation property (3.5), the third term tends to $\varepsilon^{-\alpha}$. For the first term, recall that T_y^{τ} scales the positions with $r(a_{\tau}y)^{-1}$ and scores with $(a_{\tau}y)^{-1}$. Thus,

$$T_1^{\tau} X_{\tau} = T_{r(a_{\tau})/r(a_{\tau}\varepsilon),\varepsilon^{-1}} T_{\varepsilon}^{\tau} X_{\tau}$$

By assumption, r is a regularly varying function with index β , so $r(a_{\tau})/r(a_{\tau}\varepsilon) \rightarrow \varepsilon^{-\beta}$ as $\tau \rightarrow \infty$. In particular, (6.12) and the extended continuous mapping theorem (see [6, Theorem 5.5]) imply that

$$\lim_{\tau \to \infty} \mathbf{E} \left[h([T_1^{\tau} X_{\tau}]) \mid M(X_{\tau}) > a_{\tau} \varepsilon \right] = \lim_{\tau \to \infty} \mathbf{E} \left[h(T_{r(a_{\tau})/r(a_{\tau}\varepsilon),\varepsilon^{-1}}[T_{\varepsilon}^{\tau} X_{\tau}]) \mid M(X_{\tau}) > a_{\tau} \varepsilon \right]$$
$$= \mathbf{E} \left[h(T_{\varepsilon^{-\beta},\varepsilon^{-1}}[Y]) \mid A_1(Y) = 0 \right].$$

By Proposition 4.7 and (4.19), we can rewrite the limit as

$$\begin{split} \mathbf{E}[h(T_{\varepsilon^{-\beta},\varepsilon^{-1}}[Y]) \mid A_1(Y) &= 0] &= \mathbf{E}[h(T_{\varepsilon^{-\beta},\varepsilon^{-1}}[Y]) \mid A^{\mathrm{fm}}(Y) = 0] \\ &= \int_1^\infty \mathbf{E}\left[h(T_{\varepsilon^{-\beta},\varepsilon^{-1}}[T_{u^{-\beta},u^{-1}}Q])\right] \alpha u^{-\alpha-1} \mathrm{d}u \\ &= \int_1^\infty \mathbf{E}\left[h([T_{(\varepsilon u)^{-\beta},(\varepsilon u)^{-1}}Q])\right] \alpha u^{-\alpha-1} \mathrm{d}u \\ &= \varepsilon^\alpha \int_\varepsilon^\infty \mathbf{E}\left[h([T_{y^{-\beta},y^{-1}}Q])\right] \alpha y^{-\alpha-1} \mathrm{d}y \\ &= \varepsilon^\alpha \int_0^\infty \mathbf{E}\left[h([T_{y^{-\beta},y^{-1}}Q])\right] \alpha y^{-\alpha-1} \mathrm{d}y \,. \end{split}$$

In the above we used the substitution $y = u\varepsilon$ to obtain the fourth equality. For the final equality, note that $M(T_{y^{-\beta},y^{-1}}Q) = y$ since M(Q) = 1 a.s. In particular, $h([T_{y^{-\beta},y^{-1}}Q]) = 0$ a.s. whenever $y \leq \varepsilon$ by the properties of h stated in the beginning of the proof.

Bringing everything together, (6.19) and (6.20) imply that

$$\lim_{\tau \to \infty} k_{\tau}^{d} \mathbf{E} \left[h([T_{1}^{\tau} X_{\tau}]) \right] = \lim_{\tau \to \infty} k_{\tau}^{d} \mathbf{E} \left[h([T_{1}^{\tau} X_{\tau}]) \mathbf{1} \{ M(X_{\tau}) > a_{\tau} \varepsilon \} \right]$$
$$= \varepsilon^{\alpha} \int_{0}^{\infty} \mathbf{E} \left[h([T_{y^{-\beta}, y^{-1}}Q]) \right] \alpha y^{-\alpha - 1} \mathrm{d}y \,\vartheta \,\varepsilon^{-\alpha}$$
$$= \vartheta \int_{0}^{\infty} \mathbf{E} \left[h([T_{y^{-\beta}, y^{-1}}Q]) \right] \alpha y^{-\alpha - 1} \mathrm{d}y = \nu(h) \,,$$

where ν is defined at (6.17). Since h is arbitrary, this proves (6.16).

We now prove (6.18). By [14, Lemma 4.1], it suffices to prove that $\mathbf{E}[N_{\tau}(g)] \rightarrow (\text{Leb} \times \nu)(g)$ for all $g : [0,1]^d \times \tilde{\mathcal{N}}_{01}^* \rightarrow [0,\infty)$ of the form $g(t,[\mu]) = \mathbf{1}_{(a,b]}(t)\mathbf{1}_A([\mu])$, where (a,b] is the parallelepiped in \mathbb{R}^d determined by $a = (a_1,\ldots,a_d), b = (b_1,\ldots,b_d) \in [0,1]^d$ with $a_j \leq b_j$ for all j, and $A \in \mathcal{B}^*$ (with \mathcal{B}^* defined just before Proposition 6.3) such that $\nu(\partial A) = 0$.

Recall that $X_{\tau} = X_{[0,b_{\tau}]^d}$, so $X_{\tau} = X_{\tau,1}$ and due to stationarity of X, $\mathbf{P}([X_{\tau}] \in \cdot) = \mathbf{P}([X_{\tau,i}] \in \cdot)$ for all $i \in I_{\tau} = \{1, \ldots, k_{\tau}\}^d$, where $k_{\tau} = \lfloor \tau/b_{\tau} \rfloor$. Thus, as $\tau \to \infty$,

$$\mathbf{E}[N_{\tau}(g)] = \sum_{i \in I_{\tau}} \mathbf{1}_{\{ib_{\tau}/\tau \in (a,b]\}} \mathbf{P}([T_{1}^{\tau}X_{\tau}] \in A) \sim \left(\frac{\tau}{b_{\tau}}\right)^{d} \prod_{j=1}^{d} (b_{j} - a_{j}) \mathbf{P}([T_{1}^{\tau}X_{\tau}] \in A)$$
$$\sim \prod_{j=1}^{d} (b_{j} - a_{j}) k_{\tau}^{d} \mathbf{P}([T_{1}^{\tau}X_{\tau}] \in A) \rightarrow \prod_{j=1}^{d} (b_{j} - a_{j}) \nu(A) = \operatorname{Leb} \times \nu(g) ,$$

where in the penultimate step we applied (6.16).

To finish the proof of Theorem 4.2, we need the following technical lemma.

Lemma 6.4. Assume that $(b_{\tau})_{\tau>0}$ is such that Assumptions 4.3 and 4.4 hold. Then, Assumption 4.5 implies

$$\mathbf{E}\Big[\prod_{\boldsymbol{i}\in I_{\tau}}\exp\left\{-f(\boldsymbol{i}b_{\tau}/\tau,T_{1}^{\tau}[X_{\tau,\boldsymbol{i}}])\right\}\Big] - \prod_{\boldsymbol{i}\in I_{\tau}}\mathbf{E}\Big[\exp\left\{-f\left(\boldsymbol{i}b_{\tau}/\tau,T_{1}^{\tau}[X_{\tau,\boldsymbol{i}}]\right)\right\}\Big] \to 0 \quad (6.21)$$

as $\tau \to \infty$ for every $f: [0,1]^d \times \tilde{\mathcal{N}}_{01} \to [0,\infty)$ such that

- (i) for some $\varepsilon > 0$, $M(\mu) \le \varepsilon$ implies $f(t, [\mu]) = 0$ for all $t \in [0, 1]^d$;
- (ii) f is Lipschitz, that is, for some c > 0,

$$|f(t, [\mu]) - f(s, [\nu])| \le c \max\left\{|t - s|, \tilde{\mathsf{m}}([\mu], [\nu])\right\}$$

for all $t, s \in [0, 1]^d$ and nontrivial measures $\mu, \nu \in \mathcal{N}_{01}$.

Observe that, for each $\tau > 0$, the first term on the left-hand side of (6.21) is the Laplace functional $L_f(N_{\tau})$ of the point process N_{τ} , while the second term is the Laplace functional of the point process

$$N_{\tau}^* := \left\{ \left(\boldsymbol{i} b_{\tau} / \tau \,, \, T_1^{\tau} [X_{\tau, \boldsymbol{i}}^*] \right) : \boldsymbol{i} \in I_{\tau} \right\},\,$$

where the blocks $X_{\tau,i}^*$, $i \in I_{\tau}$, are independent, and for each $i \in I_{\tau}$, $X_{\tau,i}^*$ has the same distribution as the original block $X_{\tau,i}$.

Proof of Lemma 6.4. Let f be an arbitrary function satisfying the assumptions of the lemma for some ε and c. Fix an arbitrary $\delta < \varepsilon$ and define a function $f^{\delta} : [0,1]^d \times \mathcal{N}_{01} \to [0,\infty)$ by $f^{\delta}(t,\mu) := f(t,[\mu^{\delta}])$, where μ^{δ} is the restriction of $\mu \in \mathcal{N}_{01}$ to $\mathbb{R}^d \times (\delta,\infty)$. For all $t \in [0,1]^d$ and all $\mu \in \mathcal{N}_{01}$, the Lipschitz property of f implies that

$$|f(t, [\mu]) - f^{\delta}(t, \mu)| \le c\tilde{\mathsf{m}}([\mu], [\mu^{\delta}]) \le c\mathsf{m}(\mu, \mu^{\delta}) \le ce^{-1/\delta},$$
(6.22)

see (6.9) for a similar argument which justifies the last inequality.

By the construction of f^{δ} and properties of f, Assumption 4.5 implies that (4.17) holds for the family

$$f_{\tau,\boldsymbol{i}}(\mu) := f^{\delta}(\boldsymbol{i}b_{\tau}/\tau,\mu), \quad \tau > 0, \, \boldsymbol{i} \in I_{\tau}, \, \mu \in \mathcal{N}_{01}.$$

We first show that in (4.17) one can replace the trimmed blocks $\hat{X}_{\tau,i}$ with the original ones $X_{\tau,i}$. For this, observe that

$$\begin{aligned} \mathbf{E}\Big[\prod_{\boldsymbol{i}\in I_{\tau}}\exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}X_{\tau,\boldsymbol{i}})\right\}\Big] - \mathbf{E}\Big[\prod_{\boldsymbol{i}\in I_{\tau}}\exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}\hat{X}_{\tau,\boldsymbol{i}})\right\}\Big]\Big| \\ \leq \sum_{\boldsymbol{i}\in I_{\tau}}\mathbf{E}\Big|\exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}X_{\tau,\boldsymbol{i}})\right\} - \exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}\hat{X}_{\tau,\boldsymbol{i}})\right\}\Big|,\end{aligned}$$

where we used the elementary inequality

$$\left|\prod_{i=1}^{k} a_{i} - \prod_{i=1}^{k} b_{i}\right| \le \sum_{i=1}^{k} |a_{i} - b_{i}|$$
(6.23)

valid for all k and all $a_i, b_i \in [0, 1], i = 1, ..., k$. Recall the blocks of indices $J_{\tau,i}$ and $\hat{J}_{\tau,i}$ from (4.3) and (4.16), respectively. Observe that $M(X_{J_{\tau,i}\setminus\hat{J}_{\tau,i}}) \leq a_{\tau}\delta$ implies that $f_{\tau,i}(T_1^{\tau}X_{\tau,i}) = f_{\tau,i}(T_1^{\tau}\hat{X}_{\tau,i})$. In particular, due to stationarity of X,

$$\begin{split} \sum_{\boldsymbol{i}\in I_{\tau}} \mathbf{E} \bigg| \exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}X_{\tau,\boldsymbol{i}})\right\} - \exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}\hat{X}_{\tau,\boldsymbol{i}})\right\} \bigg| \\ &\leq k_{\tau}^{d} \mathbf{P}(M(X_{[0,b_{\tau}]^{d} \setminus [l_{\tau},b_{\tau}-l_{\tau}]^{d}}) > a_{\tau}\delta) \\ &\leq k_{\tau}^{d} \mathbf{E} \bigg[\sum_{(t,s)\in X} \mathbf{1} \big\{ t \in [0,b_{\tau}]^{d} \setminus [l_{\tau},b_{\tau}-l_{\tau}]^{d}, s > a_{\tau}\delta \big\} \bigg] \\ &= k_{\tau}^{d} \operatorname{Leb}([0,b_{\tau}]^{d} \setminus [l_{\tau},b_{\tau}-l_{\tau}]^{d}) \mathbf{P}(\tilde{X}(0) > a_{\tau}\delta) \\ &\sim \operatorname{const} k_{\tau}^{d} b_{\tau}^{d-1} l_{\tau} \mathbf{P}(\tilde{X}(0) > a_{\tau}\delta) \sim \operatorname{const} \frac{l_{\tau}}{b_{\tau}} \tau^{d} \mathbf{P}(\tilde{X}(0) > a_{\tau}\delta) \to \operatorname{const} \cdot 0 \cdot \delta^{-\alpha} = 0 \end{split}$$

as $\tau \to \infty$. In the third step we used the refined Campbell's formula, in the fourth the assumption $l_{\tau}/b_{\tau} \to 0$, in the fifth the fact that $k_{\tau} \sim \tau/b_{\tau}$, and, finally, (4.2). Thus,

$$\left| \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f_{\tau,i}(T_1^{\tau} X_{\tau,i}) \right\} \Big] - \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \{ -f_{\tau,i}(T_1^{\tau} \hat{X}_{\tau,i}) \} \Big] \right| \to 0 \quad \text{as } \tau \to \infty.$$

After applying (6.23), the same arguments also imply

$$\left|\prod_{\boldsymbol{i}\in I_{\tau}} \mathbf{E}\left[\exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}X_{\tau,\boldsymbol{i}})\right\}\right] - \prod_{\boldsymbol{i}\in I_{\tau}} \mathbf{E}\left[\exp\left\{-f_{\tau,\boldsymbol{i}}(T_{1}^{\tau}\hat{X}_{\tau,\boldsymbol{i}})\right\}\right]\right| \to 0 \quad \text{as } \tau \to \infty.$$

Together with (4.17), this implies that, as $\tau \to \infty$,

$$\Delta_{\tau}^{\delta} := \left| \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f^{\delta}(ib_{\tau}/\tau, T_{1}^{\tau}X_{\tau,i}) \right\} \Big] - \prod_{i \in I_{\tau}} \mathbf{E} \Big[\exp \left\{ -f^{\delta}(ib_{\tau}/\tau, T_{1}^{\tau}X_{\tau,i}) \right\} \Big] \right|$$
$$= \left| \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f_{\tau,i}(T_{1}^{\tau}X_{\tau,i}) \right\} \Big] - \prod_{i \in I_{\tau}} \mathbf{E} \Big[\exp \left\{ -f_{\tau,i}(T_{1}^{\tau}X_{\tau,i}) \right\} \Big] \right| \to 0. \quad (6.24)$$

Using (6.23) and the inequality $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$, one gets

$$\begin{aligned} \left| \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f(ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) \right\} \Big] - \prod_{i \in I_{\tau}} \mathbf{E} \Big[\exp \left\{ -f(ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) \right\} \Big] \right| \\ &\leq \left| \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f(ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) \right\} \Big] - \mathbf{E} \Big[\prod_{i \in I_{\tau}} \exp \left\{ -f^{\delta}(ib_{\tau}/\tau, T_{1}^{\tau}X_{\tau,i}) \right\} \Big] \right| \\ &+ \left| \prod_{i \in I_{\tau}} \mathbf{E} \Big[\exp \left\{ -f(ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) \right\} \Big] - \prod_{i \in I_{\tau}} \mathbf{E} \Big[\exp \left\{ -f^{\delta}(ib_{\tau}/\tau, T_{1}^{\tau}X_{\tau,i}) \right\} \Big] \right| + \Delta_{\tau}^{\delta} \\ &\leq 2 \sum_{i \in I_{\tau}} \mathbf{E} \Big| f(ib_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,i}]) - f^{\delta}(ib_{\tau}/\tau, T_{1}^{\tau}X_{\tau,i}) \Big| + \Delta_{\tau}^{\delta} . \end{aligned}$$
(6.25)

Since $M(\mu) \leq \varepsilon$ implies $f(t, [\mu]) = f^{\delta}(t, \mu) = 0$, for all $i \in I_{\tau}$,

$$\begin{aligned} \mathbf{E} \Big| f(\mathbf{i}b_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,\mathbf{i}}]) &- f^{\delta}(\mathbf{i}b_{\tau}/\tau, T_{1}^{\tau}X_{\tau,\mathbf{i}}) \Big| \\ &= \mathbf{E} \Big| f(\mathbf{i}b_{\tau}/\tau, T_{1}^{\tau}[X_{\tau,\mathbf{i}}]) - f^{\delta}(\mathbf{i}b_{\tau}/\tau, T_{1}^{\tau}X_{\tau,\mathbf{i}}) \Big| \mathbf{1} \{ M(X_{\tau,\mathbf{i}}) > a_{\tau}\varepsilon \} \\ &\leq c e^{-1/\delta} \mathbf{P}(M(X_{\tau,\mathbf{i}}) > a_{\tau}\varepsilon) = c e^{-1/\delta} \mathbf{P}(M(X_{[0,b_{\tau}]^{d}}) > a_{\tau}\varepsilon) \,, \end{aligned}$$

where we used (6.22) in the third step. Thus, the right-hand side in (6.25) is bounded by

$$2ce^{-1/\delta}k_{\tau}^{d} \mathbf{P}(M(X_{[0,b_{\tau}]^{d}}) > a_{\tau}\varepsilon) + \Delta_{\tau}^{\delta} \le 2ce^{-1/\delta}k_{\tau}^{d}b_{\tau}^{d} \mathbf{P}(\tilde{X}(0) > a_{\tau}\varepsilon) + \Delta_{\tau}^{\delta},$$

which by (4.2) and (6.24) tends to $ce^{-1/\delta}\varepsilon^{\alpha}$ as $\tau \to \infty$. Since $\delta \in (0, \varepsilon)$ is arbitrary, letting $\delta \to 0$ finally yields (6.21).

We are finally in position to prove Theorem 4.2.

Proof of Theorem 4.2. Since the family of Lipschitz continuous functions from Lemma 6.4 is convergence determining in the sense of [3, Definition 2.1] (see [2, Proposition 4.1]), the convergence of intensities (6.18) and the asymptotic independence of blocks (6.21) imply that, as $\tau \to \infty$, N_{τ} converges in distribution to a Poisson point process N on $\mathbf{N} = [0,1]^d \times \tilde{\mathcal{N}}_{01}^*$ whose intensity measure is Leb $\times \nu$; this is [3, Theorem 2.1] which is a consequence of the classical Grigelionis theorem, see [14, Corollary 4.25]. Standard transformation results for Poisson processes now imply that N can be constructed as in (4.9), and this finishes the proof of Theorem 4.2.

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