# QUANTITATIVE INVERSE GALOIS PROBLEM FOR SEMICOMMUTATIVE FINITE GROUP SCHEMES.

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Abstract. A semicommutative finite group scheme is a finite group scheme which can be obtained from commutative finite group schemes by iterated performing semidirect products with commutative kernels and taking quotients by normal subgroups. In this article, for an étale tame semicommutative finite group scheme $G$, we give a lower bound on the number of connected $G$-torsors of bounded height (such as discriminant).

## 1. Introduction

1.1. **Inverse Galois problem for finite group schemes.** One of the most famous questions in number theory is the *inverse Galois problem*, which asks whether every finite group $G$ is realizable as the Galois group of a finite extension of the field $\mathbb{Q}$. This is a largely open question. It is classically known to admit an affirmative answer for $G$ commutative, for $G = \mathfrak{S}_n$ symmetric, for $G = \mathfrak{A}_n$ alternating, etc. Without any modifications the question can be asked for other fields and in this paper we deal with the case of a global field $F$. In this context, one has (a generalization of) the celebrated Shafarevich theorem [20, Theorem 9.6.1] which states that every solvable $G$ is a Galois group of a finite extension of $F$.

If $K/F$ is an extension, then it is a Galois extension with the Galois group $G$ if and only if $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ is a *connected* $G$-torsor. For non-constant finite group schemes $G$, one can thus ask:

**Question 1.1.1.** *Let $F$ be a global field. Does every finite $F$-group scheme $G$ admits a connected $G$-torsor?*

Although the question is a very natural one, to our knowledge, in this form it was only asked in Section “The inverse problem of Galois theory for torsors” of [21] by Cassou-Noguès, Chinburg, Morin and Taylor, where an affirmative answer is provided for the case $F$ is a number field and $G = \boldsymbol{\mu}_m$ is the finite group scheme of $m$-th roots of unity. In this article, we will always assume that, besides being finite, the group schemes are *étale* and *tame* (i.e. if the characteristic of $F$ is positive, then the cardinality of $G$ is coprime to the characteristic). In the literature one can find some other properties of a finite étale tame $F$-group scheme $G$ which imply a positive solution to Question 1.1.1 such as the following one. (Some of the next implications, even though well known to experts, were not written down explicitly in the literature unless $G$ is assumed to be constant, and we dedicate Appendix for the proofs without the assumption.)

**Definition 1.1.2.** *Denote by $BG(F)$ the set of $G$-torsors over $F$ and for a place $v$ of $F$, denote by $BG(F_v)$ the set of $G$-torsors over the completion $F_v$. We say that*

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*the weak weak approximation is valid for $G$, if there exists a finite set $S_0$ of places of $F$, such that for every finite subset $S$ of places of $F$ which is disjoint from $S_0$, we have that the canonical map*

$$BG(F) \to \prod_{v \in S} BG(F_v) \tag{1.1.3}$$

*is surjective.*

Harari defines a weaker property, the *hyperweak approximation*, which also implies the existence of a connected $G$-torsor. The property is stable for semidirect products with commutative kernel and for taking quotients by normal subgroups [13, Proposition 2 and 3], hence, the hyperweak approximation is satisfied for $G$ étale, tame and *semicommutative* (see Section 1.3). Thus Question 1.1.1 admits a positive answer for such $G$. If $F$ is a number field, it follows from a theorem of Harpaz and Wittenberg prove [14, Theorem B] that $G$ étale and *hypersolvable* (admitting composition series by finite subgroup schemes with cyclic consecutive quotients) satisfies the weak weak approximation and hence Question 1.1.1 admits an affirmative answer also for such $G$.

1.2. **Quantitative aspect.** A "quantitive" version of the inverse Galois problem is given by the *Malle conjecture*. We have a *height* function $H : BG(F) \to \mathbb{R}_{>0}$ and we count how many $X \in BG(F)$ satisfy $H(X) < B$, where $B > 0$.

**Conjecture 1.2.1** (Malle [16])**.** *Let $G$ be a non-trivial tame constant group which is embedded as a transitive subgroup of the group of permutations $\mathfrak{S}_n$ for some $n \geq 1$. Let $G_0 \subset G$ be a stabilizer of a point in $\{1, \dots, n\}$ for the action of $G$ induced by the embedding. For a $G$-torsor $X$, we write $H(X) := \Delta(X/G_0)$, where $\Delta$ denotes the norm of the discriminant. One has that*

$$\#\{X \in BG(F) | X \text{ is connected and } H(K) < B\} \asymp_{B\to\infty} B^{a(H)} \log(B)^{b(H)-1},$$

*for some explicit invariants $a = a(H)$ and $b = b(H)$.*

(This is a slightly different statement from the usual statement of the conjecture, and the equivalence with the usual one is explained in [11, Paragraph 1.1.2]). Over $\mathbb{Q}$, the conjecture is known to be true for $G$ commutative embedded in its regular representation, $G = S_3, S_4, S_5$ embedded in its standard representations, etc (see [23], [12], [2], [3]). It is significantly harder than the inverse Galois problem: e.g. it is unknown for some usual groups such as $G = \mathfrak{A}_4$ embedded in its standard or regular representation. Conjecture 1.2.1 admits counterexamples as shown by Klüners [15]. For some $G$, only upper and lower bounds on the number of $G$-torsors ($G$-extensions) of bounded height are known. For the case of a number field, based on Shafarevich theorem, Alberts establishes in [1] a lower bound of the form $\gg B^a$, with $a > 0$ for every solvable group $G$.

In our previous article [11], we proposed a version of Malle conjecture for non-constant finite group schemes $G$.

**Conjecture 1.2.2.** *Let $G$ be a non-trivial finite étale tame $F$-group scheme. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. We define invariants $a(H)$ and $b(H)$ as in Definition 2.1.1. One has that*

$$\#\{x \in BG(F) | x \text{ is secure and } H(x) \leq B\} \asymp_{B\to\infty} B^{a(H)} \log(B)^{b(H)-1}.$$

(The definition of "secure" can be found in [11, Definition 2.6.3] and the notion serves to avoid counterexamples. When $G$ is commutative, every $G$-torsor is secure.) The conjecture is a special case of a *stacky Batyrev-Manin conjecture* [10, Conjecture 9.15]. Note that we have *not* imposed a connectivity condition. Conjecture 1.2.2 has been verified in [11, Theorem 1.3.2] for $G$ commutative. However, it may happen that a positive proportion of $G$-torsors is not connected (e.g. this happens when $G = \boldsymbol{\mu}_m$ is the group scheme of $m$-th roots of unity, as remarked in [9, Remark 9.2.7.4]). Thus, *a priori*, Conjecture 1.2.2 does not imply the existence of a single connected $G$-torsor.

1.3. **Content.** The principal result of this article is a quantitative solution to the inverse Galois problem for *semicommutative* finite group schemes. These are the finite group schemes which can be obtained from finite commutative group schemes by iterated performing semidirect products with commutative kernels and taking quotients by normal subgroups (for details, see Definition 3.3.1). The constant semicommutative groups are precisely those which can be realized as Galois groups by successive solution to *split embedding problems with abelian kernels* and taking intermediate Galois extensions [17, Chapter IV, Section 2.2]. The methods of realization, however, do not work for the non-constant case.

Let us first suppose that $G$ is commutative. Then, an assertion [11, Theorem 1.3.3], which is stronger than Conjecture 1.2.2, is valid: it allows to determine the asymptotic behaviour after having fixed certain local conditions. We will show that the stronger statement, together with Lemma 3.1.2 which gives local conditions which force torsors to be connected, implies the existence of (infinitely many) connected torsors. More precisely, we obtain that:

**Theorem 1.3.1.** *Suppose that $G$ is a non-trivial commutative finite étale tame group scheme. One has that*

$$\#\{x \in BG(F) | x \text{ is connected}, H(x) \leq B\} \asymp_{B\to\infty} B^{a(H)} \log(B)^{b(H)-1}.$$

We mention that in [9, Theorem 9.2.7.3], the first named author develops the precise asymptotic behaviour (with the leading constant) for the case $F$ is a number field and $G = \boldsymbol{\mu}_m$ under additional assumption that $4 \nmid m$ or that $\sqrt{-1} \in F$.

Let us now treat the semicommutative case. A semicommutative finite étale group scheme $G$ can be written as $G = \langle A, K \rangle$, where $\iota : A \hookrightarrow G$ is normal and commutative and $K \lneq G$ is semicommutative. We establish a similar bound to Alberts' bound for solvable constant groups:

**Theorem 1.3.2.** *Suppose that $G$ is a non-trivial semicommutative finite étale and tame $F$-group scheme. Write $G = \langle A, K \rangle$ as above. There exists $C > 0$ such that*

$$\#\{x \in BG(F) | x \text{ is connected}, H(x) \leq B\} \geq CB^{a(\iota^* H)},$$

*where $\iota^* H$ is the pullback height (defined precisely in Paragraph 2.1).*

The obtained lower bounds may be as good as in the *weak Malle conjecture* (which predicts that the number grows at least as $CB^{a(H)}$ for some $C > 0$), as the following example shows. The constant alternating group $G = \mathfrak{A}_4$ is semicommutative (non-constant examples with $G(\overline{F}) = \mathfrak{A}_4$ do exist, as discussed in Example 3.3.4). Our result implies that if the characteristic of $F$ is not 2 or 3, the number of $\mathfrak{A}_4$-fields of bounded discriminant is growing at least as $CB^{\frac{1}{2}}$. For the case $F$ is a number field this was established in [1, Corollary 1.8].

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1.5. **Notations.** We will use notation $F$ for a global field. We denote by $M_F$ (respectively, by $M_F^0$ and by $M_F^\infty$) the set of its places (respectively, of its finite and infinite places).

We fix algebraic closures of $F$ and of $F_v$ for $v \in M_F$ and embeddings of the algebraic closure of $F$ in each of the algebraic closures of $F_v$. We denote by $\overline{F}$ and for $v \in M_F$ by $\overline{F_v}$ the separable closure of $F$ in and $F_v$ in the chosen algebraic closures. The notation $\Gamma_F$ and $\Gamma_v$ will be used to denote the absolute Galois group of $F$ and $F_v$, respectively. For a finite place $v$, we denote by $\Gamma_v^{\mathrm{un}}$ the Galois group $\mathrm{Gal}(F_v^{\mathrm{un}}/F_v)$, where $F_v^{\mathrm{un}}$ is the maximal unramified extension of $F_v$ and by $q_v$ the cardinality of the residue field at $v$.

Let $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be two functions, such that for $B \gg 0$ one has that $g(B) \neq 0$. We write $f \asymp_{B\to\infty} g$ if there are constants $C_1, C_2 > 0$ such that for every $B$ big enough one has that

$$C_1 g(B) \leq f(B) \leq C_2 g(B).$$

## 2. Notions

We recall some notions and results from [11, Section 2]. Let $G$ be a non-trivial finite tame $F$-group scheme.

2.1. **Heights.** Let $e$ be the exponent of $G(\overline{F})$ and let $\boldsymbol{\mu}_e$ be the group scheme of $e$-th roots of unity. The group $G(\overline{F})$ acts on the $\Gamma_F$-group $\mathrm{Hom}(\boldsymbol{\mu}_e, G(\overline{F}))$ by conjugation

$$h \cdot (x \mapsto g) := x \mapsto (hgh^{-1}).$$

The action preserves $\Gamma_F$-orbits and the identity element. We let $G_*$ be the finite pointed $F$-scheme given by the $\Gamma_F$-pointed set

$$\mathrm{Hom}(\boldsymbol{\mu}_e, G(\overline{F}))/G(\overline{F}).$$

For a closed immersion $G \hookrightarrow R$, we have a pointed morphism $G_* \to R_*$ of trivial kernel (but not necessarily injective). The following definitions are from [11, Paragraph 2.3.1].

**Definition 2.1.1.** *(1) We call a $\Gamma_F$-invariant function $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$, which satisfies that $c(x) = 0$ if and only if $x = 1_{G_*(\overline{F})}$ is the distinguished element in $G_*(\overline{F})$, a counting function.*

*(2) Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function. We define*

$$a(c) := \big(\min_{x \in G_*(\overline{F}) - 1_{G_*(\overline{F})}} c(x)\big)^{-1} \in \mathbb{R}_{>0},$$

$$b(c) := \#\big\{x \in G_*(\overline{F}) \big| c(x) = a(c)^{-1}\big\}.$$

*(3) If $\iota : G' \hookrightarrow G$ is a closed immersion of a non-trivial subgroup scheme, and $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ a counting function, then we set $\iota^* c := c \circ ((G')_* \to G_*)$ (it is a counting function).*

We denote by $BG(F)$ (respectively, for $v \in M_F$ by $BG(F_v)$) the pointed set of $G$-torsors over $F$ (respectively, over $F_v$). The $\Gamma_F$-group $G(\overline{F})$ becomes, using inclusions $\Gamma_v \hookrightarrow \Gamma_F$, a $\Gamma_v$-group for $v \in M_F$. For $K \in \{\Gamma_F\} \cup \{\Gamma_v\}_{v\in M_F}$, we denote by $Z^1(K, G(\overline{F}))$ the set of continuous crossed homomorphisms $f : K \to G(\overline{F})$. There exist canonical pointed bijections

$$\begin{aligned} BG(F) &= Z^1(\Gamma_F, G(\overline{F}))/\sim\ =: H^1(\Gamma_F, G(\overline{F})), \\ BG(F_v) &= Z^1(\Gamma_v, G(\overline{F}))/\sim\ =: H^1(\Gamma_v, G(\overline{F})), \qquad (v \in M_F) \end{aligned}$$

where $\sim$ is defined via

$$f \sim f' \iff \exists g \in G(\overline{F}) : \forall \gamma \in \Gamma_F : f'(\gamma) = g^{-1} f(\gamma)(\gamma \cdot g),$$

and analogously for $v \in M_F$. Let $\Sigma_G$ be the finite set given by the places $v$ such that $G(\overline{F})$ is ramified or not tame at $v$ (that is, $\gcd(q_v, \#G(\overline{F})) > 1$). Whenever $v \in M_F - \Sigma_G - M_F^\infty$, we have a canonical map of pointed sets

$$\Psi_v^G : BG(F_v) \to G_*(\overline{F}),$$

the kernel of which is

$$BG(\mathcal{O}_v) := H^1(\Gamma_v^{\mathrm{un}}, G(\overline{F})) \subset H^1(\Gamma_v, G(\overline{F})) = BG(F_v).$$

If $x \in BG(F)$, then for almost all finite $v$, the image of $x$ for the map $BG(F) \to BG(F_v)$ lies in $BG(\mathcal{O}_v)$, hence, for almost all finite $v$, one has that $x$ is in the kernel of the composite map

$$BG(F) = H^1(\Gamma_F, G(\overline{F})) \to H^1(\Gamma_v, G(\overline{F})) = BG(F_v) \xrightarrow{\Psi_v^G} G_*(\overline{F}).$$

**Definition 2.1.2.** *Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function. Let $M_F^\infty \cup \Sigma_G \subset \Sigma \subset M_F$ be a finite set of places. For $v \in \Sigma$, we let $c_v : BG(F_v) \to \mathbb{R}_{\geq 0}$ be functions and for $v \in M_F - \Sigma$ let us set*

$$c_v = c \circ \Psi_v^G : BG(F_v) \to \mathbb{R}_{\geq 0}.$$

*For $v \in M_F$, we denote by $H_v$ the function*

$$H_v : BG(F_v) \to \mathbb{R}_{>0} \qquad x \mapsto q_v^{c_v(x)}.$$

*The function*

$$H = H((c_v)_v) : BG(F) \to \mathbb{R}_{>0} \qquad x \mapsto \prod_{v \in M_F} H_v(x_v),$$

*where $x_v$ is the image of $x$ for the map $BG(F) \to BG(F_v)$, is called the height function defined by $(c_v)_v$ (sometimes simply the height). We say that $c$ is the type of $H$. We set*

$$\begin{aligned} a(H) &:= a(c) \\ b(H) &:= b(c). \end{aligned}$$

The quotient of two heights is a function which is bounded from above and below by positive constants. If $\iota : G' \hookrightarrow G$ is a closed immersion of a non-trivial subgroup, then we define $\iota^* H$ to be the function $BG'(F) \to BG(F) \xrightarrow{H} \mathbb{R}_{>0}$, which turns out to be a height on $BG'(F)$.

2.2. **Twists.** The references for this paragraph are [11, Paragraph 2.2.2, Lemma 2.2.6, Lemma 2.5.4]. Let $\sigma \in Z^1(\Gamma_F, G(\overline{F}))$ be a cocycle. We define ${}_\sigma G$ to be the finite group scheme which corresponds to the $\Gamma_F$-action on $G(\overline{F})$ obtained by twisting by $\sigma$:

$$\gamma \cdot g := \sigma(\gamma) g \sigma(\gamma)^{-1}, \qquad \gamma \in \Gamma_F, g \in G(\overline{F}).$$

There exists a canonical bijection $\lambda_\sigma : B({}_\sigma G)(F) \to BG(F)$, induced by

$$\Lambda_\sigma : Z^1(\Gamma_F, {}_\sigma G(\overline{F})) \to Z^1(\Gamma_F, G(\overline{F})).$$
$$f \mapsto f \cdot \sigma.$$

One has a canonical identification $({}_\sigma G)_* = G_*$. If $H : BG(F) \to \mathbb{R}_{>0}$ is a height, then $H \circ \lambda_\sigma : B({}_\sigma G)(F) \to \mathbb{R}_{>0}$ is a height. Moreover, one has that

$$a(H \circ \lambda_\sigma) = a(H)$$
$$b(H \circ \lambda_\sigma) = b(H).$$

If $R$ is another non-trivial finite étale tame $F$-group scheme and $\phi : G \hookrightarrow R$ a homomorphism which is a closed immersion, we may write ${}_\sigma R$ for ${}_{\phi(\overline{F}) \circ \sigma} R$. We have a closed immersion ${}_\sigma G \to {}_\sigma R$, and the induced morphism $({}_\sigma G)_* \to ({}_\sigma R)_*$ coincides with the morphism $G_* \to R_*$.

## 3. Semicommutative groups

In this section we prove our principal results.

3.1. **Commutative case.** We prove our main result for the commutative case.

**Lemma 3.1.1.** *Let $J$ be a finite étale $F$-group scheme. Let $X \in BJ(F)$ and let $x \in Z^1(\Gamma_F, J(\overline{F}))$ be its lift. Suppose that there exists a finite set of finite places $\{v_1, \dots, v_k\}$ of $F$ such that for $1 \le i \le k$ one has that*

*(1) the finite group scheme $J_{F_{v_i}}$ is constant;*

*(2) one has that $J(\overline{F}) = \langle x(\Gamma_{v_i}) \rangle_{i=1}^k$.*

*Then $X$ is connected.*

*Proof.* We fix a bijection $X(\overline{F}) \xrightarrow{\sim} J(\overline{F})$ and identify the set $X(\overline{F})$ with $J(\overline{F})$ via this bijection. The action on $X(\overline{F})$ is given by

$$\gamma \cdot g = (x(\gamma))(\gamma(g)) \qquad \gamma \in \Gamma_F, g \in X(\overline{F}) = J(\overline{F}).$$

One has that $X$ is connected if and only if $X(\overline{F})$ is a transitive $\Gamma_F$-set, so let us prove the latter. Let $g_1, g_2 \in X(\overline{F})$ and let $g = (g_2)(g_1)^{-1}$. By the second assumption, there exists a finite product of $\prod x(\gamma_j)$, where $\gamma_j \in \Gamma_{v_1} \cup \dots \cup \Gamma_{v_k}$ such that $\prod x(\gamma_j) = g$. Note that for $\gamma_1, \gamma_2 \in \Gamma_{v_1} \cup \dots \cup \Gamma_{v_k}$ one has that

$$x(\gamma_1\gamma_2) = (x(\gamma_1))(\gamma_1 \cdot x(\gamma_2)) = (x(\gamma_1))(x(\gamma_2)).$$

Hence, $x\big(\prod \gamma_j\big) = g$. We deduce that

$$\big(\prod \gamma_j\big) \cdot g_1 = \big(x\big(\prod \gamma_j\big)\big)\big(\big(\prod \gamma_j\big)(g_2)\big) = g g_2 = g_1.$$

The action is thus transitive and the statement follows. □

**Lemma 3.1.2.** *Let $G$ be a finite étale $F$-group scheme and let $i : BG(F) \to \prod_{v\in M_F} BG(F_v)$ be the diagonal map. Let $\Sigma$ be a finite set of places of $F$. There exists a finite subset $T \subset M_F^0 - \Sigma$, elements $y_v \in BG(\mathcal{O}_v)$ for $v \in T$, such that every $x \in BG(F)$, with*

$$i(x) \in \Big(\prod_{v\in T}\{y_v\} \times \prod_{v\in M_F - T} BG(F_v)\Big),$$

*is connected.*

*Proof.* (1) First, we prove that for every $1 \neq g \in G(\overline{F})$ we can associate a finite place $v_g$ of $F$ such that the following conditions are verified:

- for every $g \in G(\overline{F}) - \{1\}$ one has that $v_g \notin \Sigma$;
- one has that $G_{F_{v_g}}$ are constant finite group schemes;
- one has that $v_g \neq v_{g'}$, whenever $g \neq g'$.

Indeed, there exists a finite Galois extension $K/F$ contained in $\overline{F}$ such that $\Gamma_F$ acts on $G(\overline{F})$ via the Galois group $\mathrm{Gal}(K/F)$. There exist infinitely many places $v$ such that $K \subset F_v$. (We write $K = F(a)$ and let $p_a$ be the minimal polynomial of $a$ over $F$. There are infinitely many $v$ such that $v(p_a(t)) > 0$ for some $t \in \mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers of $F$. For any such $v$ which satisfies that for every coefficient $b_i$ of $p_a$ one has $v(b_i) = 0$, by Hensel's lemma [19, Chapter II, Lemma 4.6], the polynomial $p_a$ admits a root in $F_v$.) For such places $v$ one has that $G_{F_v}$ is constant. The claim follows.

(2) Now, for every $1 \neq g \in G(\overline{F})$, we fix a homomorphism

$$\Gamma_{v_g} \to \Gamma_{v_g}^{\mathrm{un}} = \widehat{\mathbb{Z}} \to \langle g\rangle \subset G(\overline{F}).$$

This defines a $G_{F_{v_g}}$-torsor $y_g$ such that $y_g \in BG(\mathcal{O}_v)$. Consider the open

$$U := \prod_{1\neq g\in G(\overline{F})}\{y_g\} \times \prod_{v\in(M_F - \{v_g | 1\neq g\in G(\overline{F})\})} BG(F_v) \subset \prod_{v\in M_F} BG(F_v),$$

which is also closed. By applying Lemma 3.1.1 to $J = G$ and to the set of places $T := \{v_g | g \in G(\overline{F}) - 1\}$, we have that if $i(x) \in U$, then $x$ is connected. □

**Theorem 3.1.3.** *Let $G$ be a commutative non-trivial finite étale and tame $F$-group scheme. Let $H$ be a height having on $BG(F)$. One has that*

$$\#\{x \in BG(F) | x \text{ is connected}\} \asymp_{B\to\infty} B^{a(H)} \log(B)^{b(H)-1}.$$

*Proof.* Note that it suffices to assume that $H$ is a normalized height, i.e. that $a(H) = 1$. Indeed, for every height $H$, one has that $H^{\frac{1}{a(H)}}$ is a normalized height and thus the claim for a normalized height then immediately implies the claim for a non-normalized height.

It follows from [20, Theorem 9.2.3 (vii)] that there exists a finite set of places $\Sigma$ such that whenever $\Sigma' \subset M_F - \Sigma$ is finite, one has that the canonical map $BG(F) \to \prod_{v\in\Sigma'} BG(F_v)$ is surjective. Let $T$ be and $(y_v)_{v\in T} \in \prod_{v\in T} BG(F_v)$ be given by applying Lemma 3.1.2 to the finite group scheme $G$ and the set of places $\Sigma$. We set

$$U := \prod_{v\in T}\{y_v\} \times \prod_{v\in M_F - T} BG(F_v).$$

By construction one has that $i(BG(F)) \cap U \neq \emptyset$. In [11, Lemma 3.5.1], we have defined a Radon measure $\omega_H$ on the product space $\prod_{v \in M_F} BG(F_v)$. We have proven in [11, Lemma 3.5.2] that

$$\mathrm{supp}(\omega_H) = \overline{i(BG(F))}.$$

As $U$ is an open neighbourhood of a point in $\mathrm{supp}(\omega_H)$, we deduce that $\omega_H(U) > 0$. Now, the statement follows by applying [11, Theorem 3.5.5] and [11, Theorem 3.5.6] to the characteristic function $\mathbf{1}_U$ of the open set with empty boundary having positive $\omega_H$-volume $U$. □

3.2. **Semidirect products.** In this subsection, we will study torsors of semidirect products of finite étale tame $F$-group schemes.

Let $A$ and $K$ be finite étale tame $F$-group schemes. Suppose we are given an $F$-homomorphism $\phi : K \to \underline{\mathrm{Aut}}(A)$, where $\underline{\mathrm{Aut}}(A)$ is the finite étale $F$-group scheme given by the $\Gamma_F$-group $\mathrm{Aut}(A(\overline{F}))$ and the following action

$$\gamma \cdot t = \gamma \circ t \circ \gamma^{-1} \qquad (\gamma \in \Gamma_F, t \in \mathrm{Aut}(A(\overline{F})).$$

We let $A \rtimes_\phi K$ be the group scheme given by the group $N(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F})$ which is endowed with the following $\Gamma_F$-action $\gamma \cdot (n_0, h_0) = (\gamma(n_0), \gamma(h_0))$.

Let $\theta \in Z^1(\Gamma_F, K(\overline{F}))$ be a crossed homomorphism and let $\Theta$ be the $K$-torsor defined by $\theta$. The image of $\theta$ for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, K(\overline{F}) \rtimes_{\phi(\overline{F})} H(\overline{F}))$ induced by

$$K(\overline{F}) \to A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F}) \qquad h \mapsto (1, h)$$

is the map $\sigma := \gamma \mapsto (1, \theta(\gamma))$. Let ${}_\sigma A$ be the group subscheme of ${}_\sigma(A \rtimes_\phi K)$ corresponding to the subgroup $A(\overline{F})$ which is $\Gamma_F$-invariant for the twisted action.

**Lemma 3.2.1.** *The canonical map*

$$u_\sigma^\phi : B({}_\sigma A)(F) \to B({}_\sigma(A \rtimes_\phi K))(F) \xrightarrow{\lambda_\sigma} B(A \rtimes_\phi K)(F)$$

*is given by* $u_\sigma^\phi(X) = X \times_F \Theta$.

*Proof.* Let $X \in B({}_\sigma A)(F)$ and let $x \in Z^1(\Gamma_F, {}_\sigma A(\overline{F}))$ be a lift of $X$. The image of $x$ under the canonical map

$$Z^1(\Gamma_F, {}_\sigma A(\overline{F})) \to Z^1(\Gamma_F, {}_\sigma(A \rtimes_\phi K))$$

is given by $\gamma \mapsto (x(\gamma), 1)$. The map $\lambda_\sigma$ is induced by the map

$$\Lambda_\sigma : Z^1(\Gamma_F, ({}_\sigma(A \rtimes_\phi K))(\overline{F})) \to Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F}))$$

which is given by $y \mapsto \big(\gamma \mapsto y(\gamma) \cdot (1, \theta(\gamma))\big)$. It follows that the image of $X$ for the map $u_\sigma^\phi$ is the $A \rtimes_\phi K$-torsor induced by the crossed homomorphism $\gamma \mapsto (x(\gamma), \theta(\gamma))$. By [22, Page 47], the $A \rtimes_\phi K$-torsor induced by $\gamma \mapsto (x(\gamma), \theta(\gamma))$ is isomorphic to $A \rtimes_\phi K$-torsor given by the group $A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F})$ and the following $\Gamma_F$-action:

$$\begin{aligned}\gamma \cdot (n_0, h_0) &= (x(\gamma), \theta(\gamma)) \cdot (\gamma(n_0), \gamma(h_0)) \\ &= \Big( (x(\gamma)) \Big( \phi(\theta(\gamma))(\gamma(n_0)) \Big), \theta(\gamma)\gamma(h_0) \Big).\end{aligned}$$

The ${}_\sigma A$-torsor $X$ is isomorphic to the ${}_\sigma A$-torsor given by the group $A(\overline{F})$ and the following $\Gamma_F$-action

$$\gamma \cdot n_0 = (x(\gamma)(\phi(\theta(\gamma)).\gamma(n_0))).$$

The $K$-torsor $\Theta$ is isomorphic to the $K$-torsor defined by the group $K(\overline{F})$ and the following $\Gamma_F$-action

$$\gamma \cdot h_0 = ((\theta(\gamma))(\gamma(h_0))).$$

By comparing the actions, we see immediately that $X \times_F \Theta = u^{\phi}_{\sigma}(X)$. The statement is proven. ☐

The following notion is a "quantitative" variant of the notion of weak weak approximation.

**Definition 3.2.2.** *Let $G$ be a non-trivial finite étale tame $F$-group scheme and let $\alpha > 0$ and $\beta \geq 0$. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. We say that $G$ is $(H, \alpha, \beta)$-saturated if the following condition is satisfied:*

- *there exists a finite subset $S \subset M_F$ such that for every finite $T \subset M_F - S$ and every $(z_v)_{v\in T} \in \prod_{v\in T} BG(F_v)$, one has that there exists $C > 0$ such that*

$$\#\{y \in BG(F) | y \text{ is connected}, \forall v \in T, y \otimes_F F_v \cong z_v, H(x) \leq B\} \geq CB^{\alpha} \log(B)^{\beta}$$

*for $B \gg 0$.*

**Remark 3.2.3.** We may drop the assumption that $y$ is connected. Indeed, it follows from Lemma 3.1.2 that one can choose finitely many local conditions at places disjoint from $S$ which will force every $G$-torsor satisfying them to be connected. We then add the corresponding places to $S$.

Clearly, for two heights $H_1$ and $H_2$ which have the same type, one has that $G$ is $(H_1, \alpha, \beta)$-saturated if and only if it is $(H_2, \alpha, \beta)$-saturated. It is well known [20, Theorem 9.2.3 (vii)] that if $G$ is commutative, then $G$ satisfies the weak weak approximation. Moreover, Theorem 3.1.3 implies that $G$ is $(H, \alpha(H), \beta(H) - 1)$-saturated.

**Proposition 3.2.4.** *Let $G$ be a non-trivial finite étale tame $F$-group scheme. We suppose that $G = \langle A, K\rangle$, where $A \leq G$ and $K \lneq G$ are closed subgroups, such that $A$ is normal in $G$ and $K$ admits a connected torsor $\Theta$. Let $\phi : K \to \underline{\mathrm{Aut}}(A)$ be the homomorphism given by the conjugation. Let $\sigma_K \in Z^1(\Gamma_F, K(\overline{F}))$ be a lift of $\Theta$ and let $\sigma$ be the image of $\sigma_K$ for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, (A \rtimes_{\phi} K)(\overline{F}))$ induced by the map*

$$K \to A \rtimes_{\phi} K, \qquad k \mapsto (1, k).$$

*Let $\alpha, \beta > 0$. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. Suppose that ${}_{\sigma}A$ is $(H \circ u^{\phi}_{\sigma}, \alpha, \beta)$-saturated. There exists $C > 0$ such that*

$$\#\{x \in BG(F) | x \text{ is connected, } H(x) \leq B\} \geq CB^{\alpha} \log(B)^{\beta}$$

*for $B \gg 0$.*

*Proof.* We split the proof in the several steps.

(1) We recall a known fact: if $G_1 \subset G_2$ is normal subgroup of a finite étale and tame $F$-group scheme $G_2$, the canonical map $B(G_2)(F) \to B(G_2/G_1)(F)$ is given by $x \mapsto (x/G_1)$. Indeed, let $\widetilde{x} \in Z^1(\Gamma_F, G_2(\overline{F}))$ be a lift of $X \in B(G_2)(F)$. Its image in $Z^1(\Gamma_F, (G_2/G_1)(\overline{F}))$ is $w \circ \widetilde{x}$, where $w : G_2(\overline{F}) \to (G_2/G_1)(\overline{F})$ is the quotient map. The element in $B(G_2/G_1)(F)$ associated to $w \circ \widetilde{x}$ is isomorphic to $\Gamma_F$-set given by $(G_2/G_1)(\overline{F})$ endowed with the following $\Gamma_F$-action, where $y \in G_2(\overline{F})$:

$$\gamma \cdot w(y) := (w(\widetilde{x}(\gamma)))\gamma(w(y))) = w(\widetilde{x}(\gamma)\gamma(y)).$$

On the other side, the quotient of $x$ by $G_2$ is isomorphic to the $\Gamma_F$-set

$$\gamma \cdot w(y) = w(\gamma \cdot y) = w(\widetilde{x}(\gamma)\gamma(y)),$$

and the claim follows.

(2) We have a map

$$A \rtimes_\phi K \to \langle A, K\rangle = G \qquad (a,k) \mapsto ak$$

and we denote by $\sigma_G$ the image of $\sigma$ for the induced map $Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F})) \to Z^1(\Gamma_F, G(\overline{F}))$. The composite map $K \to A \rtimes_\phi K \to G$ is the inclusion $K \hookrightarrow G$, hence, one has that $\sigma_G$ is precisely the image of $\sigma_K$ for the map induced by the inclusion. It is immediate that ${}_{\sigma_G}A = {}_\sigma A$ and that the homomorphism ${}_{\sigma_G}A \to {}_{\sigma_G}G$ induced by $\sigma_G$ is the homomorphism ${}_\sigma A \hookrightarrow {}_\sigma(A \rtimes_\phi K) \to {}_\sigma G = {}_{\sigma_G}G$ induced by $\sigma$. It follows that the map

$$u^\phi_\sigma : B({}_{\sigma_G}A)(F) \to B({}_{\sigma_G}G)(F) = B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F),$$

which by [11, Lemma 2.6.1] has all fibers of cardinality at most $\#G(\overline{F})$, coincides with the map

$$B({}_\sigma A)(F) \to B({}_\sigma(A \rtimes_\phi K))(F) \to B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F).$$

Now, [11, Lemma 2.2.1, Part (5)] gives that the maps coincide with the map

$$u^\phi_\sigma : B({}_\sigma A)(F) \to B({}_\sigma(A \rtimes_\phi K))(F) \xrightarrow{\lambda_\sigma} B(A \rtimes_\phi K)(F) \to BG(F).$$

By combining Part (1) together with Lemma 3.2.1, we obtain that the map $f$ is given by $x \mapsto (x \times_F \Theta)/N$, where $N$ is the kernel of $A \rtimes_\phi K \to G$. It follows, in particular, that the image $u^\phi_\sigma(x)$ is connected if $x \times_F \Theta$ is connected.

(3) By an abuse of notation, we may use the same letters for fields and corresponding spectra. Let $\widetilde{\Theta}/F$ be the Galois closure of $\Theta$. Note that in order that $x \otimes_F \Theta$ is a field it suffices that $x$ and $x \otimes_F \widetilde{\Theta}$ is a field. Let $\Theta_1, \dots, \Theta_k$ be the minimal subextensions of $\widetilde{\Theta}/F$ which are strictly larger than $F$. By [4, Chapter V, §10, n$^0$ 8, Theorem 5], if $x$ is a field, one has that $x \otimes_F \widetilde{\Theta}$ is a field if and only if $x$ does not contain any of the subfields $\Theta_1, \dots, \Theta_k$. If $\Theta_j \subset x$ then for every $v \in M^0_F$, one has that $\Theta_j \otimes_F F_v \subset x \otimes_F F_v$.

(4) For every $j = 1, \dots, k$, it follows from Čebotarev theorem [20, Theorem 9.1.3] that there exist infinitely many places $v \in M^0_F$, such that $\Theta_j$ does not have a degree 1 place over it. (We recall the implication. Let $\widetilde{\Theta}_i$ be the Galois closure of $\Theta_i$. By [19, Lemma 13.5], which is stated only for number fields, but the presented proof is valid for function fields as well, the Dirichlet density that $v$ does admit a degree 1 place over it is equal to

$$\frac{\# \bigcup_{g \in \mathrm{Gal}(\widetilde{\Theta}_i/F)} g \,\mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i) g^{-1}}{\# \mathrm{Gal}(\widetilde{\Theta}_i/F)}.$$

We verify that the last quotient is strictly less than 1. Indeed, there are at most $[\mathrm{Gal}(\widetilde{\Theta}_i/F) : \mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i)]$ conjugates of the subgroup $\mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i)$ and each of them contains the element $1 \in \mathrm{Gal}(\widetilde{\Theta}_i/F)$. Hence,

$$\begin{aligned}\# \bigcup_{g \in \mathrm{Gal}(\widetilde{\Theta}_i/F)} & g\,\mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i)g^{-1} \\ & \leq (\#\,\mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i) - 1) \cdot [\mathrm{Gal}(\widetilde{\Theta}_i/F) : \mathrm{Gal}(\widetilde{\Theta}_i/\Theta_i)] + 1 \\ & < \#\,\mathrm{Gal}(\widetilde{\Theta}_i/F).\end{aligned}$$

The claim follows.) Let $v_j$ be such a place not contained in the finite set $S \subset M_F$ which is as in the Definition 3.2.2 (recall that ${}_\sigma A$ is $(H \circ u_\sigma^\phi, \alpha, \beta)$-saturated.) One has that $\Theta_j \otimes_F F_{v_j}$ is a product fields, none of which is isomorphic to $F_{v_j}$. We set

$$U := \prod_{j=1}^{k} \{({}_\sigma A)_{F_{v_j}}\} \times \prod_{v \notin \{v_1, \dots, v_k\}} BG(F_v) \subset \prod_{v \in M_F} BG(F_v).$$

For $x \in B({}_\sigma A)(F)$ which is connected and such that $i(x) \in U$, we have that $x \not\supset \Theta_j$ because the trivial $({}_\sigma A)_{F_{v_j}}$-torsor has a component isomorphic to $\mathrm{Spec}(F_{v_j})$. Hence, for such $x$ one has that $x \otimes_F \Theta_j$ is a field. Set $T = \{v_1, \dots, v_k\}$. It follows from [11, Lemma 2.6.1] and the assumption that ${}_\sigma A$ is $(H \circ u_\sigma^\phi, \alpha, \beta)$-saturated that for $B \gg 0$ one has

$$\begin{aligned}\#\{y \in BG(F) | y & \text{ is connected and } H(x) \leq B\} \\ & \geq \#G(F) \cdot \# u_\sigma^\phi\big(\{x \in B({}_\sigma A)(F) | \\ & \qquad x \text{ is connected}, i(x) \in U, H(u_\sigma^\phi(x)) \leq B\}\big) \\ & \geq B^\alpha \log(B)^\beta,\end{aligned}$$

for some $C > 0$, where $i : B({}_\sigma A)(F) \to \prod_{v \in M_F} B({}_\sigma A)(F_v)$ is the diagonal map. The theorem has been proven.

□

**Remark 3.2.5.** We note that connected ${}_\sigma A$-torsors have $\Theta$ for a *resolvent* (that is, the Galois closure of the corresponding extensions contain the extension corresponding to $\Theta$). The question of counting extensions with a fixed resolvent has been studied in [5], [6], [7], etc.

3.3. **Semi-commutative groups.** We establish a lower bound on the number of connected torsors for *semicommutative* group schemes. A reference for the definition and basic properties for the constant case is [17, Chapter IV, Section 2.2].

**Definition 3.3.1.** *We say that a finite étale $F$-group scheme $G$ is semicommutative if there exists a finite set of commutative subgroup schemes $\{A_i\}_{i=1}^m$ such that*

$$G = \langle A_i \rangle_{i=1}^m \text{ and } A_i \leq \mathcal{N}_G(A_j) \text{ whenever } i \leq j,$$

*where $\langle K_i \rangle_{i=1}^m$ denotes the smallest closed subgroup scheme containing the subschemes $K_i$ of $G$ and $\mathcal{N}_G(K)$ denotes the normalizer of $K$, i.e. the largest closed subgroup scheme of $G$ containing the closed subgroup scheme $K$ as a normal subgroup.*

The following characterization for the constant case is due to Dentzer.

**Proposition 3.3.2.** *Let $G$ be a non-trivial finite étale $F$-group scheme. The following conditions are equivalent.*

*(1) $G$ is semicommutative.*
*(2) There exists a commutative normal subgroup $A$ of $G$ and a semicommutative closed subgroup $K \lneq G$, such that $G = \langle A, K\rangle$.*
*(3) There exist a sequence $(G_i)_{i=0}^k$ of finite étale $F$-group schemes with $G_0 = \{0\}$ and $G_k \cong G$, a sequence $(A_i)_{i=1}^{k-1}$ of commutative finite étale $F$-group schemes, a sequence of homomorphisms $(\phi_i : G_i \to \underline{\mathrm{Aut}}(A_i))_{i=0}^{k-1}$ of finite $F$-group schemes and a sequence of normal subgroup schemes $\big(N_i \subset (A_i \rtimes_{\phi_i} G_i)\big)_{i=0,\dots,k-1}$ such that for every $i = 1, \dots, k$ one has that*

$$G_i = (A_{i-1} \rtimes_{\phi_{i-1}} G_{i-1})/N_{i-1}.$$

*Proof.* The proof is identical to the constant case [17, Chapter IV, Theorem 2.7]. □

**Theorem 3.3.3.** *Suppose that $G$ is a semicommutative étale tame $F$-group scheme and write $G = \langle A, K\rangle$, with $\iota : A \hookrightarrow G$ commutative, $K \lneq G$ semicommutative. Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function and let $H : BG(F) \to \mathbb{R}_{>0}$ be a height having $c$ for its type. There exists $C > 0$ such that*

$$\#\{x \in BG(F) | x \textit{ is connected}, H(x) \leq B\} \geq CB^{a(\iota^* c)}$$

*for $B \gg 0$.*

*Proof.* The proof is by induction on the cardinality of $G$. By induction, we can suppose that there exists at least one connected $K$-torsor $\Theta$. Let $\sigma$ be the image of a lift of $\Theta$ for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F}))$. Consider the inclusion $\kappa : {}_\sigma A \to {}_\sigma G$. One has that $a(\iota^* c) = a(\kappa^* c)$, because the homomorphism $({}_\sigma A)(\overline{F}) \to ({}_\sigma G)(\overline{F})$ coincides with the homomorphism $A(\overline{F}) \to G(\overline{F})$, hence the map $({}_\sigma A)_*(\overline{F}) \to ({}_\sigma G)_*(\overline{F})$ coincides with the map $A_*(\overline{F}) \to G_*(\overline{F})$. It follows from Theorem 3.1.3, that the finite group scheme ${}_\sigma A$ is $(H \circ g, a(\iota^* c), 0)$-saturated, where $g$ is the map

$$B({}_\sigma A)(\overline{F}) \to B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F).$$

By Proposition 3.2.4, we have for $B \gg 0$ that

$$\#\{x \in BG(F) | x \text{ is connected}, H(x) \leq B\} \geq CB^{a(\iota^* c)}$$

for some $C > 0$. □

**Example 3.3.4.** Suppose that the characteristic of $F$ is not 2 or 3. The alternating group $\mathfrak{A}_4$ of order 12 has a normal commutative subgroup of order 4 which is preserved by every automorphism of $\mathfrak{A}_4$. It follows from this fact and Proposition 3.3.2 that a finite étale group scheme $G$, for which $G(\overline{F}) = \mathfrak{A}_4$, is semicommutative if and only if it contains a closed subgroup of order 3. This happens e.g. when $G = \mathfrak{A}_4$ is constant, but also for any (not necessarily constant) $G$ of the form $G = {}_\sigma(\mathfrak{A})_4$ where $\sigma : \Gamma_F \to \mathfrak{S}_4$ is such that the induced action of $\Gamma_F$ on $\{1, \dots, 4\}$ fixes an element. The natural representation $\mathfrak{A}_4 \subset \mathfrak{S}_4$ induces a counting function $c : (\mathfrak{A}_4)_* \to \mathbb{R}_{>0}$ given by $c(x) = 2$ if $x \neq 1$. We deduce, in particular, from Theorem 3.3.3 that the number of $\mathfrak{A}_4$-fields of bounded discriminant is growing as $CB^{\frac{1}{2}}$ for some $C > 0$.

## Appendix A.

Let $F$ be a global field. In this appendix, we will write down some examples of non-constant étale and tame finite $F$-group schemes for which Question 1.1.1 is known to have a positive answer, even though it is not explicitly written down in the literature.

A.1. **Hyperweak approximation.** Harari defines the following notion in [13, Section 4].

**Definition A.1.1** (Harari). *Let $G$ be étale and tame finite $F$-group scheme. We say that $G$ satisfies the hyperweak approximation, if there exists a finite set of places $S_0 \subset M_F$ such that for every finite $S \subset M_F - S_0$ one has that the image of the canonical map*

$$BG(F) \to \prod_{v \in S} BG(F_v)$$

*contains the subset $\prod_{v \in S} BG(\mathcal{O}_v)$.*

He establishes in [13, Proposition 1] that if $F$ is assumed to be a number field and $G$ to be constant étale finite group scheme satisfying the hyperweak approximation, then the inverse Galois problem has an affirmative answer for $G$. As we have said in Section 1.1, the "constant" assumption is redunant. (We thank Lucchini Arteche for indicating us this).

**Proposition A.1.2.** *Let $G$ be an étale tame finite $F$-group scheme. Suppose that $G$ satisfies the hyperweak approximation. Then $G$ admits a connected $G$-torsor.*

*Proof.* Let $S_0 \subset M_F$ be as in Definition A.1.1. By Lemma 3.1.2, it is possible to choose for $g \in G(\overline{F}) - \{1\}$ places $v_g \in M_F - S_0$ and elements $y_g \in BG(\mathcal{O}_v)$ such that $v_g \neq v_{g'}$ if $g \neq g'$ and such that whenever $x \in BG(F)$ satisfies that

$$i(x) \in \prod_{g \in G(\overline{F}) - \{1\}} \{y_g\} \times \prod_{v \in M_F - \{v_g | 1 \neq g \in G(\overline{F})\}} BG(F_v),$$

then $x$ is connected. By the assumption that $G$ satisfies the hyperweak approximation, the set of such $x$ is non-empty. The claim follows. □

We list two results due to Harari. He proves them with the assumption that $F$ is a number field, but the identical proofs work in the global field case with the tameness assumption.

**Proposition A.1.3** ([13, Harari, Proposition 2 and 3]). *Let $G$ be étale and tame finite $F$-group scheme which satisfies the hyperweak approximation.*

*(1) Let $N$ be a subgroup scheme of $G$. The finite groups scheme $G/N$ satisfies the hyperweak approximation.*
*(2) Let $A$ be a commutative étale and tame finite $F$-group scheme and let $\phi : G \to \underline{\mathrm{Aut}}(A)$ be a homomorphsm. The semidirect product $A \rtimes_\phi G$ satisfies the hyperweak approximation.*

Thus, the results of Harari are sufficient to conclude the following fact.

**Corollary A.1.4.** *Let $G$ be a semicommutative étale tame finite $F$-group scheme. Then $G$ admits a connected $G$-torsor.*

A.2. **Hypersolvable groups.** We now suppose that $F$ is a number field.

**Definition A.2.1.** *Let $G$ be an étale finite $F$-group scheme. We say that $G$ is hypersolvable if there exists composition series of finite subgroup schemes $\{1\} = G_0 \subset \cdots \subset G_k = G$ such that for every $i = 1, \ldots, k$ one has that $(G_i/G_{i-1})(\overline{F}) \cong \mathbb{Z}/r_i\mathbb{Z}$ for some $r_i \in \mathbb{Z}_{>1}$.*

The following property is an immediate consequence of [14, Theorem B] due to Harpaz and Wittenberg. (In the article, they derived it only with the assumption $G$ is constant.)

**Proposition A.2.2.** *Let $G$ be an étale finite $F$-group scheme which is hypersolvable. Then $G$ satisfies the weak weak approximation.*

*Proof.* There exists a closed embedding $G \hookrightarrow \mathrm{SL}_n$ for some $n \geq 1$ (indeed, by [18, Corollary 4.10], one can embed $G$ to $\mathrm{GL}_m$ for certain $m$ and $\mathrm{GL}_m$ can be embedded in $\mathrm{SL}_{m+1}$ via $A \mapsto A \oplus (\det(A)^{-1})$). By Hironaka theorem, there exists a proper smooth geometrically integral variety $X$ which is $F$-birational to $\mathrm{SL}_n/G$. By [14, Theorem B], one has that the Brauer-Manin obstruction is the only one for the weak approximation for $X$ (that is, the closure of the image of the canonical map $X(F) \to \prod_{v \in M_F} X(F_v)$ coincides with the vanishing locus of the Brauer-Manin pairing). The variety $X$ is unirational, thus by [8, Page 347, (6)] satisfies the conditions of [8, Lemma 13.3.13] and applying it, gives that the weak weak approximation is valid for $X$ (that is, the closure of the image of the map $X(F) \to \prod_{v \in M_F} X(F_v)$ is of the form $Z \times \prod_{v \in M_F - T} X(F_v)$ for some finite set of places $T$ and some closed subset $Z \subset \prod_{v \in T} X(F_v)$). Hence, by [8, Proposition 13.2.3], one has that the weak weak approximation is valid for $\mathrm{SL}_n/G$ and by the equivalence from [13, Page 551], we obtain that $G$ satisfies the weak weak approximation. □

Hence, it is immediate from Proposition A.1.2 that:

**Corollary A.2.3.** *Let $G$ be a hypersolvable étale finite group scheme over the number field $F$. Then $G$ admits a connected $G$-torsor.*

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