

QUANTITATIVE ALGEBRAS AND A CLASSIFICATION OF METRIC MONADS

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ABSTRACT. Quantitative algebras are Σ -algebras acting on metric spaces, where operations are nonexpanding. Mardare, Panangaden and Plotkin introduced 1-basic varieties as categories of quantitative algebras presented by quantitative equations. We prove that for the category \mathbf{UMet} of ultrametric spaces such varieties bijectively correspond to strongly finitary monads on \mathbf{UMet} . The same holds for the category \mathbf{Met} of metric spaces, provided that strongly finitary endofunctors are closed under composition.

For uncountable cardinals λ there is an analogous bijection between varieties of λ -ary quantitative algebras and monads that are strongly λ -accessible. Moreover, we present a bijective correspondence between λ -basic varieties as introduced by Mardare et al and enriched, surjections-preserving λ -accessible monads on \mathbf{Met} . Finally, for general enriched λ -accessible monads on \mathbf{Met} a bijective correspondence to generalized varieties is presented.

1. INTRODUCTION

Algebras acting on metric spaces, called *quantitative algebras*, have recently received a lot of attention in connection with the semantics of probabilistic computation, see e.g. [22, 23, 7, 25, 8]. Mardare, Panangaden and Plotkin worked in [23] with Σ -algebras in the category \mathbf{Met} of extended metric spaces and nonexpanding maps, where Σ is a signature with finite or countable arities. They introduced *quantitative equations* which are expressions $t =_\epsilon t'$ where t and t' are terms and $\epsilon \geq 0$ is a rational number. A quantitative algebra A satisfies this equation iff for every interpretation of the variables the computation of t and t' yields elements of distance at most ϵ in A . Example: almost commutative monoids are quantitative monoids in which, for a given constant ϵ , the distance of ab and ba is at most ϵ . This is presented by the quantitative equation $xy =_\epsilon yx$.

We call classes of quantitative Σ -algebras that can be presented by a set of quantitative equations *varieties* (in [23] they are called 1-basic varieties). Every variety \mathcal{V} has a free algebra on each metric space, which yields a corresponding monad $\mathbf{T}_{\mathcal{V}}$ on \mathbf{Met} . Moreover, \mathcal{V} is isomorphic to the Eilenberg-Moore category $\mathbf{Met}^{\mathbf{T}_{\mathcal{V}}}$. Can one characterize

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monads on \mathbf{Met} that correspond to varieties? One property is that such a monad is *finitary*, i.e. it preserves directed colimits. Another property is that it is enriched, that is, locally nonexpanding. Unfortunately, these conditions are not sufficient.

We work with *strongly finitary monads* \mathbf{T} introduced by Kelly and Lack [17]. This means that the endofunctor T is the enriched left Kan extension of its restriction to finite discrete spaces. More precisely, denote by $K : \mathbf{Set}_f \rightarrow \mathbf{Met}$ the full embedding of finite discrete spaces, then strong finitariness means that $T = \mathbf{Lan}_K(T \cdot K)$. We prove that for every strongly finitary monad \mathbf{T} the Eilenberg-Moore category $\mathbf{Met}^{\mathbf{T}}$ is concretely isomorphic to a variety of quantitative algebras. Unfortunately, we have not been able to solve the following

Open Problem 1.1. Are all strongly finitary endofunctors on \mathbf{Met} closed under composition?

Assuming an affirmative answer, we present a proof that for every variety \mathcal{V} of quantitative algebras the monad $\mathbf{T}_{\mathcal{V}}$ is strongly finitary (Corollary 4.31). Under this assumption we thus get a bijection

quantitative varieties \cong strongly finitary monads on \mathbf{Met}
(Theorem 5.9).

Kelly and Lack actually proved, for cartesian closed base categories, the composability of strongly finitary endofunctors. But \mathbf{Met} is not cartesian closed. We therefore also investigate quantitative algebras on the category

\mathbf{UMet}

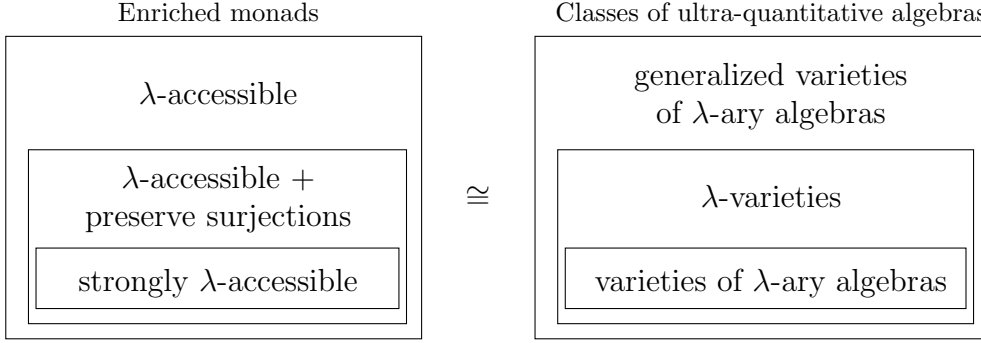
of (extended) ultrametric spaces, a full cartesian closed subcategory of \mathbf{Met} . We then speak about *ultra-quantitative algebras*. Without any extra assumption we prove a bijection

ultra-quantitative varieties \cong strongly finitary monads on \mathbf{UMet}
(Theorem 5.8).

For signatures Σ with infinitary operations, let $\lambda (\geq \aleph_1)$ be a regular cardinal larger than all arities. We get an immediate generalization: varieties of ultra-quantitative Σ -algebras correspond bijectively to *strongly λ -accessible monads* \mathbf{T} on \mathbf{UMet} . This means that $T = \mathbf{Lan}_K(T \cdot K)$ for the full embedding $K : \mathbf{Set}_{<\lambda} \hookrightarrow \mathbf{Met}$ of discrete spaces of cardinality less than λ (Corollary 7.9). But for $\lambda \geq \aleph_1$ we obtain more: bijective correspondences between

- (1) enriched λ -accessible monads and generalized λ -ary varieties (Theorem 9.16), and
- (2) λ -basic (i.e. enriched, surjections-preserving and λ -accessible) monads and λ -varieties of Mardare et al. (Theorem 8.17).

Unfortunately, none of those two results holds for $\lambda = \aleph_0$, as Example 8.13 demonstrates.



A Classification of Monads on \mathbf{UMet} (for $\lambda \geq \aleph_1$)

Related results The correspondence (1) above is based on the presentation of enriched finitary monads due to Kelly and Power [18] that we recall in Section 9. The correspondence (2) has been established, for $\lambda = \aleph_1$, in [2]. We just indicate the (very minor) modifications needed for $\lambda > \aleph_1$ in Section 8. Enriched \aleph_1 -accessible monads on \mathbf{Met} have also been semantically characterized by Rosický [26] who uses a different concept of a type of algebras. In Remark 4.11 (2) he indicates a relationship to quantitative algebras. For cartesian closed base categories Bourke and Garner [12] present a bijective correspondence between strongly λ -ary monads and varieties, where the syntax of the latter is different from the quantitative equations of Mardare et al.

For ordered Σ -algebras closely related results have been presented recently. Strongly finitary monads on \mathbf{Pos} bijectively correspond to varieties of finitary ordered Σ -algebras: see [19], for a shorter proof see [3].

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2. THE CATEGORY OF METRIC SPACES

We recall here basic properties of the category \mathbf{Met} needed throughout our paper. Objects are metric spaces, always meant in the extended sense: the distance ∞ is allowed. Morphisms in \mathbf{Met} are the nonexpanding maps $f : X \rightarrow Y$: we have $d(x, x') \geq d(f(x), f(x'))$ for all $x, x' \in X$.

Notation 2.1.

- (1) Given spaces A and B , the *tensor product* $A \otimes B$ is the cartesian product of the underlying sets equipped with the sum metric:

$$d((a, b), (a', b')) = d(a, a') + d(b, b').$$

With $\mathbf{1}$ the singleton space, \mathbf{Met} is a symmetric monoidal closed category. The internal hom-functor $[A, B]$ is given by the space $\mathbf{Met}(A, B)$ of all morphisms from A to B equipped with the supremum metric:

$$d(f, f') = \sup_{x \in X} d(f(x), f'(x')) \quad \text{for } f, f' : A \rightarrow B$$

- (2) The (categorical) product $A \times B$ is the cartesian product of the underlying sets equipped with the maximum metric:

$$d((a, b), (a', b')) = \max\{d(a, a'), d(b, b')\}.$$

- (3) \mathbf{UMet} denotes the full subcategory of *ultrametric* spaces. These are the (extended) metric spaces satisfying the stronger triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

This category is cartesian closed.

- (4) \mathbf{CUMet} denotes the full subcategory of complete (extended) ultrametric spaces. This category is also cartesian closed.

Remark 2.2 (Peter Johnstone, private communication). The categories \mathbf{Met} and \mathbf{CMet} are not cartesian closed. This can be derived from the description of exponential objects due to Clementino and Hofmann [13]. Here we present an elementary argument: the endofunctor $(-) \times Y$ for

$$Y = \{a, b\} \text{ with } d(a, b) = 2$$

does not have a right adjoint because it does not preserve coequalizers.

Let $X = \{-3, -2, 2, 3\}$ be the subspace of the real line. Consider morphisms $f, g : \mathbf{1} \rightarrow X$ representing -3 and 3 , respectively. Their coequalizer $q : X \rightarrow Q$ is the quotient map with $Q = \{-2, 2, x\}$ where $d(-2, x) = 1 = d(2, x)$ and $d(-2, 2) = 2$. Thus in $Q \times X$ all distances are at most 2. In contrast, the pair $f \times Y, g \times Y$ has a coequalizer $\bar{q} : X \times Y \rightarrow \bar{Q}$ which maps the pair $(2, a)$ and $(-2, b)$ to elements of distance 3.

Convention 2.3.

- (1) Throughout the rest of this paper all categories \mathcal{K} are understood to be enriched over \mathbf{Met} , \mathbf{UMet} or \mathbf{CUMet} : every hom-set $\mathcal{K}(X, Y)$ carries a metric and composition is a nonexpanding map from $\mathcal{K}(X, Y) \otimes \mathcal{K}(Y, Z)$ to $\mathcal{K}(X, Z)$. For a category \mathcal{K} we denote by \mathcal{K}_o the underlying ordinary category.

Also all functors $F : \mathcal{K} \rightarrow \mathcal{L}$ are understood to be enriched over \mathbf{Met} , \mathbf{UMet} or \mathbf{CUMet} : the derived maps from $\mathcal{K}(X, Y)$ to $\mathcal{L}(FX, FY)$ are nonexpanding.

- (2) Every ordinary natural transformation between enriched functors is enriched. Thus a monad on \mathbf{Met} is enriched iff its underlying endofunctor is. Analogously for \mathbf{UMet} or \mathbf{CUMet} .

- (3) Every set is considered to be the *discrete space*: all distances are 0 or ∞ .
- (4) The underlying set of a metric space X is denoted by $|X|$.

Remark 2.4.

- (1) We recall the concept of weighted colimit in a category \mathcal{K} . Given a diagram $D : \mathcal{D} \rightarrow \mathcal{K}$ (a functor with \mathcal{D} small) and a weight (a functor $W : \mathcal{D}^{op} \rightarrow \mathbf{Met}$) the *weighted colimit*

$$C = \operatorname{colim}_W D$$

is an object of \mathcal{K} together with an isomorphism

$$\psi_X : \mathcal{K}(C, X) \rightarrow [\mathcal{D}^{op}, \mathbf{Met}](W, \mathcal{K}(D-, X))$$

natural in $X \in \mathcal{K}$.

- (2) *Conical colimits* are the special case where W is the trivial weight, constant with value $\mathbf{1}$.
- (3) When \mathcal{D} is a λ -directed poset (every subset of power less than λ has an upper bound), we say that $\operatorname{colim}_W D$ is a *λ -directed colimit*. (In particular, \mathcal{D} is non-empty.)
- (4) The *unit* of $C = \operatorname{colim}_W D$ is the natural transformation $\nu : W \rightarrow \mathcal{K}(D-, C)$ given by $\nu = \psi_C(id_C)$. The unit determines ψ : for every morphism $h : C \rightarrow X$ in \mathcal{K} we obtain a natural transformation $\widehat{h} : \mathcal{K}(D-, C) \rightarrow \mathcal{K}(D-, X)$ by post-composing with h . The naturality of ψ implies that

$$\psi_X(h) = W \xrightarrow{\nu} \mathcal{K}(D-, C) \xrightarrow{\widehat{h}} \mathcal{K}(D-, X).$$

- (5) A functor $T : \mathcal{K} \rightarrow \mathcal{K}'$ *preserves* the weighted colimit if $TC = \operatorname{colim}_W TD$ with the unit having components $T\nu_d$ ($d \in \mathcal{D}$).
- (6) A category is called *cocomplete* if it has weighted colimits.
- (7) The dual concept of a weighted limit works with diagrams $D : \mathcal{D} \rightarrow \mathcal{K}$ and weights $W : \mathcal{D} \rightarrow \mathcal{K}$. The category \mathcal{K} is *complete* if it has weighted limits.

Proposition 2.5 ([6], Theorem 4.6). *The category \mathbf{Met} is complete and cocomplete.*

We now present a characterisation of directed colimits in \mathbf{Met} . First we give an example demonstrating that directed colimits are not formed on the level of the underlying sets.

Example 2.6. Let $(A_n)_{n < \omega}$ be the ω -chain of two-element spaces A_n with the underlying set $\{0, 1\}$ and the metric of A_n given by

$$d(0, 1) = 2^{-n}$$

for all $n < \omega$. Define the connecting maps to be the identity maps on $\{0, 1\}$. The colimit of this diagram is the singleton space.

Definition 2.7. An object A of a category \mathcal{K} is

- (1) λ -presentable if the hom-functor $\mathcal{K}_o(A, -) : \mathcal{K}_o \rightarrow \mathbf{Set}$ preserves λ -directed colimits, and
- (2) λ -presentable in the enriched sense if the functor $\mathcal{K}(A, -) : \mathcal{K} \rightarrow \mathbf{Met}$ preserves λ -directed colimits.
- (3) A category \mathcal{K} is locally λ -presentable in the enriched sense provided that it has weighted colimits and a set \mathcal{K}_λ of objects λ -presentable in the enriched sense such that \mathcal{K} is the closure of \mathcal{K}_λ under λ -directed colimits.

Corollary 2.8. If a metric space M is finitely presentable, then $M = \emptyset$.

Indeed, the previous example shows that the ordinary forgetful functor $|-| : \mathbf{Met}_o \rightarrow \mathbf{Set}_o$ does not preserve colimits of ω -chains. Since $|-| \cong \mathbf{Met}(\mathbf{1}, -)_o$, we see that $\mathbf{1}$ is not finitely presentable. An analogous argument holds for every nonempty space.

Proposition 2.9. Let $(D_i)_{i \in I}$ be a directed diagram in \mathbf{Met} . A cocone $c_i : D_i \rightarrow C$ is a colimit iff

- (1) it is collectively surjective:

$$C = \bigcup_{i \in I} c_i[D_i],$$

and

- (2) for every $i \in I$, given $y, y' \in D_i$ we have

$$d(c_i(y), c_i(y')) = \inf_{j \geq i} d(f_j(y), f_j(y'))$$

where $f_j : D_i \rightarrow D_j$ is the connecting morphism.

Proof. The necessity of (1) and (2) is Lemma 2.4 in [6].

To prove sufficiency, assume (1) and (2). Given a cocone

$$h_i : D_i \rightarrow X \quad (i \in I)$$

we define $h : |C| \rightarrow |X|$ by

$$h(x) = h_i(y) \text{ whenever } x = c_i(y).$$

This is well defined: by (1) such i and y exist, and by (2) the value $h_i(y)$ is independent of the choice of i and y . For the independence on i use that I is directed. For the independence on y , we now verify that given $y, y' \in D_i$, then $c_i(y) = c_i(y')$ implies $h_i(y) = h_i(y')$. Indeed, we show that $d(h_i(y), h_i(y')) < \varepsilon$ for an arbitrary $\varepsilon > 0$. Apply (2) to $c_i(y) = c_i(y')$: there exists $j \geq i$ with $d(f_j(y), f_j(y')) < \varepsilon$. Since $h_i = h_j \cdot f_j$ (by compatibility) and h_j is nonexpanding, we get

$$d(h_i(y), h_i(y')) = d(h_j \cdot f_j(y), h_j \cdot f_j(y')) < \varepsilon.$$

Thus h is well defined, and by its definition we have $h \cdot c_i = h_i$ for every $i \in I$. Moreover, it is nonexpanding. Indeed, given $x, x' \in C$ find $i \in I$

and $y, y' \in D_i$ with $x = c_i(y)$ and $x' = c_i(y')$. Then since h_i and c_i are nonexpanding, we get

$$d(h(x), h(x')) = d(h_i(y), h_i(y')) \geq d(y, y') \geq d(x, x').$$

It is easy to see that $h : C \rightarrow X$ is the unique morphism with $h \cdot c_i = h_i$ for every $i \in I$. \square

Corollary 2.10. Directed colimits in \mathbf{UMet} are characterized by the above conditions (1) and (2).

Indeed, this subcategory is closed under directed colimits in \mathbf{Met} .

Remark 2.11. Directed colimits in \mathbf{CUMet} are described analogously, see Proposition 6.3.

Example 2.12.

- (1) For every natural number n the endofunctor $(-)^n$ of \mathbf{Met} preserves directed colimits. Indeed, if a cocone (c_i) of D satisfies (1) and (2) of the above Proposition, then c_i^n clearly satisfies (1) for D^n . To verify (2), let $y, y' \in D_i^n$ be given, then for every $\varepsilon > 0$ we prove

$$d(c_i^n(y), c_i^n(y')) < \inf_{j \geq i} d(f_j^n(y), f_j^n(y')) + \varepsilon.$$

Indeed for $y = (y_k)_{k < n}$ and $y' = (y'_k)_{k < n}$, given $k < n$, condition (2) for D implies that there exists $j \geq i$ with

$$d(c_i(y_k), c_i(y'_k)) < d(f_j(y_k), f_j(y'_k)) + \varepsilon.$$

Moreover, since I is directed, we can choose our j independent of k . Then we get

$$\begin{aligned} d(c_i^n(y), c_i^n(y')) &= \max_{k < n} d(c_i(y_k), c_i(y'_k)) \\ &< \max_{k < n} d(f_j(y_k), f_j(y'_k)) + \varepsilon \\ &= d(f_j^n(y), f_j^n(y')) + \varepsilon. \end{aligned}$$

The same holds for \mathbf{UMet} and \mathbf{CUMet} .

- (2) More generally, given a regular cardinal λ , the endofunctors $(-)^n$ preserve λ -directed colimits for all cardinals $n < \lambda$.

Whereas directed colimits in \mathbf{Met} are not \mathbf{Set} -based, countably directed ones (where every countable subset of \mathcal{D} has an upper bound) are. The following is also a consequence of 2.6(1) in [6].

Corollary 2.13. The ordinary forgetful functor

$$|-| : \mathbf{Met}_o \rightarrow \mathbf{Set}_o$$

preserves countably directed colimits.

Proposition 2.14 ([6], 2.6). *For every regular cardinal $\lambda > \aleph_0$ a space is λ -presentable in \mathbf{Met} (in the ordinary or enriched sense) iff it has cardinality smaller than λ .*

Remark 2.15. A metric space is finitely presentable in the enriched sense iff it is finite and discrete. See [26], 2.5.

Definition 2.16 ([16]). A monoidal closed category \mathcal{K} is *locally λ -presentable as a closed category* if its underlying category \mathcal{K}_o is locally λ -presentable, the unit object I is λ -presentable, and the tensor product of λ -presentable objects is λ -presentable.

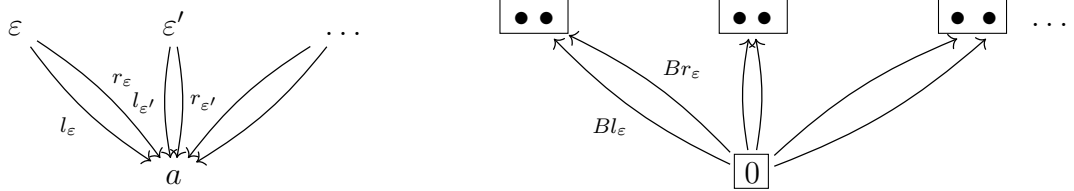
Corollary 2.17. The category \mathbf{Met} is locally λ -presentable as a closed category for every regular cardinal $\lambda > \aleph_0$. Indeed, Proposition 2.5 and 2.14 imply that \mathbf{Met}_o is locally λ -presentable: every space is the λ -filtered colimit of all its subspaces of cardinality smaller than λ . And the rest also follows from Proposition 2.14.

We now present a construction yielding every metric space as a weighted colimit of discrete spaces.

Notation 2.18. We denote by

$$B : \mathcal{B}^{op} \rightarrow \mathbf{Met}$$

the following (basic) weight. The category \mathcal{B} consists of an object a and objects ε for all rational numbers $\varepsilon > 0$. The hom-spaces of endomorphisms are trivial: $\mathcal{D}(x, x) = \mathbf{1}$. For $\varepsilon > 0$ put $\mathcal{D}(a, \varepsilon) = \emptyset$, and $\mathcal{D}(\varepsilon, a)$ consists of morphisms $l_\varepsilon, r_\varepsilon : \varepsilon \rightarrow a$ of distance ε :



The weight B assigns to a the space $\{0\}$ and to ε the space $\{l, r\}$ with $d(l, r) = \varepsilon$. Finally we define

$$Bl_\varepsilon(0) = l \quad \text{and} \quad Br_\varepsilon(0) = r.$$

Definition 2.19. By a *precongruence* of a metric space M is meant the following weighted diagram of discrete spaces

$$D_M : \mathcal{B} \rightarrow \mathbf{Met}$$

of weight B : we put $D_M a = |M|$ (the discrete space underlying M). For every ε the space

$$D_M \varepsilon \subseteq |M| \times |M|$$

consists of all pairs (x, x') with $d(x, x') \leq \varepsilon$. We have the left and right projections $\pi_l, \pi_r : D_M \varepsilon \rightarrow |M|$ and we put

$$D_M l_\varepsilon = \pi_l \quad \text{and} \quad D_M r_\varepsilon = \pi_r.$$

Proposition 2.20. *Every metric space M is the colimit of its precongruence:*

$$M = \operatorname{colim}_B D_M.$$

Proof. Let X be an arbitrary space. To give a natural transformation $\tau : B \rightarrow [D_M-, X]$ means to specify a map $f = \tau_a(0) : |M| \rightarrow X$ and maps $\tau_\varepsilon(l), \tau_\varepsilon(r) : D_M\varepsilon \rightarrow X$ making the following triangles

$$\begin{array}{ccc}
 & D_M\varepsilon & \\
 \pi_l \swarrow & & \searrow \tau_\varepsilon(l) \\
 |M| & \xrightarrow{f} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & D_M\varepsilon & \\
 \pi_r \swarrow & & \searrow \tau_\varepsilon(r) \\
 |M| & \xrightarrow{f} & X
 \end{array}$$

commutative. Thus τ is determined by f , and since τ_ε is nonexpanding and $d(l, r) = \varepsilon$, we have $d(f \cdot \pi_l, f \cdot \pi_r) \leq \varepsilon$. In other words, for every rational number $\varepsilon > 0$ from $d(x, x') \leq \varepsilon$ in M we deduce $d(f(x), f(x')) \leq \varepsilon$. This simply states that $f : M \rightarrow X$ is nonexpanding.

The desired natural isomorphism

$$\psi_X : [M, X] \rightarrow [\mathcal{B}^{op}, \mathbf{Met}](B, [D_M-, X])$$

is given by $\psi_X(f) = \tau$ for the unique natural transformation with $\tau_a(0) = f$. \square

Remark 2.21.

(1) Denote by $i_M : |M| \rightarrow M$ the identity map. The unit of the above colimit $M = \text{colim}_B D_M$ (see Remark 2.4 (4)) is the natural transformation

$$\nu : B \rightarrow [D_M-, M]$$

with components $\nu_a : 0 \mapsto i_M$ and

$$\nu_\varepsilon : l \mapsto \pi_l \text{ and } r \mapsto \pi_r.$$

(2) In the following we use the factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathbf{Met} , where \mathcal{E} is the class of surjective morphisms and \mathcal{M} is the class of isometric embeddings. Every morphism $f : X \rightarrow Y$ factorises via a surjective morphism through $f[X]$ with the metric inherited from Y . (The diagonal fill-in is clear.)

Proposition 2.22. *An endofunctor T on \mathbf{Met} preserves the colimit $M = \text{colim}_B D_M$ iff for every morphism $f : T|M| \rightarrow X$ satisfying*

(1) $d(f \cdot Tl_\varepsilon, f \cdot Tr_\varepsilon) \leq \varepsilon$ in $[TM_\varepsilon, X]$ (for each $\varepsilon \geq 0$)

there exists a unique morphism $\bar{f} : TM \rightarrow X$ with $f = \bar{f} \cdot Ti_M$.

Proof. If (1) holds, we obtain a natural transformation $\tau : B \rightarrow [TD_M-, X]$ with components

$$\tau_a : 0 \mapsto f$$

and

$$\tau_\varepsilon : l \mapsto f \cdot T\pi_l \text{ and } r \mapsto f \cdot T\pi_r.$$

This yields the desired natural isomorphism:

$$\psi_X : [TM, X] \rightarrow [B, [TD_{M-}, X]]$$

since every natural transformation $\tau : B \rightarrow [TD_{M-}, X]$ has the above form for $f = \tau_a(0)$: the component τ_ε is determined by f via naturality, and τ_ε is nonexpanding due to (1). We thus define

$$\psi_X(\bar{f}) = \tau \text{ with } \tau_a(0) = \bar{f} \cdot Ti_M$$

for all $\bar{f} : TM \rightarrow X$. It is easy to see the naturality of ψ_X .

Conversely, if T preserves the colimit, we have a natural bijection $[TM, X] \cong [B, [TD_{M-}, X]]$ with unit $T\nu$ (whose a -component is $0 \mapsto Ti_M$). Given $f : T|M| \rightarrow X$ satisfying (1), we define τ as above and put $\bar{f} = \psi_X^{-1}(\tau)$. Then $f = \bar{f} \cdot Ti_M$ follows from the unit $T\nu$. \square

Example 2.23. (1) The Hausdorff functor \mathcal{H} preserves colimits of pre-congruences. Recall that \mathcal{H} assigns to every space M the space $\mathcal{H}M$ of all compact subsets with the following metric $d_{\mathcal{H}}$:

$$d_{\mathcal{H}}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

where $d(a, B) = \inf_{b \in B} d(a, b)$. (In particular $d(A, \emptyset) = \infty$ for $A \neq \emptyset$.) On morphisms $f : M \rightarrow M'$ it is defined by $A \mapsto f[A]$. It is our task to prove that every morphism $f : \mathcal{H}|M| \rightarrow X$ satisfying (1) above is also a morphism $f : \mathcal{H}M \rightarrow X$. That is, for all $A, B \in \mathcal{H}M$ we are to verify that

$$d_{\mathcal{H}}(A, B) = \varepsilon \text{ implies } d_{\mathcal{H}}(f[A], f[B]) \leq \varepsilon.$$

For each $a \in A$, since $d(a, B) \leq \varepsilon$, there exists $b_a \in B$ with $d(a, b_a) \leq \varepsilon$. Analogously for each $b \in B$ we have $a^b \in A$ with $d(b, a^b) \leq \varepsilon$. The set $P = \{(a, b_a) \mid a \in A\} \cup \{(a^b, b) \mid b \in B\}$ lies in $\mathcal{H}(D_M \varepsilon)$ and fulfils $\mathcal{H}\pi_l(P) = A$ and $\mathcal{H}\pi_r(P) = B$. Thus, (1) implies the desired inequality

$$d_{\mathcal{H}}(f[A], f[B]) = d_{\mathcal{H}}(f \cdot \pi_l[P], f \cdot \pi_r[P]) \leq \varepsilon.$$

(2) The subfunctor \mathcal{H}_f of \mathcal{H} assigning to every space the space of all finite subsets also preserves colimits of pre-congruences, the argument is the same.

Remark 2.24. The class \mathcal{E} lies (strictly) inbetween the class of all epimorphisms, which are the dense morphisms, and all strong epimorphisms. For example, if X is a non-discrete space, the epimorphism $id : |X| \rightarrow X$ lies in \mathcal{E} but is not strong.

3. STRONGLY ACCESSIBLE FUNCTORS

Assumption 3.1. We assume throughout this section that an infinite regular cardinal λ is chosen.

Notation 3.2. We denote by

$$\mathbf{Set}_\lambda \text{ and } \mathbf{Met}_\lambda$$

full subcategories of \mathbf{Met} representing, up to isomorphism, all sets (discrete spaces) of cardinality less than λ , or all spaces of cardinality less than λ , respectively. We also denote by

$$K : \mathbf{Set}_\lambda \hookrightarrow \mathbf{Met} \text{ and } K^* : \mathbf{Met}_\lambda \hookrightarrow \mathbf{Met}$$

the full embeddings.

Recall that an endofunctor of \mathbf{Met} is λ -*accessible* iff it preserves λ -directed colimits. We characterize these functors using the concept of density presentation of Kelly [15] that we shortly recall. Let $L : \mathcal{A} \rightarrow \mathbf{Met}$ be a functor with \mathcal{A} small. This leads to a functor

$$\tilde{L} : \mathbf{Met} \rightarrow [\mathcal{A}^{op}, \mathbf{Met}], \quad X \mapsto \mathbf{Met}(L-, X).$$

A colimit of a weighted diagram in \mathbf{Met} is called L -*absolute* if \tilde{L} preserves it. A *density presentation* of L is a collection of L -absolute weighted colimits in \mathbf{Met} such that every space can be obtained from finite discrete spaces by an (iterated) application of those colimits.

Proposition 3.3. *An endofunctor T of \mathbf{Met} is λ -accessible iff it is the left Kan extension of its restriction $T \cdot K^*$ to spaces of power less than λ :*

$$T = \mathbf{Lan}_{K^*}(T \cdot K^*).$$

Proof. The functor $K^* : \mathbf{Met}_\lambda \rightarrow \mathbf{Met}$ is dense. Indeed, this follows from [15], Theorem 5.19. The functor K^* has a density presentation consisting of all λ -directed diagrams in \mathbf{Met}_λ . (Indeed, every metric space M is the λ -directed colimit of all of its subspaces of power less than λ . Now consider the corresponding λ -directed diagram in \mathbf{Met}_λ .) For every space M in \mathbf{Met}_λ the functor $[M, -]$ preserves λ -filtered colimits by Proposition 2.14. Thus λ -filtered colimits are K^* -absolute, and our proposition follows from Theorem 5.29 in [15]. \square

We now characterize strongly λ -accessible endofunctors on \mathbf{Met} . These are functors T which are (enriched) left Kan extensions of their restrictions to discrete spaces of power larger than λ . More precisely, recalling Notation 3.2 the strongly λ -accessible functors are defined as follows:

Definition 3.4. An endofunctor T of \mathbf{Met} is *strongly λ -accessible* if it is the left Kan extension of its restriction $T \cdot K$ to discrete spaces of power less than λ :

$$T = \mathbf{Lan}_K(T \cdot K).$$

If $\lambda = \aleph_0$, we speak of *strongly finitary endofunctors*.

This concept of a strongly finitary functor was introduced by Kelly and Lack [17] more generally for endofunctors of locally finitely presentable categories as enriched categories.

Recall that the above equality means that there is a natural transformation $\sigma : T \cdot K \rightarrow T \cdot K$ such that for every endofunctor F and every natural transformation $\varphi : T \cdot K \rightarrow T \cdot F$ there exists a unique $\bar{\varphi} : T \rightarrow F$ with $\varphi = (\bar{\varphi}K) \cdot \sigma$.

Example 3.5. (1) For every natural number n the functor $(-)^n$ is strongly finitary. Indeed, for every (enriched) endofunctor F to give a natural transformation $\varphi : (-)^n \cdot K \rightarrow F \cdot K$ means precisely to specify an element of F^n (since $(-)^n$ is naturally isomorphic to the hom-functor of n). And this is the same as specifying a natural transformation $\bar{\varphi} : (-)^n \rightarrow F$. Thus, $(-)^n$ is the left Kan extension with $\sigma = id$ as claimed.

(2) Directed colimits and colimits of precongruences are K -absolute. Indeed, the functor \tilde{K} assigns to X the functor of finite powers of X . And directed colimits commute with finite powers by Example 2.12, whereas reflexive coinserters do by Remark 2.21 (2).

Theorem 3.6. *An endofunctor of \mathbf{Met} is strongly λ -accessible iff it is λ -accessible and preserves colimits of precongruences.*

Proof.

The functor $K : \mathbf{Set}_\lambda \rightarrow \mathbf{Met}$ has a density presentation consisting of all λ -directed diagrams and all precongruences of spaces in \mathbf{Met}_λ . Indeed:

- (1) Every space in \mathbf{Met}_λ is a colimit of its precongruence (Proposition 2.20) which is a weighted diagram in \mathbf{Set}_λ ;
- (2) Every metric space M is a λ -directed colimit of a diagram in \mathbf{Met}_λ : consider all subspaces of M of cardinality less than λ , and form the corresponding diagram in \mathbf{Met}_λ .

Moreover, we know that both of these types of colimits are K -absolute. Our theorem then follows from Theorem 5.29 in [15]: an endofunctor T of \mathbf{Met} is strongly λ -accessible iff it preserves

- (a) λ -directed colimits and
- (b) colimits of precongruences of spaces in \mathbf{Met}_λ .

Thus it only remains to observe that (a) and (b) imply that T preserves all colimits of precongruences. Indeed, for every space M let M_i ($i \in I$) be the λ -directed diagram of all subspaces of cardinality less than λ . From $M_i = \text{colim}_B D_{M_i}$ (Proposition 2.20) we derive $TM = \text{colim}_B D_M$: the unit ν of the latter colimit is namely the λ -directed colimit of the units ν_i w.r.t M_i ($i \in I$). Indeed, recall from Remark 2.21 that the component ν_a is given by the assignment $0 \mapsto id : |M| \rightarrow M$. This is a λ -directed colimit of the components $(\nu_i)_a$ given by $0 \mapsto id : |M_i| \rightarrow M_i$. Analogously for the component ν_ε . \square

Example 3.7.

- (1) Every coproduct of strongly finitary endofunctors is strongly finitary. Indeed, more generally, in the category $[\mathbf{Met}, \mathbf{Met}]$ the strongly

finitary endofunctors are closed under weighted colimits: this follows from Definition 3.4 directly.

- (2) Let Σ be a finitary signature, i.e., a set of symbols σ with prescribed arities $\text{ar}(\sigma) \in \mathbb{N}$. The corresponding *polynomial functor* $H_\Sigma : \mathbf{Met} \rightarrow \mathbf{Met}$ given by

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

is strongly finitary. This follows from (1) and Example 3.5 (1).

- (3) The finite Hausdorff functor \mathcal{H}_f (Example 2.23) is strongly finitary. Indeed, it follows easily from Proposition 2.9 that $\mathcal{H}_\{\}$ preserves directed colimits. And we know from Example 2.23 that it preserves colimits of precongruences.

Notation 3.8.

- (1) $[\mathbf{Met}, \mathbf{Met}]$ denotes the category of endofunctors and natural transformations. It is enriched by the supremum metric: the distance of natural transformations $\alpha, \beta : F \rightarrow G$ is

$$\sup_{X \in \mathbf{Met}} d(\alpha_X, \beta_X).$$

- (2) For every category \mathcal{K} , given a non-full subcategory of \mathcal{K}_o , we can consider it as the (enriched) category \mathcal{L} by defining $\mathcal{L}(X, Y)$ as the metric subspace of $\mathcal{K}(X, Y)$. In particular, the category

$$\mathbf{Mnd}(\mathbf{Met})$$

of all monads on \mathbf{Met} is enriched as a non-full subcategory of $[\mathbf{Met}, \mathbf{Met}]$: the distance of parallel monad morphisms is the supremum of the distances of their components. Analogously for its full subcategory

$$\mathbf{Mnd}_\lambda(\mathbf{Met})$$

of all λ -accessible monads. (Notice that we are not using the view of monads as monoids here: indeed, the ordinary category of monoids in \mathcal{V} is not \mathcal{V} -enriched in general.)

- (3) We denote by $\mathbf{Mnd}_f(\mathbf{Met})$ and $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ the categories of finitary and strongly finitary monads on \mathbf{Met} , respectively, as full subcategories of $\mathbf{Mnd}(\mathbf{Met})$.

Remark 3.9. (1) For every endofunctor F of \mathbf{Met} the functor $\overline{F} = \text{Lan}_K F \cdot K$ is a strongly finitary reflection of F . Indeed, the universal natural transformation $\sigma : \overline{F} \cdot K \rightarrow F \cdot K$ yields a unique natural transformation $\tau : \overline{F} \rightarrow F$ with $\tau K = \sigma$. Given a strongly finitary functor $T = \text{Lan}_K T \cdot K$ (with its universal natural transformation $\sigma' : TK \rightarrow TK$), for every natural transformation $\alpha : T \rightarrow F$ there exists a unique natural transformation $\overline{\alpha} : T \rightarrow \overline{F}$

with $\alpha = \tau \cdot \bar{\alpha}$. Indeed, we have a unique natural transformation $\bar{\alpha} = \text{Lan}_K \alpha : \bar{T} \rightarrow \bar{F}$ making the square below commutative

$$\begin{array}{ccc} TK = \bar{T}K & \xrightarrow{\bar{\alpha}K} & \bar{F}K \\ \sigma' \downarrow & & \downarrow \sigma = \tau K \\ TK & \xrightarrow{\alpha K} & FK \end{array}$$

From $(\tau \bar{\alpha})K = (\alpha K) \cdot \sigma'$ we conclude $\tau \cdot \bar{\alpha} = \alpha$.

- (2) Analogously, for every endofunctor F the functor $\text{Lan}_{K^*} FK^*$ is a finitary reflection of F .

Open Problem 3.10. Is a composite of strongly finitary endofunctors of Met strongly finitary?

Remark 3.11. For cartesian closed locally finitely presentable categories, strongly finitary functors do compose, as proved by Kelly and Lack [17]. Thus, the corresponding answer for UMet or CUMet is affirmative. The proof in loc. cit. also works for strongly λ -ary endofunctors on locally λ -presentable categories.

Proposition 3.12. *The category $\text{Mnd}_\lambda(\text{Met})$ of λ -accessible monads is locally λ -presentable in the enriched sense, for every regular cardinal $\lambda > \aleph_0$.*

Proof.

- (1) The functor

$$W : \text{Mnd}_\lambda(\text{Met}) \rightarrow [\text{Met}_\lambda, \text{Met}]$$

assigning to every λ -accessible monad $\mathbf{T} = (T, \eta, \mu)$ the domain restriction of T to Met_λ is λ -accessible. This follows from Theorem 6.18(2) of [11]. Let $J : |\text{Met}_\lambda| \rightarrow \text{Met}_\lambda$ denote the underlying discrete category of Met_λ . The functor

$$V : [\text{Met}_\lambda, \text{Met}] \rightarrow [|\text{Met}_\lambda|, \text{Met}]$$

of precomposition with J is also λ -accessible, in fact, it has a right adjoint. Thus, the composite

$$U = V \cdot W : \text{Mnd}_\lambda(\text{Met}) \rightarrow [|\text{Met}_\lambda|, \text{Met}]$$

is λ -accessible.

- (2) U is a monadic functor. Indeed, Lack proved ([20], Corollary 3 and Remark 4) that the underlying ordinary functor U_o is monadic. Moreover, the ordinary left adjoint to U_o is easily seen to be enriched. Hence, we have an enriched adjunction $F \dashv U$ with the ordinary adjunction $F_o \dashv U_o$ being monadic. By [14] Theorem II.2.1 this means that U is monadic (in the enriched sense).

- (3) Let \mathbf{T} denote the λ -accessible monad on $[[\mathbf{Met}_\lambda], \mathbf{Met}]$, associated to $F \dashv U$. By Corollary 2.17 \mathbf{Met} is locally λ -presentable in the enriched sense, hence, so is $[[\mathbf{Met}_\lambda], \mathbf{Met}]$. By [11] Theorem 6.9, the Eilenberg-Moore category of \mathbf{T} is locally λ -presentable as an enriched category. This category is equivalent (in the enriched sense) to $\mathbf{Mnd}_\lambda(\mathbf{Met})$ by (2) above. \square

Corollary 3.13. The category $\mathbf{Mnd}_{\aleph_1}(\mathbf{Met})$ is cocomplete.

Recall the open problem 3.10.

Proposition 3.14. *Suppose that strongly finitary endofunctors on \mathbf{Met} are closed under composition. Then the category $\mathbf{Mnd}_f(\mathbf{Met})$ is cocomplete and $\mathbf{Mnd}_{sf}(\mathbf{Met})$ is closed in it under colimits.*

Proof. By Corollary 3.13 it is sufficient to prove that both of the above full subcategories of $\mathbf{Mnd}_{\aleph_1}(\mathbf{Met})$ are coreflective in it.

- (1) The coreflectivity of $\mathbf{Mnd}_{sf}(\mathbf{Met})$. Given a countably accessible monad $\mathbf{T} = (T, \mu, \eta)$, we form the coreflection of T in the category of strongly finitary endofunctors on \mathbf{Met} . By Remark 3.9 this is the functor

$$\bar{T} = \mathbf{Lan}_K T \cdot K.$$

Let $\tau : \bar{T} \rightarrow T$ be the universal natural transformation. We are going to present a canonical monad structure $(\bar{T}, \bar{\mu}, \bar{\eta})$ making τ a monad morphism. It then follows easily that $\tau : (\bar{T}, \bar{\mu}, \bar{\eta}) \rightarrow (T, \mu, \eta)$ is the desired coreflection of \mathbf{T} in $\mathbf{Mnd}_{sf}(\mathbf{Met})$.

The universal property of τ implies that $\eta : Id \rightarrow T$ factorizes as follows

$$\begin{array}{ccc} Id & \xrightarrow{\bar{\eta}} & \bar{T} \\ & \searrow \eta & \downarrow \tau \\ & & T \end{array}$$

for a unique natural transformation $\bar{\eta}$. And $\mu : T \cdot T \rightarrow T$ yields a unique natural transformation $\bar{\mu} : \bar{T} \cdot \bar{T} \rightarrow \bar{T}$ making the square below commutative (using that $\bar{T} \cdot \bar{T}$ is finitary):

$$\begin{array}{ccc} \bar{T} \cdot \bar{T} & \xrightarrow{\bar{\mu}} & \bar{T} \\ \tau * \tau \downarrow & & \downarrow \tau \\ T \cdot T & \xrightarrow{\mu} & T \end{array}$$

We verify that $(\bar{T}, \bar{\mu}, \bar{\eta})$ is a monad.

From the unit law $\mu \cdot T\eta = id$ we derive $\bar{\mu} \cdot \bar{T}\bar{\eta} = id$, using the universal property of τ and the following commutative diagram:

$$\begin{array}{ccc}
 \bar{T} \cdot \bar{T} & \xrightarrow{\bar{\mu}} & \bar{T} \\
 \downarrow \bar{T}\tau & \swarrow \bar{T}\bar{\eta} & \downarrow \tau \\
 \bar{T} \cdot T & & T \\
 \downarrow \tau T & \swarrow T\eta & \downarrow \tau \\
 T \cdot T & \xrightarrow{\mu} & T
 \end{array}$$

Indeed, the left-hand triangle and the outward square commute by definition of $\bar{\eta}$ and $\bar{\mu}$, respectively. The remaining left-hand part is the naturality of τ . Since $\tau \cdot (\bar{\mu} \cdot \bar{T}\bar{\eta}) = \tau$, we have $\bar{\mu} \cdot \bar{T}\bar{\eta} = id$. Analogously for the unit law $\mu \cdot \bar{\eta}T = id$.

From the associative law $\mu \cdot T\mu = \mu \cdot \mu T$ for \mathbf{T} we derive the one for $\bar{\mathbf{T}}$ using the universal property of τ and the following commutative

diagram:

$$\begin{array}{ccccc}
 \overline{TTT} & \xrightarrow{\overline{T\mu}} & & \xrightarrow{\overline{T\mu}} & \overline{TT} \\
 \downarrow \overline{T\tau\bar{T}} & \searrow \overline{T\tau} & & & \downarrow \overline{T\tau} \\
 & \overline{TTT} & & & \\
 & \downarrow (\tau) & & & \\
 & \overline{TT\bar{T}} & \xrightarrow{\overline{T\tau}} & \overline{TTT} & \xrightarrow{\overline{T\mu}} & \overline{TT} \\
 & \downarrow \tau T\bar{T} & (\tau) & \downarrow \tau TT & (\tau) & \downarrow \tau T \\
 \overline{\mu T} & \downarrow (\text{def } \overline{\mu}) & TT\bar{T} & \xrightarrow{TT\tau} & TTT & \xrightarrow{T\mu} & TT & \downarrow \mu & \text{(def } \overline{\mu}) & \overline{\mu} \\
 & \downarrow \mu\bar{T} & (\mu) & \downarrow \mu T & & & \downarrow \mu & & & \\
 & TT\bar{T} & \xrightarrow{T\tau} & TT & \xrightarrow{\mu} & T & & & & \\
 & \downarrow \tau\bar{T} & (\tau) & \downarrow \tau T & & & & & & \\
 & \overline{TT} & & & & & & & & \\
 \downarrow \overline{\mu T} & \downarrow \overline{T\tau} & & & & & & & & \downarrow \overline{\mu} \\
 \overline{TT} & \xrightarrow{\overline{\mu}} & & & \xrightarrow{\overline{\mu}} & \overline{T} & & & & \overline{T}
 \end{array}$$

Indeed, the parts denoted by (τ) and (μ) commute by the naturality of the corresponding transformations. Those denoted by $(\text{def } \overline{\mu})$ follow from the definition of $\overline{\mu}$. The remaining square is the associativity of μ .

We conclude that $(\overline{T}, \overline{\mu}, \overline{\eta})$ is a monad, and by its definition τ is a monad morphism.

It remains to verify that every monad morphism

$$\sigma : (S, \mu^S, \eta^S) \rightarrow \mathbf{T}$$

with S finitary factorizes through τ via a (necessarily unique) monad morphism $\overline{\sigma} : (S, \mu^S, \eta^S) \rightarrow \overline{\mathbf{T}}$. We have a unique natural transformation

$$\overline{\sigma} : S \rightarrow \overline{T} \text{ with } \sigma = \tau \cdot \overline{\sigma}.$$

It preserves the unit: we have

$$\tau \cdot (\bar{\sigma} \cdot \eta^S) = \sigma \cdot \eta^S = \eta = \tau \cdot \bar{\eta}$$

from which $\bar{\sigma} \cdot \eta^S = \bar{\eta}$ follows. And it preserves the multiplication: we verify $\bar{\sigma} \cdot \mu^S = \bar{\mu} \cdot (\bar{\sigma} * \bar{\sigma})$ analogously:

$$\begin{aligned} \tau \cdot (\bar{\sigma} \cdot \mu^S) &= \sigma \cdot \mu^S && \text{def. of } \bar{\sigma} \\ &= \mu \cdot (\sigma * \sigma) \\ &= \mu \cdot (\tau * \tau) \cdot (\bar{\sigma} * \bar{\sigma}) && \text{def. of } \bar{\sigma} \\ &= \tau \cdot \bar{\mu} \cdot (\bar{\sigma} * \bar{\sigma}) && \text{def. of } \bar{\mu} \end{aligned}$$

Since both the full embedding

$$\mathbf{Mnd}_{\text{sf}}(\mathbf{Met}) \rightarrow \mathbf{Mnd}_{\aleph_1}(\mathbf{Met})$$

and its (ordinary) right adjoint constructed above are in fact enriched, the category $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ is coreflective in $\mathbf{Mnd}_{\aleph_1}(\mathbf{Met})$.

(2) The coreflectivity of $\mathbf{Mnd}_{\text{f}}(\mathbf{Met})$ is verified completely analogously. \square

We analogously denote by

$$\mathbf{Mnd}_{\text{f}}(\mathbf{UMet}) \text{ and } \mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$$

the categories of (strongly) finitary monads on \mathbf{UMet} .

Proposition 3.15. *The category $\mathbf{Mnd}_{\text{f}}(\mathbf{UMet})$ is cocomplete and $\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$ is closed in it under colimits.*

The proof is the same as above, using Remark 3.11.

Proposition 3.16. *Let \mathbf{T} be a finitary monad on \mathbf{Met} preserving surjective morphisms. Every morphism $h : \mathbf{T} \rightarrow \mathbf{S}$ in the category $\mathbf{Mnd}_{\text{f}}(\mathbf{Met})$ has a factorization*

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{h} & \mathbf{S} \\ & \searrow e & \nearrow m \\ & \mathbf{T}' & \end{array}$$

where the components of e are surjective and those of m are isometric embeddings. The same result holds in the category $\mathbf{Mnd}_{\text{f}}(\mathbf{UMet})$.

Proof. We present a proof for $\mathbf{Mnd}_{\text{f}}(\mathbf{Met})$, the case $\mathbf{Mnd}_{\text{f}}(\mathbf{UMet})$ is analogous.

- (a) For every metric space X factorize h_X as a surjective morphism $e_X : TX \rightarrow T'X$ followed by an isometric embedding $m_X : T'X \rightarrow SX$. This defines an endofunctor T' whose action on a morphism

$f : X \rightarrow Y$ is given by the following diagonal fill-in:

$$\begin{array}{ccc}
 TX & \xrightarrow{e_X} & T'X \\
 Tf \downarrow & & \downarrow m_X \\
 TY & \xrightarrow{T'f} & SX \\
 e_Y \downarrow & & \downarrow Sf \\
 T'Y & \xrightarrow{m_Y} & SY
 \end{array}$$

It is easy to verify that T' is well defined; the fact that it is enriched follows from m_Y being an isometric embedding. The above diagram yields natural transformations $e : T \rightarrow T'$ and $m : T' \rightarrow S$ with $h = m \cdot e$.

We next prove that e and m are monad morphisms for some corresponding monad structure $\mu^{T'} : T'T' \rightarrow T'$ and $\eta^{T'} : Id \rightarrow T'$. Finally, we prove that T' is finitary.

(b) We define a natural transformation

$$\eta^{T'} = Id \xrightarrow{\eta^T} T \xrightarrow{e} T'.$$

Furthermore, recall that $e * e : T \cdot T \rightarrow T' \cdot T'$ has components

$$TTX \xrightarrow{Te_X} TT'X \xrightarrow{e_{T'X}} T'T'X$$

that are surjective because T preserves surjections. We can thus define a natural transformation

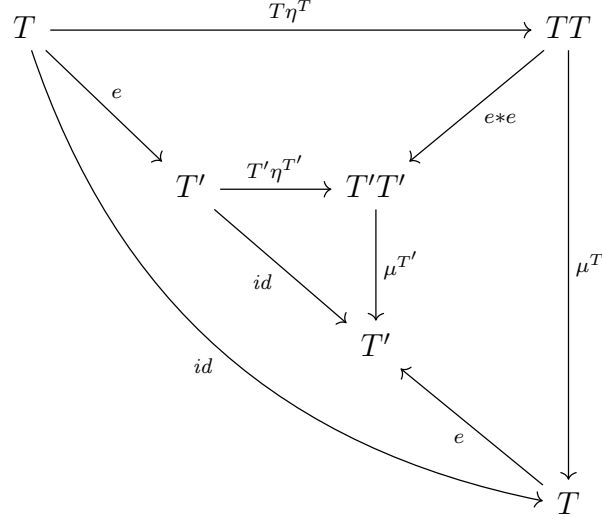
$$\mu^{T'} : T'T' \rightarrow T'$$

by components using the diagonal fill-in as follows:

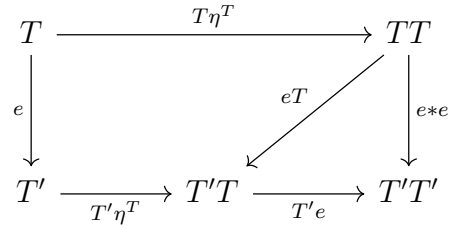
$$\begin{array}{ccc}
 TTX & \xrightarrow{(e*e)_X} & T'T'X \\
 \mu_X^T \downarrow & & \downarrow (m*m)_X \\
 TX & \xrightarrow{\mu_X^{T'}} & TT'X \\
 e_X \downarrow & & \downarrow \mu_X^S \\
 T'X & \xrightarrow{m_X} & SX
 \end{array}$$

We now verify that $(T', \mu^{T'}, \eta^{T'})$ is a monad. From the above definitions it then follows that e and m are monad morphisms.

- (i) We show that $\mu^{T'} \cdot T'\eta^{T'} = id$ holds. This follows from the diagram below:



Since \mathbf{T} is a monad, the outward triangle commutes. The right-hand part commutes by the definition of $\mu^{T'}$. We prove next that the upper part commutes, then it follows that the desired inner triangle commutes when precomposed with e . Since e has surjective components, this implies that the triangle commutes, too. For the upper part use the following diagram



The left-hand part commutes by the naturality of e , and the right-hand part is the (equivalent) definition of $e * e = T'e \cdot eT$.

- (ii) The proof of $\mu^{T'} \cdot \eta^{T'} T' = id$ is analogous.

(iii) To prove $\mu^{T'} \cdot T' \mu^{T'} = \mu^{T'} \cdot \mu^{T'} T'$ use the following diagram

$$\begin{array}{ccc}
 TTT & \xrightarrow{T\mu^T} & TT \\
 \downarrow \mu^{TT} & \searrow e*e*e & \swarrow e*e \\
 & T'T'T' & \xrightarrow{T'\mu^{T'}} & T'T' \\
 & \downarrow \mu^{T'T'} & & \downarrow \mu^{T'} \\
 & T'T' & \xrightarrow{\mu^{T'}} & T \\
 & \swarrow e*e & & \searrow e \\
 TT & \xrightarrow{\mu^T} & T
 \end{array}$$

(*)

We prove first that the upper part commutes. Since $e*e*e = T'(e*e) \cdot eTT$, this follows from the commutative diagram below:

$$\begin{array}{ccc}
 TTT & \xrightarrow{T\mu^T} & TT \\
 \downarrow eTT & & \downarrow eT \\
 T'TTT & \xrightarrow{T'\mu^T} & T'T \\
 \downarrow T'(e*e) & & \downarrow T'e \\
 T'T'T' & \xrightarrow{T'\mu^{T'}} & T'T'
 \end{array}$$

The upper square commutes by the naturality of e , and the lower one does since it is the application of T' to the left-hand part of the definition of $\mu^{T'}$.

Analogously, the left-hand part of the diagram (*) commutes. The right-hand part and the lower one follow from the definition of $\mu^{T'}$.

The outward square commutes because \mathbf{T} is a monad. We conclude that the desired inner square commutes when pre-composed by $e*e*e$. The latter transformation has surjective components. Thus, the middle square commutes.

(c) It remains to prove that T' is finitary. Let $(c_i)_{i \in I}$ be a colimit cocone of a directed diagram $(D_i)_{i \in I}$. It is our task to prove that $(T'c_i)_{i \in I}$ has properties (1) and (2) of Proposition 2.9.

Property (1): Given $x \in T'C$ we are to find $i \in I$ with $x \in T'c_i[T'C_i]$. As e_C is epic, we have $y \in TC$ with $x = e_C(y)$. Since

T preserves the colimit of (D_i) , there exists $i \in I$ and $y' \in TD_i$ with $y = Tc_i(y')$. Thus $x' = e_{D_i}(y')$ fulfils $x = T'c_i(x')$ as desired:

$$x = e_C(y) = e_C \cdot Tc_i(y') = T'c_i \cdot e_{D_i}(y') = T'c_i(x').$$

Property (2): Given $i \in I$ and $y_1, y_2 \in TD_i$, we are to prove

$$(*) \quad d(T'c_i(y_1), T'c_i(y_2)) = \inf_{j \geq i} d(T'f_j(y_1), T'f_j(y_2))$$

Put $\bar{y}_k = m_{D_i}(y_k)$ for $k = 1, 2$ and use that, since \mathbf{T} is a finitary monad, (Tc_i) is a colimit cocone of (TD_i) . Thus it has the property (2):

$$(**) \quad d(TC_i(\bar{y}_1), TC_i(\bar{y}_2)) = \inf_{j \geq i} d(Tf_j(\bar{y}_1), Tf_j(\bar{y}_2)).$$

The left-hand sides of $(*)$ and $(**)$ are equal: use that m_C is an isometric embedding to get

$$\begin{aligned} d(T'c_i(y_1), T'c_i(y_2)) &= d(m_C \cdot T'c_i(y_1), m_C \cdot T'c_i(y_2)) \\ &= d(Tc_i \cdot m_{D_i}(y_1), Tc_i \cdot m_{D_i}(y_2)) \\ &= d(Tc_i(\bar{y}_1), Tc_i(\bar{y}_2)). \end{aligned}$$

The right-hand sides are also equal: for every $j \geq i$, since m_{D_j} is an isometric embedding, we have

$$\begin{aligned} d(T'f_j(y_1), T'f_j(y_2)) &= d(m_{D_j} \cdot T'f_j(y_1), m_{D_j} \cdot T'f_j(y_2)) \\ &= d(Tf_j \cdot m_{D_i}(y_1), Tf_j \cdot m_{D_i}(y_2)) \\ &= d(Tf_j(\bar{y}_1), Tf_j(\bar{y}_2)). \end{aligned}$$

Thus, $(**)$ implies $(*)$, as desired. □

4. FROM FINITARY VARIETIES TO STRONGLY FINITARY MONADS

We present the concept of a variety of quantitative Σ -algebras, where Σ is a finitary signature. Assuming that strongly finitary endofunctors compose (see 3.10) we prove that every variety is isomorphic to $\mathbf{Met}^{\mathbf{T}}$ for a strongly finitary monad \mathbf{T} . In the subsequent section we prove the converse: $\mathbf{Met}^{\mathbf{T}}$ is isomorphic to a variety for every strongly finitary monad \mathbf{T} . For the base category \mathbf{UMet} the same result holds – but here no extra assumption is needed, see Remark 3.11.

Notation 4.1.

- (1) Throughout this section Σ denotes a finitary signature, i.e., a set of (operation) symbols σ with prescribed arities in \mathbb{N} .
- (2) For convenience, we assume that a countable infinite set V of variables has been chosen.
- (3) Throughout the rest of our paper ε and δ denote non-negative rational numbers.

The following concept was introduced in [23].

Definition 4.2. A *quantitative algebra* of signature Σ is a metric space A equipped with a nonexpanding operation $\sigma_A : A^n \rightarrow A$ for every n -ary symbol $\sigma \in \Sigma$. Thus the distance between $\sigma_A(a_i)$ and $\sigma_A(b_i)$ is at most $\max\{d(a_0, b_0), \dots, d(a_{n-1}, b_{n-1})\}$. A *homomorphism* to a quantitative algebra B is a nonexpanding function $f : A \rightarrow B$ preserving the operations:

$$f \cdot \sigma_A = \sigma_B \cdot f^n$$

for every σ of arity n in Σ .

The category of quantitative algebras is denoted by $\Sigma\text{-Met}$.

Analogously, an *ultra-quantitative algebra* is a quantitative algebra on an ultrametric space. The corresponding full subcategory of $\Sigma\text{-Met}$ is denoted by $\Sigma\text{-UMet}$.

Example 4.3. Quantitative monoids. These are monoids M acting on a metric space with nonexpansive multiplication from $M \times M$ to M . Explicitly:

$$\text{if } d(x, x') = \varepsilon_1 \text{ and } d(y, y') = \varepsilon_2, \text{ then } d(xy, x'y') \leq \max\{\varepsilon_1, \varepsilon_2\}.$$

A free quantitative monoid TM on a space M is the coproduct of powers $M^n = M \times \dots \times M$:

$$TM = \coprod_{n \in \mathbb{N}} M^n.$$

This endofunctor is strongly finitary (Example 3.7 (2)).

Remark 4.4. In contrast, the category of monoids in the monoidal category \mathbf{Met} does not yield a strongly finitary monad. If

$$M^{\otimes n} = M \otimes \dots \otimes M$$

denotes the n -th tensor power, a free monoid on a space M is given by

$$\tilde{T}M = \coprod_{n \in \mathbb{N}} M^{\otimes n}.$$

The corresponding endofunctor is not even enriched. Consider $M = \{a_1, a_2\}$ with $d(a_1, a_2) = 1$. We have the obvious morphisms $f_1, f_2 : \mathbf{1} \rightarrow M$ with $d(f_1, f_2) = 1$. However, $d(f_1^{\otimes n}, f_2^{\otimes n}) = n$, thus, $d(\tilde{T}f_1, \tilde{T}f_2) = \infty$. This indicates that Mardare et al. made the right choice when defining quantitative algebras. However, those are much more restrictive than their ‘relatives’ using the tensor product. For example both $(\mathbb{R}, +, 0)$ and $([0, 1], \cdot, 1)$ are monoids in \mathbf{Met} , but they are not quantitative monoids.

Example 4.5. In universal algebra, for every set X (of variables) the algebra $T_\Sigma X$ of *terms* is defined as the smallest set containing the variables, and such that given $\sigma \in \Sigma$ of arity n and n terms t_i ($i < n$) we obtain a (composite) term $\sigma(t_i)_{i < n}$. For metric spaces this was generalized in [7] and we recall the construction here.

- (1) Two terms t and t' in $T_\Sigma X$ are called *similar* if they have the same structure and differ only in the names of variables. That is, similarity is the smallest equivalence relation on $T_\Sigma X$ such that
 - (a) all variables in X are pairwise similar, and
 - (b) a term $\sigma(t_i)_{i < n}$ is similar to precisely those terms $\sigma(t'_i)_{i < n}$ such that t_i is similar to t'_i for each $i < n$.
- (2) Given a metric space M on a set X , we denote by $T_\Sigma M$ the quantitative algebra on $T_\Sigma X$ where the metric extends that of M as follows: the distance of non-similar terms is ∞ , and for similar terms $\sigma(t_i)_{i < n}$ and $\sigma(t'_i)_{i < n}$ the distance is $\bigvee_{i < n} d(t_i, t'_i)$.

It follows that the metric space $T_\Sigma M$ is a coproduct of powers M^k (with the supremum metric), one summand for each similarity class $[t]$ of terms t on k variables. The function $(t_i)_{i < n} \mapsto \sigma(t_i)_{i < n}$ of composite terms is clearly a nonexpanding operation $\sigma_{T_\Sigma X} : (T_\Sigma X)^n \rightarrow T_\Sigma X$.

Remark 4.6. If M is a (complete) ultrametric space, then so is $T_\Sigma M$: the categories **UMet** and **CUMet** are closed under finite products and coproducts in **Met**.

Example 4.7. For our discrete space $V = \{x_i \mid i \in \mathbb{N}\}$ of variables we get the discrete algebra $T_\Sigma V$ of terms (in the usual sense of universal algebra).

Remark 4.8.

- (1) Barr proved in [9] that for every endofunctor H of **Met** which generates a free monad \mathbf{T} the category **Met** ^{\mathbf{T}} is equivalent to the category of H -algebras. These are pairs (A, α) consisting of a metric space A and a morphism $\alpha : HA \rightarrow A$. The morphisms of H -algebras $f : (A, \alpha) \rightarrow (A', \alpha')$ are the nonexpanding maps $f : A \rightarrow A'$ satisfying $f \cdot \alpha = \alpha' \cdot Hf$.
- (2) If H is λ -accessible, then a free H -algebra exists on every metric space X . As proved in [1], the free H -algebra is the colimit of the λ -chain in **Met** whose objects W_i and connecting morphisms $w_{ij} : W_i \rightarrow W_j$ for $i \leq j < \lambda$ are given as follows:

$$\begin{aligned} W_0 &= \emptyset \\ W_{i+1} &= HW_i + X \\ W_j &= \operatorname{colim}_{i < j} W_i \text{ for limit ordinals } j < \lambda \end{aligned}$$

and

$$\begin{aligned} w_{i+1, j+1} &= Hw_{i, j} + id_X \text{ for all } i \leq j < \lambda \\ (w_{ij})_{i < \lambda} &\text{ is the colimit cocone for limit ordinals } j. \end{aligned}$$

- (3) The free monad on a λ -accessible endofunctor H is the monad of free H -algebras, see [9].

- (4) The category $\Sigma\text{-Met}$ is clearly isomorphic to the category of H_Σ -algebras for the polynomial functor (Example 3.7).

Proposition 4.9. *For every metric space X the quantitative algebra $T_\Sigma X$ is free on X w.r.t. the inclusion map $\eta_X : X \rightarrow T_\Sigma X$.*

Proof. The functor H_Σ is clearly finitary. Thus by Remark 4.8 a free H_Σ -algebra on X is a colimit of the following ω -chain W :

$$W_0 = \emptyset, \quad W_{i+1} = H_\Sigma W_i + X$$

for all $i \in \mathbb{N}$. It is easy to verify that each connecting morphism w_{ij} is an isometric embedding. Thus, the colimit is given by $\bigcup_{i < j} W_i$ with the induced metric. It is also easy to verify that W_i is isomorphic to the subspace of $T_\Sigma X$ on all terms of height less than i . Thus, $T_\Sigma X = \bigcup_{i < \omega} W_i$ is the free Σ -algebra on X . \square

Notation 4.10. For every Σ -algebra A and every morphism $f : X \rightarrow A$ we denote by $f^\# : T_\Sigma X \rightarrow A$ the unique homomorphic extension (defined in $t = \sigma(t_i)_{i < n}$ by $f^\#(t) = \sigma_A(f^\#(t_i)_{i < n})$).

Corollary 4.11. The functor T_Σ is a coproduct of functors $(-)^n$ for $n \in \mathbb{N}$. Thus the monad \mathbf{T}_Σ of free Σ -algebras is strongly finitary.

Indeed, for every space M we have seen that $T_\Sigma M$ is a coproduct of the powers M^n , one summand for each similarity equivalence class of terms on n variables. Since M itself forms just one class, we see that the similarity classes are independent of the choice of M . Thus $T_\Sigma(-) = \coprod (-)^n$ where the coproduct ranges over similarity classes of $T_\Sigma 1$. It follows that T_Σ is strongly finitary by Example 3.7.

The following definition stems from [22]. Recall that ε denotes an arbitrary non-negative rational number.

Remark 4.12. If M is an ultrametric space, then $T_\Sigma M$ is an ultra-quantitative algebra. This is the free algebra of $\Sigma\text{-UMet}$ on M ; yielding a strongly finitary endofunctor T_Σ on UMet .

Definition 4.13.

- (1) A *quantitative equation* is an expression

$$l =_\varepsilon r$$

where l and r are terms in $T_\Sigma V$. Ordinary equations $l = r$ are the special case of $\varepsilon = 0$.

- (2) A quantitative algebra A *satisfies* the quantitative equation if every interpretation of variables $f : V \rightarrow |A|$ fulfils

$$d(f^\#(l), f^\#(r)) \leq \varepsilon.$$

Example 4.14.

- (1) Our restriction to *rational* numbers ε plays no substantial role: consider $\varepsilon > 0$ irrational. Then the ‘irrational quantitative equation’ $l =_\varepsilon r$ is equivalent to the set $l =_{\varepsilon_n}$ of quantitative equations for an arbitrary decreasing sequence (ε_n) of rational numbers with $\lim \varepsilon_n = \varepsilon$ in \mathbb{R} .
- (2) *Almost commutative monoids*. For a given constant $\varepsilon > 0$ we call a quantitative monoid (Example 4.3) almost commutative if the distance of ab and ba is always at most ε . This is presented by the following quantitative equation:

$$x * y =_\varepsilon y * x.$$

- (3) *Quantitative semilattices*. Recall that a semilattice (with 0) is a poset with finite joins, or equivalently a commutative and idempotent monoid $(M, +, 0)$. (The order $x \leq y$ is defined by $x + y = y$.) A quantitative semilattice is a quantitative monoid which is commutative and idempotent.
- (4) *Almost semilattices* are almost commutative quantitative monoids which are *almost idempotent*, i.e. they satisfy

$$x * x =_\varepsilon x.$$

- (5) *Almost small spaces* are metric spaces with distances at most ε (a given constant). These are algebras with empty signature satisfying $x =_\varepsilon y$.

Definition 4.15. A *variety* of quantitative (or ultra-quantitative) Σ -algebras is a full subcategory of $\Sigma\text{-Met}$ (or UMet , resp.) specified by a set E of quantitative equations. Thus an algebra lies in the variety iff it satisfies every equation of E .

Observe that the category $\Sigma\text{-Met}$ is complete: Met is complete (Proposition 2.5) and the forgetful functor to Met creates limits. In particular, a product $\prod_{i \in I} A_i$ of Σ -algebras is the cartesian product with coordinate-wise operations and the supremum metric:

$$d((a_i), (a'_i)) = \sup_{i \in I} d(a_i, a'_i).$$

The same holds for $\Sigma\text{-UMet}$.

Lemma 4.16. *The categories $\Sigma\text{-Met}$ and $\Sigma\text{-UMet}$ have the factorization system with*

$$\begin{aligned} \mathcal{E} &= \text{surjective homomorphisms, and} \\ \mathcal{M} &= \text{homomorphisms that are isometric embeddings.} \end{aligned}$$

Proof. Given a homomorphism $f : X \rightarrow Y$ observe that $f[Y]$ is closed under the operations of Y . The diagonal fill-in property lifts from Met (or UMet) to $\Sigma\text{-Met}$. \square

Definition 4.17. Let A be a quantitative Σ -algebra. By a *subalgebra* we mean a subobject in $\Sigma\text{-Met}$ represented by an isometric embedding. By a *homomorphic image* we mean a quotient object in $\Sigma\text{-Met}$ represented by a surjective homomorphism.

Analogously for subalgebras and homomorphic images in $\Sigma\text{-UMet}$.

The following was formulated in [23], a proof can be found in [24], B19-B20:

Theorem 4.18 (Birkhoff's Variety Theorem). *A full subcategory of $\Sigma\text{-Met}$ or $\Sigma\text{-UMet}$ is a variety iff it is closed under products, subalgebras, and homomorphic images.*

In fact, loc. cit. only works for $\Sigma\text{-Met}$. Since UMet is closed under products, subspaces and quotient spaces in Met , that result immediately implies the case of $\Sigma\text{-UMet}$, too.

Corollary 4.19. For every variety \mathcal{V} the forgetful functor $U : \mathcal{V} \rightarrow \text{Met}$ has a left adjoint $F \dashv U$.

Indeed, since \mathcal{V} is closed under products and subalgebras (i.e., \mathcal{M} -subobjects), it is a reflective subcategory of $\Sigma\text{-Met}$ with reflections in \mathcal{E} ([4], Theorem 16.8). Thus, a free algebra FX of \mathcal{V} on a space X is a reflection of the free algebra $T_\Sigma X$ of Proposition 4.9.

To see that the corresponding functor $F : \text{Met} \rightarrow \mathcal{V}$ is enriched, consider morphisms $f, g : X \rightarrow Y$ in Met with $d(f, g) = \varepsilon$. To verify $d(Ff, Fg) \leq \varepsilon$, denote by $P \subseteq FX$ the set of all $p \in FX$ with $d(Ff(p), Fg(p)) \leq \varepsilon$. We prove that $P = FX$. First, P contains $\eta_X[X]$: if $p = \eta_X(x)$, then from $d(f(x), g(x)) \leq \varepsilon$ we derive $d(Ff(p), Fg(p)) = d(\eta_Y \cdot f(x), \eta_Y \cdot g(x)) \leq \varepsilon$ because η_Y is nonexpanding. Thus, to prove $P = FX$, it is sufficient to verify that P is closed under the operations. Let σ be an n -ary symbol and $p = \sigma(p_i)$ with $p_i \in P$ for each $i < n$. Since Ff and Fg are homomorphisms, we get

$$\begin{aligned} d(Ff(p), Fg(p)) &= d(\sigma_{FX}(Ff(p_i)), \sigma_{FX}(Fg(p_i))) \\ &\leq \sup_i d(Ff(p_i), Fg(p_i)) \leq \varepsilon. \end{aligned}$$

Notation 4.20. For every variety \mathcal{V} we denote by $\mathbf{T}_\mathcal{V}$ the free-algebra monad on Met , carried by UF .

Example 4.21.

- (1) The monad of quantitative monoids is the lifting of the word monad $TM = M^*$ on Set such that $M^* = \coprod_{n < \omega} M^n$ in Met . (Words of different lengths have distance ∞ .)
- (2) The monad of commutative quantitative monoids is an analogous lifting of the corresponding monad on Set : TM is a coproduct of the spaces M^n / \sim where $(x_i) \sim (y_i)$ iff there is a permutation p on n with $y_i = x_{p(i)}$ for all i .

- (3) In contrast, the monad of almost commutative monoids is a lifting of the word monad $: TM = M^*$, but the metric is more complex than in (1). For example,

$$d(x * y, y * x) = d(x, y) \wedge \varepsilon.$$

- (4) As proved in [22], the monad of quantitative semilattices is given by the finitary Hausdorff functor \mathcal{H}_f (Example 2.23 (2)).

Recall the *comparison functor*

$$K : \mathcal{V} \rightarrow \mathbf{Met}^{\mathbf{T}\mathcal{V}}$$

assigning to every algebra A in \mathcal{V} the Eilenberg-Moore algebra

$$KA \equiv T_{\mathcal{V}}A \xrightarrow{\alpha} A$$

given by the unique homomorphism α with $\alpha \cdot \eta_A = id_A$.

Example 4.22. If $\mathcal{V} = \Sigma\text{-Met}$, then $\alpha = \varepsilon_A^\sharp : T_{\mathcal{V}}A \rightarrow A$ evaluates terms in $T_{\Sigma}A$ as elements of A . The comparison functor is an isomorphism. The verification is analogous to the classical Σ -algebras, see [21], Theorem VI.8.1.

Definition 4.23. By a *concrete category* over \mathbf{Met} is meant a pair (\mathcal{A}, U) , where \mathcal{A} is an (enriched) category and $U : \mathcal{A} \rightarrow \mathbf{Met}$ an (enriched) faithful functor: $d(f, g) = d(Uf, Ug)$ holds for all parallel pairs f, g in \mathcal{A} .

A *concrete functor* from (\mathcal{A}, U) to (\mathcal{A}', U') is a functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ with $U = U' \cdot F$. If F is an isomorphism, we say that (\mathcal{A}, U) is *concretely isomorphic* to (\mathcal{A}', U') .

Proposition 4.24 (See [26], 3.8). *Every variety \mathcal{V} of algebras is concretely isomorphic to $\mathbf{Met}^{\mathbf{T}\mathcal{V}}$, the Eilenberg-Moore algebras of its free-algebra monad: the comparison functor is a concrete isomorphism.*

Remark 4.25. In the next result we use a technique due to Dubuc [14]. This makes it possible to identify algebras for a monad \mathbf{T} with monad morphisms with domain \mathbf{T} . We recall this shortly here.

- (1) Let M be a metric space. We obtain the following adjoint situation:

$$[-, M] \dashv [-, M] : \mathbf{Met} \rightarrow \mathbf{Met}^{op}.$$

The corresponding monad on \mathbf{Met} is enriched, we denote it by $\langle M, M \rangle$. Thus

$$\langle M, M \rangle X = [[X, M], M]$$

for all $X \in \mathbf{Met}$.

- (2) Let \mathbf{T} be an enriched monad on \mathbf{Met} . Every algebra $\alpha : TM \rightarrow M$ in $\mathbf{Met}^{\mathbf{T}}$ defines a natural transformation

$$\hat{\alpha} : \mathbf{T} \rightarrow \langle M, M \rangle$$

as follows. The component, for an arbitrary space X ,

$$\widehat{\alpha}_X : TX \rightarrow [[X, M], M]$$

is the adjoint transpose of the following composite

$$TX \otimes [X, M] \xrightarrow{TX \otimes T(-)} TX \otimes [TX, TM] \xrightarrow{eval} TM \xrightarrow{\alpha} M.$$

Moreover, $\widehat{\alpha}$ is a monad morphism.

Conversely, every monad morphism from \mathbf{T} to $\langle M, M \rangle$ has the form $\widehat{\alpha}$ for a unique algebra (M, α) in $\mathbf{Met}^{\mathbf{T}}$.

- (3) This bijection $(\widehat{\quad})$ is coherent for monad morphisms as is shown in the following lemma.

Lemma 4.26. *Let $\gamma : \mathbf{S} \rightarrow \mathbf{T}$ be a monad morphism. Every algebra $\alpha : TA \rightarrow A$ in $\mathbf{Met}^{\mathbf{T}}$ yields an algebra $\beta = \alpha \cdot \gamma_A : SA \rightarrow A$ in $\mathbf{Met}^{\mathbf{S}}$, for which the following triangle commutes:*

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\gamma} & \mathbf{T} \\ & \searrow \widehat{\beta} & \swarrow \widehat{\alpha} \\ & \langle A, A \rangle X & \end{array}$$

Proof. Our task is to prove that the upper triangle in the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S}X & \xrightarrow{\gamma_X} & \mathbf{T}X \\ & \searrow \widehat{\beta}_X & \swarrow \widehat{\alpha}_X \\ & \langle A, A \rangle X & \\ & \downarrow \pi_f & \\ & A & \end{array}$$

$\beta \cdot Sf$ $\alpha \cdot Tf$

For every $f \in [X, A]$ denote by $\pi_f : \langle A, A \rangle X = [[X, A], A] \rightarrow A$ the corresponding morphism. Then all π_f are clearly collectively monic. Thus, it is sufficient to prove that all π_f (for $f : X \rightarrow A$) merge the upper triangle. Indeed, we use the definition of $\langle A, A \rangle X$:

$$\beta \cdot Sf = \alpha \cdot \gamma_A \cdot Sf = \alpha \cdot Tf \cdot \gamma_X.$$

□

Notation 4.27.

- (1) For our standard set V of variables x_i ($i \in \mathbb{N}$) we put

$$V_n = \{x_i \mid i < n\}.$$

Every term $t \in T_{\Sigma}V$ lies in $T_{\Sigma}V_n$ for some $n \in \mathbb{N}$.

- (2) The endofunctor $(-)^n$ is the polynomial functor $H_{[n]}$ of the signature $[n]$ of one n -ary symbol δ (Remark 4.8). Every term $t \in T_\Sigma V_n$ yields, by Yoneda lemma, a natural transformation from $H_{[n]}$ to T_Σ , and we obtain the corresponding monad morphism

$$\bar{t} : \mathbf{T}_{[n]} \rightarrow \mathbf{T}_\Sigma.$$

It substitutes in every term of $T_{[n]}X$ all occurrences of δ by σ . More precisely, the component \bar{t}_X is identity on variables from X , and for a term $s = \delta(s_i)_{i < n}$ it is recursively given by

$$\bar{t}_X(s) = t(\bar{t}_X(s_i))_{i < n}.$$

- (3) Every n -ary symbol $\sigma \in \Sigma$ is identified with the term $\sigma(x_i)_{i < n}$ in $T_\Sigma V_n$.
(4) For monads \mathbf{T} and \mathbf{S} we denote by

$$\llbracket \mathbf{T}, \mathbf{S} \rrbracket$$

the metric space $\mathbf{Mnd}(\mathbf{Met})(\mathbf{T}, \mathbf{S})$. That is, the distance of monad morphisms $\varphi, \psi : \mathbf{T} \rightarrow \mathbf{S}$ is $\sup_{X \in \mathbf{Met}} d(\varphi_X, \psi_X)$.

Lemma 4.28. *Let l, r be terms in $T_\Sigma V_n$. A quantitative algebra expressed by $\alpha : T_\Sigma A \rightarrow A$ in $\mathbf{Met}^{\mathbf{T}_\Sigma}$ satisfies the quantitative equation $l =_\varepsilon r$ iff the composite monad morphisms $\hat{\alpha} \cdot \bar{l}$ and $\hat{\alpha} \cdot \bar{r}$:*

$$\mathbf{T}_{\Delta_n} \begin{array}{c} \xrightarrow{\bar{l}} \\ \xrightarrow{\bar{r}} \end{array} \mathbf{T}_\Sigma \xrightarrow{\hat{\alpha}} \langle A, A \rangle$$

have distance at most ε in $\llbracket \mathbf{T}_{\Delta_n}, \langle A, A \rangle \rrbracket$.

Proof.

- (1) Assuming $d(\hat{\alpha} \cdot \bar{l}, \hat{\alpha} \cdot \bar{r}) \leq \varepsilon$, we consider the V -components: they have distance at most ε in $\llbracket T_\Delta, \llbracket [V, A], A \rrbracket \rrbracket$. Applied to $\delta(x_i) \in T_\Delta V$, they yield elements of $\llbracket [V, A], A \rrbracket$ of distance at most ε . Recall that

$$\hat{\alpha} \cdot \bar{l}_V(\delta(x_i)) = \hat{\alpha}(l(x_i))$$

assigns to $f \in [V, A]$ the value $f^\#(l(x_i))$. Analogously for $\hat{\alpha} \cdot \bar{r}_V$. We thus see that

$$d(f^\#(l), f^\#(r)) \leq \varepsilon$$

for all $f \in [V, A]$. Therefore A satisfies $l =_\varepsilon r$.

- (2) Assuming that A satisfies $l =_\varepsilon r$, we prove that for every space X we have

$$d(\hat{\alpha}_X \cdot \bar{l}_X, \hat{\alpha}_X \cdot \bar{r}_X) \leq \varepsilon.$$

Our assumption is that every evaluation $f : V \rightarrow |A|$ fulfils

$$d(f^\#(l), f^\#(r)) \leq \varepsilon.$$

Our task is to prove that for every term $t \in T_\Delta V$ we have

$$d(\hat{\alpha}_X \cdot \bar{l}_X(t), \hat{\alpha}_X \cdot \bar{r}_X(t)) \leq \varepsilon.$$

Equivalently, given f , we have

$$(*) \quad d(f^\sharp(\bar{l}_X(t)), f^\sharp(\bar{r}_X(t))) \leq \varepsilon.$$

We proceed by induction on the height $h(t)$ of t (where variables have height 0 and the height of a term $t = \sigma(t_i)_{i < n}$ is $h(t) = 1 + \max_{i < n} h(t_i)$). If the height is 0, then $\bar{l}_X(t) = \bar{r}_X(t) = t$. Suppose that $t = \sigma(t_i)_{i < n}$ and for each i we have already proved

$$(**) \quad d(\hat{\alpha}_X \cdot \bar{l}_X(t_i), \hat{\alpha}_X \cdot \bar{r}_X(t_i)) \leq \varepsilon.$$

We have $\bar{l}_X(t) = \sigma(\bar{l}_X(t_i))$, and therefore

$$f^\sharp(\bar{l}_X(t)) = \sigma_A(f^\sharp(\bar{l}_X(t_i))).$$

Analogously for $f^\sharp(\bar{r}_X(t))$. Since σ_A is nonexpanding, the inequalities $(**)$ imply $(*)$, as desired. \square

Construction 4.29. For every variety \mathcal{V} we define a weighted diagram in $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ (Notation 3.8), and prove below that the free-algebra monad of \mathcal{V} is its colimit. Suppose that \mathcal{V} is specified by a collection of quantitative equations as follows:

$$l_i =_{\varepsilon_i} r_i \text{ with } l_i, r_i \in T_\Sigma V_{n(i)}$$

for $i \in I$. The domain \mathcal{D} of our diagram is the discrete category I extended by an object a and two cocones

$$\lambda_i, \rho_i : i \rightarrow a$$

Using Notation 4.27 (2), the functor $D : \mathcal{D} \rightarrow \mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ is given by

$$Di = \mathbf{T}_{[n(i)]} \text{ and } Da = \mathbf{T}_\Sigma,$$

where for morphisms we put

$$D\lambda_i = \bar{l}_i \text{ and } D\rho_i = \bar{r}_i \text{ (} i \in I \text{)}.$$

The weight $W : \mathcal{D}^{op} \rightarrow \mathbf{Met}$ is defined by

$$\begin{aligned} Wa &= \{0\} \\ Wi &= \{l, r\} \text{ with } d(l, r) = \varepsilon_i \end{aligned}$$

and on morphisms

$$W\lambda_i : 0 \mapsto l \text{ and } W\rho_i : 0 \mapsto r.$$

Theorem 4.30. *Let \mathcal{V} be a variety of quantitative algebras. Assuming that strongly finitary endofunctors on \mathbf{Met} are closed under composition, the monad $\mathbf{T}_\mathcal{V}$ is the colimit of the above diagram in $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$.*

Proof. Denote the colimit (which exists by Proposition 3.14) by \mathbf{T} . We then have $T = \text{colim}_W D$ both in $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ and in $\mathbf{Mnd}_{\text{f}}(\mathbf{Met})$: the former category is by Proposition 3.14 closed under colimits in the latter one. We prove that \mathbf{T} is the free-algebra monad of \mathcal{V} .

(1) We have an isomorphism

$$\psi_{\mathbf{S}} : \llbracket \mathbf{T}, \mathbf{S} \rrbracket \rightarrow [\mathcal{D}^{op}, \mathbf{Met}](W, \llbracket D-, \mathbf{S} \rrbracket).$$

natural in $\mathbf{S} \in \mathbf{Mnd}_f(\mathbf{Met})$. In particular, this yields the unit

$$\nu = \psi_{\mathbf{T}}(id_{\mathbf{T}}) : W \rightarrow \llbracket D-, \mathbf{T} \rrbracket.$$

Its a -component assigns to 0 a monad morphism

$$\gamma = \nu_a(0) : \mathbf{T}_{\Sigma} \rightarrow \mathbf{T}.$$

Its i -component is, due to the naturality of ν , given by

$$\nu_i : l \mapsto \gamma \cdot \bar{l}_i \text{ and } r \mapsto \gamma \cdot \bar{r}_i.$$

Since this component ν_i is nonexpanding and $d(l, r) = \varepsilon_i$ in Wi , we conclude

$$(*) \quad d(\gamma \cdot \bar{\lambda}_i, \gamma \cdot \bar{\rho}_i) \leq \varepsilon_i \text{ for } i \in I.$$

(2) For every monad morphism $\varphi : \mathbf{T} \rightarrow \mathbf{S}$ in $\mathbf{Mnd}_f(\mathbf{Met})$ the natural transformation $\psi_{\mathbf{S}}(\varphi) : W \rightarrow \llbracket D-, \mathbf{S} \rrbracket$ has the following components:

$$\psi_{\mathbf{S}}(\varphi)_a : 0 \mapsto (\mathbf{T}_{\Sigma} \xrightarrow{\gamma} \mathbf{T} \xrightarrow{\varphi} \mathbf{S})$$

and

$$\begin{aligned} \psi_{\mathbf{S}}(\varphi)_i : l \mapsto (\mathbf{T}_{\Delta} \xrightarrow{\bar{l}_i} \mathbf{T}_{\Sigma} \xrightarrow{\gamma} \mathbf{T} \xrightarrow{\varphi} \mathbf{S}) \\ r \mapsto (\mathbf{T}_{\Delta} \xrightarrow{\bar{r}_i} \mathbf{T}_{\Sigma} \xrightarrow{\gamma} \mathbf{T} \xrightarrow{\varphi} \mathbf{S}). \end{aligned}$$

This follows from the naturality of $\psi_{\mathbf{S}}$.

(3) We prove that the components of γ are surjective. Indeed, \mathbf{T}_{Σ} preserves surjective morphisms (Example 3.7 (2) and Example 4.5 (2)). Thus by Proposition 3.16 we have a factorization in $\mathbf{Mnd}_f(\mathbf{Met})$ of γ as follows

$$\gamma = \mathbf{T}_{\Sigma} \xrightarrow{e} \mathbf{S} \xrightarrow{m} \mathbf{T}$$

where e has surjective components and those of m are isometries. We will prove that m is invertible.

From the inequalities (*) we derive, since m has isometric components, that

$$d(e \cdot \bar{\lambda}_i, e \cdot \bar{\rho}_i) \leq \varepsilon_i \text{ for } i \in I.$$

Consequently, we can define a natural transformation $\delta : W \rightarrow \llbracket D-, \mathbf{S} \rrbracket$ by the following components

$$\begin{aligned} \delta_a : 0 \mapsto e, \\ \delta_i : l \mapsto e \cdot \bar{\lambda}_i, \\ r \mapsto e \cdot \bar{\rho}_i \quad (i \in I). \end{aligned}$$

Since $\psi_{\mathbf{S}}$ is invertible, we obtain a monad morphism

$$\bar{m} : \mathbf{T} \rightarrow \mathbf{S} \text{ with } \psi_{\mathbf{S}}(\bar{m}) = \delta.$$

Apply the naturality square

$$\begin{array}{ccc} \llbracket \mathbf{T}, \mathbf{S} \rrbracket & \xrightarrow{\psi_{\mathbf{S}}} & \llbracket D-, \mathbf{S} \rrbracket \\ m \cdot (-) \downarrow & & \downarrow m \cdot (-) \\ \llbracket \mathbf{T}, \mathbf{T} \rrbracket & \xrightarrow{\psi_{\mathbf{T}}} & \llbracket D-, \mathbf{T} \rrbracket \end{array}$$

to \overline{m} : the lower passage yields $\psi_{\mathbf{T}}(m \cdot \overline{m})$ and the upper one yields $m \cdot \delta$. The equality of the a -components thus implies

$$\psi_{\mathbf{T}}(m \cdot \overline{m})_a = m \cdot \delta_a = m \cdot e = \gamma.$$

Since $\gamma = \psi_{\mathbf{T}}(id_{\mathbf{T}})$ and $\psi_{\mathbf{T}}$ is invertible, we have proved $m \cdot \overline{m} = id_{\mathbf{T}}$. Thus m is monic and split epic, proving that it is invertible.

- (4) Every algebra $\alpha : T_{\Sigma}A \rightarrow A$ in \mathcal{V} defines a natural transformation

$$\delta^{\alpha} : W \rightarrow \llbracket D-, \langle A, A \rangle \rrbracket$$

with components

$$\delta_a^{\alpha} : 0 \mapsto \widehat{\alpha} : \mathbf{T}_{\Sigma} \rightarrow \langle A, A \rangle$$

(Remark 4.25) and

$$\begin{aligned} \delta_a^{\alpha} : l &\mapsto \widehat{\alpha} \cdot \overline{\lambda}_i, \\ &r \mapsto \widehat{\alpha} \cdot \overline{\rho}_i. \end{aligned}$$

Indeed, δ_i^{α} is nonexpanding because A satisfies $d(\widehat{\alpha} \cdot \overline{\lambda}_i, \widehat{\alpha} \cdot \overline{\rho}_i) \leq \varepsilon_i$ by Lemma 4.28. The naturality of δ^{α} is obvious.

- (5) We define a concrete isomorphism

$$E : \text{Met}^{\mathbf{T}} \rightarrow \mathcal{V}$$

as follows. On objects $\alpha : TA \rightarrow A$ we use that $\gamma : \mathbf{T}_{\Sigma} \rightarrow \mathbf{T}$ is a monad morphism to obtain a Σ -algebra $\alpha' = \alpha \cdot \gamma_A : T_{\Sigma}A \rightarrow A$. This algebra lies in \mathcal{V} because $\widehat{\alpha}' = \widehat{\alpha} \cdot \gamma$ (Remark 4.25) and $\widehat{\alpha}$ is nonexpanding, thus $(*)$ yields

$$d(\widehat{\alpha}' \cdot \overline{\lambda}_i, \widehat{\alpha}' \cdot \overline{\rho}_i) \leq \varepsilon_i \text{ for } i \in I.$$

Now apply Lemma 4.28. Thus we can define E on objects by

$$E(A, \alpha) = (A, \alpha').$$

We next prove that given two algebras (A, α) and (B, β) in $\text{Met}^{\mathbf{T}}$, then a nonexpanding map $f : A \rightarrow B$ is a homomorphism in $\text{Met}^{\mathbf{T}}$

iff it is a Σ -homomorphism. This follows from the diagram below:

$$\begin{array}{ccc}
 T_{\Sigma}A & \xrightarrow{\alpha'} & A \\
 \downarrow T_{\Sigma}f & \searrow \gamma_A & \nearrow \alpha \\
 & TA & \\
 & \downarrow Tf & \\
 & TB & \\
 \nearrow \gamma_B & & \searrow \beta \\
 T_{\Sigma}B & \xrightarrow{\beta'} & B \\
 & \downarrow f &
 \end{array}$$

The left-hand part is the naturality of γ , and both triangles clearly commute. Since by (3) we know that γ_A is epic, we conclude that the outward square commutes iff the right-hand part does.

We conclude that by putting $Ef = f$ we obtain a concrete, fully faithful functor. It is invertible because given an algebra $\alpha : T_{\Sigma}A \rightarrow A$ in \mathcal{V} we apply the isomorphism $\psi_{\langle A, A \rangle}$ to $\delta^{\alpha} : W \rightarrow \llbracket D-, \langle A, A \rangle \rrbracket$ to obtain a unique $\alpha_0 : TA \rightarrow A$ in $\mathbf{Met}^{\mathbf{T}}$ with $\psi_{\langle A, A \rangle}(\hat{\alpha}_0) = \delta^{\alpha}$. By (2) the a -component of $\psi_{\langle A, A \rangle}(\hat{\alpha}_0)$ is given by $0 \mapsto \hat{\alpha}_0 \cdot \gamma$. Since $\delta_a^{\alpha}(0) = \hat{\alpha}$, this proves $\hat{\alpha}_0 \cdot \gamma = \hat{\alpha}$. Lemma 4.26 yields $\alpha_0 \cdot \gamma_A = \alpha$. Thus E^{-1} is the concrete functor defined by $E^{-1}(A, \alpha) = (A, \alpha_0)$.

- (6) Since \mathcal{V} is concretely isomorphic to $\mathbf{Met}^{\mathbf{T}_{\mathcal{V}}}$, where $\mathbf{T}_{\mathcal{V}}$ is the free-algebra monad (Proposition 4.24), we conclude that the categories $\mathbf{Met}^{\mathbf{T}}$ and $\mathbf{Met}^{\mathbf{T}_{\mathcal{V}}}$ are concretely isomorphic. This proves that the monads \mathbf{T} and $\mathbf{T}_{\mathcal{V}}$ are isomorphic ([10], Theorem 3.6.3). Therefore \mathbf{T} is a free-algebra monad for \mathcal{V} as claimed. □

Corollary 4.31. Assume that strongly finitary endofunctors on \mathbf{Met} compose. Then the free-algebra monad for a finitary variety of quantitative algebras is strongly finitary.

Remark 4.32. For the base category \mathbf{UMet} we proceed completely analogously:

- (1) For every variety \mathcal{V} of ultra-quantitative algebras a free-algebra monad $\mathbf{T}_{\mathcal{V}}$ on \mathbf{UMet} exists, and \mathcal{V} is concretely isomorphic to $\mathbf{UMet}^{\mathbf{T}_{\mathcal{V}}}$. This is analogous to Corollary 4.19 and Proposition 4.24.
- (2) For a monad \mathbf{T} on \mathbf{UMet} and an ultrametric space M we have a bijection between algebras for \mathbf{T} on M and monad morphisms from \mathbf{T} to $\langle M, M \rangle$ as in Remark 4.25.

- (3) An ultra-quantitative algebra expressed by $\alpha : T_\Sigma M \rightarrow M$ in $\mathbf{UMet}^{\mathbf{T}_\nu}$ satisfies $l =_\varepsilon r$ iff $d(\widehat{\alpha} \cdot l, \widehat{\alpha} \cdot r) \leq \varepsilon$. This is proved as Lemma 4.28.
- (4) For the weighted diagram in $\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$ constructed as in Construction 4.29 the monad \mathbf{T}_ν is its colimit in $\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$. This is proved as Theorem 4.30, but here no extra assumption is needed since strongly finitary endofunctors on \mathbf{UMet} compose (Remark 3.11). We obtain the following result:

Corollary 4.33. The free-algebra monad for a finitary variety of ultra-quantitative algebras is strongly finitary.

5. FROM STRONGLY FINITARY MONADS TO VARIETIES

Throughout this section $\mathbf{T} = (T, \mu, \eta)$ denotes a strongly finitary monad on \mathbf{Met} or \mathbf{UMet} . We construct a finitary variety of quantitative (or ultra-quantitative) algebras with \mathbf{T} as its free-algebra monad. This leads to the main result: a bijection between finitary varieties and strongly finitary monads.

Remark 5.1.

- (1) Recall that (TX, μ_X) is the free algebra of $\mathbf{Met}^{\mathbf{T}}$ on $\eta_X : X \rightarrow TX$: for every algebra (A, α) and every morphism $f : X \rightarrow A$ the morphism

$$f^* = \alpha \cdot Tf : TX \rightarrow A$$

is the unique homomorphism with $f = f^* \cdot \eta_X$.

- (2) Given $(A, \alpha) = (TX, \mu_X)$, we have $\eta_X^* = id_{TX}$. And for morphisms $f : X \rightarrow TY$ and $g : Y \rightarrow TZ$ we have $g^* \cdot f^* = (g^* \cdot f)^* : TX \rightarrow TZ$. Indeed, $g^* \cdot f^*$ is a homomorphism extending $g^* \cdot f$: we have $(g^* \cdot f^*) \cdot \eta_X = g^* \cdot f$.

Notation 5.2. For the given set $V = \{x_k \mid k \in \mathbb{N}\}$ of variables we denote by V_n the set (discrete space) $\{x_k \mid k \leq n\}$.

Construction 5.3. For a strongly finitary monad \mathbf{T} on \mathbf{Met} or \mathbf{UMet} we define a signature Σ by

$$\Sigma_n = |TV_n| \text{ for all } n \in \mathbb{N}.$$

Recall that an n -ary operation symbol σ is viewed as a term $\sigma(x_i)_{i < n}$ in $T_\Sigma V_n$. Example: every variable $x_i \in V_n$ yields an n -ary symbol $\eta_{V_n}(x_i)$ which is a term.

Our variety $\mathcal{V}_{\mathbf{T}}$ is presented by three types of equations, where n and m denote arbitrary natural numbers:

- (1) $l =_\varepsilon r$ for all $l, r \in TV_n$ with $d(l, r) \leq \varepsilon$ in TV_n .
- (2) $k^*(\sigma) = \sigma(k(x_i))_{i < n}$ for all $\sigma \in TV_n$ and all maps $k : V_n \rightarrow |TV_m|$.
- (3) $\eta_{V_n}(x_i) = x_i$ for all $i < n$.

Lemma 5.4. *Every algebra $\alpha : TA \rightarrow A$ in $\mathbf{Met}^{\mathbf{T}}$ or $\mathbf{UMet}^{\mathbf{T}}$ defines a Σ -algebra A in $\mathcal{V}_{\mathbf{T}}$ with operations $\sigma_A : A^n \rightarrow A$ for $\sigma \in TV_n$ given by*

$$\sigma_A(a_i)_{i < n} = a^*(\sigma) \text{ for } a : V_n \rightarrow A, x_i \mapsto a_i.$$

Proof.

- (a) The mapping σ_A is nonexpanding: given $d((a_i)_{i < \omega}, (b_i)_{i < \omega}) = \varepsilon$ in A^ω , the corresponding maps $a, b : V_n \rightarrow A$ fulfil $d(a, b) = \varepsilon$. Since T is enriched, this yields $d(Ta, Tb) \leq \varepsilon$. Finally α is nonexpanding and $a^* = \alpha \cdot Ta$, $b^* = \alpha \cdot Tb$, thus $d(a^*, b^*) \leq \varepsilon$. In particular $d(a^*(\sigma), b^*(\sigma)) \leq \varepsilon$.
- (b) The quantitative equations (1)-(3) hold:
 Ad (1): Given $l, r \in TV_n$ with $d(l, r) \leq \varepsilon$, then for every map $a : V_n \rightarrow A$ we have $d(a^*(l), a^*(r)) \leq \varepsilon$. Thus $d(l_A(a_i), r_A(a_i)) \leq \varepsilon$ for all $(a_i) \in A^\omega$.
 Ad (2): Given $a : V_n \rightarrow A$ we prove $(k^*(\sigma))_A(a_j) = \sigma_A(k(x_i))(a_j)$. The left-hand side is $a^*(k^*(\sigma)) = (a^*k)^*(\sigma)$ by Remark 5.1 (2). The right-hand one is $a^*(\sigma_A(k(x_i))) = (a^*k)^*(\sigma)$, too.
 Ad (3): Recall that $\alpha \cdot \eta_A = id$ and $Ta \cdot \eta_{V_n} = \eta_A \cdot a$ for every map $a : V_n \rightarrow A$. Therefore

$$\begin{aligned} (\eta_{V_n}(x_i))_A(a_j) &= a^*(\eta_{V_n}(x_i)) \\ &= \alpha \cdot Ta \cdot \eta_{V_n}(x_i) \\ &= a(x_i) = a_i. \end{aligned}$$

□

Remark 5.5. Moreover, every homomorphism $h : (A, \alpha) \rightarrow (B, \beta)$ in $\mathbf{Met}^{\mathbf{T}}$ or $\mathbf{UMet}^{\mathbf{T}}$ is also a homomorphism between the corresponding Σ -algebras. Indeed, given $\sigma \in TV_n$ we verify $h \cdot \sigma_A = \sigma_B \cdot h^n : A^n \rightarrow B$. For every map $a : V_n \rightarrow A$ we want to prove $h(a^*(\sigma)) = b^*(\sigma)$ where $b : V_n \rightarrow B$ is given by $x_i \mapsto h(a_i)$, that is, $b = h \cdot a$. This follows from $h \cdot a^* = (h \cdot a)^*$: indeed, we know that $h \cdot \alpha = \beta \cdot Th$, thus

$$h \cdot a^* = h \cdot \alpha \cdot Ta = \beta \cdot Th \cdot Ta = \beta \cdot T(h \cdot a) = (h \cdot a)^*.$$

Theorem 5.6. *Every strongly finitary monad \mathbf{T} on \mathbf{Met} or \mathbf{UMet} is the free-algebra monad of the variety $\mathcal{V}_{\mathbf{T}}$.*

Proof. We present a proof for \mathbf{Met} , the case \mathbf{UMet} is completely analogous. For every metric space M we want to prove that the Σ -algebra associated with (TM, μ_M) is free in $\mathcal{V}_{\mathbf{T}}$ w.r.t. the universal map η_M . Then the theorem follows from Proposition 4.24.

We have two strongly finitary monads, \mathbf{T} and the free algebra monad of $\mathcal{V}_{\mathbf{T}}$ (Corollary 4.31). Thus, it is sufficient to prove the above for finite discrete spaces M . Then this extends to all finite spaces because we have $M = \text{colim}_{W_0} D_M$ (Proposition 2.20) and both monads preserve this colimit. Since they coincide on all finite discrete spaces, they coincide on M . Finally, the above extends to all spaces M : we have

a directed colimit $M = \operatorname{colim}_{i \in I} M_i$ of the diagram of all finite subspaces M_i ($i \in I$) which both monads preserve.

Given a finite discrete space M , we can assume without loss of generality $M = V_n$ for some $n \in \mathbb{N}$. For every algebra A in $\mathcal{V}_{\mathbf{T}}$ and a map $f : V_n \rightarrow A$, we prove that there exists a unique Σ -homomorphism $\bar{f} : TV_n \rightarrow A$ with $f = \bar{f} \cdot \eta_{V_n}$.

Existence: Define $\bar{f}(\sigma) = \sigma_A(f(x_i))_{i < n}$ for every $\sigma \in TV_n$. The equality $f = \bar{f} \cdot \eta_{V_n}$ follows since A satisfies the equations $\eta_{V_n}(x_i) = x_i$, thus the operation of A corresponding to $\eta_{V_n}(x_i)$ is the i -th projection. The map \bar{f} is nonexpanding: given $d(l, r) \leq \varepsilon$, the algebra A satisfies $l =_{\varepsilon} r$. Therefore for every n -tuple $f : V_n \rightarrow A$ we have

$$d(l_A(f(x_i)), r_A(f(x_i))) \leq \varepsilon.$$

To prove that \bar{f} is a Σ -homomorphism, take an m -ary operation symbol $\tau \in TV_m$. We prove $\bar{f} \cdot \tau_{V_m} = \tau_A \cdot \bar{f}^m$. This means that every $k : V_m \rightarrow TV_n$ fulfils

$$\bar{f} \cdot \tau_{V_m}(k(x_j))_{j < m} = \tau_A \cdot \bar{f}^m(k(x_j))_{j < m}.$$

The definition of \bar{f} yields that the right-hand side is $\tau_A(k(x_j)_A(f(x_i)))$. Due to equation (2) in Construction 5.3 with τ in place of σ this is $k^*(\tau)_A(f(x_i))$. The left-hand side yields the same result since

$$\bar{f}^m(k(x_j)) = (k(x_j))_A(f(x_i)).$$

Uniqueness: Let \bar{f} be a nonexpanding Σ -homomorphism with $f = \bar{f} \cdot \eta_{V_n}$. In TV_n the operation σ assigns to $\eta_{V_n}(x_i)$ the value σ . (Indeed, for every $a : n \rightarrow |TV_n|$ we have $\sigma_{TV_n}(a_i) = a^*(\sigma) = \mu_{V_n} \cdot Ta(\sigma)$. Thus $\sigma_{TV_n}(\eta_{V_n}(x_i)) = \mu_{V_n} \cdot T\eta_{V_n}(\sigma) = \sigma^n$.) Since \bar{f} is a homomorphism, we conclude

$$f(\sigma) = \sigma_A(\bar{f} \cdot \eta_{V_n}(x_i)) = \sigma_A(f(x_i))$$

which is the above formula. □

Recall the concept of concrete functor over \mathbf{Met} from Definition 4.23.

Notation 5.7.

- (1) The category of finitary varieties of quantitative algebras and concrete functors is denoted by

$$\mathbf{Var}_f(\mathbf{Met}).$$

Analogously $\mathbf{Var}_f(\mathbf{UMet})$ is the category of varieties of ultra-quantitative algebras.

- (2) Recall that the category of strongly finitary monads and monad morphisms is denoted by

$$\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet}) \text{ and } \mathbf{Mnd}_{\text{sf}}(\mathbf{Met}).$$

Theorem 5.8 (Main theorem). *The category $\mathbf{Var}_f(\mathbf{UMet})$ of finitary varieties of ultra-quantitative algebras is dually equivalent to the category $\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$ of strongly finitary monads on \mathbf{UMet} .*

Proof.

- (1) Given monads \mathbf{T} and \mathbf{S} on \mathbf{UMet} we recall the bijection between monad morphisms and concrete functors, see e.g. [10], Theorem 3.3. It assigns to every monad morphism $\varphi : \mathbf{S} \rightarrow \mathbf{T}$ the concrete functor $\bar{\varphi} : \mathbf{UMet}^{\mathbf{T}} \rightarrow \mathbf{UMet}^{\mathbf{S}}$ given by

$$TA \xrightarrow{\alpha} A \mapsto SA \xrightarrow{\varphi_A} TA \xrightarrow{\alpha} A.$$

The map $\varphi \mapsto \bar{\varphi}$ is bijective and preserves composition.

We denote by $(-)^*$ the inverse, assigning to a concrete functor $F : \mathbf{UMet}^{\mathbf{T}} \rightarrow \mathbf{UMet}^{\mathbf{S}}$ the corresponding monad morphism $F^* : \mathbf{S} \rightarrow \mathbf{T}$. This also preserves composition.

- (2) For every variety \mathcal{V} the comparison functor $K_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbf{UMet}^{\mathbf{T}_{\mathcal{V}}}$ is invertible (Proposition 4.24). Using Corollary 4.31, we define a functor

$$\Phi : \mathbf{Var}_f(\mathbf{UMet})^{op} \rightarrow \mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$$

on objects \mathcal{V} by

$$\Phi(\mathcal{V}) = \mathbf{T}_{\mathcal{V}}.$$

On morphisms $F : \mathcal{V} \rightarrow \mathcal{W}$ we define $\Phi(F)$ by forming the concrete functor

$$\mathbf{UMet}^{\mathbf{T}_{\mathcal{V}}} \xrightarrow{K_{\mathcal{V}}^{-1}} \mathcal{V} \xrightarrow{F} \mathcal{W} \xrightarrow{K_{\mathcal{W}}} \mathbf{UMet}^{\mathbf{T}_{\mathcal{W}}}$$

and putting

$$\Phi(F) = (K_{\mathcal{W}}FK_{\mathcal{V}}^{-1})^* : \mathbf{T}_{\mathcal{W}} \rightarrow \mathbf{T}_{\mathcal{V}}.$$

Since $(-)^*$ preserves composition, so does Φ .

- (3) Φ is obviously faithful and, since $(-)^*$ is a bijection, full. It is an equivalence functor because every monad \mathbf{T} in $\mathbf{Mnd}_{\text{sf}}(\mathbf{UMet})$ is isomorphic to $\Phi(\mathcal{V}_{\mathbf{T}})$ by Theorem 5.6.

□

For varieties of quantitative algebras the procedure is the same, but here we use Corollary 4.31, so that we need an extra assumption:

Theorem 5.9. *Assume that strongly finitary endofunctors on \mathbf{Met} compose. Then the category $\mathbf{Var}_f(\mathbf{Met})$ of finitary varieties of quantitative algebras is dually equivalent to the category $\mathbf{Mnd}_{\text{sf}}(\mathbf{Met})$ of strongly finitary monads on \mathbf{Met}*

6. MONADS ON COMPLETE ULTRAMETRIC SPACES

For the category **CUMet** of (extended) complete ultrametric spaces (a full subcategory of **Met**) we obtain the same result: strongly finitary monads bijectively correspond to varieties of complete algebras of **CUMet**. The proof is analogous, we indicate here what small changes are needed.

The category **CUMet** is closed symmetric monoidal: if A and B are complete spaces, then so are the spaces $A \otimes B$ and $[A, B]$ in Notation 2.1. In this section by 'category' and 'functor' we mean an enriched category (and functor) over **CUMet**. This category is complete and cocomplete ([6], Example 4.5).

Remark 6.1.

- (a) Recall the Cauchy completion \overline{X} of an ultrametric space X : it is the (essentially unique) complete ultrametric space containing X as a dense subspace. This embedding $X \hookrightarrow \overline{X}$ is a reflection of X in **CUMet**: for every nonexpanding map $f : X \rightarrow Y$ with Y complete there is a unique nonexpanding extension $\overline{f} : \overline{X} \rightarrow Y$. Indeed, since f is continuous, a unique continuous extension exists. To prove that \overline{f} is indeed nonexpanding, consider $a, b \in \overline{X}$ with $d(a, b) = r$. Find sequences $(a_n), (b_n)$ in X converging to a and b , respectively. Since $\overline{f}(a) = \lim_{n \rightarrow \infty} f(a_n)$ and $\overline{f}(b) = \lim_{n \rightarrow \infty} f(b_n)$, we get

$$\begin{aligned} d(\overline{f}(a), \overline{f}(b)) &= \lim_{n \rightarrow \infty} d(f(a_n), f(b_n)) \\ &\leq \lim_{n \rightarrow \infty} d(a_n, b_n) \\ &= d(a, b). \end{aligned}$$

- (b) Cauchy completion preserves finite products: the space $\overline{X} \times \overline{Y}$ is complete, and $X \times Y$ is clearly dense in it. Thus $\overline{X \times Y} \cong \overline{X} \times \overline{Y}$.
- (c) For every natural number n the endofunctor $(-)^n$ of **CUMet** preserves directed colimits. Indeed, directed colimits in **CUMet** are the Cauchy completions of the corresponding colimits in **Met**. Thus our statement follows from Example 2.12 and (b) above.
- (d) Directed colimits in **CUMet** have a characterization analogous to Proposition 2.9. However, Condition 1 must be weakened, as the following example demonstrates.

Example 6.2. In **CUMet** the subspace of the real line on the set $A = \{0\} \cup \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ is a colimit of the ω -chain of subspaces $A_n = \{2^{-k} \mid k = 1, \dots, n\}$. However, 0 does not lie in the image of any of the colimit maps.

Proposition 6.3. *Let $(D_i)_{i \in I}$ be a directed diagram in **CUMet**. A cocone $c_i : D_i \rightarrow C$ is a colimit iff*

- (1) *it is collectively dense: C is the closure of $\bigcup_{i \in I} c_i[D_i]$, and*

(2) for every $i \in I$, given $y, y' \in D_i$ we have

$$d(c_i(y), c_i(y')) = \inf_{j \geq i} d(f_j(y), f_j(y'))$$

where $f_j : D_i \rightarrow D_j$ is the connecting map.

Proof. Denote by $c'_i : D_i \rightarrow C'$ a colimit of our diagram in \mathbf{Met} . Then a colimit in \mathbf{CUMet} is given by the cocone $m \cdot c'_i : D_i \rightarrow C'$ where $m : C' \rightarrow C$ is the Cauchy completion of C' .

- (a) Sufficiency: If (1) and (2) hold, put $C' = \bigcup_{i \in I} c_i[D_i]$ and denote by $c'_i : D_i \rightarrow C'$ the codomain restriction of c_i for each $i \in I$. Consider C' as the metric subspace of C . Then we obtain a cocone (c'_i) of our diagram in \mathbf{Met} . Since by our Condition (1) the subspace C' is dense in C , we conclude that the inclusion map $m : C' \hookrightarrow C$ is a Cauchy completion of C' . Therefore, the cocone $c_i = m \cdot c'_i : D_i \rightarrow C$ is a colimit in \mathbf{CUMet} , as claimed.
- (b) Necessity: If $c_i : D_i \rightarrow C$ is a colimit cocone in \mathbf{CUMet} and $c'_i : D_i \rightarrow C'$ is the cocone of codomain restrictions to $C' = \bigcup_{i \in I} c_i[D_i]$, then we can assume without loss of generality that the inclusion map $m : C' \rightarrow C$ is a Cauchy completion of C' with $c_i = m \cdot c'_i$ for all $i \in I$. Since c'_i are collectively epic by Proposition 2.9, we conclude that c_i are collectively dense. Since c'_i fulfil Condition (2) of 2.9 and m is an isometric embedding, we conclude that c_i fulfil that condition, too.

□

Definition 6.4. By a *precongruence* of a complete ultrametric space M in \mathbf{CUMet} is meant the weighted diagram $D'_M : \mathcal{B} \rightarrow \mathbf{CUMet}$ with weight $B' : \mathcal{B}^{op} \rightarrow \mathbf{CUMet}$ where D'_M and B' are the codomain restrictions of the corresponding functors in Definition 2.19.

Proposition 6.5. *Every complete ultrametric space is the colimit of its precongruence in \mathbf{CUMet} .*

The proof is the same as that of Proposition 2.20.

Notation 6.6. Every set (= discrete space) is complete. We thus get a full embedding

$$K : \mathbf{Set}_f \hookrightarrow \mathbf{CUMet}$$

of the category of finite sets and mappings into \mathbf{CUMet} .

Definition 6.7. An endofunctor T of \mathbf{CUMet} is *strongly finitary* if it is the left Kan extension of its restriction $T \cdot K$ to finite discrete spaces:

$$T = \mathbf{Lan}_K(T \cdot K).$$

Theorem 6.8. *An endofunctor of \mathbf{CUMet} is strongly finitary iff it is finitary and preserves colimits of precongruences.*

The proof is analogous to that of Theorem 3.6. Indeed, every complete metric space is a directed colimit of its finite subspaces, and those are obtained as colimits of precongruences (using finite discrete spaces).

Notation 6.9.

- (1) Analogously to Notation 3.8 the category $\mathbf{Mnd}(\mathbf{CUMet})$ of monads on \mathbf{CUMet} is enriched via the supremum metric (inherited from $[\mathbf{CUMet}, \mathbf{CUMet}]$). Indeed, it is easy to verify that, given monads \mathbf{T} and \mathbf{S} , the subspace of $\mathbf{CUMet}(T, S)$ on all monad morphisms is closed, thus, complete.
- (2) We denote by $\mathbf{Mnd}_f(\mathbf{CUMet})$ and $\mathbf{Mnd}_{sf}(\mathbf{CUMet})$ the categories of finitary and strongly finitary monads on \mathbf{CUMet} , respectively (cf. Notation 3.8).

Proposition 6.10. *The categories $\mathbf{Mnd}_f(\mathbf{CUMet})$ and $\mathbf{Mnd}_{sf}(\mathbf{CUMet})$ are closed under weighted limits in the category of \aleph_1 -accessible monads on \mathbf{CUMet} .*

The proof is the same as for Proposition 3.14.

Remark 6.11. In \mathbf{Met} we have used the factorization system (surjective, isometric embedding) (Remark 2.21). In \mathbf{CMet} we have the factorization system (epi, strong mono): epimorphisms are the morphisms whose images are dense, and strong monomorphisms are those representing closed subspaces.

Proposition 6.12. *Let \mathbf{T} be a finitary monad on \mathbf{CUMet} preserving epimorphisms. In the category $\mathbf{Mnd}_f(\mathbf{CUMet})$ every morphism $h : \mathbf{T} \rightarrow \mathbf{S}$ has a factorization $h = m \cdot e$ where e has epic components and m has components representing closed subspaces.*

Proof. Since T preserves \mathcal{E} -morphisms of the factorization system (epi, strong mono), the fact that h has a factorization in the category of all monads as described is proved as Items (1)-(3) of Proposition 3.16. It remains to prove that the functor T' is finitary. That is, given a directed colimit $c_i : D_i \rightarrow C$ ($i \in I$) in \mathbf{CUMet} , we are to verify that the cocone $(T'c_i)_{i \in I}$ satisfies (1) and (2) of Proposition 6.3.

Ad (1): Our task is to find for every $x \in T'\overline{C}$ and every $\varepsilon > 0$ an element $y \in D_i$ (for some $i \in I$) with $d(x, T'c_i(y)) < \varepsilon$. Form a colimit $\overline{c}_i : D_i \rightarrow \overline{C}$ ($i \in I$) of the given diagram in \mathbf{Met} . Without loss of generality C is the Cauchy completion of \overline{C} and $c_i = m \cdot \overline{c}_i$ for the inclusion map $m : \overline{C} \rightarrow C$. Since $e_C : TC \rightarrow T'C$ is dense, there exists $x_0 \in T\overline{C}$ with

$$d(x, e_{\overline{C}}(x_0)) < \frac{\varepsilon}{2}.$$

The functor T is finitary, hence by Proposition 6.3 the cocone $(T\overline{c}_i)$ is collectively dense. For x_0 there thus exists $i \in I$ and $y \in TD_i$ with

$$d(x_0, T\overline{c}_i(y)) < \frac{\varepsilon}{2}.$$

Apply the nonexpanding map $e_{\overline{C}}$ and use $e_{\overline{C}} \cdot T\overline{c}_i = T'\overline{c}_i \cdot e_{D_i}$:

$$d(e_{\overline{C}}(x_0), T'\overline{c}_i \cdot e_{D_i}(y_0)) \leq \varepsilon.$$

The desired element is $y = e_{D_i}(y_0)$: since $T'c_i(y) = T\overline{c}_i \cdot e_{D_i}(y_0)$ we get

$$d(x, T'c_i(y)) \leq d(x, e_{\overline{C}}(x_0)) + d(e_{\overline{C}}(x_0), Tc_i(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Ad (2): This is the same proof as in Proposition 3.16, using $m : T' \rightarrow S$.

□

Notation 6.13. Let Σ be a finitary signature. We denote by $\Sigma\text{-CUMet}$ the category of complete ultra-quantitative algebras. This is the full subcategory of $\Sigma\text{-UMet}$ on algebras whose underlying metric is complete.

Example 6.14. For every complete space X the free Σ -algebra $T_\Sigma X$ of Example 4.5 is complete. This is then the free algebra in $\Sigma\text{-CUMet}$ w.r.t. $\eta_X : X \rightarrow T_\Sigma X$.

Indeed, we have seen in Corollary 4.11 that $T_\Sigma X$ is a coproduct of finite powers of X , thus the metric space $T_\Sigma X$ is complete. Its universal property in $\Sigma\text{-CUMet}$ thus follows from Proposition 4.9.

Corollary 6.15. The monad \mathbf{T}_Σ of free complete quantitative algebras is strongly finitary on CUMet .

The argument is the same as in Corollary 4.11.

Definition 6.16. A *variety* of complete quantitative algebras is a full subcategory of $\Sigma\text{-CMet}$ specified by a set of quantitative equations.

Example 6.17. We describe the monad \mathbf{T} of free ultra-quantitative complete semilattices on CUMet . It assigns to every complete ultrametric space M the space TM of all compact subsets with the Hausdorff metric (Example 4.21 (4)).

This holds for separable complete spaces: see [22], Theorem 9.6. To extend this result to all complete spaces, first observe that the subset Z of TM of all finite sets is dense. Indeed, every compact set $K \subseteq M$ lies in the closure of Z : given $\varepsilon > 0$, let $K_0 \subseteq K$ be a finite set such that ε -balls with centers in K_0 cover K . Then $K_0 \in Z$ and the Hausdorff distance of K_0 and K is at most ε .

Given a complete ultrametric space M , let X_i ($i \in I$) be the collection of all countable subsets. Each closure \overline{X}_i is a complete separable space, and $M = \bigcup_{i \in I} \overline{X}_i$ is a directed colimit (see Remark 2.11) preserved by T . Since $T\overline{X}_i$ is the space of all compact subsets of \overline{X}_i , and since finite subsets of M form a dense set, we conclude that TM is the space of all compact subsets of M .

Proposition 6.18. Let \mathcal{V} be a variety of complete ultra-quantitative algebras.

- (1) \mathcal{V} has free algebras: the forgetful functor $U_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbf{CUMet}$ has a left adjoint $F_{\mathcal{V}} : \mathbf{CUMet} \rightarrow \mathcal{V}$.
- (2) For the corresponding monad $\mathbf{T}_{\mathcal{V}}$ with $T_{\mathcal{V}} = U_{\mathcal{V}} \cdot F_{\mathcal{V}}$ the categories \mathcal{V} and $\mathbf{CUMet}^{\mathbf{T}_{\mathcal{V}}}$ are concretely isomorphic (via the comparison functor).

The proof is analogous to that of Corollary 4.19 and Proposition 4.24.

Construction 6.19. For every variety \mathcal{V} of complete ultra-quantitative algebras we construct a weighted diagram in $\mathbf{Mnd}_{\text{sf}}(\mathbf{CUMet})$ completely analogous to Construction 4.29. We just understand the objects $D_i = \mathbf{T}_{[n(i)]}$ and $Da = \mathbf{T}_{\Sigma}$ as monads on \mathbf{CUMet} .

Theorem 6.20. *Let \mathcal{V} be a variety of complete ultra-quantitative algebras. Then the free-algebra monad $\mathbf{T}_{\mathcal{V}}$ is the colimit of the above weighted diagram in $\mathbf{Mnd}_{\text{sf}}(\mathbf{CUMet})$.*

The proof is analogous to that of Theorem 4.30, using Proposition 6.12 and the fact that discrete spaces are complete.

Corollary 6.21. For every variety \mathcal{V} of complete ultra-quantitative algebras the monad $\mathbf{T}_{\mathcal{V}}$ is strongly finitary.

Recall the variety $\mathcal{V}_{\mathbf{T}}$ assigned to every strongly finitary monad \mathbf{T} on \mathbf{Met} in Construction 5.3. We can define a variety $\mathcal{V}_{\mathbf{T}}$ of complete ultra-quantitative algebras by the same signature and the same equations. Using the same proof as that of Theorem 5.6 we obtain the following

Theorem 6.22. *Every strongly finitary monad on \mathbf{CUMet} is the free-algebra monad of the variety $\mathcal{V}_{\mathbf{T}}$.*

Corollary 6.23. The category of finitary varieties of complete ultra-quantitative algebras (and concrete functors) is dually equivalent to the category $\mathbf{Mnd}_{\text{sf}}(\mathbf{CUMet})$ of strongly finitary monads.

7. INFINITARY ALGEBRAS

The above bijective correspondence holds, more generally, for λ -ary ultra-quantitative algebras, where λ is an arbitrary infinite regular cardinal. The proof is completely analogous, we indicate the (small) modifications needed in the present section.

Assumption 7.1. Throughout the rest of the paper λ denotes a regular infinite cardinal. And a standard set $V = \{x_i \mid i < \lambda\}$ of variables is chosen.

Let Σ be a λ -ary signature: a set of operation symbols of arities which are cardinals $n < \lambda$. A quantitative Σ -algebra is a metric space A together with nonexpanding maps $\sigma_A : A^n \rightarrow A$ for every n -ary symbol $\sigma \in \Sigma$. We again obtain the category $\Sigma\text{-Met}$ of quantitative algebras and nonexpanding homomorphisms and $\Sigma\text{-UMet}$ as its subcategory of ultra-quantitative algebras.

Example 7.2. The free algebra on a space X is the algebra $T_\Sigma X$ of terms (defined analogously to Example 4.5). If X is discrete, then so is $T_\Sigma X$.

Quantitative equations are defined precisely as in Definition 4.13. Also the definition of a variety of Σ -algebras is unchanged.

Example 7.3. Recall that a σ -semilattice is a poset with countable joins. We can express it by adding to the operations $+$ and 0 of Example 4.14(3) an operation \bigsqcup of arity ω . Besides the semilattice equations for $+$ and 0 one needs the following associativity equation:

$$y + \bigsqcup_{n < \omega} x_n = \bigsqcup_{n < \omega} (y + x_n),$$

idempotence of \bigsqcup :

$$\bigsqcup_{n < \omega} x = x,$$

and for every $k < \omega$ the equation

$$x_k + \bigsqcup_{n < \omega} x_n = \bigsqcup_{n < \omega} x_n.$$

When we then define $x \leq y$ by $x + y = y$, it is not difficult to verify that $\bigsqcup_{n < \omega} x_n$ is the join of $\{x_n \mid n < \omega\}$. A quantitative σ -semilattice is then a σ -semilattice on a metric space with $+$ and \bigsqcup nonexpanding.

The *monad of quantitative σ -semilattices* is given by the metric spaces $TM = \mathcal{P}_{\omega_1} M$ of countable subsets of M with the Hausdorff metric. The proof is analogous to the proof for semilattices in [22].

Theorem 7.4. *An enriched endofunctor T of \mathbf{Met} is strongly λ -accessible (Definition 3.4) iff it*

- (1) *is λ -accessible (preserves λ -directed colimits) and*
- (2) *preserves colimits of precongruences.*

The proof is essentially the same as that of Theorem 3.6.

Lemma 7.5. *A Σ -algebra expressed by $\alpha : T_\Sigma A \rightarrow A$ satisfies $l =_\varepsilon r$ iff $d(\widehat{\alpha} \cdot \bar{l}, \widehat{\alpha} \cdot \bar{r}) \leq \varepsilon$.*

Proof. This is completely analogous to the proof of Lemma 4.28. In the proof of the inequality (*) we proceed by transfinite induction on the height $h(t)$. This is an ordinal number defined by $h(x) = 0$ for variables x , and $h(\sigma(t_i)_{i < n}) = 1 + \bigvee_{i < n} h(t_i)$. \square

Theorem 7.6. *Every variety of λ -ary ultra-quantitative algebras is concretely isomorphic to $\mathbf{UMet}^{\mathbf{T}}$ for a strongly λ -accessible monad \mathbf{T} .*

The proof is completely analogous to that of Corollary 4.33: we construct a weighted diagram in $\mathbf{Mnd}_\lambda(\mathbf{UMet})$ as in Construction 4.29. Its colimit \mathbf{T} exists in $\mathbf{Mnd}_\lambda(\mathbf{UMet})$ by Proposition 3.12. And the proof that \mathbf{T} is the free-algebra monad for \mathcal{V} follows using Lemma 7.5, the same steps as the proof of Theorem 4.30.

Construction 7.7. Given a strongly λ -accessible monad \mathbf{T} on \mathbf{UMet} we define a λ -ary signature Σ by $\Sigma_n = |Tn|$ for all cardinals $n < \lambda$. The variety associated with \mathbf{T} is given by equations (1)-(3) of Construction 5.3 (where n and m range over cardinals smaller than λ).

Theorem 7.8. *Every strongly λ -accessible monad on \mathbf{UMet} is the free-algebra monad of its associated variety.*

The proof is analogous to that of Theorem 5.6.

Corollary 7.9. The following categories are dually equivalent:

- (1) varieties of λ -ary ultra-quantitative algebras (and concrete functors), and
- (2) strongly λ -accessible monads on \mathbf{UMet} (and monad morphisms).

Remark 7.10. Assuming that strongly λ -accessible monads on \mathbf{Met} are closed under composition, the analogous corollary holds for them: they correspond bijectively to varieties of λ -ary quantitative algebras.

8. λ -BASIC MONADS AND λ -BASIC VARIETIES

In this section we consider more general equations than those in Section 7: λ -basic quantitative equations introduced by Mardare et al. [23]. And we show that the corresponding varieties of quantitative algebras bijectively correspond to monads on \mathbf{Met} which are λ -basic, i.e. enriched, λ -accessible and preserving surjective morphisms. For $\lambda = \aleph_1$ this has been proved in [2], the proof for $\lambda > \aleph_1$ is completely analogous, we thus only shortly indicate it here. The case $\lambda = \aleph_0$ is an open problem that we discuss below.

The idea of λ -basic equations is simple and natural: in Section 7 we have considered equations $l =_\varepsilon r$ where l and r are terms in $T_\Sigma V$, the discrete free algebra on the set of variables. Why restrict ourselves to discrete free algebras? We can also consider $l =_\varepsilon r$ for pairs l, r of elements of $T_\Sigma M$ for an *arbitrary* metric space M . Now restricting to M of power less than λ (the arity of the signature Σ) does not make any difference: every metric space M_0 is a λ -directed colimit of its subspaces $M \subseteq M_0$ of power less than λ , and T_Σ preserves directed colimits. Thus every pair $l, r \in T_\Sigma M_0$ lies in $T_\Sigma M$ with $\text{card } M < \lambda$. This leads us to the following definition, formulated in [23] for $\lambda = \aleph_0$ or \aleph_1 .

Assumption 8.1. Recall Assumption 7.1. Further recall Notation 4.10 $f^\# : T_\Sigma M \rightarrow A$ for the unique homomorphism extending $f : M \rightarrow A$. For the rest of our paper λ denotes an uncountable regular cardinal. And V is a chosen set of variables of cardinality λ .

Definition 8.2 ([23]). Let Σ be a signature of arity λ .

- (1) A λ -basic quantitative equation is an expression

$$M \vdash l =_\varepsilon r$$

where M is a metric space (of variables) with $\text{card } M < \lambda$, and l, r are terms in $T_\Sigma M$. For $\lambda = \aleph_0$ we speak about ω -basic equations.

- (2) A quantitative algebra A *satisfies* this equation if every nonexpanding interpretation of variables $f : M \rightarrow A$ fulfils

$$d(f^\sharp(l), f^\sharp(r)) \leq \varepsilon.$$

Example 8.3.

- (1) *Quasi-commutative monoids* are quantitative monoids in which the commutative law holds for all pairs of distance at most 1. They can be presented by the usual monoid equations plus the following ω -basic equation

$$x =_1 y \vdash x * y = y * x.$$

More precisely: the left-hand side is the space $M = \{x, y\}$ with $d(x, y) = 1$.

- (2) *Almost quasi-commutative monoids* (compare Example 4.14 (2)): here we only request that the distance of ab and ba is at most ε for pairs of distance at most 1:

$$x =_1 y \vdash x * y =_\varepsilon y * x.$$

- (3) *Quasi-discrete spaces* (see [26]) are metric spaces with all non-zero distances greater than 1. Here we put $\Sigma = \emptyset$ and consider the ω -basic equation

$$x =_1 y \vdash x = y.$$

Remark 8.4.

- (1) The above ω -basic equations work with finite spaces M . Suppose we list all pairs of distinct elements as $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$ and denote by δ_i the distance $d(x_i, y_i)$ in M . Then, as we have seen in the above examples, we can rewrite $M \vdash l =_\varepsilon r$ in the form

$$(x_0 =_{\delta_0} y_0), \dots, (x_{n-1} =_{\delta_{n-1}} y_{n-1}) \vdash l =_\varepsilon r.$$

This is the syntax presented in [23]. In fact, in that paper the syntax is a bit more general: the left-hand side of \vdash need not be connected to a concrete space M . However, this does not influence the expressivity of the ω -basic equations, see Remark 8.20 below.

- (2) Analogously, ω_1 -basic equations, working with countable spaces M , can be rewritten in the form

$$(+) \quad (x_0 =_{\delta_0} y_0), (x_1 =_{\delta_1} y_1), (x_2 =_{\delta_2} y_2), \dots \vdash l =_\varepsilon r.$$

Definition 8.5. Given a λ -ary signature Σ , by a λ -*basic variety* is meant a full subcategory of $\Sigma\text{-Met}$ which can be presented by a set of λ -basic quantitative equations.

For $\lambda = \omega$ or ω_1 this definition stems from [23]. There it is also proved that every λ -basic variety has free algebras (their proof works for all λ). We can thus introduce the corresponding notation:

Notation 8.6. For every λ -basic variety \mathcal{V} we denote by $\mathbf{T}_{\mathcal{V}}$ the free-algebra monad on \mathbf{Met} .

The following Birkhoff Variety Theorem was also formulated in [23]. The proof there is not correct. See B19-B20 in the paper of Milius and Urbat [24] for a correct proof.

Definition 8.7 ([23]). An epimorphism $e : X \rightarrow Y$ in \mathbf{Met} is called λ -*reflexive* provided that for every subspace $Y_0 \subseteq Y$ with $\text{card } Y_0 < \lambda$ there exists a subspace $X_0 \subseteq X$ such that e restricts to an isomorphism $e_0 : X_0 \xrightarrow{\sim} Y_0$ in \mathbf{Met} .

Theorem 8.8 (Birkhoff Variety Theorem). *Let Σ be a signature of arity λ . A full subcategory of $\Sigma\text{-Met}$ is a λ -variety iff it is closed under products, subalgebras and λ -reflexive homomorphic images.*

Proposition 8.9. *Every λ -variety \mathcal{V} of quantitative algebras is concretely isomorphic to $\mathbf{Met}^{\mathbf{T}_{\mathcal{V}}}$.*

This follows from [27], Section 4.

Proposition 8.10. *For every λ -basic variety the monad $\mathbf{T}_{\mathcal{V}}$ is enriched and preserves surjective morphisms.*

Proof.

- (1) For $\mathcal{V} = \Sigma\text{-Alg}$ the corresponding monad \mathbf{T}_{Σ} has both properties. Indeed, $T_{\Sigma}M$ is the algebra of terms precisely as in the finitary case (Example 4.5). It follows that T_{Σ} is a coproduct of functors $(-)^n$ for $n < \lambda$, one for every similarity class of terms in $T_{\Sigma}V$ (compare Corollary 4.31). Due to the Birkhoff Variety Theorem for every space M there is a reflection $\varepsilon_M : T_{\Sigma}M \rightarrow T_{\mathcal{V}}M$ of the free Σ -algebra in \mathcal{V} , and ε_M is surjective.
- (2) We obtain a monad morphism $\varepsilon : \mathbf{T}_{\Sigma} \rightarrow \mathbf{T}_{\mathcal{V}}$ with surjective components. This implies that $\mathbf{T}_{\mathcal{V}}$ inherits the above two properties from \mathbf{T}_{Σ} .

□

Definition 8.11. An (enriched) monad on \mathbf{Met} is called λ -*basic* if it is λ -accessible and preserves surjective morphisms.

Theorem 8.12. *Let λ be an uncountable regular cardinal. Then the free-algebra monad of every λ -variety is λ -basic.*

Proof.

- (1) We only need to verify that $\mathbf{T}_{\mathcal{V}}$ is λ -accessible. This is easy to see in case $\mathcal{V} = \Sigma\text{-Alg}$ since every term in $T_{\Sigma}M$ contains less than λ variables. Thus the forgetful functor $U : \Sigma\text{-Alg} \rightarrow \mathbf{Met}$ preserves λ -filtered colimits, using Corollary 2.13. Since its left adjoint F preserves colimits, $T_{\Sigma} = UF$ is λ -accessible.

- (2) We verify that every λ -basic variety \mathcal{V} is closed under λ -filtered colimits in $\Sigma\text{-Met}$. To prove this, consider a single λ -basic equation $M \vdash l =_\varepsilon r$. We verify that given a λ -directed diagram of quantitative algebras A_i ($i \in I$) satisfying that equation, it follows that $A = \operatorname{colim}_{i \in I} A_i$ also satisfies it.

For every interpretation $f : M \rightarrow A$ we have, since M is λ -presentable in the enriched sense (Proposition 2.14), a factorization g through the colimit map $c_i : A_i \rightarrow A$ for some $i \in I$ in Met :

$$\begin{array}{ccc}
 & & A_i \\
 & \nearrow g & \downarrow c_i \\
 M & \xrightarrow{f} & A
 \end{array}$$

Then $f^\sharp = c_i \cdot g^\sharp$ because c_i is a morphism in $\Sigma\text{-Met}$. Since A_i satisfies $M \vdash l =_\varepsilon r$, we have $d(g^\sharp(l), g^\sharp(r)) \leq \varepsilon$. This implies $d(f^\sharp(l), f^\sharp(r)) \leq \varepsilon$ because c_i is nonexpanding.

- (3) Thus, the forgetful functor $U_{\mathcal{V}} : \mathcal{V} \rightarrow \text{Met}$ preserves λ -filtered colimits, too. Since $T_{\mathcal{V}} = U_{\mathcal{V}} \cdot F_{\mathcal{V}}$ where $F_{\mathcal{V}} \dashv U_{\mathcal{V}}$ preserves all colimits, this finishes the proof. □

Example 8.13 ([2]). Unfortunately, for ω -varieties \mathcal{V} the monads $\mathbf{T}_{\mathcal{V}}$ are not finitary in general. A simple example of this are the quasi-discrete spaces (Example 8.3 (3)). We demonstrate that $\mathbf{T}_{\mathcal{V}}$ does not preserve colimits of ω -chains of subspaces. Denote by A_n the subspace of the real line consisting of -1 and 2^{-k} for $k = 0, \dots, n$. Form the ω -chain of inclusions $A_n \hookrightarrow A_{n+1}$. Its colimit is the subspace $A = \{-1\} \cup \{2^{-k} \mid k \in \mathbb{N}\}$. The functor $T_{\mathcal{V}}$ assigns to $A_0 = \{-1, 1\}$ the 2-element space with distance 2, to $A_1 = \{-1, \frac{1}{2}, 1\}$ the 2-element space with distance $\frac{3}{2}$ etc. We see that in $T_{\mathcal{V}}A_n$ the two elements have distance $1 + 2^{-n}$. Consequently, $\operatorname{colim}_{n < \omega} T_{\mathcal{V}}A_n$ is the 2-element space with distance 1. This space does not lie in \mathcal{V} , thus it is not $T_{\mathcal{V}}(\operatorname{colim} A_n)$. In fact, the last space has one element.

Notation 8.14. In the following construction Met_{λ} is a set of metric spaces representing all spaces M with $\operatorname{card} |M| < \lambda$ up to isomorphism. We assume for simplicity that Met_{λ} contains the discrete spaces $V_k = \{x_n \mid n < k\}$ for all cardinals $k < \lambda$.

Construction 8.15 ([2]). Let λ be a regular cardinal and \mathbf{T} a λ -basic monad. We define a λ -basic variety of algebras of the following signature Σ :

$$\Sigma_n = |TV_n| \text{ for all cardinals } n < \lambda.$$

It is presented by the following λ -basic equations where n and m denote arbitrary cardinals smaller than λ (analogous to Construction 5.3):

- (1) $M \vdash l =_\varepsilon r$ for all $M \in \mathbf{Met}_\lambda$ and $l, r \in |TM|$ of distance ε in TM ,
- (2) $k^*(\sigma) = \sigma(k(x_i))$ for all $\sigma \in TV_n$ and all maps $k : V_n \rightarrow |TV_m|$,
- (3) $\eta_{V_n}(x_i) = x_i$ for all $n < \lambda$ and $i < n$.

Theorem 8.16 ([2]). *Every λ -basic monad on \mathbf{Met} is the free-algebra monad of the λ -basic variety of the above construction.*

For $\lambda = \aleph_0$ or \aleph_1 this has been proved in [2], Theorem 3.11. The general case is completely analogous.

Theorem 8.17. *For every uncountable regular cardinal λ the following categories are dually equivalent:*

- (1) λ -basic varieties of quantitative algebras (and concrete functors),
and
- (2) λ -basic monads (and monad morphisms).

This follows from Theorems 8.12 and 8.16 precisely as we have seen it in Theorem 5.8.

Remark 8.18. The development above works completely analogously for the full subcategory \mathbf{UMet} . Thus λ -basic varieties of ultra-quantitative algebras bijectively correspond to λ -basic monads on \mathbf{UMet} .

Open Problem 8.19. Which monads correspond to ω -basic varieties?

As we have seen in Example 8.13 these monads are not finitary, they can even fail to preserve directed colimits of isometric embeddings. A partial solution has been presented in [2]: ω -basic varieties are, up to isomorphism, precisely the categories $\mathbf{UMet}^{\mathbf{T}}$, where \mathbf{T} is an enriched monad preserving surjective morphisms and directed colimits of split subobjects.

Remark 8.20. We now show that, for signatures of arity \aleph_1 , as considered in [23], our ω_1 -basic equations are precisely as expressive as the formulas (+) of Remark 8.4 (2). An algebra A satisfies the latter formula provided that every interpretation $f : V \rightarrow A$ of variables for which $d(f(x_i), f(y_i)) \leq \delta_i$ holds ($i \in \mathbb{N}$) fulfils $d(f^\sharp(l), f^\sharp(r)) \leq \varepsilon$.

Below we work with (extended) *pseudometrics*: we drop the condition that $d(x, y) = 0$ implies $x = y$.

Construction 8.21. Let Σ be an \aleph_1 -ary signature. For every formula (+) of Remark 8.4 (2) denote by \widehat{d} the smallest pseudometric on the set $V = \{x_i \mid i \in \mathbb{N}\}$ of variables satisfying $\widehat{d}(x_i, x_j) \leq \delta_i$ for $i, j \in \mathbb{N}$. Let

$$e : (V, \widehat{d}) \rightarrow M$$

be the metric reflection: the quotient of (V, \widehat{d}) modulo the equivalence merging $z, z' \in V$ iff $\widehat{d}(z, z') = 0$. Put $\bar{l} = T_\Sigma e(l)$ and $\bar{r} = T_\Sigma e(r)$ to

obtain the following ω_1 -basic equation:

$$M \vdash \bar{l} =_\varepsilon \bar{r}.$$

Lemma 8.22. *An algebra A satisfies $M \vdash \bar{l} =_\varepsilon \bar{r}$ iff it satisfies the formula (+).*

Proof. Every interpretation $f : V \rightarrow A$ with $d(f(x_i), f(y_i)) \leq \delta_i$ ($i \in \mathbb{N}$) is a nonexpanding map $f : (V, \widehat{d}) \rightarrow (A, d)$. Indeed, define a pseudo-metric d_0 on V by $d_0(z, z') = d(f(z), f(z'))$. Then $\widehat{d} \leq d_0$ by the minimality of \widehat{d} . Thus

$$\widehat{d}(z, z') \leq d_0(z, z') = d(f(z), f(z')).$$

Therefore we have a unique nonexpanding map $g : M \rightarrow (A, d)$ with $f = g \cdot e$:

$$\begin{array}{ccc}
 T_\Sigma V & \xrightarrow{T_\Sigma e} & T_\Sigma M \\
 \eta_V \swarrow & & \searrow \eta_M \\
 V & \xrightarrow{e} & M \\
 f^\# \swarrow & f \searrow & \swarrow g \\
 & & A \\
 & & \nwarrow g^\#
 \end{array}$$

As $T_\Sigma e$ is a nonexpanding homomorphism, we conclude that $f^\# = g^\# \cdot T_\Sigma e$.

If A satisfies $M \vdash \bar{l} =_\varepsilon \bar{r}$, then we get $d(g^\#(\bar{l}), g^\#(\bar{r})) \leq \varepsilon$ which implies $d(f^\#(\bar{l}), f^\#(\bar{r})) \leq \varepsilon$. Conversely, if A satisfies the formula (+), then for every nonexpanding map $g : M \rightarrow A$ we put $f = g \cdot e$ and define d_0 as above. Since both e and g are nonexpanding and $\widehat{d}(x_i, y_i) \leq \delta_i$, we conclude $d(f(x_i), f(y_i)) \leq \delta_i$ for $i \in \mathbb{N}$. Thus $d(f^\#(\bar{l}), f^\#(\bar{r})) \leq \varepsilon$. From $f^\# = T_\Sigma e \cdot g^\#$ we get $d(g^\#(\bar{l}), g^\#(\bar{r})) \leq \varepsilon$. \square

9. ACCESSIBLE MONADS AND GENERALIZED VARIETIES

We now turn to general accessible enriched monads on \mathbf{Met} : they bijectively correspond to generalized varieties of quantitative algebras. Recall our standing assumption that an uncountable regular cardinal λ is given. Our signature Σ here will be more general than in Sections 7 and 8: arities are spaces, not cardinals. Our treatment in this section follows ideas of Kelly and Power [18] who showed how to present λ -accessible monads on locally λ -presentable categories. (They treated the case $\lambda = \aleph_0$ only, but their results immediately generalize to all λ .) The basic difference of our approach is that our generalized signatures Σ are collections $(\Sigma_M)_{M \in \mathbf{Met}_\lambda}$ of sets Σ_M of operation symbols of arity M , whereas in loc. cit. Σ_M are arbitrary metric spaces. This explains

why plain equations were sufficient in loc. cit., whereas we are going to apply λ -basic equations analogous to Section 8. Recall \mathbf{Met}_λ from Notation 8.14 and recall that $[M, A]$ is the space of all morphisms $f : M \rightarrow A$ with the supremum metric.

Definition 9.1.

- (1) A *generalized signature* of arity λ is a collection $\Sigma = (\Sigma_M)_{M \in \mathbf{Met}_\lambda}$ of sets Σ_M of operation symbols of arity M .
- (2) A *quantitative algebra* is a metric space A together with nonexpanding maps

$$\sigma_A : [M, A] \rightarrow A \text{ for all } \sigma \in \Sigma_M.$$

A *homomorphism* to an algebra B is a nonexpanding map $h : A \rightarrow B$ such that the squares below

$$\begin{array}{ccc} [M, A] & \xrightarrow{\sigma_A} & A \\ \downarrow h \cdot (-) & & \downarrow h \\ [M, B] & \xrightarrow{\sigma_B} & B \end{array}$$

commute for all $M \in \mathbf{Met}_\lambda$ and $\sigma \in \Sigma_M$. The category of Σ -algebras is again denoted by $\Sigma\text{-Met}$.

Example 9.2. Let $P = \{x, y\}$ be the space in \mathbf{Met}_λ with $d(x, y) = 1$. Denote by Σ the generalized signature with

$$\Sigma_P = \{\sigma\} \text{ and } \Sigma_{\mathbf{1}} = \{s\}$$

(where $\mathbf{1}$ is the singleton space), while all the other sets Σ_M are empty. An algebra is a metric space A together with an element $s_A \in A$ and a partial binary operation

$$\sigma_A : A \times A \rightarrow A$$

with $\sigma(a_1, a_2)$ defined iff $d(a_1, a_2) \leq 1$.

Remark 9.3. The forgetful functor $U : \Sigma\text{-Met} \rightarrow \mathbf{Met}$ has a left adjoint analogous to Example 4.5. For every space X we define the free algebra $T_\Sigma X$ on it as the algebra of *terms*. This is the smallest metric space such that

- (1) X is a subspace of $T_\Sigma X$, and
- (2) given σ of arity M and an $|M|$ -tuple of terms t_i ($i \in M$) such that the map $M \rightarrow T_\Sigma X$ given by $i \mapsto t_i$ is nonexpanding, we obtain a composite term $\sigma(t_i)_{i \in M}$.

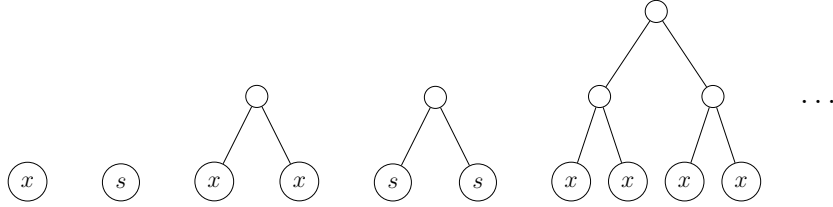
Recall the similarity of terms from Example 4.5. The distance of non-similar terms is again ∞ , and for similar terms $\sigma(t_i)_{i \in M}$ and $\sigma(t'_i)_{i \in M}$ it is the supremum of $\{d(t_i, t'_i) \mid i \in |M|\}$. The proof that $T_\Sigma X$ above

is well defined and forms the free algebra on X is analogous to Proposition 4.9: we just need to define the polynomial functor by

$$H_\Sigma X = \prod_{\sigma \in \Sigma} [M, X]$$

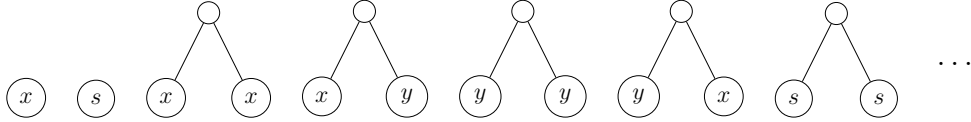
for the arity M of σ .

Example 9.4. Let Σ be the generalized signature of Example 9.2. We first describe $T_\Sigma X$ for discrete spaces X . We can identify elements $x \in X$ with singleton trees labelled by x , and composite terms $t = \sigma(t_l, t_r)$ with binary ordered trees having the maximum subtrees t_l and t_r . We see that $T_\Sigma X$ consists of all finite uniform binary trees (with the left-hand subtree similar to the right-hand one) where all the leaves are labelled by the same element x of $X + \{s\}$:



And the space $T_\Sigma X$ is discrete.

For the space $X = \{x, y\}$ with $d(x, y) = \frac{1}{2}$ the algebra $T_\Sigma X$ consists of all uniform binary trees where the leaves are either labelled by elements of X , or they all have the label s :



Definition 9.5. Let Σ be a generalized λ -ary signature. A λ -basic quantitative equation is, as in Definition 8.2, an expression $M \vdash l =_\varepsilon r$ where $M \in \text{Met}_\lambda$ and $l, r \in |T_\Sigma M|$. An algebra A satisfies it if given a nonexpanding interpretation $f : M \rightarrow A$, we have

$$d(f^\#(l), f^\#(r)) \leq \varepsilon.$$

A *generalized variety* is a full subcategory of $\Sigma\text{-Met}$ which can be presented by a set of λ -basic equations.

Example 9.6. Generalized varieties are a proper super-class of λ -basic varieties. For example, the generalized variety $\Sigma\text{-Met}$ of Example 9.2 (with no equations) is not a λ -basic variety. To see this, let \mathbf{T} be the monad of free Σ -algebras. We have seen in Proposition 8.10 that for every λ -basic variety the corresponding monad preserves surjective morphisms. This is not true for generalized varieties. In the above example let A be the discrete space on $\{x, y\}$ and B be the two-element space with distance $\frac{1}{2}$ (see Example 9.4). For the morphism $id : A \rightarrow B$ the map Tid is not surjective.

Remark 9.7. Analogously to Corollary 4.11 the free-algebra monad \mathbf{T}_Σ of a generalized signature is enriched and λ -accessible. Indeed, T_Σ is a coproduct of hom-functors of spaces of cardinality less than λ : one summand for every similarity class of terms of arity M . By Proposition 2.14 each $[M, -]$ is (enriched and) λ -accessible.

Proposition 9.8. *Every generalized variety \mathcal{V} has free algebras. The corresponding monad $\mathbf{T}_\mathcal{V}$ on \mathbf{Met} is enriched and λ -accessible.*

Proof. We prove below that \mathcal{V} is closed under products, subalgebras, and λ -directed colimits in $\Sigma\text{-Met}$. Then \mathcal{V} has free algebras; this is analogous to Corollary 4.19. Moreover, it follows that the canonical monad morphism $\mathbf{T}_\Sigma \rightarrow \mathbf{T}_\mathcal{V}$ has surjective components, thus $\mathbf{T}_\mathcal{V}$ is enriched (since \mathbf{T}_Σ is). Since \mathbf{T}_Σ is λ -accessible, closure under λ -directed colimits implies that $\mathbf{T}_\mathcal{V}$ is also λ -accessible.

- (1) Closure under products: If algebras A_i ($i \in I$) satisfy $M \vdash l =_\varepsilon r$, then so does $A = \prod_{i \in I} A_i$. Indeed, for each interpretation $f = \langle f_i \rangle_{i \in I} : M \rightarrow A$ we know that $d(f_i^\sharp(l), f_i^\sharp(r)) \leq \varepsilon$. Moreover $f^\sharp = \langle f_i^\sharp \rangle$ since every projection merges both sides. This proves

$$d(f^\sharp(l), f^\sharp(r)) = \sup_{i \in I} d(f_i^\sharp(l), f_i^\sharp(r)) \leq \varepsilon.$$

- (2) Closure under subalgebras: Let B be a subalgebra of A via an isometric embedding $m : B \rightarrow A$. If A satisfies $M \vdash l =_\varepsilon r$, so does B . Indeed, for every interpretation $f : M \rightarrow B$ we have $m \cdot f^\sharp = (m \cdot f)^\sharp$ since m is a homomorphism. We know that $d((m \cdot f)^\sharp(l), (m \cdot f)^\sharp(r)) \leq \varepsilon$. Since m preserves distances, this implies $d(f^\sharp(l), f^\sharp(r)) \leq \varepsilon$.
- (3) Closure under λ -directed colimits: Let A_i ($i \in I$) be algebras of a directed diagram over (I, \leq) with colimit A , with each of A_i satisfying $M \vdash l =_\varepsilon r$. For every interpretation $f : M \rightarrow A$, since the metric space M is λ -presentable (Proposition 2.14), there exists $i \in I$ and a nonexpanding map $\bar{f} : M \rightarrow A_i$ with $f = c_i \cdot \bar{f}$. We have $f^\sharp = c_i \cdot \bar{f}^\sharp$ because c_i is a homomorphism. And we know that $d(\bar{f}^\sharp(l), \bar{f}^\sharp(r)) \leq \varepsilon$. As c_i is nonexpanding, this proves $d(f^\sharp(l), f^\sharp(r)) \leq \varepsilon$.

□

Proposition 9.9. *Every generalized variety \mathcal{V} is concretely isomorphic to $\mathbf{Met}^{\mathbf{T}_\mathcal{V}}$, the Eilenberg-Moore category of its free-algebra monad.*

The proof is analogous to that of Proposition 4.24.

We now consider the converse direction: from accessible monads to generalized varieties. For this we use the work of Kelly and Power that we first shortly recall.

Remark 9.10. In Section 5 of [18] a presentation of enriched finitary monads on \mathcal{A} by equations is exhibited. Here \mathcal{A} is a locally

finitely presentable category (in the enriched sense). All of that section immediately generalizes to enriched λ -accessible monads on a locally λ -presentable category. For our given (uncountable) cardinal λ this specializes to \mathbf{Met} (see Corollary 2.17) as follows:

- (1) Recall \mathbf{Met}_λ from Notation 3.2 and denote by

$$\overline{K} : \overline{\mathbf{Met}_\lambda} \hookrightarrow \mathbf{Met}_\lambda$$

the full embedding of the discrete category on the same objects (\mathcal{A} is used in place of $\overline{\mathbf{Met}_\lambda}$ in [18]). We work with the functor category $[\overline{\mathbf{Met}_\lambda}, \mathbf{Met}]$. An object $B : \overline{\mathbf{Met}_\lambda} \rightarrow \mathbf{Met}$ can be considered as a collection of metric spaces $B(M)$ indexed by $M \in \overline{\mathbf{Met}_\lambda}$. In this sense every generalized signature is simply an object B of $[\overline{\mathbf{Met}_\lambda}, \mathbf{Met}]$ with all spaces $B(M)$ discrete.

- (2) The functor category $[\mathbf{Met}_\lambda, \mathbf{Met}]$ is equivalent to the category of all λ -accessible endofunctors on \mathbf{Met} . We thus have an obvious forgetful functor

$$W : \mathbf{Mnd}_\lambda(\mathbf{Met}) \rightarrow [\mathbf{Met}_\lambda, \mathbf{Met}],$$

assigning to a λ -accessible monad its underlying endofunctor. We also have the forgetful functor

$$V : [\mathbf{Met}_\lambda, \mathbf{Met}] \rightarrow [\overline{\mathbf{Met}_\lambda}, \mathbf{Met}]$$

given by precomposition with \overline{K} . The composite $V \cdot W$ assigns to a monad \mathbf{T} the collection $(TM)_{M \in \mathbf{Mnd}_\lambda}$.

- (3) The functor V has a left adjoint

$$G \dashv V$$

given by the left Kan extension along $\overline{K} : \overline{\mathbf{Met}_\lambda} \hookrightarrow \mathbf{Met}_\lambda$:

$$GB = \text{Lan}_{\overline{K}} B \overline{K}.$$

It assigns to $B : \overline{\mathbf{Met}_\lambda} \rightarrow \mathbf{Met}$ the functor

$$GB = \coprod_{M \in \mathbf{Met}_\lambda} B(M) \otimes [M, -].$$

Theorem 9.11 ([20]). *The forgetful functor*

$$V \cdot W : \mathbf{Mnd}_\lambda(\mathbf{Met}) \rightarrow [\overline{\mathbf{Met}_\lambda}, \mathbf{Met}]$$

is monadic.

Notation 9.12. The left adjoint $F \dashv V \cdot W$ assigns to $B : \overline{\mathbf{Met}_\lambda} \rightarrow \mathbf{Met}$ the free monad on the λ -accessible endofunctor of \mathbf{Met} corresponding to GB .

Corollary 9.13. Every enriched λ -accessible monad \mathbf{T} is a coequalizer, in $\mathbf{Mnd}_\lambda(\mathbf{Met})$, of a parallel pair

$$(KP1) \quad \begin{array}{ccc} & \xrightarrow{\sigma} & \\ FB' & \circlearrowright & FB \\ & \xleftarrow{\tau} & \end{array} \xrightarrow{\rho} \mathbf{T}$$

of monad morphisms σ, τ (for some $B', B : \overline{\mathbf{Met}}_\lambda \rightarrow \mathbf{Met}$).

Remark 9.14. Under the adjunction $F \dashv V \cdot W$ the monad morphisms σ, τ correspond to $\sigma', \tau' : F \rightarrow VWFB$. Kelly and Power deduce that the category $\mathbf{Met}^{\mathbf{T}}$ can be identified with the full subcategory of \mathbf{Met}^{FB} on algebras $\alpha : (FB)A \rightarrow A$ that satisfy the following quantitative equations

$$(KP2) \quad M \vdash \sigma'_M(x) = \tau'_M(x) \text{ for all } M \in \mathbf{Met}_\lambda \text{ and } x \in B'(M).$$

Satisfaction means that for every morphism $f : M \rightarrow A$ we have

$$f^\sharp(\sigma'_M(x)) = f^\sharp(\tau'_M(x)) \text{ for } f^\sharp = \alpha \cdot (FB)f : (FB)M \rightarrow A.$$

We use this to derive the corresponding presentation by generalized signatures and quantitative equations:

Theorem 9.15. *Every λ -accessible enriched monad on \mathbf{Met} is the free-algebra monad of a generalized variety.*

Proof.

- (1) We first associate with every functor $B : \overline{\mathbf{Met}}_\lambda \rightarrow \mathbf{Met}$ a generalized signature Σ whose M -ary symbols are the elements of BM :

$$\Sigma_M = |BM| \text{ for } M \in \mathbf{Met}_\lambda.$$

The free Σ -algebra $T_\Sigma X$ on a space X is, by Remark 4.8, the colimit $T_\Sigma X = \text{colim}_{i < \lambda} W_i$ of the λ -chain W_i where

$$W_{i+1} = \coprod_{M \in \mathbf{Met}_\lambda} \Sigma_M \times [M, X].$$

Whereas the free algebra $(GB)X$ is, by the same Remark, the colimit of a λ -chain W_i ($i < \lambda$) where

$$W'_{i+1} = \coprod_{M \in \mathbf{Met}_\lambda} B(M) \otimes [M, X].$$

We conclude that the underlying sets of the chains $(W_i)_{i < \lambda}$ and $(W'_i)_{i < \lambda}$ are the same, thus

$$|T_\Sigma X| = |(GB)X|.$$

- (2) For every functor $B : \overline{\mathbf{Met}}_\lambda \rightarrow \mathbf{Met}$ we now show that \mathbf{Met}^{FB} is concretely equivalent to a generalized variety of Σ -algebras. Since FB is the free monad on $H = GB$, we can work with H -algebras

in place of \mathbf{Met}^{FB} (Remark 4.8). To give a H -algebra on a space A means to give a morphism

$$\coprod_{M \in \mathbf{Met}_\lambda} B(M) \otimes [M, A] \rightarrow A.$$

Or, equivalently, a collection of morphisms

$$\alpha_M : B(M) \rightarrow [[M, A], A]$$

for $M \in \mathbf{Met}_\lambda$. Thus α_M assigns to every $\sigma \in \Sigma_M = |BM|$ an operation

$$(KP3) \quad \sigma_A = \alpha_M(\sigma) : [M, A] \rightarrow A$$

of arity M . Moreover, α_M is nonexpanding: for $\sigma, \tau \in \Sigma_M$ we have

$$d(\sigma, \tau) = \varepsilon \text{ in } B(M) \text{ implies } d(\sigma_A, \tau_A) \leq \varepsilon.$$

In other words, the Σ -algebra A satisfies the λ -basic equations

$$(KP4) \quad M \vdash \sigma =_\varepsilon \tau \text{ for } M \in \mathbf{Met}_\lambda \text{ and } d(\sigma, \tau) = \varepsilon \text{ in } B(M).$$

Conversely, every Σ -algebra A satisfying (KP3) corresponds to a unique H -algebra on the space A with α_M defined by (KP3). We conclude that \mathbf{Met}^{FB} is concretely equivalent to the generalized variety presented by (KP4).

- (3) Let \mathbf{T} be an enriched λ -accessible monad. In Remark 9.14 we have seen that $\mathbf{Met}^{\mathbf{T}}$ is the subcategory of \mathbf{Met}^{FB} presented by the equations (KP2). Due to (1) both $\sigma'_M(x)$ and $\tau'_M(x)$ are elements of $|T_\Sigma M|$, thus (KP2) are λ -basic equations for Σ in the sense of Definition 9.5. We conclude that \mathbf{T} is the free-algebra monad of the generalized variety presented by (KP2) and (KP4). □

Theorem 9.16. *For every uncountable regular cardinal λ the following categories are dually equivalent:*

- (1) *generalized varieties of λ -ary quantitative algebras (and concrete functors), and*
- (2) *λ -accessible monads on \mathbf{Met} (and monad morphisms).*

This follows from Theorem 9.15 and Propositions 9.9 precisely as in the proof of Theorem 5.8. For $\lambda = \aleph_0$ the corresponding result does not hold: see Example 8.13.

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