WIENER INDEX OF THE COZERO-DIVISOR GRAPH OF A FINITE COMMUTATIVE RING

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ABSTRACT. Let R be a ring with unity. The cozero-divisor graph of a ring R, denoted by $\Gamma'(R)$, is an undirected simple graph whose vertices are the set of all non-zero and non-unit elements of R, and two distinct vertices x and yare adjacent if and only if $x \notin Ry$ and $y \notin Rx$. In this article, we extend some of the results of [24] to an arbitrary ring. In this connection, we derive a closed-form formula of the Wiener index of the cozero-divisor graph of a finite

are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. In this article, we extend some of the results of [24] to an arbitrary ring. In this connection, we derive a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring R. As a applications, we compute the Wiener index of G'(R), when either R is the product of ring of integers modulo n or a reduced ring. At the final part of this paper, we provide a SageMath code to compute the Wiener index of the cozero-divisor graph of these class of rings including the ring \mathbb{Z}_n of integers modulo n.

1. INTRODUCTION AND PRELIMINARIES

The Wiener index is one of the most frequently used topological indices in chemistry as a molecular shape descriptor. This was first used by H. Wiener in 1947 and then the formal definition of the Wiener index was introduced by Hosaya [18]. The Wiener index of a graph is defined as the sum of the lengths of the shortest paths between all pairs of vertices in a graph. Other than the chemistry, the Wiener index was used to find various applications in quantitative structure-property relationships (see [21]). The Wiener index was also employed in crystallography, communication theory, facility location, cryptography etc. (see [14, 17, 26]). An application of the Wiener index has been established in water pipeline network which is essential for water supply management (see [15]). Other utilization of the Wiener index can be found in [16, 20, 30, 31] and reference therein.

The idea of associating a graph with a ring structure was first emerged in [12]. Then various graphs associated with rings have been studied extensively in the literature, viz. inclusion ideal graph [6], total graph [8], zero-divisor graph [9], annihilating-ideal graph [13, 19], co-maximal graph [22], etc. Afkhami et al. [1] introduced the cozero-divisor graph. The cozero-divisor graph of a ring R with unity, denoted by G'(R), is an undirected simple graph whose vertex set is the set of all non-zero and non-unit elements of the work associated with the cozero-divisor graphs of rings can be found in [3, 4, 7, 11, 23, 24, 25].

Over the recent years, the Wiener index of certain graphs associated with rings have been studied by various authors. The Wiener index of the zero divisor graph of the ring \mathbb{Z}_n of integers modulo n has been studied in [10]. Recently, Selvakumar et al. [28] calculated the Wiener index of the zero divisor graph for a finite commutative ring with unity. The Wiener index of the cozero-divisor graph of the ring \mathbb{Z}_n has been obtained in [24]. In order to extend the results of [24] to an arbitrary ring, we study the Wiener index of the cozero-divisor graph of a finite commutative

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ring with unity. First, we provide the necessary results and notations used throughout the paper. The remaining paper is arranged as follows: In Section 2, a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring with unity is presented. In Section 3, we obtain the Wiener index of the cozero-divisor graph of the product of a ring of integers modulo n. In Section 4, we calculate the Wiener index of the cozero-divisor graph of a finite commutative reduced ring. In Section 5, we derive a SageMath code to compute the Wiener index of the cozero-divisor graph of various classes of rings.

Now we recall necessary definitions, results and notations of graph theory from [29]. A graph Γ is a pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ and $E = E(\Gamma)$ are the set of vertices and edges of Γ , respectively. Let Γ be a graph. Two distinct vertices $x, y \in \Gamma$ are adjacent, denoted by $x \sim y$, if there is an edge between x and y. Otherwise, we denote it by $x \sim y$. A subgraph Γ' of a graph Γ is a graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. If $U \subseteq V(\Gamma)$ then the subgraph of Γ induced by U, denoted by $\Gamma(U)$, is the graph with vertex set U and two vertices of $\Gamma(U)$ are adjacent if and only if they are adjacent in Γ . The complement Γ of Γ is a graph with same vertex set as Γ and distinct vertices x, y are adjacent in Γ if they are not adjacent in Γ . A graph Γ is said to be complete if every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n . A path in a graph is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the next vertex of it. The graph Γ is said to be connected if there is path between every pair of vertex. The distance between any two vertices x and y of Γ , denoted by T, where T is a graph T is given by defined as the sum of all distances between every pair of vertices in the graph that is the Wiener index of a graph Γ is given by

$$W(\Gamma) = \frac{1}{2} \sum_{u \in V(\Gamma)} \sum_{v \in V(\Gamma)} d(u, v)$$

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be k pairwise disjoint graphs. Then the generalised join graph $\Gamma[\Gamma_1, \Gamma_2, \ldots, \Gamma_k]$ of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ is the graph formed by replacing each vertex u_i of Γ by Γ_i and then joining each vertex of Γ_i to every vertex of Γ_j whenever $u_i \sim u_j$ in Γ (cf. [27]). The set of zero-divisors and the set of units of the ring R is denoted by Z(R) and U(R), respectively. The set of all nonzero elements of R is denoted by R^* . For $x \in R$, we write (x) as the principal ideal generated by x. For a positive integer k, we write $[k] = \{1, 2, \ldots, k\}$.

2. Formulae for The Wiener index of the cozero-divisor graph of a finite commutative ring

The purpose of this section is to provide the closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring. Let R be a finite commutative ring with unity. Define a relation \equiv on $V(\Gamma'(R))$ such that $x \equiv y$ if and only if (x) = (y). Note that the relation \equiv is an equivalence relation. Let x_1, x_2, \ldots, x_k be the representatives of the equivalence classes of X_1, X_2, \ldots, X_k respectively, under the relation \equiv . We begin with the following lemma.

Lemma 2.1. A vertex of X_i is adjacent to a vertex of X_j if and only if $(x_i) \nsubseteq (x_j)$ and $(x_j) \nsubseteq (x_i)$.

Proof. Suppose $a \in X_i$ and $b \in X_j$. Then $(a) = (x_i)$ and $(b) = (x_j)$ in R. If $a \sim b$ in $\Gamma'(R)$, then $(a) \not\subset (b)$ and $(b) \not\subset (a)$. It follows that $(x_i) \not\subset (x_j)$ and $(x_j) \not\subset (x_i)$. The converse holds by the definition of $\Gamma'(R)$.

Corollary 2.2. (i) For $i \in \{1, 2, ..., k\}$, the induced subgraph $\Gamma'(X_i)$ of $\Gamma'(R)$ is isomorphic to $\overline{K}_{|X_i|}$. (ii) For distinct $i, j \in \{1, 2, ..., k\}$, a vertex of X_i is adjacent to either all or none of the vertices of X_j .

Define a subgraph $\Upsilon'(R)$ (or Υ') induced by the set $\{x_1, x_2, \dots, x_k\}$ of representatives of the respective equivalence classes X_1, X_2, \dots, X_k under the relation \equiv .

Lemma 2.3. The graph $\Upsilon'(R)$ is connected if and only if the cozero-divisor graph $\Gamma'(R)$ is connected. Moreover, for $a, b \in V(\Gamma'(R))$, we have

$$d_{\Gamma'(R)}(a,b) = \begin{cases} 2 & \text{if } a,b \in X_i, \\ d_{\Upsilon'(R)}(x_i,x_j) & \text{if } a \in X_i, b \in X_j \text{ and } i \neq j. \end{cases}$$

Proof. First suppose that $\Upsilon'(R)$ is connected. Let a,b be two arbitrary vertices of $\Gamma'(R)$. We may now suppose that $a \in X_i$ and $b \in X_j$. If i = j, then $a \nsim b$ in $\Gamma'(R)$. Since $\Upsilon'(R)$ is connected, we have $x_t \in X_t$ such that $x_i \sim x_t$ in $\Gamma'(R)$. Consequently, $a \sim x_t \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a,b) = 2$. If $a \sim b$, then there is nothing to prove. Let $a \nsim b$ in $\Gamma'(R)$. Connectedness of $\Upsilon'(R)$ implies that there exists a path $x_i \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim x_j$, where $i \neq j$. It follows that $a \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a,b) = d_{\Upsilon'(R)}(x_i,x_j)$. Therefore, $\Gamma'(R)$ is connected. The converse is straightforward.

In view of Corollary 2.2, we have the following proposition.

Proposition 2.4. Let Γ'_i be the subgraph induced by the set X_i in $\Gamma'(R)$. Then $\Gamma'(R) = \Upsilon'[\Gamma'_1, \Gamma'_2, \dots, \Gamma'_k]$.

Let R be a finite commutative ring with unity. As a consequence of Lemma 2.3 and Proposition 2.4, we have the following theorem.

Theorem 2.5. The Wiener index of the cozero-divisor graph $\Gamma'(R)$ of a finite commutative ring with unity is given by

$$W(\Gamma'(R)) = 2\sum_{\substack{i \neq j \\ 1 \le i < j \le k}} |X_i| |X_j| d_{\Upsilon'(R)}(x_i, x_j),$$

where, x_i is a representative of the equivalence class X_i under the relation \equiv .

In the subsequent sections, we use Theorem 2.5 to derive the Wiener index of the cozero-divisor graph $\Gamma'(R)$ of various class of rings.

3. Wiener Index of the cozero-divisor graph of the product of ring of integers modulo n

In this section, we obtain the Wiener index of the cozero-divisor graph $\Gamma'(R)$, when $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ or $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. For a positive integer n, let d_1, d_2, \ldots, d_t be the proper divisors of n. Define $\mathcal{A}_{d_i} = \{x \in V(\Gamma'(\mathbb{Z}_n) : \gcd(x, n) = d_i\}$. Moreover, $|\mathcal{A}_{d_i}| = \phi(\frac{n}{d_i})$ (cf. [32]) where ϕ is the Euler-totient function. Observe that \mathcal{A}_{d_i} 's are the equivalence classes of the relation \equiv for the ring \mathbb{Z}_n . Now for each $\mathbb{Z}_{p_i^{m_i}}$, let $X_i^0, X_i^1, \ldots, X_i^{m_i}$ be the corresponding equivalence classes, where $X_i^0 = \{0\}$, $X_i^1 = U(\mathbb{Z}_{p_i^{m_i}})$ and $X_i^j = \mathcal{A}_{p^{j-1}}$ for $2 \leq j \leq m_i$. Now we have

$$|X_i^j| = \begin{cases} 1 & \text{if } j = 0, \\ p_i^{m_i} - p_i^{m_i - 1} & \text{if } j = 1, \\ p_i^{m_i - j + 1} - p_i^{m_i - j} & \text{if } 2 \le j \le m_i. \end{cases}$$

Let $x = (x_1, x_2, \dots, x_r, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_r, \dots, y_k) \in R$. Notice that (x) = (y) if and only if $(x_i) = (y_i)$ for each i. It follows that the equivalence classes of the ring R is of the form $X_1^{j_1} \times X_2^{j_2} \times \dots \times X_k^{j_k}$. Consequently, $|X_1^{j_1} \times X_2^{j_2} \times \dots \times X_k^{j_k}| = \prod_{i=1}^k X_i^{j_i}$.

Lemma 3.1. Let $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and let $x = (x_1, x_2, \dots, x_r, \dots, x_k), y = (y_1, y_2, \dots, y_r, \dots, y_k) \in R$. Define $S_r = \{(x, y) : x_r, y_r \in Z(\mathbb{Z}_{n_r})^* \text{ and } (x_r) \subseteq (y_r), x_i = 0, y_i \in U(\mathbb{Z}_{n_i}) \text{ for each } i \neq r\}$. Then

$$d_{\Gamma'(R)}(x,y) = \begin{cases} 1 & \text{if } x \sim y, \\ 2 & \text{if } x \nsim y \text{ and } (x,y) \notin S_r, \\ 3 & \text{if } (x,y) \in S_r. \end{cases}$$

Proof. To prove the result, we discuss the following cases.

Case-1. $x_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then d(x,y) = 1. Otherwise, either $(x) \subseteq (y)$ or $(y) \subseteq (x)$. Suppose that $y_i \in Z(\mathbb{Z}_{n_i})^*$, for each $i \in [k]$. Then for $z = (1,0,\ldots,0) \in R$, we obtain $x \sim z \sim y$ in $\Gamma'(R)$. It follows that d(x,y) = 2. Now assume that $y_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $y_j = 0$ for some $j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $(y_i) \subseteq (x_i)$ for each i. Choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $y_i \in Z(\mathbb{Z}_{n_i})^*$ and $z_j \in U(\mathbb{Z}_{n_j})$ whenever $y_j = 0$, for some $i, j \in [k]$. Consequently, $x \sim z \sim y$ in $\Gamma'(R)$. It follows that d(x,y) = 2. If $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j = 0$, for some $i, j \in [k]$, then note that d(x,y) = 1. Now, let $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_i})^*$, for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $(x_i) \subseteq (y_i)$ for each $i \in [k]$. Choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $y_i \in U(\mathbb{Z}_{n_i})^*$, and $z_j \in U(\mathbb{Z}_{n_j})$ whenever $y_j \in Z(\mathbb{Z}_{n_j})^*$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$ and so d(x,y) = 2. Further, assume that $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Then $x \sim y$ in $\Gamma'(R)$ and so d(x,y) = 1. Case-2. $x_i \in U(\mathbb{Z}_{n_i})$ and $x_j = 0$ for some $i, j \in [k]$. Suppose $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j = 0$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then d(x,y) = 1. Otherwise, choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{when both } x_i = y_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that d(x,y)=2. Further, suppose that $y_i\in Z(\mathbb{Z}_{n_i})$ for each $i\in [k]$ and $y_j=0$ for some $j\in [k]$. If $x\nsim y$ in $\Gamma'(R)$, then choose $z=(z_1,z_2,\ldots,z_k)$ such that $z_i\in U(\mathbb{Z}_{n_i})$ whenever $y_i=0$, and $z_j=0$ whenever $y_j\in Z(\mathbb{Z}_{n_j})^*$. Consequently, d(x,y)=2. Suppose that $y_i\in Z(\mathbb{Z}_{n_i})^*$ and $y_j\in U(\mathbb{Z}_{n_j})$ for some $i,j\in [k]$. If $x\sim y$ in $\Gamma'(R)$, then d(x,y)=1. Otherwise, consider $z=(z_1,z_2,\ldots,z_k)\in R$ such that

$$z_i = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

Note that $x \sim z \sim y$ in $\Gamma'(R)$. It follows that d(x,y) = 2. Assume that $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i \in U(\mathbb{Z}_{n_i})$ whenever $x_i = 0$, and $z_j = 0$ whenever $x_j \in U(\mathbb{Z}_{n_j})$, for some $i, j \in [k]$. Consequently, d(x, y) = 2.

Case-3. $x_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $x_j = 0$ for some $j \in [k]$. Suppose $y_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $y_j = 0$ for some $j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then d(x,y) = 1. Let $x \nsim y$ in $\Gamma'(R)$. Then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$, whenever $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 1$, whenever $x_j = 0$ for some $i, j \in [k]$. It follows that $x \sim z \sim y$ and so d(x,y) = 2. Next, assume that $y_i \in Z(\mathbb{Z}_{n_i})$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ when $x_i = 0$, and $z_j = 0$ when $x_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i, j \in [k]$. Consequently, we have $x \sim z \sim y$ in $\Gamma'(R)$. It implies that d(x,y) = 2. Further, assume that $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i, j \in [k]$. Let $x \nsim y$ in $\Gamma'(R)$. Suppose that there exists $r \in [k]$ such that $x_r \in Z(\mathbb{Z}_{n_r})^*$ and $x_i = 0$ for each $i \in [k] \setminus \{r\}$. Also, $y_i \in U(\mathbb{Z}_{n_i})$ and $y_r \in Z(\mathbb{Z}_{n_r})^*$ for each $i \in [k] \setminus \{r\}$. Then $(x_r) \subsetneq (y_r)$. If there exists $a = (a_1, a_2, \ldots, a_r, \ldots, a_k)$ such that $a \sim y$, then $(y_r) \subsetneq (a_r)$. It follows that $(x_r) \subsetneq (y_r) \subsetneq (a_r)$. Consequently, $a \nsim x$ in $\Gamma'(R)$. Therefore, d(x,y) > 2. Consider $z = (z_1, z_2, \ldots, z_k)$ and $z' = (z'_1, z'_2, \ldots, z'_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{if } x_i = 0, \\ 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \end{cases}$$

and

$$z_i' = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

It follows that $x \sim z \sim z' \sim y$ in $\Gamma'(R)$. Therefore, d(x,y) = 3. Next, we claim that if there exist t and $r \in [k]$ such that $x_t \in Z(\mathbb{Z}_{n_t})^*$, $x_r \in Z(\mathbb{Z}_{n_r})^*$ then $d(x,y) \leq 2$. If $x \sim y$ in $\Gamma'(R)$, then d(x,y) = 1. Next, assume that $x \nsim y$ in $\Gamma'(R)$. Since $x \nsim y$, we have $(x) \subsetneq (y)$. If there exists $i_1 \in [k]$ such that $x_{i_1}, y_{i_1} \in Z(\mathbb{Z}_{n_{i_1}})^*$ then take $r = i_1$. Now consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_t = 0, z_r = 1$ and, for $i \neq \{t, r\}$ whenever $y_i \in U(\mathbb{Z}_{n_i})$ take $z_i = 0$ and, whenever $y_i \in Z(\mathbb{Z}_{n_i})^*$ then choose $z_i = 1$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$. Therefore, $d(x, y) \leq 2$.

Case-4. $x_i \in Z(\mathbb{Z}_{n_i})^*$ and $x_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Let $y_i \in Z(\mathbb{Z}_{n_i})^*$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then d(x,y) = 1. Let $x \nsim y$ in $\Gamma'(R)$. Then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $x_i \in U(\mathbb{Z}_{n_i})$, and $z_j = 1$ whenever $x_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i, j \in [k]$. It follows that d(x,y) = 2. Next, let $y_i \in Z(\mathbb{Z}_{n_i})$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ whenever $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 0$ whenever $x_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Therefore, d(x,y) = 2.

Case-5. $x_i \in Z(\mathbb{Z}_{n_i})$ and $x_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Assume that $y_i \in Z(\mathbb{Z}_{n_i})$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then d(x, y) = 1. Otherwise, choose $z = (z_1, z_2, \ldots, z_k) \in R$ as follows

$$z_i = \begin{cases} 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \text{ and } x_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } x_i = 0. \end{cases}$$

Then $x \sim z \sim y$ in $\Gamma'(R)$. It follows that d(x,y) = 2.

In view of Lemma 3.1, now we calculate the Wiener index of $\Gamma'(R)$. Let $x=(x_1^{j_1},x_2^{j_2},\ldots,x_k^{j_k})$ and $y=(y_1^{l_1},y_2^{l_2},\ldots,y_k^{l_k})$ be the representatives of two distinct equivalence classes $X_1^{j_1}\times X_2^{j_2}\times\cdots\times X_k^{j_k}$ and $X_1^{l_1}\times X_2^{l_2}\times\cdots\times X_k^{l_k}$, respectively.

Theorem 3.2. The Wiener index of the cozero-divisor graph $\Gamma'(R)$, where $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, is given below:

$$\begin{split} W(\Gamma'(R)) &= 2 \sum_{\substack{(x_1^{j_1}, x_2^{j_2}, \dots x_k^{j_k}) \in \Upsilon' \\ (x, y) \notin S_r}} \binom{\prod_{i=1}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i})}{2} + \sum_{x \sim y} \left(\prod_{\substack{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - l_i})\right) \begin{pmatrix} \prod_{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \end{pmatrix} + \sum_{x \sim y} \left(\prod_{\substack{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - l_i})\right) \begin{pmatrix} \prod_{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \end{pmatrix} + \sum_{x \sim y} \left(\prod_{\substack{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i})\right) \begin{pmatrix} \prod_{\substack{i=1 \\ j_i \geq 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \end{pmatrix} + \sum_{x \sim y} \left(\prod_{\substack{i=1 \\ j_i > 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i})\right) \begin{pmatrix} \prod_{\substack{i=1 \\ j_i > 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \end{pmatrix} \right) \begin{pmatrix} \prod_{i=1 \\ j_i > 1}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \end{pmatrix}. \end{split}$$

Thus, the Wiener index of the cozero-divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$ is

 $\{\{Y_1,Y_2\}, \{Y_1,Y_5\}, \{Y_1,Y_6\}, \{Y_1,Y_{10}\}, \{Y_1,Y_{15}\}, \{Y_2,Y_5\}, \{Y_2,Y_6\}, \{Y_2,Y_7\}, \{Y_2,Y_8\}, \{Y_2,Y_{10}\}, \{Y_2,Y_{11}\}, \{Y_2,Y_{15}\}, \{Y_2,Y_{16}\}, \{Y_3,Y_4\}, \{Y_3,Y_5\}, \{Y_3,Y_{12}\}, \{Y_3,Y_{13}\}, \{Y_4,Y_5\}, \{Y_4,Y_6\}, \{Y_4,Y_7\}, \{Y_4,Y_8\}, \{Y_4,Y_{12}\}, \{Y_4,Y_{13}\}, \{Y_4,Y_{14}\}, \{Y_4,Y_{16}\}, \{Y_5,Y_6\}, \{Y_5,Y_7\}, \{Y_5,Y_8\}, \{Y_6,Y_8\}, \{Y_6,Y_{15}\}, \{Y_7,Y_8\}, \{Y_7,Y_{13}\}, \{Y_8,Y_{13}\}, \{Y_8,Y_{15}\}, \{Y_8,Y_{16}\}, \{Y_9,Y_{10}\}, \{Y_9,Y_{11}\}, \{Y_9,Y_{12}\}, \{Y_9,Y_{14}\}, \{Y_9,Y_{15}\}, \{Y_9,Y_{16}\}, \{Y_{10},Y_{11}\}, \{Y_{10},Y_{15}\}, \{Y_{11},Y_{13}\}, \{Y_{11},Y_{15}\}, \{Y_{11},Y_{16}\}, \{Y_{12},Y_{13}\}, \{Y_{12},Y_{14}\}, \{Y_{13},Y_{16}\}, \{Y_{14},Y_{15}\}, \{Y_{14},Y_{16}\}, \{Y_{15},Y_{16}\}, \{Y_{15},Y$

 $W(\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9)) = 2 \times \frac{1}{2} [30 + 2 + 2 + 0 + 132 + 30 + 12 + 2 + 0 + 30 + 2 + 2 + 12 + 0 + 30 + 2]$ + [6(2 + 1 + 4 + 2 + 1 + 2 + 2 + 4 + 1 + 2) + 2(2 + 1 + 1 + 2 + 1) + 2(6 + 2 + 1 + 6 + 2 + 1 + 6 + 2) + (1 + 6 + 2) + 12(1 + 6 + 2 + 2 + 4 + 1 + 6 + 2) + 6(4 + 1 + 6 + 2 + 2 + 4 + 1 + 2) + 4(1 + 6 + 2 + 2 + 1 + 6 + 2) + 2(1 + 6 + 2 + 2 + 1) + (0) + 6(2 + 4 + 1 + 2) + 2(2 + 1) + 2(6 + 2) + 4(6) + (0)] + 2[6(2 + 12 + 6 + 6 + 6) + 2(12 + 6 + 4 + 2 + 6 + 2 + 6 + 2) + 2(1 + 12 + 4 + 2 + 4) + (12 + 6 + 4 + 2 + 2 + 4 + 1 + 2) + 12(6 + 4 + 2) + 6(2 + 6) + 4(2 + 4) + 2(4 + 6 + 2) + (6 + 2 + 2 + 4 + 1 + 6 + 2) + 6(2 + 6) + 2(4 + 6 + 2) + 2(4 + 1) + 4(1 + 2) + (6 + 2) + 6(2)] $+ 3[(2 \times 4) + (1 \times 6)]$ = 2611

4. The Wiener Index of the Cozero-Divisor graph of reduced ring

In this section, we obtain the Wiener index of the cozero-divisor graph of a finite commutative reduced ring. Let R be a reduced ring i.e. $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_k}$ with $k \geq 2$, where F_q is a finite field with q elements. Notice that, for $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k) \in R$ such that (x) = (y), we have $x_i = 0$ if and only if $y_i = 0$ for each i. For $i_1, i_2, \dots, i_r \in [k]$, define

$$X_{\{i_1,i_2,\ldots,i_r\}} = \{(x_1,x_2,\ldots,x_k) \in R : \text{only } x_{i_1},x_{i_2},\ldots,x_{i_r} \text{ are non-zero}\}.$$

Note that the sets X_A , where A is a non-empty proper subset of [k], are the equivalence classes under the relation \equiv . We write x_A by the representative of equivalence class X_A . Now we obtain the possible distances between the vertices of $\Upsilon'(R)$.

Lemma 4.1. For the distinct vertices x_A and x_B of $\Upsilon'(R)$ we have

$$d_{\Upsilon'(R)}(x_A, x_B) = \begin{cases} 1 & \text{if } A \nsubseteq B \text{ and } B \nsubseteq A \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First assume that $A \nsubseteq B$ and $B \nsubseteq A$. Then $(x_A) \nsubseteq (x_B)$ and $(x_B) \nsubseteq (x_A)$. It follows that $d_{\Upsilon'}(x_A, x_B) = 1$. Now without loss of generality let $A \subsetneq B$. Then there exists $i \in [k]$ such that $i \notin B$ and so $i \notin A$. Then by Lemma 2.1, we have $x_A \sim x_{\{i\}} \sim x_B$. Thus, $d_{\Upsilon'}(x_A, x_B) = 2$.

For distinct $A, B \subseteq [k]$, we define $D_1 = \{\{A, B\} : A \subseteq B\}$ and $D_2 = \{\{A, B\} : A \subseteq B\}$. Using Theorem 2.5 and the sets D_1 and D_2 , we obtain the Wiener index of the cozero-divisor $\Gamma'(R)$ of a reduced ring R in the following theorem.

Theorem 4.2. The Wiener index of the cozero-divisor graph of a finite commutative reduced ring $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_k}$, $k \geq 2$, is given by

$$W(\Gamma'(R)) = 2 \sum_{A \subset [k]} \left(\prod_{i \in A} (q_i - 1) \right) + \sum_{\{A, B\} \in D_1} \left(\prod_{i \in A} (q_i - 1) \right) \left(\prod_{j \in B} (q_j - 1) \right) + 2 \sum_{\{A, B\} \in D_2} \left(\prod_{i \in A} (q_i - 1) \right) \left(\prod_{j \in B} (q_j - 1) \right).$$

Proof. The proof follows from Lemma 4.1.

Example 4.3. [24, Corollary 6.2] Let $R = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$, where p, q are distinct prime numbers. Then we have two distinct equivalence classes, $X_{\{1\}} = \{(a,0) : a \in \mathbb{Z}_p \setminus \{0\}\}$ and $X_{\{2\}} = \{(0,b) : b \in \mathbb{Z}_q \setminus \{0\}\}$, of the equivalence relation \equiv . Moreover, $D_1 = \{\{\{1\}, \{2\}\}\}\}$ and $D_2 = \{\}$. Note that $|X_{\{1\}}| = p - 1$ and $|X_{\{2\}}| = q - 1$. Consequently, by Theorem 4.2, we get $W(\Gamma'(\mathbb{Z}_{pq})) = (p-1)(p-2) + (q-1)(q-2) + (p-1)(q-1) = p^2 + q^2 - 4p - 4q + pq + 5$.

Example 4.4. Let $R = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p,q,r are distinct prime numbers. For $a \in \mathbb{Z}_p \setminus \{0\}$, $b \in \mathbb{Z}_q \setminus \{0\}$ and $c \in \mathbb{Z}_q \setminus \{0\}$, we have the equivalence classes : $X_{\{1\}} = \{(a,0,0)\}$, $X_{\{2\}} = \{(0,b,0)\}$, $X_{\{3\}} = \{(0,0,c)\}$, $X_{\{1,2\}} = \{(a,b,0)\}$, $X_{\{1,3\}} = \{(a,0,c)\}$, $X_{\{2,3\}} = \{(0,b,c)\}$. Moreover, $D_1 = \{\{\{1\},\{2\}\},\{\{1\},\{3\}\},\{\{2\},\{3\}\},\{\{1,2\},\{1,3\}\},\{\{1,2\},\{2,3\}\},\{\{1\},\{2,3\}\},\{\{1\},\{2,3\}\},\{\{2\},\{1,3\}\}\}$ and $D_2 = \{\{\{1\},\{1,2\}\},\{\{1\},\{1,3\}\},\{\{2\},\{1,2\}\},\{\{2\},\{2,3\}\},\{\{3\},\{1,3\}\},\{\{3\},\{2,3\}\}\}$. Also, $|X_{\{1\}}| = (p-1)$, $|X_{\{2\}}| = (q-1)$, $|X_{\{3\}}| = (r-1)$, $|X_{\{1,2\}}| = (p-1)(q-1)$, $|X_{\{1,3\}}| = (p-1)(r-1)$. Then, by Theorem 4.2, the Wiener index of Γ'(R) is given by

$$\begin{split} W(\Gamma'(\mathbb{Z}_{pqr})) &= 2\binom{p-1}{2} + 2\binom{q-1}{2} + 2\binom{r-1}{2} + 2\binom{(p-1)(q-1)}{2} + 2\binom{(p-1)(r-1)}{2} + 2\binom{(p-1)(r-1)}{2} + 2\binom{(q-1)(r-1)}{2} \\ &+ (p-1)(q-1) + (p-1)(r-1) + (q-1)(r-1) + (p-1)(q-1)(p-1)(r-1) + (p-1)(q-1)(q-1)(r-1) \\ &+ (p-1)(r-1)(q-1)(r-1) + (p-1)(q-1)(r-1) + (q-1)(p-1)(r-1) + (r-1)(p-1)(q-1) \\ &+ 2(p-1)\left[(p-1)(q-1)\right] + 2(p-1)\left[(p-1)(r-1)\right] + 2(q-1)\left[(p-1)(q-1)\right] + 2(q-1)\left[(q-1)(r-1)\right] \\ &+ 2(r-1)\left[(p-1)(r-1)\right] + 2(r-1)\left[(q-1)(r-1)\right]. \end{split}$$

simplifying this expression, we get

$$W(\Gamma'(\mathbb{Z}_{par})) = pqr(p+q+r-3) + p^2q^2 + p^2r^2 + q^2r^2 - p^2(q+r) - q^2(p+r) - r^2(p+q) - 2(pq+pr+qr) + 4(p+q+r) - 3.$$

Remark: For the ring \mathbb{Z}_n of integers modulo n, the equivalence classes with respect to the relation \equiv are the sets \mathcal{A}_{d_i} , where d_i 's are the proper divisors of n and $\mathcal{A}_{d_i} = \{x \in \mathbb{Z}_n : \gcd(x,n) = d_i\}$. It is known that $|\mathcal{A}_{d_i}| = \phi(\frac{n}{d_i})$ for $1 \le i \le k$, where ϕ is Euler totient function (see [32]). Thus, we have the following corollaries of Theorem 2.5.

Corollary 4.5. [24, Theorem 6.1] For $n = p_1 p_2 \cdots p_k$, where p_i 's are distinct primes and $2 \le k \in \mathbb{N}$, we have

$$W(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{2^k - 2} \phi(\frac{n}{d_i}) \left(\phi(\frac{n}{d_i}) - 1 \right) + \frac{1}{2} \sum_{\substack{d_i \nmid d_j \\ d_i \nmid d_i}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_j}) + 2 \sum_{\substack{d_i \mid d_j \\ i \neq j}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_j}),$$

where d_i 's are the proper divisors of n.

Let $\tau(n)$ be the number of divisors of n and let $D = \{d_1, d_2, \dots, d_{\tau(n)-2}\}$ be the set of all proper divisors of $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdots p_k^{n_k}$ with $k \geq 2$. If $d_i \mid d_j$, then define

$$A = \{(d_i, d_j) \in D \times D \mid d_i \neq p_r^s\};$$

$$B = \{(d_i, d_j) \in D \times D \mid d_i = p_r^s \text{ and } \frac{n}{d_j} \neq p_r^t\};$$

$$C = \{(d_i, d_j) \in D \times D \mid d_i = p_r^s \text{ and } \frac{n}{d_i} = p_r^t\}.$$

Corollary 4.6. [24, Theorem 6.3] With the above defined notations, for $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdots p_k^{n_k}$ with $k \geq 2$ and p_i 's are distinct primes, we have

$$W(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{\tau(n)-2} \phi(\frac{n}{d_i}) \left(\phi(\frac{n}{d_i}) - 1 \right) + \frac{1}{2} \sum_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_j}) + 2 \sum_{\substack{(d_i, d_j) \in A}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_j}) + 2 \sum_{\substack{(d_i, d_j) \in B}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_j}) + 2 \sum_{\substack{(d_i, d_j) \in G}} \phi(\frac{n}{d_i}) \phi(\frac{n}{d_i}) + 2 \sum_{\substack{(d_i, d_j) \in G}} \phi(\frac{n}{d_i})$$

5. SageMath Code

In this section, we produce a SAGE code to compute the Wiener index of the cozero-divisor graph of ring classes considered in this paper including the ring \mathbb{Z}_n of integers modulo n. On providing the value of integer n, the following SAGE code computes the Wiener index of the graph $\Gamma'(\mathbb{Z}_n)$.

```
cozero_divisor_graph=Graph()
E = []
n = 72
for i in range(n):
     for j in range(n):
         if(i!=i):
              p=\gcd(i,n)
              q = \gcd(j, n)
              if (p\%q!=0 \text{ and } q\%p!=0):
                  E. append ((i,j))
cozero_divisor_graph.add_edges(E)
if(E==[]):
    V=[]
    for i in range (1,n):
         if (gcd(i,n)!=1):
              V. append(i)
```

cozero_divisor_graph.add_vertices(V)

```
W=cozero_divisor_graph.wiener_index();

if (W=oo):
    print("Wiener_Index_undefined_for_Null_Graph")

else :
    print("Wiener_Index:", W)
```

Using the given code, in the Table 1, we obtain the Wiener index of $\Gamma'(\mathbb{Z}_n)$ for some values of n.

| n | 100 | 500 | 1000 | 1500 | 2000 | 2500 |
|----------------------------|------|-------|--------|--------|---------|---------|
| $W(\Gamma'(\mathbb{Z}_n))$ | 2954 | 77174 | 306202 | 930248 | 1222530 | 1946274 |

Table 1. Wiener index of $\Gamma'(\mathbb{Z}_n)$

Let R be a reduced ring i.e. $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_n}$, where F_{q_i} is a field with q_i elements. The following code determines the Wiener index of $\Gamma'(R)$ on providing the values of the field size q_i $(1 \le i \le n)$.

```
field_orders = [3,5,7]
P=Subsets(range(len(field_orders)))[1:-1]
P = [Set(i) \text{ for } i \text{ in } P]
D1 = []
D2 = []
for i in P:
    for j in P:
         if (not(i.issubset(j) or j.issubset(i)) and P.index(i) > P.index(j)):
             D1. append ([i, j])
         if (i.issubset(j) \text{ and } i!=j):
             D2. append ([i,j])
partial_sum=0
for i in P:
    sum_pp=1
    for j in i:
         sum_pp *= field_orders[j]-1
    partial\_sum +=((sum\_pp*(sum\_pp-1))/2)
D1_{sum}=0
for i in D1:
    D1_pp=1
    for j in i [0]:
```

```
D1_pp *= field_orders[j]-1

for k in i[1]:
    D1_pp *= field_orders[k]-1

D1_sum += D1_pp

D2_sum=0

for i in D2:
    D2_pp=1
    for j in i[0]:
        D2_pp *= field_orders[j]-1

for k in i[1]:
        D2_pp *= field_orders[k]-1

D2_sum += D2_pp

W = 2*partial_sum + D1_sum + 2*D2_sum

print("Wiener_Index:", W)
```

Using the given code, in the following tables, we obtain the Wiener index of the cozero-divisor graphs of the reduced rings $F_{q_1} \times F_{q_2}$ (see Table 2) and $F_{q_1} \times F_{q_2} \times F_{q_3}$ (see Table 3), respectively.

| (q_1,q_2) | (9, 25) | (49, 81) | (101, 121) | (125, 139) | (163, 169) | (289, 343) |
|--------------------------------------|---------|----------|------------|------------|------------|------------|
| $W(\Gamma'(F_{q_1} \times F_{q_2}))$ | 800 | 12416 | 36180 | 51270 | 81354 | 297774 |

Table 2. Wiener index of $\Gamma'(F_{q_1} \times F_{q_2})$

| (q_1,q_2,q_3) | (7, 8, 13) | (9, 25, 49) | (53, 64, 81) | (83, 101, 121) | (125, 131, 169) | (289, 343, 361) |
|---|------------|-------------|--------------|----------------|-----------------|-----------------|
| $W(\Gamma'(F_{q_1} \times F_{q_2} \times F_{q_3}))$ | 35196 | 2500400 | 108637254 | 620456582 | 2355211790 | 71251552134 |

Table 3. Wiener index of $\Gamma'(F_{q_1} \times F_{q_2} \times F_{q_3})$

Let $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. Then the following SAGE code gives the value of $W(\Gamma'(R))$ after providing the values of $p_i^{m_i} (1 \le i \le k)$, where each p_i is a prime.

```
orders = [2,4,9]
A = cartesian_product([range(i) for i in orders]).list()
units = [{i for i in range(1,j) if gcd(i,j) == 1} for j in orders]

def contQ(lst1, lst2):
    flag = True
    for i in range(len(orders)):
        p=gcd(lst1[i], orders[i])
```

```
q=gcd(lst2[i], orders[i])
    if(not(lst1[i]==0 or {lst2[i]}.issubset(units[i]) or p%q==0)):
        flag = False
    return flag

E=[]
for i in A:
    if(not(contQ(i,j) or contQ(j,i))and A.index(i) > A.index(j)):
        E.append([i,j])

G = Graph()
G.add_edges(E)
W=G.wiener_index()
print("Wiener_Index:", W)
```

Using the given code, we obtain the Wiener index of the cozero-divisor graph of the ring $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_n^{m_k}}$ (see Table 4).

| R | $W(\Gamma'(R))$ |
|--|-----------------|
| $\mathbb{Z}_4 \times \mathbb{Z}_9$ | 420 |
| $\mathbb{Z}_9 \times \mathbb{Z}_{25}$ | 8808 |
| $\mathbb{Z}_{16} \times \mathbb{Z}_{25}$ | 48870 |
| $\mathbb{Z}_{27} \times \mathbb{Z}_{49}$ | 268022 |
| $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ | 521 |
| $\mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$ | 14948 |
| $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{16}$ | 167769 |
| $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$ | 327394 |
| $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_9$ | 232937 |
| $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_8$ | 333963 |

Table 4. Wiener index of $\Gamma'(R)$

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