

ORDINARY REPRESENTATIONS AND COHOMOLOGY

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ABSTRACT. We can associate p -adic admissible unitary representation of $GL_2(\mathbb{Q}_p)$ to every local Galois representation. We prove if local Galois representations is ordinary then there exists a sub representation of this representation of $GL_2(\mathbb{Q}_p)$ that appears in ordinary parts of the cohomology. We give a positive answer to a question raised by Chojecki [Cho18].

1. INTRODUCTION

Fix a prime p and E a finite extension of \mathbb{Q}_p with ring of integers O_E , uniformiser ω and maximal ideal \mathfrak{m} with residue field $\kappa(\mathfrak{m}) := O_E/\mathfrak{m}$. Let $\rho : G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(E)$ be a *pro-modular* global Galois representation as in [Eme11, p. 3]. In other words $\rho \simeq \rho_f$ for a cuspidal p -adic modular eigenform of possibly *non-integral* weights with associated Galois representation ρ_f . Let $\rho_p := \rho|_{G_p} : G_p \rightarrow GL_2(E)$ local Galois representation obtained by the restriction of the Galois representation to the decomposition group $G_p := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. We also assume that the residual Galois representation $\bar{\rho} : G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\kappa(\mathfrak{m}))$ is absolutely irreducible. Now, ramified primed will give rise to levels of the p -adic modular forms, we are starting with. We assume that the levels of p -adic modular forms are of the form $\Gamma(p^m) \cap \Gamma_1(N)$ with $(N, p) = 1$.

This Galois representation ρ_p can be very complicated and most importantly *are vast*. However, thanks to the work pioneered by Fontaine on p -adic Hodge theory and developed by several mathematicians (including Colmez, Breuil, and Berger), we have a better understanding of these local Galois representation ρ_p . Assume that these local Galois representations satisfy the hypothesis of Fontaine-Mazur conjecture [Eme11, p. 4, Theorem 1.2.4], namely ρ_p is de-Rham and hence potentially semi-stable [FO08, §6.5.2]. For this article, we further assume that ρ_p is potentially crystalline [FO08]. Now Fontaine-Mazur conjecture has been proved in a great generality [Kis09] (see also [Kis03], [SW99], and [Pan22]). Hence, we conclude that

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$\rho \simeq \rho_f$ for a global Galois representation ρ_f associated to an elliptic modular form f of level $\Gamma(p^m) \cap \Gamma_1(N)$ with $(N, p) = 1$.

By a program initiated by Breuil starting from the beginning of this century, we can associate a p -adic admissible, unitary automorphic representation $B(\rho_p)$ of the group $G := \mathrm{GL}_2(\mathbb{Q}_p)$ to these local Galois representations. By [Eme11], these Galois representations appear in the completed cohomology groups of the modular curves. Thanks to the work of Scholze [Sch15], the modular curves break into the ordinary and supersingular parts. It is natural to ask under what condition of ρ_p , the corresponding automorphic representation $B(\rho_p)$ will appear in the completed cohomology of the ordinary or supersingular part of the cohomology. This article aims to investigate this condition. We show that ordinary representations appear in the ordinary parts of the cohomologies.

For ρ_p absolutely irreducible, a similar theorem was proved by Chojecki [Cho15, Theorem 6.3] for mod p situation and [Cho18] for p -adic situation. In [Cho18, p. 469], Chojecki asked if the above theorem can be generalized to the situation when ρ_p is reducible, non-split. Our theorem answers the question raised by Chojecki. For totally real fields, we generalize Chojecki's theorem [BR22] in the mod p situation again under the assumption that ρ_p is absolutely irreducible. Our theorem substantiates that ordinary representations will appear in the ordinary part of the cohomology representation similar to Chojecki whose theorem tells that the supersingular representations appear in the supersingular part of the cohomology.

Recall that for any schemes X and any ring A , $H^1(X)_A := H_{\text{ét}}^1(X, A)$ and H_{ord}^1 denotes the cohomology of the ordinary part of the modular curve (see § 2.1). We also denote by $\widehat{H}_{\text{ord}}^1$, the completed cohomology of the ordinary locus. Let $B(\rho_p)$ be the automorphic representation associated by the p -adic Langlands correspondences to ρ_p (cf § 3). Let ϵ be the p -adic cyclotomic character with \mathbb{B}_{st} the semi-stable period ring of Fontaine. Note that this period ring \mathbb{B}_{st} contains $\mathbb{C}_p := \widehat{\mathbb{Q}_p}$.

Theorem 1.1. *Let $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(E)$ be a pro-modular Galois representation with the corresponding local representation $\rho_p \simeq \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with $\star \neq 0$, $\eta_1, \eta_2 : \mathbb{Q}_p^\times \rightarrow O_E^\times$ integral characters and $\eta : G_p \rightarrow E^\times$. We also assume that $\eta_1 \cdot \eta_2 \notin \{\epsilon^{\pm 1}\}$. We assume that ρ_p is potentially crystalline reducible non-split with distinct Hodge-Tate between $(0, k-1)$. Let $B(\rho_p)$ be the automorphic representation associated by the p -adic local Langlands correspondence. Then, there exists a principal series sub representation π_1 of $B(\rho_p)$ such that*

$$\rho_p \otimes_{\mathbb{C}_p} \pi_1 \subset \widehat{H}_{\mathbb{B}_{st}, \text{ord}}^1.$$

We also have the inclusion of global representations $\rho \subset \mathrm{Hom}_G(\pi_1, \widehat{H}_{\mathbb{B}_{st}, \text{ord}}^1)$.

Here, we take the principal series representations in the category of smooth representations. Note that the representation H_{ord}^1 is smooth but not admissible as $\text{GL}_2(\mathbb{Q}_p)$ representation. We prove our main theorem in § 6 and the main ingredient is a recent result due to Colmez-Niziol-Dospinescu [CDN20, Theorem 5.8] and vanishing of certain Ext^1 proved in § 5 by closely following Emerton.

2. MODULAR CURVES AND DRINFELD TOWERS

2.1. Modular curves of infinite level. Recall some notation about modular curve at infinity following [Cho18, p. 460]. Let \mathbb{A}_f be the set of all finite adèles over \mathbb{Q} . Let W_p (respectively WD_p) be the Weil group (respectively the Weil-Deligne group). For a compact open subgroup $K = K_p \times K^p \subset \text{GL}_2(\mathbb{A}_f)$, denote by

$$Y(K) := \text{GL}_2(\mathbb{Q}) \backslash (\mathbb{C} - \mathbb{R}) \times \text{GL}_2(\mathbb{A}_f) / K.$$

This is a canonical model defined over \mathbb{Q} . Following Scholze, we consider these curves as adic spaces over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$. Consider the compactification $X(K)$ of $Y(K)$ as an adic space over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$. Recall the notion of equivalence \sim following [SW13, Definition 2.4.1]. For sufficiently small compact prime to p level $K^p \subset \text{GL}_2(\mathbb{A}_f)$, by now famous theorem due to Scholze [Sch15, Theorem III.1.2] there exist adic space $Y(K^p)$ and $X(K^p)$ such that

$$Y(K^p) \sim \varprojlim_{K_p} Y(K_p K^p); X(K^p) \sim \varprojlim_{K_p} X(K_p K^p).$$

Now, we have a notion of supersingular $Y(K)^{\text{ss}}$ and ordinary part $Y(K)^{\text{ord}}$ (respectively $X(K)^{\text{ss}}$ and $X(K)^{\text{ord}}$) on the special fiber of these modular curves (respectively on the compactified curves).

First, assume $K_p = \text{GL}_2(\mathbb{Z}_p)$, we define the ordinary (respectively supersingular part) of the modular surface $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ord}}$ (respectively $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ss}}$) to be the inverse image of the ordinary (respectively supersingular) part of the special fiber of $Y(\text{GL}_2(\mathbb{Z}_p)K^p)$.

By [SW13, Proposition 2.4.3], there exist adic spaces $Y^{\text{ss}}, Y^{\text{ord}}$ and $X^{\text{ss}}, X^{\text{ord}}$ over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$ such that

$$\begin{aligned} Y^{\text{ss}} &\sim \varprojlim_{K_p} Y(K_p K^p)^{\text{ss}}; X(K^p)^{\text{ss}} \sim \varprojlim_{K_p} X(K_p K^p)^{\text{ss}}; \\ Y^{\text{ord}} &\sim \varprojlim_{K_p} Y(K_p K^p)^{\text{ord}}; X(K^p)^{\text{ord}} \sim \varprojlim_{K_p} X(K_p K^p)^{\text{ord}}. \end{aligned}$$

For an arbitrary compact open $K_p \subset \text{GL}_2(\mathbb{Z}_p)$, we define $Y(K^p)^{\text{ord}}$ (respectively $Y(K^p)^{\text{ss}}$) as the pullback of $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ord}}$ (respectively $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ss}}$).

For a fixed tame (prime to p) level K^p , write $Y = Y(K^p)$ and $X = X(K^p)$. Let $\mathbb{P}^{1,ad}$ be the adic projective space of dimension 1. Recall that Scholze [Sch15, p. 1012, Theorem 3.3.18] defined a $GL_2(\mathbb{Q}_p)$ -equivariant Hodge Tate period map

$$\pi_{HT} : X \rightarrow \mathbb{P}^{1,ad}$$

and $X^{\text{ord}} = \pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$ and $X^{\text{ss}} = \pi_{HT}^{-1}(\mathbb{P}^{1,ad} - \mathbb{P}^1(\mathbb{Q}_p))$.

Following [SW13, Chapter 6], denote by LT_{K^p} the Lubin-Tate space for $GL_2(\mathbb{Q}_p)$ at the level K_p with K_p a compact open subgroup of $GL_2(\mathbb{Q}_p)$. These are the local analogs of global objects like modular curves. By [SW13, Theorem 6.3.4], there exists a perfectoid space LT_∞ over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$ such that

$$LT_\infty \sim \varinjlim_{K_p} LT_{K_p}.$$

Now, these two spaces are connected by the p -adic uniformization theorem [Sch15, p. 972]:

$$X_{ss} \simeq LT_\infty.$$

Following [CDN22], we replace the group $G = GL_2(\mathbb{Q}_p)$ by the group $G' = G / \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{\mathbb{Z}}$. Let $X(K^p)^p, Y(K^p)^p$ (respectively LT_∞^p) be the quotient of the curves $X(K^p)^p, Y(K^p)^p$ and LT_∞ by the the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$.

2.2. Drinfeld tower for $F = \mathbb{Q}$. Recall now the construction of the Drinfeld tower [CDN20, §0.1]. For $l \neq p$, by the work of Faltings, Fargues, Harris and Taylor, the étale cohomology groups of the Drinfeld tower encode the classical Langlands and classical Jacquet-Langlands for $GL_2(\mathbb{Q}_p)$. It is expected that the p -adic étale cohomology groups also encode the hypothetical p -adic local Langlands. Let $G = GL_2(\mathbb{Q}_p)$ and \check{G} be the group of invertible elements of the quaternion algebra D with center \mathbb{Q}_p . Let $\Omega_{Dr,p} := \mathbb{P}_{\mathbb{Q}_p}^{1,ad} - \mathbb{P}^1(\mathbb{Q}_p)$ the Drinfeld's p -adic upper half plane. In [Dri76], Drinfeld defined certain covering $\check{\mathcal{M}}_n$ of $\Omega_{Dr,p}$. This covering is defined over $\check{\mathbb{Q}}_p := \widehat{\mathbb{Q}_p^{nr}}$. Note that the action of W_p is compatible with the natural action of $\check{\mathbb{Q}}_p$. There is a natural covering map $\check{\mathcal{M}}_{n+1} \rightarrow \check{\mathcal{M}}_n \rightarrow \Omega_{Dr,p}$ compatible with the natural action of G and \check{G} . Denote by $\mathcal{M}_n := C \times_{\check{\mathbb{Q}}_p} \check{\mathcal{M}}_n$ and \mathcal{M}_∞ is the projective limit of all \mathcal{M}_n . Denote by $\widehat{\mathcal{M}}_\infty$ the p -adic completion of \mathcal{M}_∞ . This is now a perfectoid space in the sense of Scholze. By [CDN20, p. 316], \mathcal{M}_n posses a $G \times \check{G} \times W_p$ equivariant semi-stable model over O_K with K a finite extension of $\check{\mathbb{Q}}_p$. Again by Scholze [Sch15, p. 972], $LT_\infty \simeq \mathcal{M}_\infty$. These towers realize both Jacquet Langlands and classical Langlands even for equal characteristics (for details see [Str05], [Str08]).

2.3. Étale sheaves and exact sequences. Let X be a scheme and $j : U \hookrightarrow X$ be an open immersion. Let $Z = X \setminus U$ and $i : Z \rightarrow X$ be the inclusion map. Let \mathcal{F} be an étale sheaf on X , we get the following exact sequence of sheaves on X

$$0 \rightarrow j_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*i^*\mathcal{F} \rightarrow 0,$$

which gives the following long exact sequence of cohomology groups

$$\cdots \rightarrow H^0(X, i_*i^*\mathcal{F}) \rightarrow H^1(X, j_!j^*\mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, i_*i^*\mathcal{F}) \rightarrow \cdots$$

The cohomology with compact support is defined by $H_c^r(U, \mathcal{G}) := H^r(X, j_!\mathcal{G})$, for $r \geq 0$ and any étale sheaf \mathcal{G} on U . Also, by definition of i^* we have $H^i(X, i_*i^*\mathcal{F}) = H^i(Z, i^*\mathcal{F})$ for $i = 0, 1$. Therefore we get

$$\cdots \rightarrow H^0(Z, i^*\mathcal{F}) \rightarrow H_c^1(U, j^*\mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(Z, i^*\mathcal{F}) \rightarrow \cdots$$

We will also consider cohomology groups with support on Z . For this let Z be a closed subvariety (or subscheme) of X . For any étale sheaf \mathcal{F} on X we have the following long exact sequence of cohomology groups

$$\cdots \rightarrow H_Z^r(X, \mathcal{F}) \rightarrow H^r(X, \mathcal{F}) \rightarrow H^r(U, \mathcal{F}) \rightarrow H^{r+1}(X, \mathcal{F}) \rightarrow \cdots$$

By the general formalism of six operations for Berkovich spaces (see [Ber93]) and by the comparison results of étale cohomology of schemes and its analytification (see [Ber95]) we can use the above two long exact sequence (with compact support and with support on Z) for the modular curves settings.

We follow the notation of [Cho15, Section 2.1] in this section and write down the above exact sequence for the elliptic modular curves. Let $\mathcal{X}_1(Np^m)$ be the Katz-Mazur compactification of the modular curve associated with the moduli problem $\Gamma(p^m) \cap \Gamma_1(N)$ with $(N, p) = 1$. This is the model defined over \mathbb{Q} (see also [BE10, §3] for adelic definitions of level that are the same because the class number is 1).

Consider $X = \mathcal{X}_1(Np^m)^{an}$, $U = \mathcal{X}_1(Np^m)^{ss}$ and $Z = \mathcal{X}_1(Np^m)^{ord}$ and we get the following long exact sequence with \mathcal{F} constant sheaves \mathbb{Q}_p and $\overline{\mathbb{F}}_p$.

$$\cdots \rightarrow H^0(\mathcal{X}_1(Np^m)^{ss}) \rightarrow H_{\mathcal{X}_1(Np^m)^{ord}}^1(\mathcal{X}_1(Np^m)^{an}) \rightarrow H^1(\mathcal{X}_1(Np^m)^{an}) \rightarrow H^1(\mathcal{X}_1(Np^m)^{ss}) \rightarrow \cdots$$

$$\begin{aligned} 0 \rightarrow H_c^0(\mathcal{X}_1(Np^m)^{ss}) \rightarrow H^0(\mathcal{X}_1(Np^m)) \cdots \rightarrow H^0(\mathcal{X}_1(Np^m)^{ord}) \\ \rightarrow H_c^1(\mathcal{X}_1(Np^m)^{ss}) \rightarrow H^1(\mathcal{X}_1(Np^m)) \rightarrow H^1(\mathcal{X}_1(Np^m)^{ord}) \rightarrow \cdots, \end{aligned}$$

Define, the *ordinary* and *supersingular* part of the cohomology groups as

$$H_{ord}^1(Np^m) := H_{\text{ét}}^1(\mathcal{X}_1(Np^m)^{ord}, \mathbb{Q}_p), H_{ss}^1(Np^m) := H_{\text{ét}}^1(\mathcal{X}_1(Np^m)^{ss}, \mathbb{Q}_p).$$

Using [Cho18, p. 462], we have an explicit description

$$H_{\text{ord}}^1 = \text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)} H^0(\{\infty\}, \mathcal{R}^1 \pi_{HT, \star}(\mathbb{Q}_p)).$$

For an irreducible principal series representation π_p , Scholze [Sch18] defined $S^1(\pi_p)$ and there is a close relation between $S^1(\pi_p)$ and $H_{\text{ord}}^1[\pi_p]$ [CDN22]. Recall that Judith Ludwig [Lud17] proved that $S^2(\pi_p) = 0$ and $S^1(\pi_p) \neq 0$. Let A be a \mathbb{Q}_p algebra. Using [Sch12, Theorem 7.17] and $X \sim \varinjlim_{K_p} X(K_p K^p)$, we define the p -adic completed cohomology of X to be

$$(2.1) \quad \widehat{H}_{A, \text{ord}}^1 := \varprojlim_n \varinjlim_m H_{\text{ord}}^1(Np^m, \mathbb{Z}/p^n \mathbb{Z}) \otimes A.$$

3. p -ADIC AND MOD p LOCAL LANGLANDS FOR $\text{GL}_2(\mathbb{Q}_p)$

Following [BB10] and [Bre10a], we recall some basic facts about p -adic and mod p local Langlands. This theory is for $\ell = p$ and there are certain similarities and differences with the classical local Langlands with $\ell \neq p$. Fix a finite extension E of \mathbb{Q}_p and a vector space V over E . According to the p -adic local Langlands correspondence, for every p -adic representation $\rho_p : G_p \rightarrow \text{GL}(V)$, we can associate an admissible unitary Banach space representation $B(\rho_p)$.

Now, the category of p -adic Galois representations is big. According to Fontaine, there are the following categories of p -adic representations with the inclusions as follows: Crystalline \subset Semi-stable \subset De-Rham. Explicit construction of the Banach space $B(V)$ associated with V can also be found in [CDP14], [BE10], [Eme06b, Conj. 3.3.1, p. 297]. These Banach space representations satisfy the following properties:

- (1) For two representations V, V' of G_p , we have $V \simeq V'$ if and only if as a $\text{GL}_2(\mathbb{Q}_p)$ representation, we have topological isomorphism ($\text{GL}_2(\mathbb{Q}_p)$ equivariant) between $B(V)$ and $B(V')$. In [Col10b] (see also [Col10a]), Colmez defined the now famous Montreal or magical functor MF . The property can be deduced using the Montreal functor.
- (2) If V has a determinant χ then $B(V)$ has central character $\chi \cdot \epsilon$.
- (3) For any continuous character $\chi : G_p \rightarrow E^\times$, there is a topological isomorphism of vector spaces:

$$B(V \otimes \chi) \simeq B(V) \otimes (\chi \circ \det).$$

- (4) The map $V \rightarrow B(V)$ is compatible with the extension of scalars.
- (5) If V is irreducible then $B(V)$ is topologically irreducible.

(6) Recall that mod p and p -adic local Langlands are compatible. In other words, there is a commutative diagram [GG15]:

$$\begin{array}{ccc} V & \longrightarrow & B(V) \\ & \searrow \text{dashed} & \downarrow \\ & & \overline{B(V)}. \end{array}$$

Under the above maps,

- If $B(V)$ is a principal series representation, then $\overline{B(V)}$ will again be a principal series representation.
- If $B(V)$ is special then $\overline{B(V)}$ is a special representation.
- If $B(V)$ is supercuspidal then $\overline{B(V)}$ is a supersingular representation.

Let (ρ, V) be a two dimensional crystalline representation of the local Galois group G_p with distinct Hodge Tate weights between $(0, k-1)$. For $i \in \{1, 2\}$, let $\chi_i : \mathbb{Q}_p^\times \rightarrow O_E^\times$ be integral characters. For a character $\chi = \chi_1 \otimes \chi_2$ of the torus $T(\mathbb{Q}_p)$, denote by $(\text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi))^{\text{e}0}$ the set of all continuous (equivalently locally analytic) functions $h : \text{GL}_2(\mathbb{Q}_p) \rightarrow E$ such that $h\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)h(g)$. On these Banach spaces, the group $\text{GL}_2(\mathbb{Q}_p)$ acts by right translation and makes them unitary $\text{GL}_2(\mathbb{Q}_p)$ Banach spaces. Each parabolic P determines a modulus character δ_P on the torus T with values in \mathbb{Q}_p^\times . Note that continuous functions are equivalent to locally analytic functions following [BB10].

Consider the group $G = \text{GL}_2(\mathbb{Q}_p)$ and recall that we define $I_P^G(\chi) := (\text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi))^{\text{e}0}$. In the next proposition, we consider Hom in the category of continuous (equivalently locally analytic) representation of the group $G = \text{GL}_2(\mathbb{Q}_p)$.

Recall that there are few possibilities for $B(\rho_{f,p})$ ([BE10] [Eme06b, §6] (see also [Col14]):

Proposition 3.1. (1) (absolutely reducible) Let $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with η_1, η_2 integral characters and $\eta : G_p \rightarrow E^\times$ continuous character. In this case,

$$B(\rho_{f,p}) \simeq \text{Ind}_B^G(\eta_1 \otimes \eta_2 \epsilon^{-1})^{\text{e}0} \otimes \eta \bigoplus \text{Ind}_B^G(\eta_2 \otimes \eta_1 \epsilon^{-1})^{\text{e}0} \otimes \eta.$$

(2) (reducible non-split, case I)

If $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with η_1, η_2, η as above. We assume that $\star \neq 0$ and $\eta_1 \cdot \eta_2^{-1} \neq \epsilon^{\pm 1}$, then the corresponding automorphic representation $B(\rho_{f,p})$ satisfies the exact sequence:

$$0 \rightarrow \pi_1 \otimes \eta \rightarrow B(\rho_{f,p}) \rightarrow \pi_2 \otimes \eta \rightarrow 0;$$

with $\pi_1 := \text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\eta_2 \otimes \eta_1)^{\text{e}0}$ and $\pi_2 := \text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\eta_1 \otimes \eta_2)^{\text{e}0}$.

- (3) (*reducible non-split, case II*) If $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with η_1, η_2 and η as above. Suppose that $\star \neq 0$ and $\eta_1 \cdot \eta_2^{-1} = \epsilon^{\pm 1}$ then the corresponding automorphic representation $B(\rho_{f,p})$ has a Jordan-Hölder filtration $0 \subset \pi_1 \subset \pi_2 \subset \pi$ with $\pi_1 \simeq (\chi \circ \det) \circ St \otimes \eta$ and $\frac{\pi}{\pi_2} \simeq \text{Ind}_B^G(\eta_2 \otimes \eta_1 \epsilon^{-1}) \otimes \eta$.
- (4) If $\rho_{f,p}$ is absolutely irreducible the $B(\rho_{f,p})$ is irreducible.

The reducible non-split case I is the analog of principle series representation, while case II is the analog of the twists of Steinberg or special representations of the classical local Langlands correspondences. Note that since case II is of interest to us, we analyze the same following [BE10, §2.3]. Recall that by our assumption, ρ_p is potentially crystalline. We write

$$\rho_{f,p} \simeq \begin{pmatrix} \chi_1 | \cdot |^{1-k} \epsilon^{k-2} & 0 \\ 0 & \chi_2 | \cdot |^{k-1} \epsilon^{1-k} \end{pmatrix} \otimes \eta;$$

for a continuous character $\eta : G_p \rightarrow E^\times$ and a unique natural number $k > 1$. Here, $\chi_1 \otimes \chi_2$ is classical of weight $k > 1$. Note that χ_1, χ_2 are locally constant characters such that $v_p(\chi_1(p)) = 1 - k$ and $v_p(\chi_2(p)) = k - 1$. Here,

$$\eta_1 := \chi_1 | \cdot |^{1-k} \epsilon^{k-2} = \chi_1 | \cdot |^{-1} z^{k-2}; \eta_2 = \chi_2 | \cdot |^{k-1} \epsilon^{2-k} = \chi_2 z^{2-k}.$$

3.1. The Banach space $B(\rho_{f,p})$. Recall that functions $f : \mathbb{Z}_p \rightarrow E$ is of class C^{k-1} if the Mahler series development

$$f(z) = \sum_{n=0}^{\infty} a_n(f) \binom{z}{n}$$

is such that $n^{k-1} |a_n(f)| \rightarrow 0$ as $n \rightarrow \infty$. Here, $\binom{z}{0} = 1, \binom{z}{n} := \frac{z(z-1)\dots(z-n+1)}{n!}$ if $n > 0$. Let $C^{k-1}(\mathbb{Z}_p, E)$ the E vector space of all functions. It is a Banach space with norm $\|f\| := \text{Sup}_n n^{k-1} |a_n(f)|$.

Suppose V is the L vector space of functions $f : \mathbb{Q}_p \rightarrow L$ such that $f_1(z) := f(pz)$ and $f_2(z) := (\chi_2 \chi_1^{-1})(z) f(\frac{1}{z})$ is of class $C^{k-1}(\mathbb{Z}_p, E)$. It is a Banach space with norm $\text{Sup}(\|f_1\|, \|f_2\|)$.

For $0 \leq j \leq k-2$ and $a \in \mathbb{Q}_p$, the functions $f(z) = z^j$ and $f(z) = (z-a)^{-j} (\chi_2 \chi_1^{-1})(z-a)$ are in V . We define W to be L vector space generated by these functions. Recall [BE10, Theorem 2.2.2, p. 14], the Banach space quotient V/W with the induced action of G is the universal unitary completion [Eme05] of the locally analytic space $(\text{Ind}_{B(\mathbb{Q}_p)}^G(\chi_1 \otimes \chi_2))^{an}$. Our p -adic automorphic representation $B(\rho_{f,p})$ is the twist by η of the universal unitary completion of the locally analytic induction $(\text{Ind}_{B(\mathbb{Q}_p)}^G(\chi_1 \otimes \chi_2))^{an}$. Now this representation $(\text{Ind}_{B(\mathbb{Q}_p)}^G(\chi_1 \otimes \chi_2))^{an}$ is of topological length 2 and it is non-trivial extension of $\text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\eta_1 \epsilon \otimes \eta_2 \epsilon^{-1})^{\epsilon^0}$ by

$\text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\eta_2 \otimes \eta_1)^{\text{c}^0}$ [BE10, Theorem 2.2.2, p. 14]. In [Eme10a, p. 362]. Emerton studied the following categories of $\text{GL}_2(\mathbb{Q}_p)$ representations:

$$\text{Admissible} \hookrightarrow \text{Locally Admissible} \hookrightarrow \text{Smooth}.$$

Thanks to [BB10], we know that $B(\rho_{f,p})$ (and hence π_1) is a *non-zero*, admissible representation.

4. LOCAL GALOIS REPRESENTATIONS AND COHOMOLOGIES OF MODULAR CURVES

In this section, we prove Theorem 1.1. For any de Rham representation V , we can associate a two-dimensional filtered (ϕ, N, G_p) module M to V . Now with this M thanks to recent development due to Breuil, Berger, Colmez, Paskunas, and Dospinescu, we can associate p -adic local Langlands correspondence $B(V) := LL(M)$. We defined modular curves at infinity in Section 2.1. For a representation π of G , we denote by π^* the dual representation of π and let us recall $nr_{\alpha,H}$ as in [CDN20, p. 346].

Consider the group $G = \text{GL}_2(\mathbb{Q}_p)$ and recall that we define $I_P^G(\chi) := (\text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi))^{\text{c}^0}$. In the next proposition, we consider Hom in the category of smooth representation of the group $G = \text{GL}_2(\mathbb{Q}_p)$.

Proposition 4.1. *Let $B(\rho_p)$ as in Theorem 1.1. Then, we have*

$$\text{Hom}_{\mathbb{C}_p[G]}(B(\rho_p), \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathbf{H}_{et}^1(LT_\infty, \mathbb{Q}_p)) = 0.$$

Proof. First, we show that $\text{Hom}_{\mathbb{C}_p[G]}(B(\rho_p), \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathbf{H}_{et}^1(LT_\infty^p, \mathbb{Q}_p)) = 0$. If possible, there exists a non-zero $\phi : \pi \rightarrow \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbf{H}_{et}^1(LT_n^p, \mathbb{Q}_p)$ for some n . By [Niz21] and [Ben22, Theorem 13.4.10, p. 177], there exists a period isomorphism

$$\alpha_{st} : \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathbf{H}_{et}^1(LT_n^p, \mathbb{Q}_p) \simeq \mathbf{H}_{dR}^1(LT_n^p) \otimes_{\mathbb{Q}_p^{nr}} \mathbb{B}_{st}.$$

Hence, we get a non-zero homomorphism $\pi \rightarrow \mathbf{H}_{dR,c}^1(LT_n^p) \otimes_{\mathbb{Q}_p^{nr}} \mathbb{B}_{st}$.

We use [CDN20, p. 352] and follow the notation of loc. cit. that says that

$$(4.1) \quad \mathbf{H}_{dR,c}^1(LT_n^p) = \bigoplus_{M \in \Phi N^\omega} JL^1(M) \bigotimes_C WD^1(M) \bigotimes_C (LL^1(M))^*.$$

In Equation 4.1 the direct summation is over modules M which are indecomposable as the Weil Deligne representations and most importantly of rank 2. In particular, there will not be any contribution in equation 4.1 from $B(\rho_{f,p})$ as in Proposition 3.1(2). By our assumption for all modules M , there exists a homomorphism

$$B(\rho_p) \rightarrow (LL^1(M))^* \otimes_{\mathbb{Q}_p^{nr}} \mathbb{B}_{st}.$$

By §3, there doesn't exist a homomorphism as above.

Finally, we use [CDN20, p. 346] given by $\phi \rightarrow \phi \otimes nr_{\alpha, G}$ with inverse given by $\psi \rightarrow \psi \otimes nr_{\alpha, G}^{-1}$ to get

$$\begin{aligned} & \text{Hom}_G(B(\rho_p), H_{dR, c}^1(LT_\infty) \otimes_{\mathbb{Q}_p^{nr}} \mathbb{B}_{st}) = \\ & \text{Hom}_G(B(\rho_p) \otimes nr_{\alpha, G}, H_{dR, c}^1(LT_\infty^p) \otimes_{\mathbb{Q}_p^{nr}} \mathbb{B}_{st}) \otimes nr_{\alpha, G}^{-1} \otimes nr_{\alpha, W_p}^{-1}. \end{aligned}$$

That completes the proof of the proposition. □

The integral version of the above comparison isomorphism theorem is proved by a recent work due to Scholze-Bhatt-Morrow [BMS18].

5. EMERTON'S RESULTS ON THE VANISHING OF Ext^1

Consider the parabolic subgroup P of a reductive group G with Levi decomposition $P = M \cdot N$. Let \bar{P} be the opposite parabolic of P . Let \mathcal{G} (respectively \mathcal{M}) be the category of smooth G representations (respectively smooth M representations). The Jacquet module [Cas93] of V is the set of all N co-invariants V_N . The Jacquet functor $J_P : \mathcal{G} \rightarrow \mathcal{M}$ is the functor $J_P(V) = V_N$. This functor has the following important properties:

- The functor J_P is exact (both left and right).
- If $V \neq 0$ and irreducible then the Jaquet module $J_P(V) \neq 0$ if and only if V appears as sub-representations of the parabolically induced representations.

Emerton generalized this functor to slightly general (locally analytic or equivalently continuous) classes of representations in [Eme06a] and [Eme07]. Soon after, Emerton invented the functor Ord_P [Eme10b], [Eme10c] that works for representations over Artinian rings (rather than only over fields) and that is again an adjoint of induced representations. Now, the Jacquet functor and Ord_P functors are closely related [Sor17].

We have left exact additive functors $\mathfrak{F} : \mathcal{G} \rightarrow \mathcal{M}$ given by $V \rightarrow \mathfrak{F}(V) := Ord_P(V)$ (Ordinary modules of V). We have another exact sequence from the category of M representations to the M representations given by $\mathfrak{G}(V) := \text{Hom}_M(U, V)$ this is again left exact. Note that $\mathfrak{F}(V)$ takes injective objects to G acyclic objects.

We consider Ext^1 in the category of smooth representations over an Artinian ring $R_n := O/\omega^n$ for some n . By the Grothendieck spectral sequence and Frobenius reciprocity, we have [Eme10c, p. 429]

$$E_2^{i, j} := \text{Ext}_M^i(U, R^j Ord_P(V)) \Rightarrow \text{Ext}_G^{i+j}(I_P^G(U), V).$$

In particular, it gives rise to the exact sequence:

$$(5.1) \quad 0 \rightarrow \text{Ext}_M^1(U, Ord_P(V)) \rightarrow \text{Ext}_G^1(I_P^G(U), V) \rightarrow \text{Hom}_M(U, R^1(Ord_P(V))) \rightarrow \dots$$

Recall that the cohomology group $H_c^1(X^{\text{ss}})$ is a smooth representation of $\text{GL}_2(\mathbb{Q}_p)$ [DLB17, Proposition 3.6] but not admissible. Using these functors, Emerton proved the following Proposition 5.1 regarding the vanishing of Ext groups.

Recall the following vanishing results due to Emerton. Note that although, Emerton wrote the proof for $n = 1$ the same proof works for arbitrary n [BH15, Propostion B.2]. Hauseux generalized these theorems [Hau16] and [Hau17] regarding the vanishing of Ext groups to more general reductive groups. By interchanging P and \bar{P} in the short exact sequence 5.1, we get the following Proposition:

Proposition 5.1. (Emerton) [Eme10c, p. 443]

(1) (Lemma 4.3.12) If α_1, α_2, χ are characters $\mathbb{Q}_p^\times \rightarrow R_n$. If $(\alpha_1, \alpha_2) \neq (\chi, \chi)$ then

$$\text{Ext}_G^1(I_P^G(\alpha \otimes \alpha_2), \chi \circ \det \circ \text{St}) = 0;$$

(2) (Lemma 4.3.12) If α_1, α_2 , are characters $\mathbb{Q}_p^\times \rightarrow R_n$. If V is a supersingular representation of G then

$$\text{Ext}_G^1(I_P^G(\alpha \otimes \alpha_2), V) = 0;$$

In the following Lemma, by suitable twisting we assume that the principal series representations are coming from *integral* characters. Suppose V (respectively W) be an admissible, unitary representation of G with unit balls V_0 (resp. W_0). Let V_n (resp. W_n) be reduction of U_0 (resp. V_0) modulo ω^n .

Lemma 5.2. Let π_1 be a principal series sub representation of $B(\rho_p)$. For any sub-representation W of $H_c^1(X^{\text{ss}})$, we have

$$\text{Ext}_G^1(\pi_1, W) = 0.$$

Proof. Since $\eta_1 \cdot \eta_2 \neq \epsilon^{\pm 1}$ and $\eta_1 \neq \eta_2$, let $N \in \mathbb{N}$ be the maximal integer such that $\eta_1 \cdot \eta_2 = \epsilon^2 \pmod{\omega^N}$ and $\eta_1 = \eta_2 \pmod{\omega^N}$. We now consider all the representations modulo ω^n [Eme10c, Lemma 4.3. 10] (see also [BH15, Propostion B.2]).

First assume that $W = H_c^1(X^{\text{ss}})$. By [Hau16, Propostion B.2, p. 267], we have

$$\dim_E(\text{Ext}_G^1(V, W)) \leq \dim_{R_n}(\text{Ext}_G^1(V_n, W_n)).$$

We show that $\text{Ext}_G^1(I_P^G(\chi)_n, W_n) = 0$ for a sub-representation W of $H_c^1(X^{\text{ss}})$.

By 4.1 as G representation, we have

$$H_c^1(X^{\text{ss}}) \simeq \bigoplus U_i$$

with U_i either supercuspidal or Steinberg. By [DF04, p. 793, ex. 10], we have

$$\text{Ext}_G^1(V, \bigoplus_i U_i) = \bigoplus_i \text{Ext}_G^1(V, U_i).$$

By Proposition 5.1, we know that $\text{Ext}^1(V, U_i) = 0$. Note that by our assumption $\chi_1 \neq \chi_2$.

We now prove our assertion for a proper subspace W of $H_c^1(X^{\text{ss}})$. Suppose, we have an exact sequence:

$$0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0.$$

This gives rise to a long exact sequence

$$0 \rightarrow \text{Hom}_G(\pi_1, W) \rightarrow \text{Hom}_G(\pi_1, V) \rightarrow \text{Hom}_G(\pi_1, V/W) \rightarrow \text{Ext}_G^1(\pi_1, W) \rightarrow \text{Ext}_G^1(\pi_1, V) \dots;$$

By above $\text{Hom}_G(\pi_1, W) = \text{Hom}_G(\pi_1, V) = \text{Ext}_G^1(\pi_1, V) = 0$. Now, since $V = \bigoplus W_i$ with W_i irreducible, hence W that in turn, V/W is again the direct sum of W_i 's. But there is no *non-zero* map from $\pi_1 \rightarrow W_i$. Hence, $\text{Hom}_G(\pi_1, V/W) = 0$ that in turn implies $\text{Ext}_G^1(\pi_1, W) = 0$. \square

6. MAIN THEOREM

We now proceed to prove our main theorem. Consider the following subgroups of the cohomology groups [BE10]. Let $c \in G_{\mathbb{Q}}$ be the complex conjugation. For any cohomology group H , denote by H^{\pm} the \pm eigenspace of c . Let f be a p -adic cusp form of weight $k \geq 2$, level $N = Mp^m$ with $m \geq 1$ and $(M, p) = 1$ and character $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \rightarrow \overline{\mathbb{Q}}^{\times}$ defined over a finite extension E of \mathbb{Q}_p contained in $\overline{\mathbb{Q}_p}$. Assume that f is a Hecke eigenform with $T_l f = a_l f$ for all $l \in \mathbb{N}$. For any Hecke module X , denote by

$$X^f := \{x \mid x \in X; T_l x = a_l x\}.$$

We now prove Theorem 1.1.

Proof. Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(E)$ be a global Galois representation (continuous, irreducible representation associated with a modular form that is unramified outside a finite set of primes) with $\rho_p : G_p \rightarrow \text{GL}_2(E)$ local Galois representation obtained by the restriction of the Galois representation to the decomposition group $G_p := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. As discussed in the introduction, we may assume $\rho = \rho_f$ for some elliptic Hecke eigenform f for some modular form of level Np^m . Denote the modular curve over \mathbb{Q} by $X = \mathcal{X}_1(Np^m)$ (cf. § 2.3).

Since ρ_p is reducible and non-split with the corresponding characters satisfying the assumption of the theorem, we have an inclusion (cf. Proposition 3.1),

$$\pi_1 \hookrightarrow B(\rho_p).$$

Recall that $\text{Hom}_G(B(\rho_p), \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{H}^1(LT_\infty)) = 0$ by Proposition 4.1 and observe that $X_{ss} \simeq LT_\infty$ (cf. §2.1). By the long exact sequence in § 2.3, we have

$$\begin{aligned} 0 \rightarrow \mathbb{H}_c^0(X^{ss})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} &\rightarrow \mathbb{H}^0(X)^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} \\ &\dots \rightarrow \mathbb{H}^0(X^{\text{ord}})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} \\ \rightarrow \mathbb{H}_c^1(X^{ss})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} &\xrightarrow{f_1} \mathbb{H}^1(X)^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} \xrightarrow{f_2} \mathbb{H}^1(X^{\text{ord}})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} \rightarrow \dots \end{aligned}$$

From the long exact sequence, we get a short exact sequence

$$0 \rightarrow \text{Ker}(f_1) \rightarrow \mathbb{H}_c^1(X^{ss})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st} \rightarrow \text{Im}(f_1) \rightarrow 0.$$

Applying the left exact Hom functor, we get

$$\begin{aligned} 0 \rightarrow \text{Hom}_G(\pi_1, \text{ker}(f_1)) &\rightarrow \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes \mathbb{H}_c^1(X^{ss})^{f,\pm}) \\ &\rightarrow \text{Hom}_G(\pi_1, \text{Im}(f_1)) \rightarrow \text{Ext}_G^1(\pi_1, \text{ker}(f_1)) \\ &\quad \text{Ext}_G^1(\pi_1, \mathbb{H}_c^1(X^{ss})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st}) \dots \end{aligned}$$

Note that $\text{Hom}_G(\pi_1, \mathbb{H}_c^1(X^{ss})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st}) = 0$ and hence

$$\text{Hom}_G(\pi_1, \text{Im}(f_1)) \hookrightarrow \text{Ext}_G^1(\pi_1, \text{ker}(f_1)).$$

By Lemma 5.2, we have $\text{Ext}_G^1(\pi_1, \text{ker}(f_1)) = 0$ and hence

$$\text{Hom}_G(\pi_1, \text{Im}(f_1)) = 0.$$

We also have an exact sequence

$$0 \rightarrow \text{Im}(f_1) \rightarrow \mathbb{H}^1(X)^{f,\pm} \otimes \mathbb{B}_{st} \rightarrow \text{Im}(f_2) \rightarrow 0.$$

By applying Hom functor [DF04, Theorem 10, p. 785, Chapter 17], we get

$$\begin{aligned} 0 \rightarrow \text{Hom}_G(\pi_1, \text{Im}(f_1)) &\rightarrow \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{C}_p} \mathbb{H}^1(X)^{f,\pm}) \\ &\rightarrow \text{Hom}_G(\pi_1, \text{Im}(f_2)) \rightarrow \text{Ext}_{\mathbb{C}_p[G]}^1(\pi_1, \text{Im}(f_1)) \\ &\rightarrow \text{Ext}_G^1(\pi_1, \mathbb{B}_{st} \otimes \mathbb{H}^1(X)^{f,\pm}) \rightarrow \text{Ext}_G^1(\pi_1, \text{Im}(f_2)) \dots \end{aligned}$$

We have inclusions of cohomology groups:

$$\text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{C}_p} \mathbb{H}^1(X)^{f,\pm}) \hookrightarrow \text{Hom}_G(\pi_1, \text{Im}(f_2)) \hookrightarrow \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes \mathbb{H}^1(X^{\text{ord}})).$$

As a consequence, we have

$$\pi_1 \otimes \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{C}_p} \mathrm{H}^1(X)^{f,\pm}) \hookrightarrow \pi_1 \otimes \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathrm{H}^1(X^{\mathrm{ord}})).$$

Now, π_1 is irreducible as G representation (cf. §3). We deduce that

$$(6.1) \quad \pi_1 \otimes \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes \mathrm{H}^1(X)^{f,\pm}) \subset \mathbb{B}_{st} \otimes \mathrm{H}^1(X^{\mathrm{ord}}).$$

As f is a cusp form, we have [Bre10b, Lemma 2.1.4]

$$\mathrm{H}_c^1(X)^{f,\pm} \simeq \mathrm{H}^1(X)^{f,\pm}.$$

On the other hand by [BE10, Proof of Theorem 5.7.3], we have

- (1) $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_f, \mathrm{Hom}_G(B(\rho_p), \mathrm{H}_c^1(X)^{f,\pm})) \simeq E$.
- (2) $\mathrm{Hom}_{\mathbb{C}_p[G]}(\pi_1, \mathrm{H}_c^1(X)^{f,\pm}) \simeq \mathrm{Hom}_{\mathbb{C}_p[G]}(\pi, \mathrm{H}_c^1(X)^{f,\pm})$.

From the above, we have an inclusion $\rho_f \subset \mathrm{Hom}_G(B(\rho_p), \mathrm{H}_{\acute{e}t}^1(X)^{f,\pm}) \simeq \mathrm{Hom}_{\mathbb{C}_p[G]}(\pi_1, \mathrm{H}_c^1(X)^{f,\pm})$. By 6.1, we deduce that (as the field \mathbb{C}_p is flat)

$$\rho_f \otimes_{\mathbb{C}_p} \pi_1 \subset \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes \mathrm{H}^1(X)^{f,\pm}) \otimes \pi_1 \subset \mathbb{B}_{st} \otimes_{\mathbb{C}_p} \mathrm{H}^1(X^{\mathrm{ord}})^{f,\pm} \subset \mathbb{B}_{st} \otimes_{\mathbb{C}_p} \mathrm{H}^1(X^{\mathrm{ord}});$$

proving the first part of our theorem by Proposition 2.1.

We now proceed to prove the second part. Since $\mathrm{Im}(f_2) \subset \mathrm{H}^1(X^{\mathrm{ord}})^{f,\pm}$, we have

$$\begin{aligned} \rho_f &\hookrightarrow \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathrm{H}^1(X)^{f,\pm}) \hookrightarrow \\ &\mathrm{Hom}_G(\pi_1, \mathrm{Im}(f_2)) \hookrightarrow \mathrm{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} \mathrm{H}^1(X^{\mathrm{ord}})). \end{aligned}$$

Again, we complete the proof by Proposition 2.1. □

The Main theorem of Chojecki [Cho18] and our Theorem 1.1 together give a nice dichotomy:

- Corollary 1.**
- If ρ_p is absolutely irreducible then $\rho \otimes_{\mathbb{C}_p} B(\rho_{f,p}) \hookrightarrow \widehat{H}_{\mathbb{B}_{st}, \mathrm{ss}}^1$.
 - If ρ_p is reducible and non-split then there exists a principal series subrepresentation π_1 of $B(\rho_p)$ such that $\rho_p \otimes_{\mathbb{C}_p} \pi_1 \subset \widehat{H}_{\mathbb{B}_{st}, \mathrm{ord}}^1$.

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