Foxby equivalence relative to C- fp_n -injective and C- fp_n -flat modules

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Abstract

Let R and S be rings, $C = {}_{S}C_{R}$ a (faithfully) semidualizing bimodule, and n a positive integer or $n = \infty$. In this paper, we introduce the concepts of C- fp_n -injective R-modules and C- fp_n -flat S-modules as a common generalization of some known modules such as C- FP_n -injective (resp. C-weak injective) R-modules and C- FP_n -flat (resp. C-weak flat) S-modules. Then we investigate C- fp_n -injective and C- fp_n -flat dimensions of modules, where the classes of these modules, namely $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$, respectively. We study Foxby equivalence relative to these classes, and also the existence of $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ preenvelopes and covers. Finally, we study the exchange properties of these classes, as well as preenvelopes (resp. precovers) and Foxby equivalence, under almost excellent extensions of rings.

Keywords: C- fp_n -flat modules; C- fp_n -injective modules; Foxby equivalence; semidualizing bimodules.

2020 Mathematics Subject Classification: 16E10; 16E30; 16E65; 16P70.

1 Introduction

Throughout this paper, n is a positive integer, R and S are fixed associative rings with unites, and all R- or S-modules are understood to be unital left R- or S-modules (unless specified otherwise). ${}_{S}M$ (resp. M_{R}) is used to denote that M is a left S-module (resp. right R-module). Also, ${}_{S}M_{R}$ is used to denote that M is an (S, R)-bimodule which means that M is both a left S-module and a right R-module, and these structures are compatible. Right R- or S-modules are identified with left modules over the opposite rings R^{op} and S^{op} . Wei and Zhang in [21] introduced the notion of fp_n -injective (resp. fp_n -flat) modules as a generalization of fp-injective and FP_n -injective (resp. fp-flat and FP_n -flat) modules, where fp-injective and fp-flat modules introduced by Garkusha and Generalov in [10], and also FP_n -injective and FP_n -flat modules introduced by Bravo and Pérez in [4].

Over a commutative Noetherian ring R, a semidualizing module for R is a finitely generated R-module C with $\operatorname{Hom}_R(C, C)$ canonically isomorphic to R and $\operatorname{Ext}_R^i(C, C) = 0$ for all $i \ge 1$. Semidualizing modules (under different names) were independently studied by Foxby in [8], Vasconcelos in [20], and Golod in [15]. Araya et al., in [1], extended the notion of semidualizing modules to a pair of non-commutative, but Noetherian rings. Holm and White, in [17], generalized the notion of a semidualizing module to general associative rings, and defined and studied Auslander and Bass classes under a semidualizing bimodule C, and then introduced the notions of C-flat, C-projective, and C-injective modules, where $C = {}_{S}C_{R}$ stands for a semidualizing bimodule.

In [22], Wu and Gao introduced the notion of C- FP_n -injective (resp. C- FP_n -flat) modules as a common generalization of some known modules such as C-injective, C-FP-injective and C-weak injective (resp. C-flat and C-weak flat) modules (see [14, 17, 25]). Furthermore, they investigated Foxby equivalence relative to C- FP_n -injective R-modules and C- FP_n -flat S-modules, proved that the classes $\mathcal{FI}_C^n(R)$ and $\mathcal{FF}_C^n(S)$ are preenveloping and covering, and found that when these classes are closed under extensions, cokernels of monomorphisms, and kernels of epimorphisms, where $\mathcal{FI}_C^n(R)$ and $\mathcal{FF}_C^n(S)$ are the classes of C- FP_n -injective R-modules and C- FP_n -flat S-modules, respectively.

Recently, the homological theory for injective modules and flat modules with respect to semidualizing bimodules has became an important area of research (see for example [2, 3, 11, 14, 17, 22]). In this paper, we introduce and study the notion of C- fp_n -injective (resp. C- fp_n -flat) modules as a common generalization of C-weak injective and C- FP_n -injective (resp. C-weak flat and C- FP_n -flat) modules.

In Section 2, we state some fundamental notions and some preliminary results. Then we present some features of the Auslander and Bass classes, and modules of fp_n -injective and fp_n -flat dimension at most k. In Section 3, first we introduce $C-fp_n$ -injective R-modules and $C-fp_n$ -flat S-modules, and then we give some homological relationships between the classes $fp_nI(S)_{\leq k}$, $fp_nF(R)_{\leq k}$, $Cfp_nI(R)_{\leq k}$, $Cfp_nF(S)_{\leq k}$, $\mathcal{A}_C(R)$, and $\mathcal{B}_C(S)$, where these classes are the class of S-modules with fp_n -injective dimension at most k, the class of R-modules with $f p_n$ -flat dimension at most k, the class of R-modules with C- fp_n -injective dimension at most k, the class of S-modules with C- fp_n -injective dimension at most k, the Auslander class, and the Bass class under faithfully semidualizing bimodules C, respectively. Among other results, we prove that (i) Foxby equivalence relative to these classes, (ii) for an R-module M (resp. S-module N), $M \in Cfp_nI(R)_{\leq k}$ (resp. $N \in Cfp_nF(S)_{\leq k}$) if and only if $M \in \mathcal{A}_C(R)$ (resp. $N \in \mathcal{B}_C(S)$) and $C \otimes_R M \in fp_n I(S)_{\leq k}$ (resp. $\operatorname{Hom}_S(C, N) \in fp_n F(R)_{\leq k}$), and (iii) the classes $Cfp_nI(R)$ and $Cfp_nF(S)$ are preenveloping and covering. Section 4 considering faithfully semidualizing modules C is devoted to the exchange properties of these classes under change of rings. For example, let $S \ge R$ be an almost excellent extension. Then we show that (i) the classe $(C \otimes_R S) fp_n I(R)$ and $(C \otimes_R S) fp_n F(R)$ are preenveloping and precovering, (ii) if $M \in \mathcal{A}_C(R)$, then $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S)$; if $M \in \mathcal{B}_C(R)$, then $\operatorname{Hom}_R(S, M) \in \mathcal{B}_{C \otimes_R S}(S)$, and (ii) existence Foxby equivalence relative to the classes $(C \otimes_R S) fp_n I(R)$, $(C \otimes_R S) fp_n F(R)$, $\mathcal{A}_{C \otimes_R S}(S)$ and $\mathcal{B}_{C \otimes_R S}(S)$.

2 Preliminaries

In this section, some fundamental concepts and notations are stated.

Definition 2.1. (see [4, Section 1 and Definitions 2.2, 3.1, and 3.2], [27, Definitions 2.8 and 2.18], [13, Definition 2.1], [12, Definition 2.1], [14, 1.2, 1.3, and 1.4], [22, Definition 3.1], [14, Definition 2.1] and [21, Definition 2.1])

(i) An R-module M is called *finitely* n-presented if there exists an exact sequence

 $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$

where each F_i is a finitely generated free (equivalently, finitely generated projective) *R*-module. A ring *R* is called *n*-coherent if every finitely *n*-presented *R*-module is finitely (n + 1)-presented;

- (ii) An *R*-module *M* is called FP_n -injective or (n, 0)-injective (resp. FP_n -flat or (n, 0)-flat) if $\operatorname{Ext}_R^1(L, M) = 0$ (resp. $\operatorname{Tor}_1^R(L, M) = 0$) for any finitely *n*-presented *R*-module (resp. R^{op} -module) *L*. \mathcal{FP}_n -Inj(R) and \mathcal{FP}_n -Flat(R) denote the class of FP_n -injective *R*-modules and the class of FP_n -Flat *R*-modules, respectively;
- (iii) The FP_n -injective dimension (or (n, 0)-injective dimension) and the FP_n -flat dimension (or (n, 0)-flat dimension) of an R-module M are defined by

$$\mathcal{FP}_n$$
. id_R(M) = inf{k : Ext^{k+1}_R(L, M) = 0 for every finitely *n*-presented *R*-module L}

and

$$\mathcal{FP}_n$$
. fd_R(M) = inf{k : Tor^R_{k+1}(L, M) = 0 for every finitely *n*-presented R^{op}-module L},

respectively;

(iv) A degreewise finite projective resolution (or, super finitely presented) of an R-module M is a projective resolution of M:

 $\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0,$

where each P_i is a finitely generated projective (equivalently, finitely generated free) *R*-module;

- (v) An *R*-module *M* is called *weak injective* (resp. *weak flat*) if $\operatorname{Ext}_{R}^{1}(U, M) = 0$ (resp. $\operatorname{Tor}_{1}^{R}(U, M) = 0$) for any super finitely presented *R*-module (resp. R^{op} -module) *U*.
- (vi) An (S, R)-bimodule $C = {}_{S}C_{R}$ is semidualizing if the following conditions hold:
 - (a_1) _SC admits a degreewise finite S-projective resolution;
 - (a_2) C_R admits a degreewise finite R^{op} -projective resolution;
 - (b_1) The homothety map $_S\gamma: {}_SS_S \longrightarrow \operatorname{Hom}_{R^{op}}(C,C)$ is an isomorphism;
 - (b₂) The homothety map $\gamma_R : {}_RR_R \longrightarrow \operatorname{Hom}_S(C, C)$ is an isomorphism;
 - (c_1) Extⁱ_S(C, C) = 0 for all $i \ge 1$;

(c₂) $\operatorname{Ext}_{R^{op}}^{i}(C,C) = 0$ for all $i \geq 1$.

A semidualizing bimodule ${}_{S}C_{R}$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_{S}N$ and M_{R} :

- (1) If $\text{Hom}_{S}(C, N) = 0$, then N = 0;
- (2) If $\text{Hom}_{R^{op}}(C, M) = 0$, then M = 0.

By [17, Proposition 3.2], there exist many examples of faithfully semidualizing bimodules were provided over a wide class of non-commutative rings;

- (vii) The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying the following conditions:
 - (A₁) $\operatorname{Tor}_{i}^{R}(C, M) = 0$ for all $i \geq 1$;
 - (A_2) Extⁱ_S $(C, C \otimes_R M) = 0$ for all $i \ge 1$;
 - (A₃) The natural evaluation homomorphism $\mu_M : M \longrightarrow \operatorname{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of *R*-modules).

The Bass class $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying the following conditions:

- (B₁) $\operatorname{Ext}_{S}^{i}(C, N) = 0$ for all $i \geq 1$;
- (B₂) $\operatorname{Tor}_{i}^{R}(C, \operatorname{Hom}_{S}(C, N)) = 0$ for all $i \geq 1$;
- (B₃) The natural evaluation homomorphism $\nu_N : C \otimes_R \operatorname{Hom}_S(C, N) \longrightarrow N$ is an isomorphism (of S-modules).

It is an important property of Auslander and Bass classes that they are equivalent under the pair of functors:

$$\mathcal{A}_C(R) \xrightarrow[]{C\otimes_R -}{\sim} \mathcal{B}_C(S)$$

$$\underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\leftarrow}} \mathcal{B}_C(S)$$

(see [17, Proposition 4.1]);

- (viii) An *R*-module is called *C*-weak injective if it has the form $\text{Hom}_S(C, X)$ for some weak injective *S*-module *X*. An *S*-module is called *C*-weak flat if it has the form $C \otimes_R Y$ for some weak flat *R*-module *Y*;
- (ix) An *R*-module is called C- FP_n -injective if it has the form $\operatorname{Hom}_S(C, X)$ for some FP_n -injective S-module X. An S-module is called C- FP_n -flat if it has the form $C \otimes_R Y$ for some FP_n -flat *R*-module Y;
- (x) An *R*-module *M* is called fp_n -injective (resp. fp_n -flat) if for every exact sequence $0 \longrightarrow K \longrightarrow L$ with *K* and *L* are finitely *n*-presented *R*-modules (resp. R^{op} -modules), the induced sequence $\operatorname{Hom}_R(L, M) \longrightarrow \operatorname{Hom}_R(K, M) \longrightarrow 0$ (resp. $0 \longrightarrow K \otimes_R M \longrightarrow L \otimes_R M$) is exact. $fp_n I(R)$ and $fp_n F(R)$ denote the class of fp_n -injective *R*-modules and the class of fp_n -Flat *R*-modules, respectively.

By [4, Proposition 1.7(1)], every FP_n -injective (resp. FP_n -flat) module is fp_m -injective (resp. fp_m -flat) for any $m \ge n$. But not conversely, see Example 3.3. The Proposition 2.4 shows that the converse is also true over *n*-coherent rings.

Definition 2.2. Let $\mathcal{Y} = \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} U \longrightarrow 0$, be an exact sequence of projective *R*-modules F_i . Then \mathcal{Y} is called \mathcal{Y} -finitely presented (equivalently, super finitely presented in [13]) if U and Ker f_i are finitely presented for any $i \geq 0$.

Proposition 2.3. Let C be a semidualizing module. Then the following assertions hold true:

- (i) $M \in \mathcal{A}_C(R)$ if and only if $M^* \in \mathcal{B}_C(R^{op})$;
- (ii) $M \in \mathcal{B}_C(R)$ if and only if $M^* \in \mathcal{A}_C(R^{op})$.

Proof. (i). (\Rightarrow) Assume that $M \in \mathcal{A}_C(R)$. Then by [19, Lemma 3.53], there is a $(-\otimes_R M)^*$ -exact exact sequence $\mathcal{Y} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$, with each F_i is finitely generated and free. So by [19, Theorem 2.76], it is easy to check that $0 = \operatorname{Tor}_i^R(C, M)^* \cong \operatorname{Ext}_{R^{op}}^i(C, M^*)$ for any $i \ge 1$.

On the other hand, we have $\operatorname{Ext}_{R}^{i}(C, C \otimes_{R} M) = 0$, and so $\operatorname{Hom}_{R}(\mathcal{Y}, C \otimes_{R} M)$ is exact. Since \mathcal{Y} is \mathcal{Y} -finitely presented, then by [19, Lemmas 3.53 and 3.55 and Theorem 2.76], we deduce that $\operatorname{Hom}_{R}(\mathcal{Y}, C \otimes_{R} M)$ -exact if and only if $\operatorname{Hom}_{R^{op}}(\mathcal{Y}, C \otimes_{R} M)^{*}$ -exact if and only if $\mathcal{Y} \otimes_{R^{op}} (C \otimes_{R} M)^{*}$ exact if and only if $\mathcal{Y} \otimes_{R^{op}} \operatorname{Hom}_{R^{op}}(C, M^{*})$ -exact. Hence $\operatorname{Tor}_{i}^{R^{op}}(C, \operatorname{Hom}_{R^{op}}(C, M^{*})) = 0$ for all $i \geq 1$. Also, we have $M \cong \operatorname{Hom}_{R}(C, C \otimes_{R} M)$. So by [19, Lemma 3.55], $M^{*} \cong \operatorname{Hom}_{R^{op}}(C, C \otimes_{R} M)^{*} \cong$ $C \otimes_{R^{op}} (C \otimes_{R} M)^{*} \cong C \otimes_{R^{op}} \operatorname{Hom}_{R^{op}}(C, M^{*})$. Then, it follows that $M^{*} \in \mathcal{B}_{C}(R^{op})$.

(\Leftarrow) Let $M^* \in \mathcal{B}_C(\mathbb{R}^{op})$. Then there is a $\operatorname{Hom}_{\mathbb{R}^{op}}(-, M^*)$ -exact exact sequence $\mathcal{Y} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$, with each F_i is finitely generated and free. So by [19, Theorem 2.76 and Lemma 3.53], $\operatorname{Hom}_{\mathbb{R}^{op}}(\mathcal{Y}, M^*)$ -exact if and only if $(\mathcal{Y} \otimes_{\mathbb{R}^{op}} M)^*$ -exact if and only if $(\mathcal{Y} \otimes_{\mathbb{R}} M)$ -exact. So $\operatorname{Tor}_i^R(C, M) = 0$ for any $i \ge 1$. Also, we have $\operatorname{Tor}_i^{\mathbb{R}^{op}}(\mathcal{Y}, \operatorname{Hom}_{\mathbb{R}^{op}}(C, M^*)) = 0$ for any $i \ge 1$. Then since \mathcal{Y} is \mathcal{Y} -finitely presented, $\mathcal{Y} \otimes_{\mathbb{R}^{op}} \operatorname{Hom}_{\mathbb{R}^{op}}(C, M^*)$ -exact if and only if $\mathcal{Y} \otimes_{\mathbb{R}^{op}} (C \otimes_{\mathbb{R}} M)^*$ -exact if and only Hom $_{\mathbb{R}^{op}}(\mathcal{Y}, C \otimes_{\mathbb{R}} M)^*$ -exact if and only if Hom $_{\mathbb{R}^{op}}(\mathcal{X}, C \otimes_{\mathbb{R}} M)$ -exact, and so $\operatorname{Ext}_R^i(C, C \otimes_{\mathbb{R}} M) = 0$ for any $i \ge 1$. We have $M^* \cong C \otimes_{\mathbb{R}^{op}} \operatorname{Hom}_{\mathbb{R}^{op}}(C, M^*) \cong C \otimes_{\mathbb{R}^{op}} (C \otimes_{\mathbb{R}} M)^* = \operatorname{Hom}_{\mathbb{R}}(C, C \otimes_{\mathbb{R}} M)^*$, and so $M \cong \operatorname{Hom}_{\mathbb{R}}(C, C \otimes_{\mathbb{R}} M)$. Consequently, $M \in \mathcal{A}_C(\mathbb{R})$.

(ii). This is similar to that of (i).

Proposition 2.4. Let R (resp. R^{op}) be an n-coherent ring and M an R-module. Then M is fp_m -injective (resp. fp_m -flat) if and only if M is FP_n -injective (resp. FP_n -flat) for any $m \ge n$.

Proof. Assume that M is an fp_m -injective (resp. fp_m -flat) R-module and L is a finitely n-presented R-module (resp. R^{op} -module). Since R (resp. R^{op}) is an n-coherent ring, there is an exact sequence

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow L \longrightarrow 0$$

of *R*-modules (resp. R^{op} -modules), where K_0 and F_0 are finitely *m*-presented. Thus we get $\operatorname{Ext}^1_R(L,M) = 0$ (resp. $\operatorname{Tor}^R_1(L,M) = 0$) by applying the derived functors of $\operatorname{Hom}_R(-,M)$ (resp. $-\otimes_R M$) to the above short exact sequence. Hence *M* is an *FP_n*-injective (resp. *FP_n*-flat) *R*-module.

The following lemmas will be useful in the proof of the first main result of this section.

Lemma 2.5. Suppose that M is an fp_n -injective (resp. fp_n -flat) R-module and that

$$0 \longrightarrow K \longrightarrow F \longrightarrow L \longrightarrow 0$$

is a short exact sequence of R-modules (resp. R^{op} -modules) such that K is finitely n-presented and F is finitely generated and free. Then $\operatorname{Ext}_{R}^{1}(L, M) = 0$ (resp. $\operatorname{Tor}_{1}^{R}(L, M) = 0$).

Proof. By applying the derived functors of $\text{Hom}_R(-, M)$ (resp. $-\otimes_R M$) to the above short exact sequence, the assertion follows.

Lemma 2.6. Suppose that M is an fp_n -injective (resp. fp_n -flat) R-module and that

 $0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{j-1} \longrightarrow X_j \longrightarrow X_{j+1} \longrightarrow \cdots$

is an exact sequence of R-modules (resp. R^{op} -modules) such that X_j is finitely n-presented for all $j \ge 0$. Then the sequence

$$\cdots \longrightarrow \operatorname{Hom}_{R}(X_{j}, M) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{R}(X_{2}, M) \longrightarrow \operatorname{Hom}_{R}(X_{1}, M) \longrightarrow \operatorname{Hom}_{R}(X_{0}, M) \longrightarrow 0$$

(resp.

$$0 \longrightarrow X_0 \otimes_R M \longrightarrow X_1 \otimes_R M \longrightarrow X_2 \otimes_R M \longrightarrow \cdots \longrightarrow X_j \otimes_R M \longrightarrow \cdots)$$

is exact.

Proof. Assume that $C_j = \operatorname{Coker}(X_{j-1} \longrightarrow X_j)$ for all $j \ge 1$. Then there exist short exact sequences

$$0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow C_1 \longrightarrow 0$$

and

$$0 \longrightarrow C_{j-1} \longrightarrow X_j \longrightarrow C_j \longrightarrow 0,$$

for all $j \ge 2$. Thus C_j is finitely *n*-presented for all $j \ge 1$ from [4, Proposition 1.7(1)] and using an induction argument on j. Now, by applying the functor $\operatorname{Hom}_R(-, M)$ (resp. $-\otimes_R M$) to the above exact sequences, the assertion follows.

Definition 2.7. The fp_n -injective dimension of an S-module M is defined that fp_n .id_S(M) $\leq k$ if and only if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

of S-modules with each $I_i \in fp_n I(S)$ for all $0 \le i \le k$. Also, the fp_n -flat dimension of an R-module N is defined that fp_n .fd_R(N) \le k if and only if there exists an exact sequence

 $0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$

of *R*-modules with each $F_i \in fp_n F(R)$ for all $0 \le i \le k$.

It is clear that $fp_n.id_S(M) \leq 0$ if and only if M is an fp_n -injective S-module, and $fp_n.fd_R(N) \leq 0$ if and only if N is an fp_n -flat R-module.

For convenience, we set

- $fp_n I(S)_{\leq k}$ = the class of S-modules of fp_n -injective dimension at most k.
- $fp_nF(R)_{\leq k}$ = the class of *R*-modules of fp_n -flat dimension at most *k*.

In the next lemma, we show that the Bass class $\mathcal{B}_C(S)$ contains all S-modules with finite fp_n -injective dimension and the Auslander class $\mathcal{A}_C(R)$ contains all R-modules with finite fp_n -flat dimension.

Lemma 2.8. Let $C = {}_{S}C_{R}$ be a faithfully semidualizing bimodule. Then the following assertions hold true:

- (i) $fp_n I(S)_{\leq k} \subseteq \mathcal{B}_C(S);$
- (ii) $fp_nF(R)_{\leq k} \subseteq \mathcal{A}_C(R).$

Proof. (i). First, we show that for k = 0, $fp_nI(S) \subseteq \mathcal{WI}(S)$. Cosider \mathcal{Y} -finitely presented $\mathcal{Y} = \cdots \longrightarrow F_j \longrightarrow F_{j-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0$ of S-modules. Then there is an exact sequence $0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow U \longrightarrow 0$, where K_0, F_0 and U are finitely n-finitely presented. So if $X \in fp_nI(S)$, then by Lemma 2.6, $\operatorname{Hom}_S(F_0, X) \longrightarrow \operatorname{Hom}_S(K_0, X) \longrightarrow 0$ is exact. Hence by Lemma 2.5, $\operatorname{Ext}^1_S(U, X) = 0$, and then $X \in \mathcal{WI}(S)$. Consequently, $fp_nI(S) \subseteq \mathcal{B}_C(S)$ from [14, Theorem 2.2]. So for $M \in fp_nI(S)_{\leq k}$ there exists an exact sequence

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow 0$$

of S-modules, where each $X_i \in \mathcal{B}_C(S)$ for all $0 \le i \le k$. Then by [17, Corollary 6.3], we deduce that $M \in \mathcal{B}_C(S)$.

(ii). Let $N \in fp_n F(R)_{\leq k}$. Then there is an exact sequence

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

of *R*-modules with each $F_i \in fp_n F(R)$ for all $0 \le i \le k$. Then by [19, Lemma 3.53],

$$0 \longrightarrow N^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{k-1}^* \longrightarrow F_k^* \longrightarrow 0$$

of R^{op} -modules, where each $F_i^* \in fp_n I(R^{op})$ by [21, Proposition 2.4(2)]. By (i), $F_i^* \in \mathcal{B}_C(R^{op})$, and then [17, Corollary 6.3] and Proposition 2.3, we deduce that $N^* \in \mathcal{B}_C(R^{op})$ if and only if $N \in \mathcal{A}_C(R)$.

3 C- fp_n -injective and C- fp_n -flat modules

Definition 3.1. An *R*-module is called C- fp_n -injective if it has the form $\text{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. An *S*-module is called C- fp_n -flat if it has the form $C \otimes_R Y$ for some $Y \in fp_nF(R)$. We denote the class of C- fp_n -injective *R*-modules by $Cfp_nI(R)$ and the class of C- fp_n -flat *S*-modules by $Cfp_nF(S)$. Therefore

 $Cfp_nI(R) = \{ \operatorname{Hom}_S(C, X) : X \in fp_nI(S) \}$

and

$$Cfp_nF(S) = \{C \otimes_R Y : Y \in fp_nF(R)\}$$

- **Remark 3.2.** (i) Every C- FP_n -injective (resp. C- FP_n -flat) module is C- fp_m -injective (resp. C- fp_m -flat) module for any $m \ge n$ (see [4, Proposition 1.7(1)]). But not conversely, see Example 3.3.
- (ii) Over n-coherent rings, every C- fp_n -injective (resp. C- fp_n -flat) module is also C- FP_m -injective (resp. C- FP_m -flat) module for any $m \ge n$ (see Proposition 2.4);
- (iii) Every C-fp_n-injective (resp. C-fp_n-flat) module is C-fp_m-injective (resp. C-fp_m-flat) for all m≥ n, and so we have

$$Cfp_1I(R) \subseteq Cfp_2I(R) \subseteq \cdots \subseteq Cfp_nI(R) \subseteq Cfp_{n+1}I(R) \subseteq \cdots$$

and

 $Cfp_1F(S) \subseteq Cfp_2F(S) \subseteq \cdots \subseteq Cfp_nF(S) \subseteq Cfp_{n+1}F(S) \subseteq \cdots$

(iv) An R-module M (resp. S-module) is $C-fp_{\infty}$ -injective (resp. $C-fp_{\infty}$ -flat) if and only if weak injective (resp. weak flat).

Recall that a ring R is said to be an (n, 0)-ring or n-regular ring if every finitely n-presented R-module is projective (see [18, 27]).

Example 3.3. Let K be a field, E a K-vector space with infinite rank, and A a Noetherian ring of global dimension 0. Set $B = K \ltimes E$ the trivial extension of K by E and $R = A \times B$ the direct product of A and B. By [18, Theorem 3.4(3)], R is a (2,0)-ring which is not a (1,0)-ring. Thus, for every R-module M and every finitely 2-presented R-module L, $\operatorname{Ext}_{R}^{1}(L, M) = 0$ (resp. $\operatorname{Tor}_{1}^{R}(L, M) = 0$). Hence every R-module is FP_{2} -injective (resp. FP_{2} -flat), and so every R-module is fp_{2} -injective (resp. FP_{2} -flat). On the other hand, there exists an R-module which is not FP_{1} -injective (resp. FP_{1} -flat), since if every R-module is FP_{1} -injective (resp. FP_{1} -flat), [27, Theorem 3.9] implies that R is (1,0)-ring, contradiction. Therefore, if C = R = S, then every R-module is C- fp_{2} -injective and C- fp_{2} -flat, and there exists an R-module which is not C- FP_{1} -flat).

Definition 3.4. Let $C = {}_{S}C_{R}$ be a faithfully semidualizing bimodule. The C- fp_{n} -injective dimension of an R-module M is defined that Cfp_{n} .id $_{R}(M) \leq k$ if and only if there exists an exact sequence

 $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$

of *R*-modules with each $I_i \in Cfp_nI(R)$ for all $0 \le i \le k$. Also, the *C*-fp_n-flat dimension of an *S*-module *N* is defined that $Cfp_n.fd_S(N) \le k$ if and only if there exists an exact sequence

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

of S-modules with each $F_i \in Cfp_nF(S)$ for all $0 \le i \le k$.

It is clear that $Cfp_n.id_R(M) \leq 0$ if and only if M is a $C-fp_n$ -injective R-module, and $Cfp_n.fd_S(N) \leq 0$ if and only if N is a $C-fp_n$ -flat S-module.

For convenience, we set

• $Cfp_nI(R)_{\leq k}$ = the class of *R*-modules of *C*- fp_n -injective dimension at most *k*.

• $Cfp_nF(S)_{\leq k}$ = the class of S-modules of C- fp_n -flat dimension at most k.

The following lemma is needed in the proof of the first main result of this section.

Lemma 3.5. Then the following assertions hold true:

- (i) $Cfp_nI(R)_{\leq k} \subseteq \mathcal{A}_C(R);$
- (ii) $Cfp_nF(S)_{\leq k} \subseteq \mathcal{B}_C(S).$

Proof. (i). Assume that $N \in Cfp_nI(R)$. Then $N = \operatorname{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. By Lemma 2.8(i), $X \in \mathcal{B}_C(S)$ and so $N \in \mathcal{A}_C(R)$ from [14, Lemma 2.9(1)]. Now, if $M \in Cfp_nI(R)_{\leq k}$, then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

of *R*-modules with each $I_i \in Cfp_nI(R)$ for all $0 \le i \le k$, and also any $I_i \in \mathcal{A}_C(R)$. Hence by [17, Corollary 6.3], $M \in \mathcal{A}_C(R)$.

(ii). This is similar to the first part.

In the following, we investigate Foxby equivalence relative to the classes $Cfp_nI(R)$ and $Cfp_nF(S)$ as a generalization of Foxby equivalence relative to the classes $\mathcal{FI}^n_C(R)$ and $\mathcal{FF}^n_C(S)$ in [22].

Proposition 3.6. Then we have the following equivalences of categories:

(i)
$$Cfp_nI(R)_{\leq k} \xrightarrow[]{\sim} fp_nI(S)_{\leq k}$$
;
Hom_S(C,-)

(ii)
$$fp_n F(R)_{\leq k} \xrightarrow[]{C \otimes_R -} Cfp_n F(S)_{\leq k}$$

Hom_S(C,-)

Proof. (i). Let $M \in Cfp_nI(R)_{\leq k}$. There exists an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

of *R*-modules with each $I_i \in Cfp_nI(R)$ for all $0 \leq i \leq k$. Thus, $I_i = \operatorname{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. By Lemma 2.8(i), $X \in \mathcal{B}_C(S)$, and then $C \otimes_R \operatorname{Hom}_S(C, X) \cong X$. So $C \otimes_R I_i \in fp_nI(S)$ and also by Lemma 3.5(i), $I_i \in \mathcal{A}_C(R)$, and so $\operatorname{Tor}_j^R(C, I_i) = 0$ for all $j \geq 1$. By Lemma 3.5(i), $M \in \mathcal{A}_C(R)$ and hence $\operatorname{Tor}_j^R(C, M) = 0$ for all $j \geq 1$. Therefore, by applying the functor $C \otimes_R -$ to the above exact sequence, we obtain the exact sequence

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R I_0 \longrightarrow C \otimes_R I_1 \longrightarrow \cdots \longrightarrow C \otimes_R I_{k-1} \longrightarrow C \otimes_R I_k \longrightarrow 0$$

of S-modules which shows that $C \otimes_R M \in fp_n I(S)_{\leq k}$. Now, let $N \in fp_n I(S)_{\leq k}$. There exists an exact sequence

 $0 \longrightarrow N \longrightarrow I'_0 \longrightarrow I'_1 \longrightarrow \cdots \longrightarrow I'_{k-1} \longrightarrow I'_k \longrightarrow 0$

of S-modules with each $I'_i \in fp_n I(S)$ for all $0 \le i \le k$. For all $0 \le i \le k$, from Lemma 2.8(i), $I'_i \in \mathcal{B}_C(S)$, and so $\operatorname{Ext}^j_S(C, I'_i) = 0$ for all $j \ge 1$. Also, by Lemma 2.8(i), $N \in \mathcal{B}_C(S)$ and hence

 $\operatorname{Ext}_{S}^{j}(C,N) = 0$ for all $j \geq 1$. Therefore, by applying the functor $\operatorname{Hom}_{S}(C,-)$ to the above exact sequence, we obtain the exact sequence

$$0 \to \operatorname{Hom}_{S}(C, N) \to \operatorname{Hom}_{S}(C, I'_{0}) \to \operatorname{Hom}_{S}(C, I'_{1}) \to \cdots \to \operatorname{Hom}_{S}(C, I'_{k-1}) \to \operatorname{Hom}_{S}(C, I'_{k}) \to 0$$

of *R*-modules which shows that $\operatorname{Hom}_{S}(C, N) \in Cfp_{n}I(R)_{\leq k}$. Note that, if $M \in Cfp_{n}I(R)_{\leq k}$, then by Lemma 3.5(iii), $M \in \mathcal{A}_{C}(R)$, and if $N \in fp_{n}I(S)_{\leq k}$, then from Lemma 2.8(i), $N \in \mathcal{B}_{C}(S)$. Hence we have the natural isomorphisms $M \cong \operatorname{Hom}_{S}(C, C \otimes_{R} M)$ and $C \otimes_{R} \operatorname{Hom}_{S}(C, N) \cong N$.

(ii). This is similar to that of (i).

Corollary 3.7. Let $C = {}_{S}C_{R}$ be a semidualizing bimodule. Then we have the following equivalences of categories:

(i)
$$Cfp_nI(R) \xrightarrow[Hom_S(C,-)]{C\otimes_{R^-}} fp_nI(S)$$

(ii)
$$fp_n F(R) \xrightarrow[]{C\otimes_R -} Cfp_n F(S)$$
.

Proof. Put k = 0 in Proposition 3.6.

By using Lemma 3.5, Proposition 3.6, and Corollary 3.7, we get the first main result of this section. **Theorem 3.8.** (Foxby Equivalence) *Then we have the following equivalences of categories:*

$$\begin{array}{c} & \xrightarrow{C\otimes_{R}-} \\ fp_{n}F(R) & \xrightarrow{\sim} \\ & \xrightarrow{} \\ fp_{n}F(R) \leq k \\ & \xrightarrow{\sim} \\ fp_{n}F(R) \leq k \\ & \xrightarrow{\sim} \\ & \xrightarrow{C} \\ fp_{n}F(R) \leq k \\ & \xrightarrow{\sim} \\ & \xrightarrow{C} \\ &$$

We are now ready to state and prove the first main result of Theorem 3.8 (Foxby Equivalence).

Corollary 3.9. Let M be an R-module, and N an S-module. Then the following assertions hold true:

- (i) $M \in Cfp_nI(R)_{\leq k}$ if and only if $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in fp_nI(S)_{\leq k}$;
- (ii) $N \in Cfp_nF(S)_{\leq k}$ if and only if $N \in \mathcal{B}_C(S)$ and $\operatorname{Hom}_S(C,N) \in fp_nF(R)_{\leq k}$.

Proof. (i). (\Rightarrow) This follows from Lemma 3.5(i) and Theorem 3.8.

(⇐) If $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in fp_n I(S)_{\leq k}$, then $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$ and, by Theorem 3.8, $\operatorname{Hom}_S(C, C \otimes_R M) \in Cfp_n I(R)_{\leq k}$. Thus $M \in Cfp_n I(R)_{\leq k}$.

(ii). This is similar to the first part.

Corollary 3.10. Let X be an S-module and Y an R-module. Then the following statements hold true:

- (i) $\operatorname{Hom}_{S}(C, X) \in Cfp_{n}I(R)_{\leq k}$ if and only if $X \in fp_{n}I(S)_{\leq k}$.
- (ii) $C \otimes_R Y \in Cfp_nF(S)_{\leq k}$ if and only if $Y \in fp_nF(R)_{\leq k}$;

Proof. (i). Let $\operatorname{Hom}_S(C, X) \in Cfp_nI(R)_{\leq k}$. Then, by Corollary 3.9(i), $\operatorname{Hom}_S(C, X) \in \mathcal{A}_C(R)$. Therefore, from [14, Lemma 2.9(1)], $X \in \mathcal{B}_C(S)$ and hence $C \otimes_R \operatorname{Hom}_S(C, X) \cong X$. Thus $X \in fp_nI(S)_{\leq k}$ by Theorem 3.8.

(ii). This is similar to that of (i).

In the course of the remaining parts of the paper, we denote the character module of M by $M^* := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ [19, Page 135].

Proposition 3.11. Let M be an R-module and N an S-module. Then the following statements hold:

- (i) $M \in Cfp_nI(R)_{\leq k}$ if and only if $M^* \in Cfp_nF(R^{op})_{\leq k}$;
- (ii) $N \in Cfp_nF(S)_{\leq k}$ if and only if $N^* \in Cfp_nI(S^{op})_{\leq k}$.

Proof. (i). Assume that $M \in Cfp_nI(R)_{\leq k}$. We proceed by induction on k. (\Rightarrow) If k = 0, then $M = \operatorname{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. From [21, Proposition 2.4(1)], $X^* \in fp_nF(S^{op})$. Thus $M^* \in Cfp_nF(R^{op})$ because $M^* = \operatorname{Hom}_S(C, X)^* \cong C \otimes_{S^{op}} X^*$ by [19, Lemma 3.55 and Proposition 2.56]. (\Leftarrow) Assume that $M^* \in Cfp_nF(R^{op})$. Then, from Corollary 3.9(ii), $M^* \in \mathcal{B}_C(R^{op})$ and $\operatorname{Hom}_{R^{op}}(C, M^*) \in fp_nF(S^{op})$. Also, by [19, Proposition 2.56 and Theorem 2.76], $(C \otimes_R M)^* \cong \operatorname{Hom}_{R^{op}}(C, M^*)$ and so $C \otimes_R M \in fp_nI(S)$ from [21, Proposition 2.4(1)]. Since $M^* \in \mathcal{B}_C(R^{op})$, $M^* \cong C \otimes_{S^{op}} \operatorname{Hom}_{R^{op}}(C, M^*) \cong C \otimes_{S^{op}} (C \otimes_R M)^* \cong \operatorname{Hom}_S(C, C \otimes_R M)^*$ from [19, Proposition 2.56, Theorem 2.76, and Lemma 3.55]. Hence $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$ by [19, Lemma 3.53]. Thus $M \in Cfp_nI(R)$.

Assume that $M \in Cfp_n I(R)_{\leq k}$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow Y \longrightarrow L \longrightarrow 0,$$

where $Y \in Cfp_nI(R)$ and $L \in Cfp_nI(R)_{\leq k-1}$. Since $Y^* \in Cfp_nF(R^{op})$, and by [19, Lemma 3.53],

$$0 \longrightarrow L^* \longrightarrow Y^* \longrightarrow M^* \longrightarrow 0$$

is an exact sequence, we deduce that $M \in Cfp_nI(R)_{\leq k}$ if and only if $L \in Cfp_nI(R)_{\leq k-1}$ if and only if $L^* \in Cfp_nF(R^{op})_{\leq k-1}$ if and only if $M^* \in Cfp_nF(R^{op})_{\leq k}$.

(ii). This is similar to the first part.

Corollary 3.12. Let M be an R-module and N an S-module. Then the following assertions hold:

- (i) $M \in Cfp_nI(R)_{\leq k}$ if and only if $M^{**} \in Cfp_nI(R)_{\leq k}$;
- (ii) $N \in Cfp_nF(S)_{\leq k}$ if and only if $N^{**} \in Cfp_nF(S)_{\leq k}$.

Proof. This follows by Proposition 3.11.

In the next proposition, we prove that the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are closed under direct summands, direct products, and direct sums.

Proposition 3.13. The following assertions hold:

- (i) The class $Cfp_nI(R)_{\leq k}$ is closed under direct summands, direct products, and direct sums;
- (ii) The class $Cfp_nF(S)_{\leq k}$ is closed under direct summands, direct products, and direct sums.

Proof. (i). Let $M \in Cfp_nI(R)_{\leq k}$ and let M' be a summand of M. Then, by Corollary 3.9(i), $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in fp_nI(S)_{\leq k}$, and also there exists an R-module M'' such that $M \cong M' \oplus M''$. From [17, Proposition 4.2(a)], it follows that $M' \in \mathcal{A}_C(R)$. Also, by [19, Theorem 2.65], we have $C \otimes_R M \cong (C \otimes_R M') \oplus (C \otimes_R M'')$ which shows from [21, Proposition 2.3(1)] that $C \otimes_R M' \in fp_nI(S)_{\leq k}$. Thus $M' \in Cfp_nI(R)_{\leq k}$ by Corollary 3.9(i).

Now, let $\{M_j\}_{j\in J}$ be a family of *R*-modules of C- fp_n -injective dimension at most k. Then, by Corollary 3.9(i), $M_j \in \mathcal{A}_C(R)$ and $C \otimes_R M_j \in fp_n I(S)_{\leq k}$ for all $j \in J$. Hence, from [17, Proposition 4.2(a)], $\prod_{j\in J} M_j \in \mathcal{A}_C(R)$ (resp. $\bigoplus_{j\in J} M_j \in \mathcal{A}_C(R)$). On the other hand, there exists an exact sequence

$$0 \longrightarrow C \otimes_R M_j \longrightarrow I_{0j} \longrightarrow I_{1j} \longrightarrow \cdots \longrightarrow I_{k-1j} \longrightarrow I_{kj} \longrightarrow 0$$

of S-modules with each $I_{ij} \in fp_n I(S)$ for all $0 \le i \le k$. So we have the exact sequence

$$0 \longrightarrow \prod_{j \in J} (C \otimes_R M_j) \longrightarrow \prod_{j \in J} I_{0j} \longrightarrow \prod_{j \in J} I_{1j} \longrightarrow \cdots \longrightarrow \prod_{j \in J} I_{k-1j} \longrightarrow \prod_{j \in J} I_{kj} \longrightarrow 0$$

of S-modules, where by [21, Proposition 2.3(1)], $\prod_{j\in J} I_{ij} \in fp_n I(S)$ for all $0 \leq i \leq k$, and so $\prod_{j\in J}(C\otimes_R M_j) \in fp_n I(S)_{\leq k}$. Similarly, $\bigoplus_{j\in J}(C\otimes_R M_j) \in fp_n I(S)_{\leq k}$. Since C is finitely presented, from [6, Lemma 2.10(2)] we have $C\otimes_R(\prod_{j\in J} M_j) \cong \prod_{j\in J}(C\otimes_R M_j)$, and then $C\otimes_R(\prod_{j\in J} M_j) \in fp_n I(S)_{\leq k}$. Also, $C\otimes_R(\bigoplus_{j\in J} M_j) \in fp_n I(S)$ by [19, Theorem 2.65]. Thus $\prod_{j\in J} M_j \in Cfp_n I(R)_{\leq k}$ (resp. $\bigoplus_{j\in J} M_j \in Cfp_n I(R)_{\leq k}$) by Corollary 3.9(i).

(ii). By using [19, Theorem 2.30 and Corollary 2.32] and [6, Lemma 2.9], the proof is similar to that of (i). $\hfill \Box$

Let \mathcal{F} be a class of R-modules and let M be an R-module. A morphism $f: F \longrightarrow M$ (resp. $f: M \longrightarrow F$) with $F \in \mathcal{F}$ is called an \mathcal{F} -precover (resp. \mathcal{F} -preenvelope) of M when $\operatorname{Hom}_R(F', F) \longrightarrow$ $\operatorname{Hom}_R(F', M) \longrightarrow 0$ (resp. $\operatorname{Hom}_R(F, F') \longrightarrow \operatorname{Hom}_R(M, F') \longrightarrow 0$) is exact for all $F' \in \mathcal{F}$. Assume that $f: F \longrightarrow M$ (resp. $f: M \longrightarrow F$) is an \mathcal{F} -precover (resp. \mathcal{F} -preenvelope) of M. Then f is called an \mathcal{F} -cover (resp. \mathcal{F} -envelope) of M if every morphism $g: F \longrightarrow F$ such that fg = f (resp. gf = f) is an isomorphism. The class \mathcal{F} is called (pre)covering (resp. (pre)enveloping) if each R-module has an \mathcal{F} -(pre)cover (resp. \mathcal{F} -(pre)envelope) (see [7, Definitions 5.1.1 and 6.1.1]).

A duality pair over R is a pair $(\mathcal{M}, \mathcal{N})$, where \mathcal{M} is a class of R-modules and \mathcal{N} is a class of R^{op} -modules, subject to the following conditions:

- (i) For an *R*-module M, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{N}$;
- (ii) \mathcal{N} is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{N})$ is called *(co)product-closed* if the class \mathcal{M} is closed under *(co)products* in the category of all *R*-modules (see [16, Definition 2.1]).

Corollary 3.14. $(Cfp_nI(R)_{\leq k}, Cfp_nF(R^{op})_{\leq k})$ and $(Cfp_nF(S)_{\leq k}, Cfp_nI(S^{op})_{\leq k})$ are duality pairs.

Proof. By Proposition 3.11, an *R*-module *M* (resp. *S*-module *N*) is in $Cfp_nI(R)_{\leq k}$ (resp. $Cfp_nF(S)_{\leq k}$) if and only if M^* (resp. N^*) is in $Cfp_nF(R^{op})_{\leq k}$) (resp. $Cfp_nI(S^{op})_{\leq k}$). Also, from Proposition 3.13, $Cfp_nF(R^{op})_{\leq k}$ (resp. $Cfp_nI(S^{op})_{\leq k}$) is closed under direct summands and direct sums. Thus the assertions follow.

Assume that M' is an R-submodule of M. We say that M' is a *pure submodule* of M, M/M' is a *pure quotient* of M, and M is a *pure extension* of M' and M/M' if

$$0 \longrightarrow A \otimes_R M' \longrightarrow A \otimes_R M \longrightarrow A \otimes_R M/M' \longrightarrow 0$$

is an exact sequence for all R^{op} -modules A, equivalently, if

$$0 \longrightarrow \operatorname{Hom}_{R}(B, M') \longrightarrow \operatorname{Hom}_{R}(B, M) \longrightarrow \operatorname{Hom}_{R}(B, M/M') \longrightarrow 0$$

is an exact sequence for all finitely 1-presented R-modules B [7, Definition 5.3.6].

Wei and Zhang proved in [21, Proposition 2.4(2)] that the classes $fp_nI(R)$ and $fp_nF(R)$ are closed under pure submodules and pure quotients. The next corollary shows that the classes $Cfp_nI(R)$ and $Cfp_nF(S)$ are also closed under pure submodules, pure quotients, and pure extensions.

Corollary 3.15. Let M' be a pure submodule of R-module M and let N' be a pure submodule of S-module N. Then the following statements hold true:

- (i) $M \in Cfp_nI(R)$ if and only if $M' \in Cfp_nI(R)$ and $M/M' \in Cfp_nI(R)$;
- (ii) $N \in Cfp_nF(S)$ if and only if $N' \in Cfp_nF(S)$ and $N/N' \in Cfp_nF(S)$.

Proof. The assertion follows by Corollary 3.14 and [16, Theorem 3.1].

In the second main result of this section, by the use of duality pairs, we show that $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering.

Theorem 3.16. The classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering.

Proof. By Corollary 3.14, $(Cfp_nI(R)_{\leq k}, Cfp_nF(R^{op})_{\leq k})$ and $(Cfp_nF(S)_{\leq k}, Cfp_nI(S^{op})_{\leq k})$ are duality pairs. Also, from Proposition 3.13, the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are closed under direct products and direct sums. Therefore, from [16, Theorem 3.1], the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{< k}$ are preenveloping and covering.

4 $C-fp_n$ -injective and $C-fp_n$ -flat dimension of modules with respect to change of rings

We assume $S \leq R$ is a unitary ring extension. The ring S is called right R-projective, [24, 26] in case, for any right S-module M_S with an S module N_S , $N_R \mid M_R$ implies $N_S \mid M_S$, where $N \mid M$ means N is a direct summand of M. S is called a finite normalizing extension of R if there exist elements $a_1, \dots, a_n \in S$ such that $a_1 = 1$, $S = Ra_1 + \dots + Ra_n$. A finite normalizing extension $S \leq R$ is called an almost excellent extension in case RS is flat, S_R is projective, and the ring S is right R-projective. An almost excellent extension $S \leq R$ is an excellent extension in case both RS and S_R are free modules with a common basis $\{a_1, \dots, a_n\}$.

In this section, we investigat modules of C- fp_n -injective dimension at most k and also, modules of C-flat dimension at most k under an almost excellent extension of rings, where C is a faithfully semidualizing R-module.

Lemma 4.1. Let $S \ge R$ be an almost excellent extension. Then the following assertions hold:

- (i) If $X \in fp_n I(R)_{\leq k}$, then $\operatorname{Hom}_R(S, X) \in fp_n I(S)_{\leq k}$.
- (ii) If $X \in fp_n F(R)_{\leq k}$, then $(S \otimes_R X) \in fp_n F(S)_{\leq k}$.

Proof. (i). Consider, the exact sequence $0 \longrightarrow K \longrightarrow L$, where K and L are finitely *n*-presented S-modules. By [23, Theorem 5], K and L are finitely *n*-presented R-modules. If k = 0, then $X \in fp_nI(R)$. We show that $\operatorname{Hom}_R(S, X) \in fp_nI(S)$. We have the commutative diagram

$$\begin{split} \operatorname{Hom}_{S}(L,\operatorname{Hom}_{R}(S,X)) & \longrightarrow \operatorname{Hom}_{S}(K,\operatorname{Hom}_{R}(S,X)) \\ & \downarrow \cong & \downarrow \cong \\ \operatorname{Hom}_{R}(L,X) & \longrightarrow \operatorname{Hom}_{R}(K,X) & \longrightarrow 0 \end{split}$$

and so, the sequence $\operatorname{Hom}_{S}(L, \operatorname{Hom}_{R}(S, X)) \longrightarrow \operatorname{Hom}_{S}(K, \operatorname{Hom}_{R}(S, X)) \longrightarrow 0$ is exact and hence $\operatorname{Hom}_{R}(S, X) \in fp_{n}I(S)$.

Now, let $X \in fp_n I(R)_{\leq k}$. Then there exists an exact sequence

 $0 \longrightarrow X \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k \longrightarrow 0$

of *R*-modules with each $X_i \in fp_n I(R)$ for all $0 \le i \le k$. Since S_R is projective, there exists an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R}(S, X) \longrightarrow \operatorname{Hom}_{R}(S, X_{0}) \longrightarrow \operatorname{Hom}_{R}(S, X_{1}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{R}(S, X_{k}) \longrightarrow 0$$

of S-modules with each $\operatorname{Hom}_R(S, X_i) \in fp_n I(S)$ for all $0 \le i \le k$. Thus, $\operatorname{Hom}_R(S, X) \in fp_n I(S)_{\le k}$.

(ii). By Definition 2.7 and [21, Proposition 2.4(1)], it follows that for an *R*-module *Y*, $Y \in fp_n I(R)_{\leq k}$ if and only if $Y^* \in fp_n F(R^{op})_{\leq k}$ and $Y \in fp_n F(R)_{\leq k}$ if and only if $Y^* \in fp_n I(R^{op})_{\leq k}$. So if $X \in fp_n F(R)_{\leq k}$, then $X^* \in fp_n I(R^{op})_{\leq k}$. Hence by (1) and [19, Proposition 2.56 and Theorem 2.76], $(S \otimes_R X)^* \cong \operatorname{Hom}_R(S, X^*) \in fp_n I(S^{op})_{\leq k}$, and then $(S \otimes_R X) \in fp_n F(S)_{\leq k}$.

Lemma 4.2. Let $S \ge R$ be an almost excellent extension and C a (faithfully) semidualizing R-module. Then $C \otimes_R S$ is a faithfully semidualizing S-module. *Proof.* Let C a faithfully semidualizing R-module. Then by [21, Lemma 3.4], $C \otimes_R S$ is a semidualizing S-module. Let $\operatorname{Hom}_S(C \otimes_R S, N) = 0$ for a S-module N. Then $0 = \operatorname{Hom}_S(C \otimes_R S, N) \cong \operatorname{Hom}_R(C, \operatorname{Hom}_S(C, N)) \cong \operatorname{Hom}_R(C, N)$, and so N = 0.

Proposition 4.3. Let $S \ge R$ be an almost excellent extension. Then the following assertions hold true:

- (i) If $M \in Cfp_nI(R)_{\leq k}$, then $\operatorname{Hom}_R(S, M) \in (C \otimes_R S)fp_nI(S)_{\leq k}$;
- (ii) If $M \in Cfp_nF(R)_{\leq k}$, then $(S \otimes_R M) \in (C \otimes_R S)fp_nF(S)_{\leq k}$.

Proof. (i). Let $M \in Cfp_nI(R)_{\leq k}$. If k = 0, then $M = \operatorname{Hom}_R(C, X)$ for some $X \in fp_nI(R)$. We have

$$\operatorname{Hom}_{R}(S, M) \cong \operatorname{Hom}_{R}(S, \operatorname{Hom}_{R}(C, X))$$
$$\cong \operatorname{Hom}_{R}(C \otimes_{R} S, X)$$
$$\cong \operatorname{Hom}_{R}(C \otimes_{R} S \otimes_{S} S, X)$$
$$\cong \operatorname{Hom}_{S}(C \otimes_{R} S, \operatorname{Hom}(S, X)).$$

Since by Lemma 4.1, $\operatorname{Hom}_R(S, X) \in fp_nI(S)$ and by Lemma 4.2, $C \otimes_R S$ is semidualizing S-module, we deduce that $\operatorname{Hom}_S(C \otimes_R S, \operatorname{Hom}(S, X)) \in (C \otimes_R S)fp_nI(S)$. So, it follows that $\operatorname{Hom}_R(S, M) \in (C \otimes_R S)fp_nI(S)$.

(ii). This is similar to that of (i).

In the following, we give equivalent conditions with modules of C- fp_n -injective dimension at most k and also, modules of C- fp_n -flat dimension at most k under almost excellent extension of rings.

Proposition 4.4. Let $S \ge R$ be an almost excellent extension and M an S-module. Then the following assertions are equivalent:

- (i) $M \in Cfp_nI(R)_{\leq k}$;
- (ii) $\operatorname{Hom}_R(S, M) \in (C \otimes_R S) fp_n I(S)_{\leq k};$
- (iii) $M \in (C \otimes_R S) fp_n I(S)_{\leq k}$.

Proof. (i) \Longrightarrow (ii). Let $M \in Cfp_nI(R)_{\leq k}$. Then by Proposition 4.3(1), $\operatorname{Hom}_R(S, M) \in (C \otimes_R S)fp_nI(S)_{\leq k}$.

(ii) \Longrightarrow (iii). By [24, Lemma 1.1], $_{S}M$ is isomorphic to a direct summand of S-module $\operatorname{Hom}_{R}(S, M)$. Then by (2) and Proposition 3.13(1), $M \in (C \otimes_{R} S) fp_{n}I(S)_{\leq k}$.

(iii) \Longrightarrow (i). Let k = 0. Then $M \in (C \otimes_R S) fp_n I(S)$, and so $M = \text{Hom}_S(C \otimes_R S, X)$ for some $X \in fp_n I(S)$. We have $M = \text{Hom}_S(C \otimes_R S, X) \cong \text{Hom}_R(C, \text{Hom}_S(S, X)) \cong \text{Hom}_R(C, X)$. We show that $X \in fp_n I(R)$. Let $0 \longrightarrow K \longrightarrow L$ be an exact sequence of R-modules, where K and L are finitely n-presented R-modules. Since S is flat R-module, we have that

 $0 \longrightarrow K \otimes_R S \longrightarrow L \otimes_R S$ is an exact sequence of S-modules, where $K \otimes_R S$ and $K \otimes_R S$ are finitely n-presented S-modules by [23, Lemma 4]. We have the commutative diagram

So, the sequence $\operatorname{Hom}_R(L, X) \longrightarrow \operatorname{Hom}_R(K, X) \longrightarrow 0$ is exact, and then $X \in fp_nI(R)$. Therefore, we get that $M \in Cfp_nI(R)$. Also, if $M \in (C \otimes_R S)fp_nI(S)_{\leq k}$, it simply follows that $M \in Cfp_nI(R)_{\leq k}$.

Proposition 4.5. Let $S \ge R$ be an almost excellent extension and M an S-module. Then the following assertions are equivalent:

- (i) $M \in Cfp_nF(R)_{\leq k}$;
- (ii) $(S \otimes_R M) \in (C \otimes_R S) fp_n F(S)_{\leq k};$
- (iii) $M \in (C \otimes_R S) fp_n F(S)_{\leq k}$.

Proof. By Propositions 4.4 and 3.11 and [19, Proposition 2.56 and Theorem 2.76], $M \in Cfp_nF(R)_{\leq k}$ if and only if $M^* \in Cfp_nI(R^{op})_{\leq k}$ if and only if $\operatorname{Hom}_R(S, M^*) \in (C \otimes_R S)fp_nI(S^{op})_{\leq k}$ if and only if $(S \otimes_R M)^* \in (C \otimes_R S)fp_nI(S^{op})_{\leq k}$ if and only if $(S \otimes_R M) \in (C \otimes_R S)fp_nF(S)_{\leq k}$. ALso, $M \in Cfp_nF(R)_{\leq k}$ if and only if $M^* \in Cfp_nI(R^{op})_{\leq k}$ if and only if $M^* \in (C \otimes_R S)fp_nI(S^{op})_{\leq k}$ if and only if $M \in (C \otimes_R S)fp_nF(S)_{\leq k}$.

Corollary 4.6. Let $S \ge R$ be an almost excellent extension and R an n-coherent ring. Then the following assertions hold true:

- (i) The class $(C \otimes_R S) fp_n I(S)_{\leq k}$ is closed under extentions and cokernels of monomorphisms.
- (ii) The class $(C \otimes_R S) fp_n F(S)_{\leq k}$ is closed under extentions and kernels of epimorphisms.

Proof. (i). Consider the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, of S-modules, where A and C are in $(C \otimes_R S) fp_n I(S)_{\leq k}$. Then by Proposition 4.4, A and C are in $C fp_n I(R)_{\leq k}$. So by Remark 3.2(ii) and [22, Theorem 4.9], B is in $C fp_n I(R)_{\leq k}$, and then B is in $(C \otimes_R S) fp_n I(S)_{\leq k}$ from Proposition 4.4. Similarly, if B and C are in $(C \otimes_R S) fp_n I(S)_{< k}$, then A is in $(C \otimes_R S) fp_n I(S)_{< k}$.

(ii). This is similar to that of (i) by using Proposition 4.5 and [22, Theorem 4.8].

Theorem 4.7. Let $S \ge R$ be an almost excellent extension. Then the class $(C \otimes_R S) fp_n I(S)_{\le k}$ is preenveloping and precovering.

Proof. Let M is an S-module. We show that M has a $(C \otimes_R S)fp_nI(S)_{\leq k}$ -preenvelope. Since M is an R-module, then by Theorem 3.16, M has a $Cfp_nI(R)_{\leq k}$ -preenvelope. Let R-homomorphism $\alpha : M \longrightarrow N$ be a $Cfp_nI(R)_{\leq k}$ -preenvelope of M. Then by Proposition 4.3(1), $\operatorname{Hom}_R(S,N) \in (C \otimes_R S)fp_nI(S)_{\leq k}$. We prove that $\alpha_*\lambda_M : M \longrightarrow \operatorname{Hom}_R(S,N)$ is a $(C \otimes_R S)fp_nI(S)_{\leq k}$ -preenvelope of S-module M, where $\lambda_M : M \longrightarrow \operatorname{Hom}_R(S,M)$ and $\alpha_* : \operatorname{Hom}_R(S,M) \longrightarrow \operatorname{Hom}_R(S,N)$. If $L \in (C \otimes_R S)fp_nI(S)_{\leq k}$, and $\beta : M \longrightarrow L$ is an S-homomorphism, then by Proposition 4.4, $L \in Cfp_nI(R)_{\leq k}$, and so there exists R-homomorphism $\gamma : N \longrightarrow L$ such that $\beta = \gamma \alpha$. Thus, we have the following commutative diagram:

$$SM \xrightarrow{\lambda_{M}} \operatorname{Hom}_{R}(S, M) \xrightarrow{\alpha_{*}} \operatorname{Hom}_{R}(S, N)$$

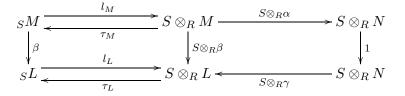
$$\downarrow^{\beta} \xrightarrow{\pi_{M}} \downarrow^{\beta_{*}} \qquad \qquad \downarrow^{1}$$

$$SL \xrightarrow{\pi_{L}} \operatorname{Hom}_{R}(S, L) \xleftarrow{\gamma_{*}} \operatorname{Hom}_{R}(S, N)$$

So, we have $(\pi_L \gamma_*)(\alpha_* \lambda_M) = \pi_L(\gamma_* \alpha_*)\lambda_M = \pi_L(\gamma \alpha)_*\lambda_M = \pi_L(\beta)_*\lambda_M = \pi_L\lambda_L\beta = \beta$. Therefore, we get that every S-module M has a $(C \otimes_R S)fp_nI(S)_{\leq k}$ -preenvelope. Similarly, it is proved that the class $(C \otimes_R S)fp_nI(S)_{\leq k}$ is precovering.

Theorem 4.8. Let $S \ge R$ be an almost excellent extension. Then the class $(C \otimes_R S) fp_n F(S)_{\le k}$ is preenveloping and precovering.

Proof. Let M is an S-module. We show that M has a $(C \otimes_R S)fp_nF(S)_{\leq k}$ -preenvelope. Since M is an R-module, then by Theorem 3.16, M has a $Cfp_nF(R)_{\leq k}$ -preenvelope. Let R-homomorphism $\alpha : M \longrightarrow N$ be a $Cfp_nF(R)_{\leq k}$ -preenvelope of M. Then by Proposition 4.3(2), $(S \otimes_R N) \in (C \otimes_R S)fp_nF(S)_{\leq k}$. We prove that $(S \otimes_R \alpha)l_M : M \longrightarrow S \otimes_R N$ is a $(C \otimes_R S)fp_nF(S)_{\leq k}$ -preenvelope of S-module M, where $l_M : M \longrightarrow (S \otimes_R M)$ and $S \otimes_R \alpha : S \otimes_R M \longrightarrow S \otimes_R N$. If $L \in (C \otimes_R S)fp_nF(S)_{\leq k}$, and $\beta : M \longrightarrow L$ is an S-homomorphism, then by Proposition 4.5, $L \in Cfp_nF(R)_{\leq k}$, and so there exists R-homomorphism $\gamma : N \longrightarrow L$ such that $\beta = \gamma \alpha$. Thus, we have the following commutative diagram:



Thus, we have $\tau_L(S \otimes_R \gamma)(S \otimes_R \alpha)l_M = \tau_L(S \otimes_R \gamma \alpha)l_M = \tau_L l_L \beta = \beta$, and so every S-module M has a $(C \otimes_R S)fp_nF(S)_{\leq k}$ -preenvelope. Similarly, it is proved that the class $(C \otimes_R S)fp_nF(S)_{\leq k}$ is precovering.

Corollary 4.9. Let $S \ge R$ be an almost excellent extension. Then the following assertions are equivalent:

- (i) Every S-module has a monic $(C \otimes_R S) fp_n I(S)_{\leq k}$ -cover;
- (ii) Every S^{op} -module has an epic $(C \otimes_R S) fp_n F(S^{op})_{\leq k}$ -envelope;
- (iii) Every quotient in $(C \otimes_R S) fp_n I(S)_{\leq k}$ is in $(C \otimes_R S) fp_n I(S)_{\leq k}$;
- (iv) Every submodule of $(C \otimes_R S) fp_n F(S^{op}) \leq k$ is in $(C \otimes_R S) fp_n F(S^{op}) \leq k$.

Moreover, if R is an n-coherent ring, then the above conditions are also equivalent to:

- (v) The kernel of any $Cfp_nI(R)$ -precover of any R-module is in $Cfp_nI(R)$;
- (vi) The cokernel of any $Cfp_nF(R^{op})$ -preenvelope of any R^{op} -module is in $Cfp_nF(R^{op})$.

Proof. (i) \Leftrightarrow (iii). First, we show that the class $(C \otimes_R S)fp_nI(S)_{\leq k}$ is closed under direct sums. Let $\{M_j\}_{j\in J}$ be a family of S-modules such that every $M_j \in (C \otimes_R S)fp_nI(S)_{\leq k}$. Then by Proposition 4.4, $M_j \in Cfp_nI(R)_{\leq k}$, and then by Proposition 3.13(i), $\bigoplus_{j\in J} M_j \in Cfp_nI(R)_{\leq k}$, and so by Proposition 4.4, $\bigoplus_{j\in J} M_j \in (C \otimes_R S)fp_nI(S)_{\leq k}$. So [9, Proposition 4] shows that (i) and (iii) are equivalent.

(ii) \Leftrightarrow (iv). The proof is similar to that of (i) \Leftrightarrow (iii) by using Propositions 3.13(ii), 4.5 and [5, Theorem 2].

(iii) \Rightarrow (iv). Let $N \in (C \otimes_R S) fp_n F(S^{op})_{\leq k}$ and N' be a submodule of N. From the short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0,$$

we get the short exact sequence

$$0 \longrightarrow (N/N')^* \longrightarrow N^* \longrightarrow N'^* \longrightarrow 0.$$

By Propositions 4.5 and 3.11(ii), $N \in Cfp_nF(R^{op})_{\leq k}$ if and only if $N^* \in Cfp_nI(R)_{\leq k}$ if and only if $N^* \in (C \otimes_R S) fp_n I(S)_{\leq k}$. Then by (iii) and Propositions 4.4, $N'^* \in (C \otimes_R S) fp_n I(S)_{\leq k}$ if and only if $N'^* \in Cfp_nI(R)_{\leq k}$, and consequently by Propositions 3.11(i) and 4.5, $N' \in Cfp_nF(R^{op})_{\leq k}$ if and only if $N' \in (C \otimes_R S) fp_n F(S^{op})_{\leq k}$.

 $(iv) \Rightarrow (iii)$. This is similar to that of $(iii) \Rightarrow (iv)$.

 $(i) \Rightarrow (v)$. Assume that M is an S-module and that, by Theorem 4.7, $f: F \longrightarrow M$ is a $(C \otimes_R f)$ $S)fp_nI(S)_{\leq k}$ -precover of M. Assume also that $g: E \longrightarrow M$ is a monic $(C \otimes_R S)fp_nI(S)_{\leq k}$ -cover of M. Then [7, Lemma 8.6.3] implies that $\operatorname{Ker}(f) \oplus E \cong F$. By Proposition 4.4, $F \in Cfp_n I(R)_{\leq k}$, and so by Proposition 3.13(i), $\operatorname{Ker}(f) \in Cfp_n I(R)_{\leq k}$. Then $\operatorname{Ker}(f) \in (C \otimes_R S)fp_n I(S)_{\leq k}$ from Proposition 4.4.

(ii) \Rightarrow (vi). The proof is similar to that of (i) \Rightarrow (v) by using the dual of [7, Lemma 8.6.3].

(vi) \Rightarrow (iv). Assume that $N \in (C \otimes_R S) fp_n F(S^{op})_{\leq k}$ and that N' is a submodule of N. Assume also that, by Theorem 4.8, $f: N' \longrightarrow F$ is a $(C \otimes_R S) f p_n F(S^{op})_{\leq k}$ -preenvelope of N'. Then we have the following commutative diagram

with exact rows. In particular, the sequence

0

$$0 \longrightarrow N' \longrightarrow F \longrightarrow \operatorname{Coker}(f) \longrightarrow 0$$

is exact, and then by Remark 3.2(ii) and Corollary 4.6(ii), $N' \in (C \otimes_R S) f p_n F(S^{op})_{\leq k}$.

 $(v) \Rightarrow (iii)$. The proof is similar to that of $(vi) \Rightarrow (iv)$ by using Corollary 4.6(i).

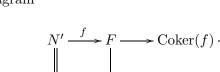
In next proposition, we investigate the homological behavior of Auslander and Bass classes under almost excellent extension of rings.

Proposition 4.10. Let $S \geq R$ be an almost excellent extension. Then the following assertions hold:

- (i) If $M \in \mathcal{A}_C(R)$, then $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S)$;
- (ii) If $M \in \mathcal{B}_C(R)$, then $\operatorname{Hom}_R(S, M) \in \mathcal{B}_{C \otimes_R S}(S)$.

Proof. (i). There exists an exact sequence of *R*-modules

$$\cdots \longrightarrow F_{j+1} \longrightarrow F_j \longrightarrow F_{j-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0,$$



where each F_j is finitely generated and free for all $j \ge 0$. Since $M \in \mathcal{A}_C(R)$, we have the following exact sequence

$$\cdots \longrightarrow F_{j+1} \otimes_R M \longrightarrow F_j \otimes_R M \longrightarrow F_{j-1} \otimes_R M \longrightarrow \cdots \longrightarrow F_1 \otimes_R M \longrightarrow F_0 \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0,$$

and since S is flat R-module, we have the following commutative diagram

and so $\operatorname{Tor}_{j}^{S}(C \otimes_{R} S, S \otimes_{R} M) = 0$ for any $j \geq 0$.

On the other hand, $C \otimes_R M \in \mathcal{B}_C(R)$ by [17, Proposition 4.1]. So there exists the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R}(C, C \otimes_{R} M) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{R}(F_{j}, C \otimes_{R} M) \longrightarrow \operatorname{Hom}_{R}(F_{j+1}, C \otimes_{R} M) \longrightarrow \cdots,$$

and hence by [19, Lemma 4.86], we have the following commutative diagram:

Therefore, we deduce that $\operatorname{Ext}_{S}^{j}(C \otimes_{R} S, (C \otimes_{R} S) \otimes_{S} (S \otimes_{R} M)) = 0$, and also

$$S \otimes_R M \cong S \otimes_R \operatorname{Hom}_R(C, C \otimes_R M) \cong \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R S) \otimes_S (S \otimes_R M)).$$

Hence, it follows that $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S)$.

(ii). Let $M \in \mathcal{B}_C(R)$. Then by Proposition 2.3(ii), $M^* \in \mathcal{A}_C(R^{op})$. So $(S \otimes_{R^{op}} M^*) \in \mathcal{A}_{C \otimes_{R^{op}} S}(S^{op})$ by (i). Since S is a finitely presented R-module, [19, Lemma 3.55] implies that $\operatorname{Hom}_{R^{op}}(S, M)^* \in \mathcal{A}_{C \otimes_{R^{op}} S}(S^{op})$. Consider the exact sequence $\mathcal{Y} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$ of R-modules, where each F_j is finitely generated and free for all $j \geq 0$. Then by Lemma 4.2, $\mathcal{Y} \otimes_R S$ is a $\mathcal{Y} \otimes_R S$ -finitely presented, and then similar to the proof of Proposition 2.3(ii), $\operatorname{Hom}_R(S, M) \in \mathcal{B}_{C \otimes_R S}(S)$.

Corollary 4.11. Let $S \ge R$ be an almost excellent extension. Then the following assertions hold:

- (i) $fp_n F(S)_{\leq k} \subseteq \mathcal{A}_{C \otimes_R S}(S);$
- (ii) $fp_n I(S)_{\leq k} \subseteq \mathcal{B}_{C \otimes_R S}(S).$

Proof. (i). Let $M \in fp_n F(S)_{\leq k}$. Then there exists an exact sequence

$$0 \longrightarrow Y_k \longrightarrow Y_{k-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

of S-modules with each $X_i \in fp_n F(S)$ for all $0 \leq i \leq k$. By [21, Proposition 3.2], $X_i \in fp_n F(R)$. So we obtain that $M \in fp_n F(R)_{\leq k}$. Thus by Lemma 2.8(ii), $M \in \mathcal{A}_C(R)$, and so by Proposition 4.10(i), $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S)$. By [24, Lemma 1.1], we see that S-module M is isomorphic to a direct summand of $S \otimes_R M$. Then [17, Proposition 4.2] implies that $M \in \mathcal{A}_{C \otimes_R S}(S)$.

(ii). This is similar to the proof of (i).

Lemma 4.12. Let $S \ge R$ be an almost excellent extension. Then the following assertions hold true:

(i)
$$(C \otimes_R S) f p_n I(S)_{\leq k} \subseteq \mathcal{A}_{C \otimes_R S}(S);$$

(ii)
$$(C \otimes_R S) f p_n F(S)_{\leq k} \subseteq \mathcal{B}_{C \otimes_R S}(S).$$

Proof. (i). Assume that $M \in (C \otimes_R S) fp_n I(S)_{\leq k}$. Then by Proposition 4.4, $M \in C fp_n I(R)_{\leq k}$, and so $M \in \mathcal{A}_C(R)$ by Lemma 3.5(i). Thus by Proposition 4.10(i), $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S)$. By [24, Lemma 1.1], M is isomorphic to a direct summand of $S \otimes_R M$, and consequently by [17, Proposition 4.2], $M \in \mathcal{A}_{C \otimes_R S}(S)$.

(ii). This is similar to the first part.

In the following, we investigate Foxby equivalence relative to the class $(C \otimes_R S) fp_n I(S)_{\leq k}$ with the class $fp_n I(S)_{\leq k}$ and the class $(C \otimes_R S) fp_n F(S)_{\leq k}$ with the class $fp_n F(S)_{\leq k}$, where $S \geq R$ is an almost excellent extension.

Proposition 4.13. Let $S \ge R$ be an almost excellent extension. Then we have the following equivalences of categories:

(i)
$$(C \otimes_R S) fp_n I(S)_{\leq k} \xrightarrow[Hom_S(C \otimes_R S, -)]{(C \otimes_R S) \otimes_S -} fp_n I(S)_{\leq k};$$

(ii)
$$fp_nF(S)_{\leq k} \xrightarrow[]{(C\otimes_R S)\otimes_S -} \\[-1.5ex] \sim \\[-1.5ex] Hom_S(C\otimes_R S, -) \\[-1.5ex] (C\otimes_R S)fp_nF(S)_{\leq k}.$$

Proof. (i). Let $M \in (C \otimes_R S) fp_n I(S)_{\leq k}$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

of S-modules with each $I_i \in (C \otimes_R S) fp_n I(S)$ for all $0 \leq i \leq k$. By Proposition 4.4, each $I_i \in Cfp_n I(R)$, and so by Proposition 3.6(i) and [21, Proposition 3.2], $C \otimes_R I_i \in fp_n I(R)$ if and only if $C \otimes_R I_i \in fp_n I(S)$. On the other hand, by Proposition 4.4, $M \in Cfp_n I(R)_{\leq k}$, and then by Lemma 3.5(i), M and I_i are in $\mathcal{A}_C(R)$. So, there exists exact sequence

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R I_0 \longrightarrow C \otimes_R I_1 \longrightarrow \cdots \longrightarrow C \otimes_R I_{k-1} \longrightarrow C \otimes_R I_k \longrightarrow 0$$

of S-modules with each $C \otimes_R I_i \in Cfp_nI(S)$ for all $0 \le i \le k$, and hence $(C \otimes_R S) \otimes_S M \cong C \otimes_R M \in fp_nI(S)_{\le k}$.

Also, $M \in \mathcal{A}_{C\otimes_R S}(S)$ by Lemma 4.12(i). So we have $M \cong \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R S) \otimes_S M)$. Now, let $N \in fp_n I(S)_{\leq k}$. Then there exists an exact sequence

$$0 \longrightarrow N \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow 0$$

of S-modules with each $X_i \in fp_nI(S)$ for all $0 \leq i \leq k$. By [21, Proposition 3.2], $X_i \in fp_nI(R)$. So we get that $N \in fp_nI(R)_{\leq k}$. Thus by Proposition 3.6(i), $\operatorname{Hom}_R(C, N) \in Cfp_nI(R)_{\leq k}$. We have $\operatorname{Hom}_S(C \otimes_R S, N) \cong \operatorname{Hom}_R(C, \operatorname{Hom}_S(S, N)) \cong \operatorname{Hom}_R(C, N)$. Hence $\operatorname{Hom}_S(C \otimes_R S, N) \in Cfp_nI(R)_{\leq k}$, and then by Proposition 4.4, $\operatorname{Hom}_S(C \otimes_R S, N) \in (C \otimes_R S)fp_nI(S)_{\leq k}$.

(ii). This is similar to that of (i).

In the following, we give equivalent conditions with modules of the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ under almost excellent extension of rings.

Proposition 4.14. Let $S \ge R$ be an almost excellent extension and M an S-module. Then the following assertions are equivalent:

- (i) $M \in \mathcal{A}_C(R)$;
- (ii) $(S \otimes_R M) \in \mathcal{A}_{C \otimes_R S}(S);$
- (iii) $M \in \mathcal{A}_{C \otimes_R S}(S)$.

Proof. (i) \Longrightarrow (ii). It is clear by Proposition 4.10(1).

(ii) \Longrightarrow (iii). By [24, Lemma 1.1], $_{S}M$ is isomorphic to a direct summand of S-module $S \otimes_{R} M$. Then by [17, Proposition 4.2(1)], $M \in \mathcal{A}_{C \otimes_{R} S}(S)$.

(iii) \Longrightarrow (i). Let $M \in \mathcal{A}_{C\otimes_R S}(S)$. Then $\operatorname{Tor}_j^S(C \otimes_R S, M) = 0$ for any $j \ge 1$. So, we have the following commutative diagram:

where the first line is exact by (iii), and so the second line is also exact, and then $\operatorname{Tor}_{j}^{R}(C, M) = 0$ for any $j \geq 1$.

On the other hand, $\operatorname{Ext}_{S}^{j}(C \otimes_{R} S, (C \otimes_{R} S) \otimes_{S} M) = 0$ for any $j \geq 1$. Then, we have the following commutative diagram:

where the first and second lines are exact by (iii), and so the third line is also exact, and then $\operatorname{Ext}_{R}^{j}(C, M) = 0$ for any $j \geq 1$.

Also by (iii) and [19, Theorem 2.75], we have

$$M \cong \operatorname{Hom}_{S}(C \otimes_{R} S, (C \otimes_{R} S) \otimes_{S} M) \cong \operatorname{Hom}_{S}(C \otimes_{R} S, C \otimes_{R} M) \cong \operatorname{Hom}_{R}(C, C \otimes_{R} M).$$

Consequently, $M \in \mathcal{A}_C(R)$.

Proposition 4.15. Let $S \ge R$ be an almost excellent extension and M an S-module. Then the following assertions are equivalent:

- (i) $M \in \mathcal{B}_C(R)$;
- (ii) $\operatorname{Hom}_R(S, M) \in \mathcal{B}_{C \otimes_R S}(S);$
- (iii) $M \in \mathcal{B}_{C \otimes_B S}(S)$.

Proof. This is similar to the proof of Proposition 4.14.

Under chang of rings, Auslander and Bass classes are equivalent under the pair of functors.

Proposition 4.16. Let $S \ge R$ be an almost excellent extension. Then there are equivalences of categories:

$$\mathcal{A}_{C\otimes_R S}(S) \xrightarrow[]{(C\otimes_R S)\otimes_S -} \mathcal{B}_{C\otimes_R S}(S)$$

Proof. By Proposition 4.14, $M \in \mathcal{A}_{C\otimes_R S}(S)$ if and only if $M \in \mathcal{A}_C(R)$. Then by [17, Proposition 4.1], $(C \otimes_R M) \in \mathcal{B}_C(R)$, and so $(C \otimes_R S) \otimes_S M \cong (C \otimes_R M) \in \mathcal{B}_{C\otimes_R S}(S)$ by Proposition 4.15. Also, we have $M \cong \operatorname{Hom}_R(C, C \otimes_R M) \cong \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R S) \otimes_S M)$.

On the other hand, By Proposition 4.15, $N \in \mathcal{B}_{C\otimes_R S}(S)$ if and only if $N \in \mathcal{B}_C(R)$. Thus by [17, Proposition 4.1], $\operatorname{Hom}_R(C, N) \in \mathcal{A}_C(R)$, and so $\operatorname{Hom}_S(C \otimes_R S, N) \cong \operatorname{Hom}_R(C, N) \in \mathcal{A}_{C\otimes_R S}(S)$ by Proposition 4.14 and [19, Theorem 2.75]. Also, we have

$$N \cong C \otimes_R \operatorname{Hom}_R(C, N) \cong (C \otimes_R S) \otimes_S \operatorname{Hom}_S(C \otimes_R S, N).$$

By using Corollary 4.11, Lemma 4.12 and Propositions 4.10, 4.13, 4.16, we get Foxby Equivalence under an almost excellent extension:

Theorem 4.17. (Foxby Equivalence under almost excellent extension of rings) Then we have the

following equivalences of categories:

$$fp_{n}F(S) \xrightarrow{(C \otimes_{R}S) \otimes_{S}-} (C \otimes_{R}S) fp_{n}F(S)$$

$$\xrightarrow{Hom_{S}(C \otimes_{R}S,-)} (C \otimes_{R}S) fp_{n}F(S) \leq_{k}$$

$$fp_{n}F(S) \leq_{k} \xrightarrow{\sim} (C \otimes_{R}S) fp_{n}F(S) \leq_{k}$$

$$\xrightarrow{(C \otimes_{R}S) \otimes_{S}-} (C \otimes_{R}S) fp_{n}F(S) \leq_{k}$$

$$\xrightarrow{(C \otimes_{R}S) \otimes_{S}-} \mathcal{B}_{C \otimes_{R}S}(S)$$

$$\xrightarrow{(C \otimes_{R}S) fp_{n}I(S) \leq_{k}} \xrightarrow{\sim} fp_{n}I(S) \leq_{k}$$

$$\xrightarrow{(C \otimes_{R}S) fp_{n}I(S)} \xrightarrow{(C \otimes_{R}S,-)} fp_{n}I(S)$$

$$(C \otimes_{R}S) fp_{n}I(S) \xrightarrow{\sim} fp_{n}I(S)$$

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