Mock theta functions and indefinite modular forms II

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Abstract

In this paper, we compute the Zwegers's modification of the mock theta functions $\Phi^{[m,0]*}$ and study the modular transformation properties of the indefinite modular forms which appear in the explicit formula for the modified functions $\widetilde{\Phi}^{[m,0]*}$.

Contents

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1 Introduction

In the previous paper [17] we studied indefinite modular forms obtained from the Zwegers' modification of the mock theta function $\Phi^{(-)[m,\frac{1}{2}]}$ for $m \in \frac{1}{2}\mathbf{N}_{\rm odd}$. In the current paper, we study the case $m \in \mathbf{N}$ where, in order that the Zwegers' modification works, we consider the function

$$\Phi^{(\pm)[m,s]\,*} \ := \ \Phi_1^{(\pm)[m,s]} + \Phi_2^{(\pm)[m,s]} \tag{1.1}$$

with $\Phi_i^{(\pm)[m,s]}$ $(i=1,2)$ defined in [17].

Comparing the results in the current paper with those in the previous paper [17], there appear strange difference between the case $m \in \mathbf{N}$ and the case $m \in \frac{1}{2}\mathbf{N}_{\rm odd}$. In the case $m \in \frac{1}{2}\mathbf{N}_{\rm odd}$ in [17], we need 3 series of functions $g_k^{(i)[m,p]}(\tau)$ $(i=1,2,3)$ to get $SL_2(\mathbf{Z})$-invariant family for each $m \in \frac{1}{2}\mathbf{N}_{\rm odd}$ whereas, in the case $m \in \mathbf{N}$ in the current paper, only one series of functions $g_k^{(1)[m,p]\,*}(\tau)$ span $SL_2(\mathbf{Z})$-invariant spaces for each $m \in \mathbf{N}$.

This paper is organized as follows.

In section 2, we make preparation on basic properties of $\Phi^{(\pm)[m,s]\,*}$. In section 3, we derive the explicit formula for $\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$ by using the Kac-Peterson's identity just in the similar way with [19]. In section 4, we deduce the formula for $\Phi^{[m,0]\,*}(\tau,z_1+p\tau,z_2-p\tau,0)$ in the case when

$$(z_1,z_2) = \Big(\frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}, \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}\Big) \quad \text{and} \quad (z_1,z_2) = \Big(\frac{z}{2}+\frac{\tau}{2}, \frac{z}{2}-\frac{\tau}{2}\Big). \tag{1.2}$$

In section 5, we compute the Zwegers's correction function $\Phi_{\rm add}^{[m,0]\,*}(\tau,z_1,z_2,0)$ for (z_1,z_2) given by (1.2). In section 6, using the results obtained in §5, we make detail investigation on the relation between $\Phi^{[m,0]\,*}(\tau,z_1+p\tau,z_2-p\tau,0)$ and the modified function $\widetilde{\Phi}^{[m,0]\,*}(\tau,z_1+p\tau,z_2-p\tau,0)$ when (z_1,z_2) is given by (1.2). In section 7, we obtain the explicit formula for $\widetilde{\Phi}^{[m,0]\,*}(\tau,z_1,z_20)$ in the case when (z_1,z_2) is given by (1.2). In section 8, we introduce functions $\Xi^{(i)[m,p]\,*}(\tau,z)$ and $\Upsilon^{(i)[m,p]\,*}(\tau,z)$ for $p \in \mathbf{Z}$ such that $0 \leq p \leq 2m$ and $i \in \{1,2\}$, and compute the modular transformation properties of these functions. In section 9, we define the functions $G^{(i)[m,p]*}(\tau,z)$ and $g_k^{(i)[m,p]\,*}(\tau)$ by

$$\begin{aligned} G^{(i)[m,p]*}(\tau,z) &:= \Xi^{(i)[m,p]\,*}(\tau,z) - \Upsilon^{(i)[m,p]\,*}(\tau,z) \\ &= \sum_{\substack{k \in \mathbf{Z} \\ 0 \leq k \leq m}} g_k^{(i)[m,p]\,*}(\tau)[\theta_{k,m}+\theta_{-k,m}](\tau,z) \end{aligned}$$

and compute modular transformation of these functions. In section 10, using the relation between $g_k^{(1)[m,p]\,*}(\tau)$ and $g_k^{(2)[m,p]\,*}(\tau)$, we define the indefinite modular forms $g_k^{[m,p]\,*}(\tau)$ and obtain their modular transformation properties.

2 Preliminaries

Lemma 2.1. *Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \frac{1}{2}\mathbf{Z}$ and $a \in \mathbf{Z}$. Then*

$$\Phi^{(\pm)[m,s]*}(\tau, z_1 + a\tau, z_2 - a\tau, t) = (\pm 1)^a \, e^{2\pi i m a (z_1 - z_2)} q^{ma^2} \Phi^{(\pm)[m,s-2am]*}(\tau, z_1, z_2, t) \tag{2.1}$$

Proof. This follows immediately from Lemma 2.3 in [17] and definition of $\Phi^{(\pm)[m,s]*}$. □

Lemma 2.2. *Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \frac{1}{2}\mathbf{Z}$ and $a \in \mathbf{Z}_{\geq 0}$. Then*

$$\begin{aligned} &\Phi^{(\pm)[m,s]*}(\tau,\, z_1 + a\tau,\, z_2 - a\tau,\, 0) \\ &= (\pm 1)^a e^{2\pi i m a (z_1 - z_2)} q^{ma^2} \Big\{ \Phi^{(\pm)[m,s]*}(\tau, z_1, z_2, 0) \\ &\quad + \sum_{\substack{k \in \mathbf{Z} \\ 1 \leq k \leq 2am}} e^{-\pi i (k-s)(z_1 - z_2)} \, q^{-\frac{1}{4m}(k-s)^2} \big[\theta^{(\pm)}_{k-s,m} + \theta^{(\pm)}_{-(k-s),m} \big](\tau, z_1 + z_2) \Big\} \end{aligned} \tag{2.2}$$

Proof. This lemma can be shown in the similar way with the proof of Lemma 2.4 in [17] as folloes. By Lemma 2.1 in [16] with Remark 2.1 in [17], we have

$$\begin{aligned} &\Phi^{(\pm)[m,s-a]*}(\tau, z_1, z_2, 0) \; - \; \Phi^{(\pm)[m,s]*}(\tau, z_1, z_2, 0) \\ &= \sum_{\substack{k \in \mathbf{Z} \\ -a \leq k \leq -1}} e^{\pi i (s+k)(z_1 - z_2)} \, q^{-\frac{(s+k)^2}{4m}} \big[\theta^{(\pm)}_{s+k,\,m} + \theta^{(\pm)}_{-(s+k),\,m} \big](\tau, z_1 + z_2) \end{aligned}$$

Letting $a \to 2am$, we have

$$\begin{aligned} &\Phi^{(\pm)[m,s-2am]*}(\tau, z_1, z_2, 0) \; - \; \Phi^{(\pm)[m,s]*}(\tau, z_1, z_2, 0) \\ &= \sum_{\substack{k \in \mathbf{Z} \\ -2am \leq k \leq -1}} e^{\pi i (s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \big[\theta^{(\pm)}_{s+k,\,m} + \theta^{(\pm)}_{-(s+k),\,m} \big](\tau, z_1 + z_2) \\ &\underset{\substack{\uparrow \\ k \to -k}}{=} \sum_{\substack{k \in \mathbf{Z} \\ 1 \leq k \leq 2am}} e^{\pi i (s-k)(z_1 - z_2)} \, q^{-\frac{(s-k)^2}{4m}} \big[\theta^{(\pm)}_{k-s,\,m} + \theta^{(\pm)}_{-(k-s),\,m} \big](\tau, z_1 + z_2) \end{aligned} \tag{2.3}$$

Substituting this equation (2.3) into (2.1), we obtain (2.2), proving Lemma 2.2. □

Lemma 2.3. *Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \frac{1}{2}\mathbf{Z}$ and $p \in \mathbf{Z}$ such that $mp \in \mathbf{Z}$. Then*

1) *if* $p \geq 0$,

$$\begin{aligned} \Phi^{(\pm)[m,s]*}(\tau,\, z_1,\, z_2 + p\tau,\, 0) \;&=\; e^{-2\pi i m p z_1} \, \Phi^{(\pm)[m,s]*}(\tau,\, z_1,\, z_2,\, 0) \\ &- \; e^{-2\pi i m p z_1} \sum_{k=0}^{mp-1} e^{\pi i (s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \, \big[\theta^{(\pm)}_{s+k,m} + \theta^{(\pm)}_{-(s+k),m} \big](\tau, z_1 + z_2) \end{aligned}$$

2) *if* $p<0$,

$$\Phi^{(\pm)[m,s]\,*}(\tau,\, z_1,\, z_2+p\tau,\, 0) \;=\; e^{-2\pi impz_1}\,\Phi^{(\pm)[m,s]\,*}(\tau,\, z_1,\, z_2,\, 0)$$
$$+\; e^{-2\pi impz_1}\sum_{\substack{k\in\mathbf{Z} \\ mp\le k<0}} e^{\pi i(s+k)(z_1-z_2)}q^{-\frac{(s+k)^2}{4m}}\,\big[\theta^{(\pm)}_{s+k,m}+\theta^{(\pm)}_{-(s+k),m}\big](\tau, z_1+z_2)$$

Proof. These formulas follow immediately from Lemma 2.5 in [16] and definition of $\Phi^{(\pm)[m,s]*}$. □

Following formulas for theta functions can be shown easily by using Lemma 1.1 in [17] and Note 1.1 in [17], and will be used in the proof of Lemmas 4.1 and 4.2.

Note 2.1. *For $m\in\mathbf{N}$ and $p\in\mathbf{Z}$, the following formulas hold:*

1) $$\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \frac{m(2p+1)\tau-m}{m+\frac12}\Big) \;=\; e^{-\frac{\pi im}{2m+1}}\,q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\,\theta^{(-)}_{2mp+(m+\frac12),\,m+\frac12}(\tau,0)$$

2) $$\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \frac{m(2p+1)\tau}{m+\frac12}\Big) \;=\; q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\,\theta^{(-)}_{2mp+(m+\frac12),\,m+\frac12}(\tau,0)$$

Note 2.2. *For $m\in\frac12\mathbf{N}$ and $p\in\mathbf{Z}$, the following formulas hold:*

1) (i) $$\theta^{(-)}_{-\frac12,m+\frac12}\Big(\tau,\ \ z+\frac{(2p+1)\tau-1}{2m+1}\Big) \;=\; e^{\frac{\pi i}{2(2m+1)}}\,q^{-\frac{1}{16(m+\frac12)}(2p+1)^2}\,e^{-\frac{\pi i}{2}(2p+1)z}\,\theta_{p,m+\frac12}(\tau,z)$$

(ii) $$\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ z-\frac{(2p+1)\tau-1}{2m+1}\Big) \;=\; e^{\frac{\pi i}{2(2m+1)}}\,q^{-\frac{1}{16(m+\frac12)}(2p+1)^2}\,e^{\frac{\pi i}{2}(2p+1)z}\,\theta_{-p,m+\frac12}(\tau,z)$$

2) (i) $$\theta^{(-)}_{-\frac12,m+\frac12}\Big(\tau,\ \ z+\frac{(2p+1)\tau}{2m+1}\Big) \;=\; q^{-\frac{1}{16(m+\frac12)}(2p+1)^2}\,e^{-\frac{\pi i}{2}(2p+1)z}\,\theta^{(-)}_{p,m+\frac12}(\tau,z)$$

(ii) $$\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ z-\frac{(2p+1)\tau}{2m+1}\Big) \;=\; q^{-\frac{1}{16(m+\frac12)}(2p+1)^2}\,e^{\frac{\pi i}{2}(2p+1)z}\,\theta^{(-)}_{-p,m+\frac12}(\tau,z)$$

Note 2.3. *For $p\in\mathbf{Z}$, the following formulas hold:*

1) (i) $$\vartheta_{11}\Big(\tau,\ \frac{z}{2}+\frac{(2p+1)\tau-1}{2}\Big) \;=\; q^{-\frac18(2p+1)^2}\,e^{-\frac{\pi i}{2}(2p+1)z}\,\theta_{0,\frac12}(\tau,z)$$

(ii) $$\vartheta_{11}\Big(\tau,\ \frac{z}{2}-\frac{(2p+1)\tau-1}{2}\Big) \;=\; -\,q^{-\frac18(2p+1)^2}\,e^{\frac{\pi i}{2}(2p+1)z}\,\theta_{0,\frac12}(\tau,z)$$

2) (i) $$\vartheta_{11}\Big(\tau,\ \frac{z}{2}+\Big(p+\frac12\Big)\tau\Big) \;=\; -\,i\,(-1)^p\,q^{-\frac18(2p+1)^2}\,e^{-\frac{\pi i}{2}(2p+1)z}\,\theta^{(-)}_{0,\frac12}(\tau,z)$$

(ii) $$\vartheta_{11}\Big(\tau,\ \frac{z}{2}-\Big(p+\frac12\Big)\tau\Big) \;=\; i\,(-1)^p\,q^{-\frac18(2p+1)^2}\,e^{\frac{\pi i}{2}(2p+1)z}\,\theta^{(-)}_{0,\frac12}(\tau,z)$$

3 Explicit formula for $\Phi^{[m,0]\,*}(\tau, z_1, z_2, 0)$

Proposition 3.1. *For $m \in \frac{1}{2}\mathbf{N}$, $\Phi^{[m,0]\,*}(\tau, z_1, z_2, 0)$ is given by the following formula:*

$$\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ \frac{m(z_1-z_2)}{m+\frac{1}{2}}\Big)\Phi^{[m,0]\,*}(\tau, z_1, z_2, 0)$$

$$= \ - \ \Bigg[\sum_{\substack{j,\,k\,\in \mathbf{Z} \\ 0<k\leq 2mj}} - \sum_{\substack{j,\,k\,\in \mathbf{Z} \\ 2mj<k\leq 0}} \Bigg] (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}}$$

$$\times \ e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i k(z_1-z_2)} \, \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z_1+z_2)$$

$$+ \ i\,\eta(\tau)^3 \Bigg\{ - \frac{\theta^{(-)}_{-\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ z_1+z_2+\dfrac{z_1-z_2}{2m+1}\Big)}{\vartheta_{11}(\tau, z_1)} + \frac{\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ z_1+z_2-\dfrac{z_1-z_2}{2m+1}\Big)}{\vartheta_{11}(\tau, z_2)} \Bigg\} \tag{3.1}$$

Proof. Lettig $z_1 = z_2$ in the formulas (3.1a) and (3.1b) in [18] and making their sum, we have

$$\sum_{j\in\mathbf{Z}} (-1)^j q^{(m+\frac{1}{2})j^2+\frac{1}{2}j} \, e^{-2\pi i j m(z_1-z_3)} \, \big[\underbrace{\Phi_1^{[m,0]} + \Phi_2^{[m,0]}}_{\substack{|| \\ \Phi^{[m,0]\,*}}}\big](\tau, \, z_1, \, -z_3-2j\tau, \, 0)$$

$$= \ e^{-\pi i z_1} \sum_{k\in\mathbf{Z}} (-1)^k q^{(m+\frac{1}{2})k^2-\frac{1}{2}k} \, e^{2\pi i k m(z_1-z_3)} \, \Phi_1^{(-)[\frac{1}{2},\frac{1}{2}]}(\tau, \, z_1, \, -z_1-2k\tau, \, 0)$$

$$+ \ e^{-\pi i z_3} \sum_{k\in\mathbf{Z}} (-1)^k q^{(m+\frac{1}{2})k^2-\frac{1}{2}k} \, e^{-2\pi i k m(z_1-z_3)} \, \Phi_1^{(-)[\frac{1}{2},\frac{1}{2}]}(\tau, \, z_3, \, -z_3-2k\tau, \, 0)$$

Letting $(j, k) \to (-j, -k)$ and changing the notation $-z_3 \to z_2$, this formula becomes:

$$\sum_{j\in\mathbf{Z}} (-1)^j q^{(m+\frac{1}{2})j^2-\frac{1}{2}j} \, e^{2\pi i j m(z_1+z_2)} \, \Phi^{[m,0]\,*}(\tau, \, z_1, \, z_2+2j\tau, \, 0)$$

$$= \ e^{-\pi i z_1} \sum_{k\in\mathbf{Z}} (-1)^k q^{(m+\frac{1}{2})k^2+\frac{1}{2}k} \, e^{-2\pi i k m(z_1+z_2)} \, \Phi_1^{(-)[\frac{1}{2},\frac{1}{2}]}(\tau, \, z_1, \, -z_1+2k\tau, \, 0)$$

$$+ \ e^{\pi i z_2} \sum_{k\in\mathbf{Z}} (-1)^k q^{(m+\frac{1}{2})k^2+\frac{1}{2}k} \, e^{2\pi i k m(z_1+z_2)} \, \Phi_1^{(-)[\frac{1}{2},\frac{1}{2}]}(\tau, \, -z_2, \, z_2+2k\tau, \, 0) \tag{3.2}$$

First we compute the LHS of this equation (3.2):

LHS of (3.2)

$$= \ \underbrace{q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{j\in\mathbf{Z}_{\geq 0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{2\pi i j m(z_1+z_2)} \Phi^{[m,0]\,*}(\tau, \, z_1, \, z_2+2j\tau, \, 0)}_{\text{(I)}}$$

$$+ \underbrace{q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{j\in\mathbf{Z}_{<0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{2\pi ijm(z_1+z_2)} \Phi^{[m,0]\,*}(\tau,\ z_1,\ z_2+2j\tau,\ 0)}_{\text{(II)}}$$

where (I) and (II) are computed by using Lemma 2.3 as folloows:

$$\begin{aligned}
\text{(I)} \ &= \ q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{j\in\mathbf{Z}_{\geq 0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ijm(z_1-z_2)} \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) \\
&- \ q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{j\in\mathbf{Z}_{\geq 0}} \sum_{k=0}^{2mj-1} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}} e^{-2\pi ijm(z_1-z_2)} e^{\pi ik(z_1-z_2)} \\
&\quad \times \ \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z_1+z_2)
\end{aligned}$$

$$\begin{aligned}
\text{(II)} \ &= \ q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{j\in\mathbf{Z}_{<0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ijm(z_1-z_2)} \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) \\
&+ \ q^{-\frac{1}{16(m+\frac{1}{2})}} \sum_{\substack{j,\,k\,\in\mathbf{Z} \\ 2mj\,\leq\, k\,<\,0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}} e^{-2\pi ijm(z_1-z_2)} e^{\pi ik(z_1-z_2)} \\
&\quad \times \ \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z_1+z_2)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\text{LHS of (3.2)} \ = \ \text{(I)}+\text{(II)} \\
&= \ q^{-\frac{1}{16(m+\frac{1}{2})}} \underbrace{\sum_{j\in\mathbf{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ijm(z_1-z_2)}}_{\substack{|| \\ e^{\frac{-\pi im}{2m+1}(z_1-z_2)}\,\theta^{(-)}_{\frac{1}{2},\,m+\frac{1}{2}}\left(\tau,\ \frac{m}{m+\frac{1}{2}}(z_1-z_2)\right)}} \ \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) \\
&- \ q^{-\frac{1}{16(m+\frac{1}{2})}} \Bigg[\sum_{\substack{j,\,k\,\in\mathbf{Z} \\ 0\,\leq\, k\,<\,2mj}} - \sum_{\substack{j,\,k\,\in\mathbf{Z} \\ 2mj\,\leq\, k\,<\,0}} \Bigg] \\
&\quad \times \ (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}} e^{-2\pi ijm(z_1-z_2)} e^{\pi ik(z_1-z_2)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z_1+z_2)
\end{aligned} \tag{3.3a}$$

Next we compute the RHS of (3.2) by using Lemma 2.1 in [19]:

$$\begin{aligned}
&\text{RHS of (3.2)} \\
&= \ e^{-\pi iz_1} \sum_{k\in\mathbf{Z}} (-1)^k q^{(m+\frac{1}{2})k^2+\frac{1}{2}k} \, e^{-2\pi ikm(z_1+z_2)} \underbrace{\Phi_1^{(-)[\frac{1}{2},\frac{1}{2}]}(\tau,\ z_1,\ -z_1+2k\tau,\ 0)}_{\substack{|| \\ -\,i\,e^{-2\pi ikz_1}\,\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,\,z_1)}}}
\end{aligned}$$

$$+\ e^{\pi i z_2}\sum_{k\in\mathbf{Z}}(-1)^k q^{(m+\frac12)k^2+\frac12 k}\, e^{2\pi i k m(z_1+z_2)}\underbrace{\Phi_1^{(-)[\frac12,\frac12]}(\tau,\,-z_2,\,z_2+2k\tau,\,0)}_{\substack{\parallel \\ -i\,e^{2\pi i k z_2}\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,-z_2)}}}$$

$$= \ -\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2(m+\frac12)}(z_1-z_2)}$$

$$\times\ \underbrace{\sum_{k\in\mathbf{Z}}(-1)^k\,q^{(m+\frac12)(k+\frac{1}{4(m+\frac12)})^2}\,e^{-2\pi i(m+\frac12)(k+\frac{1}{4(m+\frac12)})\cdot\frac{m(z_1+z_2)+z_1}{m+\frac12}}}_{\substack{\parallel \\ \theta^{(-)}_{\frac12,m+\frac12}\left(\tau,\ -\frac{m(z_1+z_2)+z_1}{m+\frac12}\right)}}\ \frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_1)}$$

$$+\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2(m+\frac12)}(z_1-z_2)}$$

$$\times\ \underbrace{\sum_{k\in\mathbf{Z}}(-1)^k\,q^{(m+\frac12)(k+\frac{1}{4(m+\frac12)})^2}\,e^{2\pi i(m+\frac12)(k+\frac{1}{4(m+\frac12)})\cdot\frac{m(z_1+z_2)+z_2}{m+\frac12}}}_{\substack{\parallel \\ \theta^{(-)}_{\frac12,m+\frac12}\left(\tau,\ \frac{m(z_1+z_2)+z_2}{m+\frac12}\right)}}\ \frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_2)}$$

$$= \ -\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2m+1}(z_1-z_2)}\theta^{(-)}_{-\frac12,m+\frac12}\Big(\tau,\ \ z_1+z_2+\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_1)}$$

$$+\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2m+1}(z_1-z_2)}\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ z_1+z_2-\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_2)} \tag{3.3b}$$

Then by (3.3a) and (3.3b), we have

$$q^{-\frac{1}{16(m+\frac12)}}\,e^{\frac{-\pi i m}{2m+1}(z_1-z_2)}\,\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ \frac{m(z_1-z_2)}{m+\frac12}\Big)\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$-\ q^{-\frac{1}{16(m+\frac12)}}\Bigg[\sum_{\substack{j,k\in\mathbf{Z}\\ 0\le k<2mj}}-\sum_{\substack{j,k\in\mathbf{Z}\\ 2mj\le k<0}}\Bigg]$$

$$\times\ (-1)^j q^{(m+\frac12)(j-\frac{1}{4(m+\frac12)})^2}q^{-\frac{k^2}{4m}}\,e^{\pi i(k-2jm)(z_1-z_2)}\,\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z_1+z_2)$$

$$= \ -\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2m+1}(z_1-z_2)}\theta^{(-)}_{-\frac12,m+\frac12}\Big(\tau,\ \ z_1+z_2+\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_1)}$$

$$+\ i\,q^{-\frac{1}{16(m+\frac12)}}\,e^{-\frac{\pi i m}{2m+1}(z_1-z_2)}\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ z_1+z_2-\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_2)}$$

Multiplying $q^{\frac{1}{16(m+\frac12)}}\,e^{\frac{\pi i m}{2m+1}(z_1-z_2)}$ to both sides, we have

$$\theta^{(-)}_{\frac12,m+\frac12}\Big(\tau,\ \ \frac{m(z_1-z_2)}{m+\frac12}\Big)\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$= \Bigg[\sum_{\substack{j,k \in \mathbf{Z} \\ 0 \le k < 2mj}} - \sum_{\substack{j,k \in \mathbf{Z} \\ 2mj \le k < 0}} \Bigg] (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}}$$

$$\times \ e^{-2\pi i m (j-\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} e^{\pi i k(z_1-z_2)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z_1+z_2)$$

$$- \ i\,\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}\Big(\tau, \ \ z_1+z_2+\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_1)}$$

$$+ \ i\,\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}\Big(\tau, \ \ z_1+z_2-\frac{z_1-z_2}{2m+1}\Big)\frac{\eta(\tau)^3}{\vartheta_{11}(\tau,z_2)} \tag{3.4a}$$

The 1st term in the RHS of this equation (3.4a) is rewritten by putting $j=-j'$ and $k=-k'$ as follows:

the 1st term in the RHS of (3.4a)

$$= \Bigg[\sum_{\substack{j',k' \in \mathbf{Z} \\ 2mj' < k' \le 0}} - \sum_{\substack{j',k' \in \mathbf{Z} \\ 0 < k' \le 2mj'}} \Bigg] (-1)^{j'} q^{(m+\frac{1}{2})(j'+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k'^2}{4m}}$$

$$\times \ e^{2\pi i m (j'+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} e^{-\pi i k'(z_1-z_2)} \big[\theta_{-k',m}+\theta_{k',m}\big](\tau, z_1+z_2) \tag{3.4b}$$

Then by (3.4a) and (3.4b) we obtain (3.1), proving Proposition 3.1. □

In order to rewrite the formula (3.1), we use the following:

Note 3.1. *For $m \in \frac{1}{2}\mathbf{N}$, the following formula holds:*

$$\Bigg[\sum_{\substack{j,k \in \mathbf{Z} \\ 0 < k \le 2mj}} - \sum_{\substack{j,k \in \mathbf{Z} \\ 2mj < k \le 0}} \Bigg] (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{k^2}{4m}} \ e^{2\pi i m (j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)}$$

$$\times \ e^{-\pi i k(z_1-z_2)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z_1+z_2)$$

$$= \Bigg[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 < r \le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j < r \le 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 \le s \le m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{(2mr-s)^2}{4m}}$$

$$\times \ e^{2\pi i m (j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} e^{-\pi i(2mr-s)(z_1-z_2)} \big[\theta_{s,m}+\theta_{-s,m}\big](\tau, z_1+z_2)$$

$$+ \Bigg[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j \le r < 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 < s < m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{(2mr+s)^2}{4m}}$$

$$\times \ e^{2\pi i m (j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} e^{-\pi i(2mr+s)(z_1-z_2)} \big[\theta_{s,m}+\theta_{-s,m}\big](\tau, z_1+z_2) \tag{3.5}$$

Proof. In order to prove (3.5), we need only to note the following for $j \in \mathbf{Z}$:

$$\Big\{k \in \mathbf{Z} \ ; \ 0 < k \le 2mj\Big\} \ = \ \left\{k = 2mr - s \ ; \ r, s \in \mathbf{Z} \ \text{ and } \ \begin{matrix} 0 < r \leqq j \\ 0 \le s < 2m \end{matrix} \right\}$$

$$\Big\{k \in \mathbf{Z} \ ; \ 2mj < k \le 0\Big\} \ = \ \left\{k = 2mr - s \ ; \ r, s \in \mathbf{Z} \ \text{ and } \ \begin{matrix} j < r \le 0 \\ 0 \le s < 2m \end{matrix} \right\}$$

Then the LHS of (3.5) is rewritten as follows:

$$\text{LHS of (3.5)} \ = \ \text{(I)} + \text{(II)}$$

where

$$\begin{aligned} \text{(I)} \ :=& \ \Bigg[\sum_{\substack{j,r \in \mathbf{Z} \\ 0<r\le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j<r\le 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 \le s \le m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2mr-s)^2}{4m}} \\ & \times \ e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i(2mr-s)(z_1-z_2)} \, \big[\theta_{-s,m} + \theta_{s,m}\big](\tau, z_1+z_2) \end{aligned} \tag{3.6a}$$

$$\begin{aligned} \text{(II)} \ :=& \ \Bigg[\sum_{\substack{j,r \in \mathbf{Z} \\ 0<r\le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j<r\le 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ m < s < 2m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2mr-s)^2}{4m}} \\ & \times \ e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i(2mr-s)(z_1-z_2)} \, \big[\theta_{-s,m} + \theta_{s,m}\big](\tau, z_1+z_2) \end{aligned}$$

Putting $s' := 2m - s$, we have

$$m \ < \ s \ < \ 2m \quad \Longleftrightarrow \quad 0 \ < \ s' \ < \ m$$

so (II) is rewritten as follows:

$$\begin{aligned} \text{(II)} \ =& \ \Bigg[\sum_{\substack{j,r \in \mathbf{Z} \\ 0<r\le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j<r\le 0}} \Bigg] \sum_{\substack{s' \in \mathbf{Z} \\ 0 < s' < m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2m(r-1)+s')^2}{4m}} \\ & \times \ e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i(2m(r-1)+s')(z_1-z_2)} \, \big[\theta_{s',m} + \theta_{-s',m}\big](\tau, z_1+z_2) \\ \underset{\substack{\uparrow \\ r-1=r'}}{=}& \ \Bigg[\sum_{\substack{j,r' \in \mathbf{Z} \\ 0\le r'< j}} - \sum_{\substack{j,r' \in \mathbf{Z} \\ j\le r'< 0}} \Bigg] \sum_{\substack{s' \in \mathbf{Z} \\ 0 < s' < m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2mr'+s')^2}{4m}} \\ & \times \ e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i(2mr'+s')(z_1-z_2)} \, \big[\theta_{s',m} + \theta_{-s',m}\big](\tau, z_1+z_2) \end{aligned} \tag{3.6b}$$

Then by (3.6a) and (3.6b), we have

$$\text{(I)} + \text{(II)} \ = \ \text{RHS of (3.5)}\,, \qquad \text{proving Note 3.1.}$$

□

By Note 3.1, the formula (3.1) in Proposition 3.1 is rewritten as follows:

Proposition 3.2. *For* $m \in \frac{1}{2}\mathbf{N}$, $\Phi^{[m,0]\,*}(\tau, z_1, z_2, 0)$ *is given by the following formula:*

$$\begin{aligned}
& \theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ \frac{m(z_1-z_2)}{m+\frac{1}{2}}\Big) \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) \\
&= \ i\,\eta(\tau)^3 \Bigg\{ - \frac{\theta^{(-)}_{-\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ z_1+z_2+\dfrac{z_1-z_2}{2m+1}\Big)}{\vartheta_{11}(\tau, z_1)} + \frac{\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}\Big(\tau, \ z_1+z_2-\dfrac{z_1-z_2}{2m+1}\Big)}{\vartheta_{11}(\tau, z_2)} \Bigg\} \\
&\quad - \Bigg[\sum_{\substack{j, r \in \mathbf{Z} \\ 0<r\leq j}} - \sum_{\substack{j, r \in \mathbf{Z} \\ j<r\leq 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 \leq s \leq m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2mr-s)^2}{4m}} \\
&\quad \times \ e^{2\pi i m (j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i (2mr-s)(z_1-z_2)} \, \big[\theta_{s,m}+\theta_{-s,m}\big](\tau, z_1+z_2) \\
&\quad - \Bigg[\sum_{\substack{j, r \in \mathbf{Z} \\ 0\leq r< j}} - \sum_{\substack{j, r \in \mathbf{Z} \\ j\leq r< 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 < s < m}} (-1)^j \, q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} \, q^{-\frac{(2mr+s)^2}{4m}} \\
&\quad \times \ e^{2\pi i m (j+\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)} \, e^{-\pi i (2mr+s)(z_1-z_2)} \, \big[\theta_{s,m}+\theta_{-s,m}\big](\tau, z_1+z_2)
\end{aligned} \tag{3.7}$$

4 $\Phi^{[m,0]\,*}(\tau, z_1, z_2, t) \sim$ the case $z_1 - z_2 = 2a\tau + 2b$

4.1 $\Phi^{[m,0]\,*}(\tau, z_1, z_2, t) \sim$ the case $z_1 - z_2 = (1+2p)\tau - 1$

Lemma 4.1. *For* $m \in \mathbf{N}$ *and* $p \in \mathbf{Z}$, *the following formula holds:*

$$\begin{aligned}
& \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \, \Phi^{[m,0]\,*}\Big(\tau, \ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}+p\tau, \ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}-p\tau, \ 0\Big) \\
&= \ q^{\frac{m}{4}(2p+1)^2} \, \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau, z)} \cdot \big[\theta_{p, m+\frac{1}{2}} + \theta_{-p, m+\frac{1}{2}}\big](\tau, 0) \\
&\quad - \Bigg[\sum_{\substack{j, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\leq r< j}} - \sum_{\substack{j, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j\leq r< 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \\
&\qquad\qquad \times \ q^{-\frac{1}{4m}[2mr+k+2mp]^2 + \frac{m}{4}(2p+1)^2} \, \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z) \\
&\quad - \Bigg[\sum_{\substack{j, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\leq r\leq j}} - \sum_{\substack{j, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j< r< 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \leq k \leq m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \\
&\qquad\qquad \times \ q^{-\frac{1}{4m}[2mr-k+2mp]^2 + \frac{m}{4}(2p+1)^2} \, \big[\theta_{k,m}+\theta_{-k,m}\big](\tau, z)
\end{aligned}$$

$$+\ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le m}} (-1)^k q^{-\frac{1}{4m}k^2+\frac{k}{2}(2p+1)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \tag{4.1}$$

Proof. Letting $\begin{cases} z_1 &= \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}+p\tau \\ z_2 &= \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}-p\tau \end{cases}$ namely $\begin{cases} z_1+z_2 &= z \\ z_1-z_2 &= (2p+1)\tau-1 \end{cases}$ in the formula (3.7) in Proposition 3.2, we have

$$\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\ \frac{m((2p+1)\tau-1)}{m+\frac{1}{2}}\Big)\,\Phi^{[m,0]*}(\tau,z_1,z_2,0)$$

$$= \ i\,\eta(\tau)^3\Bigg\{\underbrace{-\frac{\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\ z+\frac{(2p+1)\tau-1}{2m+1}\Big)}{\vartheta_{11}(\tau,z_1)}+\frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\ z-\frac{(2p+1)\tau-1}{2m+1}\Big)}{\vartheta_{11}(\tau,z_2)}}_{\substack{||\ put \\ \text{(I)}}}\Bigg\}$$

$$- \Big[\sum_{\substack{j,r\in\mathbf{Z}\\0<r\le j}}-\sum_{\substack{j,r\in\mathbf{Z}\\j<r\le 0}}\Big]\sum_{\substack{s\in\mathbf{Z}\\0\le s\le m}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2}\,q^{-\frac{(2mr-s)^2}{4m}}$$

$$\times\ e^{2\pi im(j+\frac{1}{4(m+\frac{1}{2})})((2p+1)\tau-1)}\,e^{-\pi i(2mr-s)((2p+1)\tau-1)}\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$- \Big[\sum_{\substack{j,r\in\mathbf{Z}\\0\le r<j}}-\sum_{\substack{j,r\in\mathbf{Z}\\j\le r<0}}\Big]\sum_{\substack{s\in\mathbf{Z}\\0<s<m}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2}\,q^{-\frac{(2mr+s)^2}{4m}}$$

$$\times\ e^{2\pi im(j+\frac{1}{4(m+\frac{1}{2})})((2p+1)\tau-1)}\,e^{-\pi i(2mr+s)((2p+1)\tau-1)}\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$= \ i\,\eta(\tau)^3\times\text{(I)}$$

$$- \ e^{-\frac{\pi im}{2m+1}}\Big[\sum_{\substack{j,r\in\mathbf{Z}\\0<r\le j}}-\sum_{\substack{j,r\in\mathbf{Z}\\j<r\le 0}}\Big]\sum_{\substack{s\in\mathbf{Z}\\0\le s\le m}}(-1)^{j+s}\,q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2+m(j+\frac{1}{4(m+\frac{1}{2})})(2p+1)}$$

$$\times\ q^{-\frac{(2mr-s)^2}{4m}-\frac{1}{2}(2mr-s)(2p+1)}\,\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$- \ e^{-\frac{\pi im}{2m+1}}\Big[\sum_{\substack{j,r\in\mathbf{Z}\\0\le r<j}}-\sum_{\substack{j,r\in\mathbf{Z}\\j\le r<0}}\Big]\sum_{\substack{s\in\mathbf{Z}\\0<s<m}}(-1)^{j+s}\,q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2+m(j+\frac{1}{4(m+\frac{1}{2})})(2p+1)}$$

$$\times\ q^{-\frac{(2mr+s)^2}{4m}-\frac{1}{2}(2mr+s)(2p+1)}\,\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z) \tag{4.2}$$

The LHS of this equation (4.2) becomes by Note 2.1 as follows:

$$\text{LHS of (4.2)} \ = \ e^{-\frac{\pi im}{2m+1}}q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\cdot\Phi^{[m,0]*}(\tau,z_1,z_2,0)$$

Also (I) is computed by using Notes 2.2 and 2.3 as follows:

$$(\mathrm{I}) \;=\; -\,\frac{\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\;\; z+\dfrac{(2p+1)\tau-1}{2m+1}\Big)}{\vartheta_{11}\Big(\tau,\; \dfrac{z}{2}+\dfrac{(2p+1)\tau-1}{2}\Big)} \;+\; \frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\;\; z-\dfrac{(2p+1)\tau-1}{2m+1}\Big)}{\vartheta_{11}\Big(\tau,\; \dfrac{z}{2}-\dfrac{(2p+1)\tau-1}{2}\Big)}$$

$$= \;-\;\frac{e^{\frac{\pi i}{2(2m+1)}}\, q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2}\; e^{-\frac{\pi i}{2}(2p+1)z}\theta_{p,m+\frac{1}{2}}(\tau,0)}{q^{-\frac{1}{8}(2p+1)^2}e^{-\frac{\pi i}{2}(2p+1)z}\theta_{0,\frac{1}{2}}(\tau,z)}$$

$$+\;\frac{e^{\frac{\pi i}{2(2m+1)}}\, q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2}\; e^{\frac{\pi i}{2}(2p+1)z}\theta_{-p,m+\frac{1}{2}}(\tau,0)}{-\,q^{-\frac{1}{8}(2p+1)^2}e^{\frac{\pi i}{2}(2p+1)z}\theta_{0,\frac{1}{2}}(\tau,z)}$$

$$= \;-\,e^{\frac{\pi i}{2(2m+1)}}\, q^{\frac{m}{8(m+\frac{1}{2})}(2p+1)^2}\;\frac{1}{\theta_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta_{p,m+\frac{1}{2}}+\theta_{-p,m+\frac{1}{2}}\big](\tau,0)$$

Then substituting these into (4.2) and multiplying $e^{\frac{\pi i m}{2m+1}}$ and rewriting the 2nd terms in the RHS of (4.2) by using

$$\sum_{\substack{j,r\in\mathbf{Z}\\ 0<r\leq j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r\leq 0}} \;=\; \Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\leq r\leq j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg] - \sum_{\substack{r=0\\ j\in\mathbf{Z}}} \tag{4.3}$$

the above formula (4.2) becomes as follows:

$$q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\,\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$= \;\; q^{\frac{m}{8(m+\frac{1}{2})}(2p+1)^2}\;\frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta_{p,m+\frac{1}{2}}+\theta_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$-\;\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\leq r\leq j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg]\sum_{\substack{s\in\mathbf{Z}\\ 0\leq s\leq m}}(-1)^{j+s}\,q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2-\frac{m^2}{2(2m+1)}(2p+1)^2}$$

$$\times\;q^{-\frac{1}{4m}[2m(r+\frac{1}{2})-s+2mp]^2+\frac{m}{4}(2p+1)^2}\,\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$+\;\underbrace{\sum_{j\in\mathbf{Z}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2}}_{\substack{||\\ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)}}\,q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}$$

$$\times\;\sum_{\substack{s\in\mathbf{Z}\\ 0\leq s\leq m}}(-1)^s q^{-\frac{1}{4m}s^2+\frac{s}{2}(2p+1)}\,\big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$- \Bigg[\sum_{\substack{j,\, r \in \mathbf{Z} \\ 0 \le r < j}} - \sum_{\substack{j,\, r \in \mathbf{Z} \\ j \le r < 0}} \Bigg] \sum_{\substack{s \in \mathbf{Z} \\ 0 < s < m}} (-1)^{j+s} \, q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2 - \frac{m^2}{2(2m+1)}(2p+1)^2}$$

$$\times \; q^{-\frac{1}{4m}[2m(r+\frac{1}{2})+s+2mp]^2 + \frac{m}{4}(2p+1)^2} \big[\theta_{s,m} + \theta_{-s,m}\big](\tau, z) \tag{4.4}$$

Multiplying $q^{\frac{m^2}{2(2m+1)}(2p+1)^2}$ and replacing $\sum_{j,r \in \mathbf{Z}}$ with $\sum_{j',r' \in \frac{1}{2}\mathbf{Z}_{\rm odd}}$ by putting $j + \frac{1}{2} = j'$ and $r + \frac{1}{2} = r'$ in the above formula (4.4), we obtain

$$\theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \, \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0)$$

$$= \; q^{\frac{m}{4}(2p+1)^2} \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau, z)} \cdot \big[\theta_{p, m+\frac{1}{2}} + \theta_{-p, m+\frac{1}{2}}\big](\tau, 0)$$

$$- \Bigg[\sum_{\substack{j',\, r' \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r' \le j'}} - \sum_{\substack{j',\, r' \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j' < r' < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j'-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr'-k+2mp]^2 + \frac{m}{4}(2p+1)^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

$$+ \; \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^k q^{-\frac{1}{4m}k^2 + \frac{k}{2}(2p+1)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

$$- \Bigg[\sum_{\substack{j',\, r' \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r' < j'}} - \sum_{\substack{j',\, r' \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j' \le r' < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j'-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr'+k+2mp]^2 + \frac{m}{4}(2p+1)^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

proving Lemma 4.1. $\square$

4.2 $\Phi^{[m,0]\,*}(\tau, z_1, z_2, t) \sim$ the case $z_1 - z_2 = (1+2p)\tau$

Lemma 4.2. *For $m \in \mathbf{N}$ and $p \in \mathbf{Z}$, the following formula holds:*

$$\theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \, \Phi^{[m,0]\,*}\Big(\tau, \;\; \frac{z}{2} + \frac{\tau}{2} + p\tau, \;\; \frac{z}{2} - \frac{\tau}{2} + p\tau, \;\; 0\Big)$$

$$= \;\; (-1)^p \, q^{\frac{m}{4}(2p+1)^2} \frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau, z)} \cdot \big[\theta^{(-)}_{p, m+\frac{1}{2}} + \theta^{(-)}_{-p, m+\frac{1}{2}}\big](\tau, 0)$$

$$- \Bigg[\sum_{\substack{j,\, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r < j}} - \sum_{\substack{j,\, r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr+k+2mp]^2 + \frac{m}{4}(2p+1)^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

$$- \Big[\sum_{\substack{j,r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac12 \mathbf{Z}_{\rm odd} \\ j < r < 0}} \Big] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac12} \, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2} \times \, q^{-\frac{1}{4m}[2mr-k+2mp]^2 + \frac{m}{4}(2p+1)^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$+ \, \theta^{(-)}_{2mp+m+\frac12, m+\frac12}(\tau,0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} q^{-\frac{1}{4m}k^2 + \frac{k}{2}(2p+1)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z) \tag{4.5}$$

Proof. Letting $\begin{cases} z_1 &= \frac{z}{2} + \frac{\tau}{2} + p\tau \\ z_2 &= \frac{z}{2} - \frac{\tau}{2} - p\tau \end{cases}$ namely $\begin{cases} z_1 + z_2 &= z \\ z_1 - z_2 &= (2p+1)\tau \end{cases}$ in the formula (3.7) in Proposition 3.2, we have

$$\theta^{(-)}_{\frac12, m+\frac12}\Big(\tau, \ \frac{m(2p+1)\tau}{m+\frac12}\Big) \, \Phi^{[m,0]*}(\tau, z_1, z_2, 0)$$

$$= \ i\,\eta(\tau)^3 \Bigg\{ \underbrace{- \frac{\theta^{(-)}_{-\frac12, m+\frac12}\Big(\tau, \ z + \dfrac{(2p+1)\tau}{2m+1}\Big)}{\vartheta_{11}(\tau, z_1)} + \frac{\theta^{(-)}_{\frac12, m+\frac12}\Big(\tau, \ z - \dfrac{(2p+1)\tau}{2m+1}\Big)}{\vartheta_{11}(\tau, z_2)}}_{\rm (I)} \Bigg\}$$

$$- \Big[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 < r \le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j < r \le 0}} \Big] \sum_{\substack{s \in \mathbf{Z} \\ 0 \le s \le m}} (-1)^j \, q^{(m+\frac12)(j+\frac{1}{4(m+\frac12)})^2} \, q^{-\frac{(2mr-s)^2}{4m}} \times \, e^{2\pi i m (j+\frac{1}{4(m+\frac12)})(2p+1)\tau} \, e^{-\pi i (2mr-s)(2p+1)\tau} \, \big[\theta_{s,m} + \theta_{-s,m}\big](\tau,z)$$

$$- \Big[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j \le r < 0}} \Big] \sum_{\substack{s \in \mathbf{Z} \\ 0 < s < m}} (-1)^j \, q^{(m+\frac12)(j+\frac{1}{4(m+\frac12)})^2} \, q^{-\frac{(2mr+s)^2}{4m}} \times \, e^{2\pi i m (j+\frac{1}{4(m+\frac12)})(2p+1)\tau} \, e^{-\pi i (2mr+s)(2p+1)\tau} \, \big[\theta_{s,m} + \theta_{-s,m}\big](\tau,z)$$

$$= \ i\,\eta(\tau)^3 \, \times \, {\rm (I)}$$

$$- \Big[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 < r \le j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j < r \le 0}} \Big] \sum_{\substack{s \in \mathbf{Z} \\ 0 \le s \le m}} (-1)^j \, q^{(m+\frac12)(j+\frac{1}{4(m+\frac12)})^2 + m(j+\frac{1}{4(m+\frac12)})(2p+1)} \times \, q^{-\frac{(2mr-s)^2}{4m} - \frac12 (2mr-s)(2p+1)} \, \big[\theta_{s,m} + \theta_{-s,m}\big](\tau,z)$$

$$- \Big[\sum_{\substack{j,r \in \mathbf{Z} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \mathbf{Z} \\ j \le r < 0}} \Big] \sum_{\substack{s \in \mathbf{Z} \\ 0 < s < m}} (-1)^j \, q^{(m+\frac12)(j+\frac{1}{4(m+\frac12)})^2 + m(j+\frac{1}{4(m+\frac12)})(2p+1)}$$

$$\times\ q^{-\frac{(2mr+s)^2}{4m}-\frac{1}{2}(2mr+s)(2p+1)}\ \big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z) \tag{4.6}$$

The LHS of this equation (4.6) becomes by Note 2.1 as follows

$$\text{LHS of (4.6)} \ = \ q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\,\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

Also (I) is computed by using Notes 2.2 and 2.3 as follows:

$$\text{(I)} \ = \ -\frac{\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\ \ z+\dfrac{(2p+1)\tau}{2m+1}\Big)}{\vartheta_{11}\Big(\tau,\ \dfrac{z}{2}+\dfrac{(2p+1)\tau}{2}\Big)} + \frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}\Big(\tau,\ \ z-\dfrac{(2p+1)\tau}{2m+1}\Big)}{\vartheta_{11}\Big(\tau,\ \dfrac{z}{2}-\dfrac{(2p+1)\tau}{2}\Big)}$$

$$= \ -\ \frac{q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2}\,e^{-\frac{\pi i}{2}(2p+1)z}\theta^{(-)}_{p,m+\frac{1}{2}}(\tau,0)}{-i\,(-1)^pq^{-\frac{1}{8}(2p+1)^2}e^{-\frac{\pi i}{2}(2p+1)z}\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} + \frac{q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2}\,e^{\frac{\pi i}{2}(2p+1)z}\theta^{(-)}_{-p,m+\frac{1}{2}}(\tau,0)}{i\,(-1)^pq^{-\frac{1}{8}(2p+1)^2}e^{\frac{\pi i}{2}(2p+1)z}\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}$$

$$= \ -\,i\,(-1)^p\,q^{\frac{m}{8(m+\frac{1}{2})}(2p+1)^2}\,\frac{1}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta^{(-)}_{p,m+\frac{1}{2}}+\theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

Then substituting these into (4.6) and rewriting the 2nd term in the RHS of (4.6) by using (4.3), the above formula (4.6) becomes as follows:

$$q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\,\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$= \ \ (-1)^p\,q^{\frac{m}{8(m+\frac{1}{2})}(2p+1)^2}\,\frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta^{(-)}_{p,m+\frac{1}{2}}+\theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$-\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r\le j}}-\sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg]\sum_{\substack{s\in\mathbf{Z}\\ 0\le s\le m}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2-\frac{m^2}{4(m+\frac{1}{2})}(2p+1)^2}$$

$$\times\ q^{-\frac{1}{4m}[2m(r+\frac{1}{2})-s+2mp]^2+\frac{m}{4}(2p+1)^2}\ \big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$+\ \underbrace{\sum_{j\in\mathbf{Z}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2}}_{\substack{||\\ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)}}\ q^{-\frac{m^2}{4(m+\frac{1}{2})}(2p+1)^2}$$

$$\times\ \sum_{\substack{s\in\mathbf{Z}\\ 0\le s\le m}}q^{-\frac{1}{4m}s^2+\frac{s}{2}(2p+1)}\ \big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z)$$

$$-\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r<j}}-\sum_{\substack{j,r\in\mathbf{Z}\\ j\le r<0}}\Bigg]\sum_{\substack{s\in\mathbf{Z}\\ 0<s<m}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{1}{2}+\frac{2pm}{2m+1})^2-\frac{m^2}{2(2m+1)}(2p+1)^2}$$

$$\times \; q^{-\frac{1}{4m}[2m(r+\frac{1}{2})+s+2mp]^2+\frac{m}{4}(2p+1)^2} \big[\theta_{s,m}+\theta_{-s,m}\big](\tau,z) \tag{4.7}$$

Multiplying $q^{\frac{m^2}{2(2m+1)}(2p+1)^2}$ and replacing $\sum_{j,r\in\mathbf{Z}}$ with $\sum_{j',r'\in\frac{1}{2}\mathbf{Z}_{\rm odd}}$ by putting $j+\frac{1}{2}=j'$ and $r+\frac{1}{2}=r'$ in the above formula (4.7), we obtain

$$\begin{aligned}
&\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\,\Phi^{[m,0]\,*}(\tau,z_1,z_2,0) \\
&= (-1)^p\,q^{\frac{m}{4}(2p+1)^2}\,\frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta^{(-)}_{p,m+\frac{1}{2}}+\theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0) \\
&\quad - \Bigg[\sum_{\substack{j',r'\in\frac{1}{2}\mathbf{Z}_{\rm odd}\\ 0\le r'<j'}} - \sum_{\substack{j',r'\in\frac{1}{2}\mathbf{Z}_{\rm odd}\\ j'\le r'<0}}\Bigg]\sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}}(-1)^{j'-\frac{1}{2}}\,q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2} \\
&\qquad\qquad \times\; q^{-\frac{1}{4m}[2mr'+k+2mp]^2+\frac{m}{4}(2p+1)^2}\,\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\
&\quad - \Bigg[\sum_{\substack{j',r'\in\frac{1}{2}\mathbf{Z}_{\rm odd}\\ 0\le r'\le j'}} - \sum_{\substack{j',r'\in\frac{1}{2}\mathbf{Z}_{\rm odd}\\ j'<r'<0}}\Bigg]\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le m}}(-1)^{j'-\frac{1}{2}}\,q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2} \\
&\qquad\qquad \times\; q^{-\frac{1}{4m}[2mr'-k+2mp]^2+\frac{m}{4}(2p+1)^2}\,\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\
&\quad + \underbrace{\sum_{j\in\mathbf{Z}}(-1)^j\,q^{(m+\frac{1}{2})(j+\frac{m+\frac{1}{2}+2pm}{2m+1})^2}}_{\substack{||\\ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)}}\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le k\le m}} q^{-\frac{1}{4m}k^2+\frac{k}{2}(2p+1)}\,\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)
\end{aligned}$$

proving Lemma 4.2. □

5 $\Phi^{[m,0]\,*}_{\rm add}(\tau,z_1,z_2,t)\sim$ the case $z_1-z_2=2a\tau+2b$

Lemma 5.1. *Let $m\in\frac{1}{2}\mathbf{N}$, $j,a\in\frac{1}{2}\mathbf{Z}$ and $b\in\mathbf{Q}$ such that $4mb\in\mathbf{Z}$. Then the functions $P_{j,m}(\tau,z):=P^{(+)}_{j,m}(\tau,z)$ and $Q_{j,m}(\tau,z):=Q^{(+)}_{j,m}(\tau,z)$ defined by the formulas (4.1a) and (4.1b) in [17] satisfy the following:*

1) $P_{j,m}(\tau,a\tau+b)+e^{4\pi ijb}\,e^{8\pi imab}\,P_{-j,m}(\tau,a\tau+b) \;=\; 0$

2) $$\begin{aligned}
&Q_{j,m}(\tau,a\tau+b) + e^{4\pi ijb}\,e^{8\pi imab}Q_{-j,m}(\tau,a\tau+b) \\
&\quad= e^{2\pi ijb}\,e^{8\pi imab}\sum_{\substack{k,\in\mathbf{Z}\\ 0\le k\le 2a}} e^{4\pi imbk}q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \\
&\qquad + e^{2\pi ijb}\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le 2a}} e^{4\pi imbk}q^{-\frac{1}{4m}(j-2m(2a-k))(j+2mk)}
\end{aligned}$$

Proof. The claim 1) follows from (4.4) in [17] and the claim 2) follows from (4.6b) in [17], since $e^{-8\pi imab} = e^{8\pi imab}$. □

Lemma 5.2. *Let $m \in \mathbf{N}$, $a, b \in \frac{1}{2}\mathbf{Z}$ and $j \in \mathbf{Z}$. Then*

$$1)\quad R_{j,m}(\tau,\, a\tau+b) + R_{2m-j,m}(\tau,\, a\tau+b) \;=\; 2\,e^{2\pi ijb} \sum_{\substack{k\in\mathbf{Z} \\ 1\le k\le 2a}} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)}$$

$$2)\quad R_{0,m}(\tau,\, a\tau+b) \;=\; \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2a}} q^{mk(2a-k)}$$

Proof. The claim 1) is obtained immediately from Lemma 4.1 in [17].

To prove the claim 2) we note, by letting $j = 0$ in Lemma 5.1, that

$$P_{0,m}(\tau,\, a\tau+b) \;=\; 0$$

and that

$$\begin{aligned} Q_{0,m}(\tau,\, a\tau+b) \;&=\; \frac{1}{2} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2a}} \Big\{ \underbrace{q^{-\frac{1}{4m}\cdot 2m(2a-k)(-2mk)} + q^{\frac{1}{4m}\cdot 2m(2a-k)\,2mk}}_{\substack{\| \\ 2\,q^{mk(2a-k)}}} \Big\} \\ &=\; \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2a}} q^{mk(2a-k)} \end{aligned}$$

Thus we have

$$R_{0,m}(\tau,\, a\tau+b) \;=\; P_{0,m}(\tau,\, a\tau+b) + Q_{0,m}(\tau,\, a\tau+b) \;=\; \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2a}} q^{mk(2a-k)}$$

proving Lemma 5.2. □

Using the above Lemmas 5.1 and 5.2, the Zwegers' additional function

$$\Phi^{[m,0]\,*}_{\text{add}}(\tau, z_1, z_2, 0) \;:=\; \Phi^{[m,0]}_{1,\,\text{add}}(\tau, z_1, z_2, 0) + \Phi^{[m,0]}_{2,\,\text{add}}(\tau, z_1, z_2, 0)$$

is obtained as follows:

Proposition 5.1. *Let $m \in \mathbf{N}$ and $a, b \in \frac{1}{2}\mathbf{Z}$. Then, for z_1 and z_2 satisfying $z_1 - z_2 = 2a\tau + 2b$, the correction function $\Phi^{[m,0]\,*}_{\text{add}}(\tau, z_1, z_2, 0)$ is given by the following formula:*

$$\begin{aligned} \Phi^{[m,0]\,*}_{\text{add}}(\tau, z_1, z_2, 0) \;&=\; - \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2a}} q^{mk(2a-k)}\, \theta_{0,m}(\tau, z_1+z_2) \\ &- \frac{1}{2} \sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} e^{2\pi ijb} \sum_{\substack{k\in\mathbf{Z} \\ 1\le k\le 2a}} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \big[\theta_{j,m} + \theta_{-j,m}\big](\tau, z_1+z_2) \end{aligned} \tag{5.1}$$

Proof. By the formula for $\Phi^{[m,0]}_{i,\mathrm{add}}$ in [17], we have

$$\Phi^{[m,0]*}_{\mathrm{add}}(\tau,z_1,z_2,0) \;=\; -\frac{1}{2}\underbrace{\sum_{\substack{j\in\mathbf{Z}\\ 0\le j<2m}}}_{\substack{\| \\ \sum_{j=0}+\sum_{\substack{j\in\mathbf{Z}\\ 0<j<2m}}}} R_{j,m}(\tau,a\tau+b)\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z_1+z_2)$$

$$= \; -\frac{1}{2}\;\underbrace{R_{0,m}(\tau,a\tau+b)}_{\substack{\| \\ \sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le 2a}} q^{mk(2a-k)} \ \text{by Lemma 5.2}}} \times\; 2\,\theta_{0,m}(\tau,z_1+z_2)$$

$$-\frac{1}{4}\sum_{\substack{j\in\mathbf{Z}\\ 0<j<2m}} R_{j,m}(\tau,a\tau+b)\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z_1+z_2)$$

$$-\frac{1}{4}\sum_{\substack{j\in\mathbf{Z}\\ 0<j<2m}} R_{2m-j,m}(\tau,a\tau+b)\,\big[\underbrace{\theta_{2m-j,m}+\theta_{-(2m-j),m}}_{\substack{\| \\ \theta_{j,m}+\theta_{-j,m}}}\big](\tau,z_1+z_2)$$

$$= \; -\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le 2a}} q^{mk(2a-k)}\,\theta_{0,m}(\tau,z_1+z_2)$$

$$-\frac{1}{4}\sum_{\substack{j\in\mathbf{Z}\\ 0<j<2m}} \big\{R_{2m-j,m}(\tau,a\tau+b)+R_{j,m}(\tau,a\tau+b)\big\}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z_1+z_2)$$

Then, using Lemma 5.2, this is rewritten as follows:

$$= \; -\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le 2a}} q^{mk(2a-k)}\,\theta_{0,m}(\tau,z_1+z_2)$$

$$-\frac{1}{4}\sum_{\substack{j\in\mathbf{Z}\\ 0<j<2m}} 2\,e^{2\pi ijb}\sum_{\substack{k\in\mathbf{Z}\\ 1\le k\le 2a}} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z_1+z_2)$$

$$= \; -\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le 2a}} q^{mk(2a-k)}\,\theta_{0,m}(\tau,z_1+z_2)$$

$$-\frac{1}{2}\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}} e^{2\pi ijb}\sum_{\substack{k\in\mathbf{Z}\\1\le k\le 2a}} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)}\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z_1+z_2)$$

proving Proposition 5.1. □

Using the above Proposition 5.1, the Zwegers's additional functions $\Phi^{[m,0]*}_{\rm add}(\tau,z_1,z_2,0)$ for $(z_1,z_2)=\frac{z}{2}+\frac{\tau}{2}-\frac{1}{2},\,\frac{z}{2}-\frac{\tau}{2}+\frac{1}{2})$ and $(z_1,z_2)=(\frac{z}{2}+\frac{\tau}{2},\,\frac{z}{2}-\frac{\tau}{2})$ are obtained as follows:

Lemma 5.3. *For $m\in\mathbf{N}$, the following formulas hold:*

1) $$\Phi^{[m,0]*}_{\rm add}\Big(\tau,\,\frac{z}{2}+\frac{\tau}{2}-\frac{1}{2},\,\frac{z}{2}-\frac{\tau}{2}+\frac{1}{2},\,0\Big) \;=\; -\sum_{\substack{j\in\mathbf{Z}\\0\le j\le 2m}}(-1)^j\,q^{-\frac{1}{4m}j(j-2m)}\theta_{j,m}(\tau,z)$$

2) $$\Phi^{[m,0]*}_{\rm add}\Big(\tau,\,\frac{z}{2}+\frac{\tau}{2},\,\frac{z}{2}-\frac{\tau}{2},\,0\Big) \;=\; -\sum_{\substack{j\in\mathbf{Z}\\0\le j\le 2m}} q^{-\frac{1}{4m}j(j-2m)}\theta_{j,m}(\tau,z)$$

Proof. 1) In the case $\left\{\begin{array}{lcl} z_1 &=& \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}\\ z_2 &=& \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}\end{array}\right.$, letting $\left\{\begin{array}{lcr} 2a &=& 1\\ 2b &=& -1\end{array}\right.$ in (5.1), we have

$$\begin{aligned}
\Phi^{[m,0]*}_{\rm add}(\tau,z_1,z_2,0) &= -\underbrace{\sum_{\substack{k\in\mathbf{Z}\\0\le k\le 1}} q^{mk(1-k)}}_{2}\;\theta_{0,m}(\tau,z_1+z_2)\\
&\quad -\frac{1}{2}\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}} e^{-\pi ij}\sum_{\substack{k\in\mathbf{Z}\\1\le k\le 1}} q^{-\frac{1}{4m}(j+2m(1-k))(j-2mk)}\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)\\
&= -2\,\theta_{0,m}(\tau,z_1+z_2)\;-\;\frac{1}{2}\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}}(-1)^j\,q^{-\frac{1}{4m}j(j-2m)}\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)\\
&= -2\theta_{0,m}(\tau,z)-\frac{1}{2}\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}}(-1)^j q^{-\frac{1}{4m}j(j-2m)}\theta_{j,m}(\tau,z)-\frac{1}{2}\underbrace{\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}}(-1)^j q^{-\frac{1}{4m}j(j-2m)}\theta_{-j,m}(\tau,z)}_{\substack{||\;\leftarrow\;2m-j=j'\\ \sum_{j\in\mathbf{Z},\,0<j'<2m}(-1)^{j'}q^{-\frac{1}{4m}j'(j'-2m)}\theta_{j',m}(\tau,z)}}\\
&= -2\,\theta_{0,m}(\tau,z)\;-\;\sum_{\substack{j\in\mathbf{Z}\\0<j<2m}}(-1)^j\,q^{-\frac{1}{4m}j(j-2m)}\theta_{j,m}(\tau,z)
\end{aligned}$$

proving 1).

2) In the case $\begin{cases} z_1 &= \frac{z}{2}+\frac{\tau}{2} \\ z_2 &= \frac{z}{2}-\frac{\tau}{2} \end{cases}$, letting $\begin{cases} 2a &= 1 \\ 2b &= 0 \end{cases}$ in (5.1), we have

$$\Phi^{[m,0]\,*}_{\rm add}(\tau, z_1, z_2, 0) = - \underbrace{\sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le 1}} q^{mk(1-k)}}_{2} \theta_{0,m}(\tau, z_1+z_2)$$

$$- \frac{1}{2} \sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} \underbrace{\sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 1}} q^{-\frac{1}{4m}(j+2m(1-k))(j-2mk)}}_{\substack{|| \\ q^{-\frac{1}{4m}j(j-2m)}}} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$$

$$= -2\,\theta_{0,m}(\tau, z_1+z_2) - \frac{1}{2} \sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} q^{-\frac{1}{4m}j(j-2m)} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$$

$$= -2\,\theta_{0,m}(\tau,z) - \frac{1}{2} \sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} q^{-\frac{1}{4m}j(j-2m)} \theta_{j,m}(\tau,z) - \frac{1}{2} \underbrace{\sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} q^{-\frac{1}{4m}j(j-2m)} \theta_{-j,m}(\tau,z)}_{\substack{|| \ \leftarrow \ 2m-j=j' \\ \sum_{\substack{j\in\mathbf{Z} \\ 0<j'<2m}} q^{-\frac{1}{4m}j'(j'-2m)}\theta_{j',m}(\tau,z)}}$$

$$= -2\,\theta_{0,m}(\tau,z) - \sum_{\substack{j\in\mathbf{Z} \\ 0<j<2m}} q^{-\frac{1}{4m}j(j-2m)} \theta_{j,m}(\tau,z)$$

proving 2). □

6 $\Phi^{[m,0]\,*}(\tau, z_1+p\tau, z_2-p\tau, 0)$

6.1 $\Phi^{[m,0]\,*}(\tau, z_1+p\tau, z_2-p\tau, t) \sim$ the case $z_1 - z_2 = \tau - 1$

Lemma 6.1. *For $m \in \mathbf{N}$ and $p \in \mathbf{Z}$, the following formula holds:*

$$\Phi^{[m,0]\,*}\Big(\tau, \ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}+p\tau, \ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}-p\tau, \ 0\Big)$$

$$= q^{mp(p+1)} \Big\{ \widetilde{\Phi}^{[m,0]\,*}\Big(\tau, \ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}, \ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}, \ 0\Big)$$

$$+ \sum_{\substack{r\in\mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0 \le k < m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$- \ 2 \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)} \theta_{0,m}(\tau,z) \ + \ (-1)^m \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)} \, \theta_{m,m}(\tau,z) \Big\} \tag{6.1}$$

Proof. Letting $\begin{cases} z_1 &= \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2} \\ z_2 &= \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2} \end{cases}$ namely $\begin{cases} z_1 - z_2 &= \tau - 1 \\ z_1 + z_2 &= z \end{cases}$ in the formula (2.2) in Lemma 2.2, we have

$$\begin{aligned}
& \Phi^{[m,0]*}(\tau, \, z_1 + p\tau, \, z_2 - p\tau, \, 0) \\
= \ & e^{2\pi i m p(\tau-1)} q^{mp^2} \Big\{ \Phi^{[m,0]*}(\tau, z_1, z_2, 0) + \sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 2pm}} e^{-\pi i k(\tau-1)} q^{-\frac{1}{4m}k^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z) \Big\} \\
= \ & q^{mp(p+1)} \Big\{ \Phi^{[m,0]*}(\tau, z_1, z_2, 0) + \underbrace{\sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 2pm}} (-1)^k q^{-\frac{1}{4m}(k^2+2mk)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)}_{\text{(I)}} \Big\} \qquad (6.2)
\end{aligned}$$

We compute (I) by putting $k = 2mr + k' \quad (0 \le r < p, \ \ 1 \le k' \le 2m)$ as follows:

$$\begin{aligned}
\text{(I)} = & \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} (-1)^{2mr+k'} q^{-\frac{1}{4m}(2mr+k')(2mr+k'+2m)} \big[\theta_{2mr+k',m} + \theta_{-(2mr+k'),m}\big](\tau,z) \\
= & \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} (-1)^{k'} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \, \theta_{k',m}(\tau,z)}_{(A)} \\
& + \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} (-1)^{k'} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \, \theta_{-k',m}(\tau,z)}_{(B)}
\end{aligned}$$

where (A) is computed by using

$$\sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 2m}} = \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} - \sum_{k=0} + \sum_{k=2m} \tag{6.3}$$

as follows:

$$(A) \ = \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \, \theta_{k,m}(\tau,z)$$

$$- \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \underbrace{q^{-\frac{1}{4m} 2mr \cdot 2m(r+1)}}_{\substack{\| \\ q^{-mr(r+1)}}} \theta_{0,m}(\tau,z) + \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} q^{-\frac{1}{4m}(2mr+2m)(2m(r+1)+2m)}}_{\substack{\| \\ \sum_{\substack{r \in \mathbf{Z} \\ 1 \le r \le p}} q^{-mr(r+1)}}} \underbrace{\theta_{2m,m}}_{\substack{\| \\ \theta_{0,m}}}(\tau,z)$$

$$= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau,z)$$

$$+ \Bigg\{ - \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} + \sum_{\substack{r \in \mathbf{Z} \\ 1 \le r \le p}} \Bigg\} q^{-mr(r+1)} \theta_{0,m}(\tau,z)$$

$$= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau,z) + \big\{ q^{-mp(p+1)} - 1 \big\} \theta_{0,m}(\tau,z)$$

And (B) becomes by putting $k' = 2m - k$ and $r + 2 = -r'$ as follows:

$$(B) = \sum_{\substack{r' \in \mathbf{Z} \\ -p-1 \le r' \le -1}} \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k' < 2m}} (-1)^{k'} q^{-\frac{1}{4m}(2mr'+k')(2m(r'+1)+k')} \theta_{k',m}(\tau,z)$$

$$- \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k' < 2m}} (-1)^{k'} q^{-\frac{1}{4m}(-2m+k')k'} \theta_{k',m}(\tau,z)$$

Then (I) becomes as follows:

$$(\mathrm{I}) = (A) + (B)$$

$$= \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} (-1)^k \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ -p-1 \le r < p}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)}}_{\substack{\| \leftarrow \; r+1=r' \\ \sum_{\substack{r' \in \mathbf{Z} \\ -p \le r' \le p}} q^{-\frac{1}{4m}(2m(r'-1)+k)(2mr'+k)}}} \theta_{k,m}(\tau,z)$$

$$\underbrace{- \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} (-1)^k q^{-\frac{1}{4m}(-2m+k)k} \theta_{k,m}(\tau,z) - \theta_{0,m}(\tau,z)}_{\substack{\| \\ - \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le 2m}} (-1)^k q^{-\frac{1}{4m}(-2m+k)k} \theta_{k,m}(\tau,z)}} + q^{-mp(p+1)} \theta_{0,m}(\tau,z) \tag{6.4}$$

Then substituting (6.4) into (6.2), we have

$$\Phi^{[m,0]\,*}(\tau,\, z_1+p\tau,\, z_2-p\tau,\, 0) \;=\; q^{mp(p+1)}\Big\{\Phi^{[m,0]\,*}(\tau,z_1,z_2,0) \;+\; (\mathrm{I})\Big\}$$

$$= \; q^{mp(p+1)}\Big\{\Phi^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$+ \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\,\theta_{k,m}(\tau,z)$$

$$\underbrace{- \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le 2m}} (-1)^k q^{-\frac{1}{4m}k(k-2m)}\theta_{k,m}(\tau,z)}_{\substack{|| \\ \Phi^{[m,0]\,*}_{\mathrm{add}}(\tau,z_1,z_2,0) \quad \text{by Lemma 5.3}}} \;+\; q^{-mp(p+1)}\,\theta_{0,m}(\tau,z)\Big\}$$

$$= \; q^{mp(p+1)}\Big\{\widetilde{\Phi}^{[m,0]\,*}(\tau,z_1,z_2,0) + \underbrace{\sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)}_{(\mathrm{II})}$$

$$+ \; q^{-mp(p+1)}\,\theta_{0,m}(\tau,z)\Big\} \tag{6.5}$$

We go further to compute

$$(\mathrm{II}) \;=\; (\mathrm{II})_A + (\mathrm{II})_B$$

where

$$(\mathrm{II})_A \;:=\; \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

$$(\mathrm{II})_B \;:=\; \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ m<k<2m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

First we compute $(\mathrm{II})_A$:

$$(\mathrm{II})_A \;=\; \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \underbrace{\sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le m}}}_{\substack{|| \\ \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} + \sum_{k=m}}} (-1)^k\, q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

$$= \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} (-1)^k\, q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

$$+\ (-1)^m \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\, \theta_{m,m}(\tau,z) \tag{6.6a}$$

Next, $(\mathrm{II})_B$ is rewritten as follows by putting $k = 2m-k'$:

$$\begin{aligned}
(\mathrm{II})_B &= \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k' \in \mathbf{Z} \\ 0<k'<m}} (-1)^{k'} q^{-\frac{1}{4m}(2m(r+1)-k')(2mr-k')} \theta_{-k',m}(\tau,z) \\
&\underset{r=-r'}{\overset{\uparrow}{=}} \sum_{\substack{r' \in \mathbf{Z} \\ -p \le r' \le p}} \underbrace{\sum_{\substack{k' \in \mathbf{Z} \\ 0<k'<m}}}_{\substack{\| \\ \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k'<m}} - \sum_{k'=0}}} (-1)^{k'} q^{-\frac{1}{4m}(2m(r'-1)+k')(2mr'+k')} \theta_{-k',m}(\tau,z) \\
&= \sum_{\substack{r' \in \mathbf{Z} \\ -p \le r' \le p}} \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k'<m}} (-1)^{k'} q^{-\frac{1}{4m}(2m(r'-1)+k')(2mr'+k')} \theta_{-k',m}(\tau,z) \\
&\quad \underbrace{- \sum_{\substack{r' \in \mathbf{Z} \\ -p \le r' \le p}} q^{-mr'(r'-1)}\, \theta_{0,m}(\tau,z)}_{\substack{\| \\ -2 \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)}\, \theta_{0,m}(\tau,z)\ -\ q^{-mp(p+1)}\theta_{0,m}(\tau,z)}}
\end{aligned} \tag{6.6b}$$

Then by (6.6a) and (6.6b), we have

$$\begin{aligned}
&(\mathrm{II}) + q^{-mp(p+1)}\theta_{0,m}(\tau,z) \\
&= \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k<m}} (-1)^k\, q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\
&\quad -2 \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)}\theta_{0,m}(\tau,z)\ +\ (-1)^m \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\, \theta_{m,m}(\tau,z)
\end{aligned} \tag{6.7}$$

Substituting this equation (6.7) into (6.2), we obtain

$$\begin{aligned}
\Phi^{[m,0]*}(\tau,\, z_1+p\tau,\, z_2-p\tau,\, 0) \ &= \ q^{mp(p+1)} \Big\{ \widetilde{\Phi}^{[m,0]*}(\tau,z_1,z_2,0) \\
&+ \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k<m}} (-1)^k\, q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)
\end{aligned}$$

$$- \ 2 \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \ + \ (-1)^m \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)} \, \theta_{m,m}(\tau, z) \Big\}$$

proving Lemma 6.1. ☐

6.2 $\Phi^{[m,0]\,*}(\tau, z_1 + p\tau, z_2 - p\tau, t) \sim$ the case $z_1 - z_2 = \tau$

Lemma 6.2. *For $m \in \mathbf{N}$ and $p \in \mathbf{Z}$, the following formula holds:*

$$\begin{aligned}
&\Phi^{[m,0]\,*}\Big(\tau, \ \ \frac{z}{2} + \frac{\tau}{2} + p\tau, \ \ \frac{z}{2} - \frac{\tau}{2} - p\tau, \ \ 0\Big) \\
&= \ q^{mp(p+1)} \Big\{ \widetilde{\Phi}^{[m,0]\,*}\Big(\tau, \ \ \frac{z}{2} + \frac{\tau}{2}, \ \ \frac{z}{2} - \frac{\tau}{2}, \ \ 0\Big) \\
&\quad + \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \ \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z) \\
&\quad - \ 2 \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \ + \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)} \, \theta_{m,m}(\tau, z) \Big\}
\end{aligned} \tag{6.8}$$

Proof. Letting $\begin{cases} z_1 &= \frac{z}{2} + \frac{\tau}{2} \\ z_2 &= \frac{z}{2} - \frac{\tau}{2} \end{cases}$ namely $\begin{cases} z_1 - z_2 &= \tau \\ z_1 + z_2 &= z \end{cases}$ in the formula (2.2) in Lemma 2.2, we have

$$\begin{aligned}
&\Phi^{[m,0]\,*}(\tau, \ z_1 + p\tau, \ z_2 - p\tau, \ 0) \\
&= \ e^{2\pi i m p\tau} q^{mp^2} \Big\{ \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) + \sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 2pm}} e^{-\pi i k\tau} q^{-\frac{1}{4m}k^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z) \Big\} \\
&= \ q^{mp(p+1)} \Big\{ \Phi^{[m,0]\,*}(\tau, z_1, z_2, 0) + \underbrace{\sum_{\substack{k \in \mathbf{Z} \\ 1 \le k \le 2pm}} q^{-\frac{1}{4m}(k^2 + 2mk)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)}_{\text{(I)}} \Big\}
\end{aligned} \tag{6.9}$$

We compute (I) by putting $k = 2mr + k' \quad (0 \le r < p, \ \ 1 \le k' \le 2m)$ as follows:

$$\begin{aligned}
\text{(I)} &:= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \ \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} q^{-\frac{1}{4m}(2mr+k')(2mr+k'+2m)} \big[\theta_{2mr+k',m} + \theta_{-(2mr+k'),m}\big](\tau, z) \\
&= \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \ \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \, \theta_{k',m}(\tau, z)}_{(A)}
\end{aligned}$$

$$+ \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k' \in \mathbf{Z} \\ 1 \le k' \le 2m}} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \theta_{-k',m}(\tau, z)}_{(B)}$$

where (A) is computed by using (6.3) as follows:

$$\begin{aligned}
(A) &= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) \\
&\quad - \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \underbrace{q^{-\frac{1}{4m}2mr \cdot 2m(r+1)}}_{\substack{\| \\ q^{-mr(r+1)}}} \theta_{0,m}(\tau, z) + \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} q^{-\frac{1}{4m}(2mr+2m)(2m(r+1)+2m)}}_{\substack{\| \\ \sum_{\substack{r \in \mathbf{Z} \\ 1 \le r \le p}} q^{-mr(r+1)}}} \underbrace{\theta_{2m,m}}_{\substack{\| \\ \theta_{0,m}}}(\tau, z) \\
&= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) \\
&\quad + \Bigg\{ - \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} + \sum_{\substack{r \in \mathbf{Z} \\ 1 \le r \le p}} \Bigg\} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \\
&= \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r < p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) + \left\{ q^{-mp(p+1)} - 1 \right\} \theta_{0,m}(\tau, z)
\end{aligned}$$

And (B) becomes by putting $k' = 2m - k$ and $r + 2 = -r'$ as follows:

$$\begin{aligned}
(B) &= \sum_{\substack{r' \in \mathbf{Z} \\ -p-1 \le r' \le -1}} \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k' < 2m}} q^{-\frac{1}{4m}(2mr'+k')(2m(r'+1)+k')} \theta_{k',m}(\tau, z) \\
&\quad - \sum_{\substack{k' \in \mathbf{Z} \\ 0 \le k' < 2m}} q^{-\frac{1}{4m}(-2m+k')k'} \theta_{k',m}(\tau, z)
\end{aligned}$$

Then (I) becomes as follows:

$$(\mathrm{I}) = (A) + (B)$$

$$= \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ -p-1 \le r < p}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)}}_{\substack{|| \ \leftarrow \ r+1=r' \\ \sum_{\substack{r' \in \mathbf{Z} \\ -p \le r' \le p}} q^{-\frac{1}{4m}(2m(r'-1)+k)(2mr'+k)}}} \theta_{k,m}(\tau, z)$$

$$\underbrace{- \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(-2m+k)k} \, \theta_{k,m}(\tau, z) \ - \ \theta_{0,m}(\tau, z)}_{\substack{|| \\ - \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le 2m}} q^{-\frac{1}{4m}(-2m+k)k} \, \theta_{k,m}(\tau, z)}} + q^{-mp(p+1)} \, \theta_{0,m}(\tau, z) \tag{6.10}$$

Then substituting (6.10) into (6.9), we have

$$\Phi^{[m,0]*}(\tau, \ z_1 + p\tau, \ z_2 - p\tau, \ 0) \ = \ q^{mp(p+1)} \Big\{ \Phi^{[m,0]*}(\tau, z_1, z_2, 0) \ + \ (\mathrm{I}) \Big\}$$

$$= \ q^{mp(p+1)} \Big\{ \Phi^{[m,0]*}(\tau, z_1, z_2, 0) \tag{6.11}$$

$$+ \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \, \theta_{k,m}(\tau, z)$$

$$\underbrace{- \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le 2m}} q^{-\frac{1}{4m}k(k-2m)} \theta_{k,m}(\tau, z)}_{\substack{|| \\ \Phi^{[m,0]*}_{\mathrm{add}}(\tau, z_1, z_2, 0) \quad \text{by Lemma 5.3}}} \quad + \quad q^{-mp(p+1)} \, \theta_{0,m}(\tau, z) \Big\}$$

$$= \ q^{mp(p+1)} \Big\{ \widetilde{\Phi}^{[m,0]*}(\tau, z_1, z_2, 0) + \underbrace{\sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < 2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)}_{(\mathrm{II})}$$

$$+ \ q^{-mp(p+1)} \, \theta_{0,m}(\tau, z) \Big\} \tag{6.12}$$

We go further to compute

$$(\mathrm{II}) \ = \ (\mathrm{II})_A + (\mathrm{II})_B$$

where

$$(\mathrm{II})_A \ := \ \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)$$

$$(\mathrm{II})_B \;:=\; \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} \;\sum_{\substack{k\in\mathbf{Z}\\ m<k<2m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

First we compute $(\mathrm{II})_A$:

$$(\mathrm{II})_A \;=\; \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} \underbrace{\sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le m}}}_{\sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}}+\sum_{k=m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

$$= \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} \;\sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\theta_{k,m}(\tau,z)$$

$$+ \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\,\theta_{m,m}(\tau,z) \qquad (6.13a)$$

Next, $(\mathrm{II})_B$ is rewritten as follows by putting $k=2m-k'$:

$$(\mathrm{II})_B \;=\; \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} \;\sum_{\substack{k'\in\mathbf{Z}\\ 0<k'<m}} q^{-\frac{1}{4m}(2m(r+1)-k')(2mr-k')}\theta_{-k',m}(\tau,z)$$

$$\underset{r=-r'}{\overset{\uparrow}{=}} \sum_{\substack{r'\in\mathbf{Z}\\ -p\le r'\le p}} \underbrace{\sum_{\substack{k'\in\mathbf{Z}\\ 0<k'<m}}}_{\sum_{\substack{k'\in\mathbf{Z}\\ 0\le k'<m}}-\sum_{k'=0}} q^{-\frac{1}{4m}(2m(r'-1)+k')(2mr'+k')}\theta_{-k',m}(\tau,z)$$

$$= \sum_{\substack{r'\in\mathbf{Z}\\ -p\le r'\le p}} \;\sum_{\substack{k'\in\mathbf{Z}\\ 0\le k'<m}} q^{-\frac{1}{4m}(2m(r'-1)+k')(2mr'+k')}\theta_{-k',m}(\tau,z)$$

$$\underbrace{- \sum_{\substack{r'\in\mathbf{Z}\\ -p\le r'\le p}} q^{-mr'(r'-1)}\,\theta_{0,m}(\tau,z)}_{-2\sum_{\substack{r\in\mathbf{Z}\\ 0\le r\le p-1}} q^{-mr(r+1)}\,\theta_{0,m}(\tau,z)\;-\;q^{-mp(p+1)}\theta_{0,m}(\tau,z)} \qquad (6.13b)$$

Then by (6.13a) and (6.13b), we have

$$(\mathrm{II}) + q^{-mp(p+1)}\theta_{0,m}(\tau,z)$$

$$= \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-2 \sum_{\substack{r\in\mathbf{Z} \\ 0\le r\le p-1}} q^{-mr(r+1)}\theta_{0,m}(\tau,z) + \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\,\theta_{m,m}(\tau,z) \tag{6.14}$$

Substituting this equation (6.14) into (6.9), we obtain

$$\Phi^{[m,0]\,*}(\tau,\,z_1+p\tau,\,z_2-p\tau,\,0) \;=\; q^{mp(p+1)}\bigg\{\widetilde{\Phi}^{[m,0]\,*}(\tau,z_1,z_2,0)$$

$$+ \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} (-1)^k\, q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-\;2 \sum_{\substack{r\in\mathbf{Z} \\ 0\le r\le p-1}} q^{-mr(r+1)}\theta_{0,m}(\tau,z) + \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\,\theta_{m,m}(\tau,z)\bigg\}$$

proving Lemma 6.2. □

7 Modified function $\widetilde{\Phi}^{[m,0]\,*}$ with specialization

Proposition 7.1. *For $m\in\mathbf{N}$ and $p\in\mathbf{Z}_{\ge 0}$, the following formulas hold:*

1) $$(-1)^p\, q^{-\frac{m}{4}}\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)\;\;\widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\;\; \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2},\;\; \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2},\;\; 0\Big)$$

$$= \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau,z)}\cdot\big[\theta_{p,m+\frac{1}{2}}+\theta_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$-\Bigg[\sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r<j}} - \sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ j\le r<0}}\Bigg] \sum_{\substack{k\in\mathbf{Z} \\ 0<k<m}} (-1)^{j-\frac{1}{2}+k}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times\;\; q^{-\frac{1}{4m}\,[2mr+k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-\Bigg[\sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r\le j}} - \sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ j<r<0}}\Bigg] \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le m}} (-1)^{j-\frac{1}{2}+k}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times\;\; q^{-\frac{1}{4m}\,[2mr-k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-(-1)^p\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)\sum_{\substack{r\in\mathbf{Z} \\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}\,(m(2r-1)+k)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+\;2\,(-1)^p\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)\sum_{\substack{r\in\mathbf{Z} \\ 0\le r\le p-1}} q^{-m(r+\frac{1}{2})^2}\,\theta_{0,m}(\tau,z)$$

$$- \ (-1)^{m+p}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p<r<p}} q^{-mr^2}\, \theta_{m,m}(\tau,z) \tag{7.1a}$$

2) $$(-1)^p\, q^{-\frac{m}{4}}\, \theta^{(-)}_{-p+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \ \ \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \ \frac{z}{2}+\frac{\tau}{2},\ \ \frac{z}{2}-\frac{\tau}{2},\ \ 0\Big)$$

$$= \ (-1)^p\, \frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r<j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j\le r<0}}\Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \times\ q^{-\frac{1}{4m}\,[2mr+k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r\leqq j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j<r<0}}\Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0\le k\le m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \times\ q^{-\frac{1}{4m}\,[2mr-k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$- \ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p<r\le p}} \ \sum_{\substack{k \in \mathbf{Z} \\ 0\le k<m}} q^{-\frac{1}{4m}\,(m(2r-1)+k)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+ \ 2\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ 0\le r\le p-1}} q^{-m(r+\frac{1}{2})^2}\, \theta_{0,m}(\tau,z)$$

$$- \ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p<r<p}} q^{-mr^2}\, \theta_{m,m}(\tau,z) \tag{7.1b}$$

Proof. 1) In the case $\left\{\begin{array}{lcl} z_1 &=& \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2} \\ z_2 &=& \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}\end{array}\right.$, substituting (6.1) into (4.1), we have

$$\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \ \times \ \Bigg\{q^{mp(p+1)}\, \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2},\ \ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2},\ \ 0\Big)$$

$$+ \ q^{mp(p+1)} \sum_{\substack{r \in \mathbf{Z} \\ -p\le r\le p}} \ \sum_{\substack{k \in \mathbf{Z} \\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$- \ 2\, q^{mp(p+1)} \sum_{\substack{r \in \mathbf{Z} \\ 0\le r\le p-1}} q^{-mr(r+1)}\, \theta_{0,m}(\tau,z)$$

$$+ \ (-1)^m\, q^{mp(p+1)} \sum_{\substack{r \in \mathbf{Z} \\ -p\le r\le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\, \theta_{m,m}(\tau,z)\Bigg\}$$

$$= \; q^{\frac{m}{4}(2p+1)^2} \, \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \; q^{\frac{m}{4}(2p+1)^2} \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$
$$\times \; q^{-\frac{1}{4m}[2mr+k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \; q^{\frac{m}{4}(2p+1)^2} \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$
$$\times \; q^{-\frac{1}{4m}[2mr-k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$+ \; \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^k q^{-\frac{1}{4m}k^2 + \frac{k}{2}(2p+1)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

Multiplying $q^{-\frac{m}{4}(2p+1)^2} = q^{-mp(p+1)-\frac{m}{4}}$ to both sides, we have

$$q^{-\frac{m}{4}} \, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \, \widetilde{\Phi}^{[m,0]\,*}\Big(\tau, \;\; \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}, \;\; \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}, \;\; 0\Big)$$

$$+ \; q^{-\frac{m}{4}} \, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)$$
$$\times \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < m}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \; 2\, q^{-\frac{m}{4}} \, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)} \, \theta_{0,m}(\tau,z)$$

$$+ \; (-1)^m \, q^{-\frac{m}{4}} \, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)} \, \theta_{m,m}(\tau,z)$$

$$= \; \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$
$$\times \; q^{-\frac{1}{4m}[2mr+k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times\ q^{-\frac{1}{4m}[2mr-k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+\ \theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\sum_{\substack{k\in\mathbf{Z}\\0\le k\le m}}(-1)^k q^{-\frac{1}{4m}[k-m(2p+1)]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

namely

$$q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\,\widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \frac{z}{2}+\frac{\tau}{2}-\frac12,\ \frac{z}{2}-\frac{\tau}{2}+\frac12,\ 0\Big)$$

$$=\ \frac{\eta(\tau)^3}{\theta_{0,\frac12}(\tau,z)}\cdot\big[\theta_{p,m+\frac12}+\theta_{-p,m+\frac12}\big](\tau,0)$$

$$-\Bigg[\sum_{\substack{j,r\in\frac12\mathbf{Z}_{\rm odd}\\0\le r<j}}-\sum_{\substack{j,r\in\frac12\mathbf{Z}_{\rm odd}\\j\le r<0}}\Bigg]\sum_{\substack{k\in\mathbf{Z}\\0<k<m}}(-1)^{j-\frac12+k}\,q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2}$$
$$\times\ q^{-\frac{1}{4m}[2mr+k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-\Bigg[\sum_{\substack{j,r\in\frac12\mathbf{Z}_{\rm odd}\\0\le r\le j}}-\sum_{\substack{j,r\in\frac12\mathbf{Z}_{\rm odd}\\j<r<0}}\Bigg]\sum_{\substack{k\in\mathbf{Z}\\0\le k\le m}}(-1)^{j-\frac12+k}\,q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2}$$
$$\times\ q^{-\frac{1}{4m}[2mr-k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+\ \underbrace{\theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\sum_{\substack{k\in\mathbf{Z}\\0\le k\le m}}(-1)^k q^{-\frac{1}{4m}[k-m(2p+1)]^2}\ \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}_{\rm (I)_A}$$

$$\underbrace{-\,q^{-\frac{m}{4}}\theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\sum_{\substack{r\in\mathbf{Z}\\-p\le r\le p}}\ \sum_{\substack{k\in\mathbf{Z}\\0\le k<m}}(-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}_{\rm (I)_B}$$

$$+\ 2\,q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\sum_{\substack{r\in\mathbf{Z}\\0\le r\le p-1}}q^{-mr(r+1)}\,\theta_{0,m}(\tau,z)$$

$$-\ (-1)^m\,q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac12,m+\frac12}(\tau,0)\sum_{\substack{r\in\mathbf{Z}\\-p\le r\le p}}q^{-mr^2+\frac{m}{4}}\,\theta_{m,m}(\tau,z) \tag{7.2a}$$

Since

$$-\frac{1}{4m}(2mr+k)(2m(r-1)+k)\ -\ \frac{m}{4}\ =\ -\frac{1}{4m}\big(m(2r-1)+k\big)^2$$

$(\mathrm{I})_B$ becomes as follows:

$$(\mathrm{I})_B = -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \underbrace{\sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}}}_{\substack{\| \\ \sum_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} + \sum_{r=-p}}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}(m(2r-1)+k)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$= -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$\underbrace{-\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}\left(-m(2p+1)+k\right)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}_{\substack{\uparrow\\ -(\mathrm{I})_A}}$$

so

$$(\mathrm{I})_A+(\mathrm{I})_B = -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}(m(2r-1)+k)^2} \times \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \tag{7.2b}$$

Substituting (7.2b) into (7.2a) and noticing that

$$\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) = (-1)^p\,\theta^{(-)}_{-p+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) = (-1)^p\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0) \tag{7.3}$$

we obtain (7.1a), proving the claim 1).

2) In the case $\begin{cases} z_1 &= \frac{z}{2}+\frac{\tau}{2} \\ z_2 &= \frac{z}{2}-\frac{\tau}{2} \end{cases}$, substituting (6.8) into (4.5), we have

$$\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \times \Bigg\{ q^{mp(p+1)}\,\widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \frac{z}{2}+\frac{\tau}{2},\ \frac{z}{2}-\frac{\tau}{2},\ 0\Big)$$

$$+\ q^{mp(p+1)} \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$-\ 2\,q^{mp(p+1)} \sum_{\substack{r\in\mathbf{Z}\\ 0\le r\le p-1}} q^{-mr(r+1)}\,\theta_{0,m}(\tau,z)$$

$$+\ q^{mp(p+1)} \sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\,\theta_{m,m}(\tau,z)\Bigg\}$$

$$= \quad (-1)^p\, q^{\frac{m}{4}(2p+1)^2}\, \frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \; q^{\frac{m}{4}(2p+1)^2} \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr+k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \; q^{\frac{m}{4}(2p+1)^2} \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr-k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$+ \; \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} q^{-\frac{1}{4m}k^2+\frac{k}{2}(2p+1)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

Multiplying $q^{-\frac{m}{4}(2p+1)^2} = q^{-mp(p+1)-\frac{m}{4}}$ to both sides, we have

$$q^{-\frac{m}{4}}\, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\, \widetilde{\Phi}^{[m,0]*}\Big(\tau, \;\; \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}, \;\; \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}, \;\; 0\Big)$$

$$+ \; q^{-\frac{m}{4}}\, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)$$

$$\times \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \; 2\, q^{-\frac{m}{4}}\, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)}\, \theta_{0,m}(\tau,z)$$

$$+ \; q^{-\frac{m}{4}}\, \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-\frac{m}{4}(2r+1)(2r-1)}\, \theta_{m,m}(\tau,z)$$

$$= \quad (-1)^p\, \frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\mathrm{odd}} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr+k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau,z)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \times \, q^{-\frac{1}{4m}\,[2mr-k+2mp]^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+ \ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} q^{-\frac{1}{4m}[k-m(2p+1)]^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

namely

$$q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0)\,\widetilde{\Phi}^{[m,0]\,*}\Big(\tau, \ \ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2}, \ \ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2}, \ \ 0\Big)$$

$$= \ (-1)^p \ \frac{\eta(\tau)^3}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} \cdot \big[\theta^{(-)}_{p,m+\frac{1}{2}}+\theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau,0)$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j \le r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 < k < m}} (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \times \, q^{-\frac{1}{4m}\,[2mr+k+2mp]^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$\underbrace{\begin{aligned} &- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \times \, q^{-\frac{1}{4m}\,[2mr-k+2mp]^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\ &+ \ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} q^{-\frac{1}{4m}[k-m(2p+1)]^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \end{aligned}}_{\rm (I)_A}$$

$$\underbrace{- q^{-\frac{m}{4}}\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < m}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}_{\rm (I)_B}$$

$$+ \ 2\,q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-mr(r+1)}\,\theta_{0,m}(\tau,z)$$

$$- \ q^{-\frac{m}{4}}\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-mr^2+\frac{m}{4}}\,\theta_{m,m}(\tau,z) \tag{7.4a}$$

where $(\mathrm{I})_B$ becomes as follows:

$$(\mathrm{I})_B = -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \underbrace{\sum_{\substack{r\in\mathbf{Z}\\ -p\le r\le p}}}_{\substack{\| \\ \sum\limits_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} + \sum\limits_{r=-p}}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$= -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$\underbrace{-\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}\left(-m(2p+1)+k\right)^2} \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}_{\substack{\| \\ -(\mathrm{I})_A}}$$

so

$$(\mathrm{I})_A+(\mathrm{I})_B = -\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z}\\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z}\\ 0\le k<m}} q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2} \times \big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \tag{7.4b}$$

Substituting (7.4b) into (7.4a) and using (7.3), we obtain (7.1b), proving the claim 2). □

8 Modular transformation of $\widetilde{\Phi}^{[m,0]\,*}$

8.1 $\widetilde{\psi}^{(i)[m]\,*}(\tau,z)$

For $m\in\mathbf{N}$ and $i\in\{1,2\}$, we consider functions $\widetilde{\psi}^{(i)[m]\,*}(\tau,z)$ defined by

$$\begin{aligned}
\widetilde{\psi}^{(1)[m]\,*}(\tau,z) &:= \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \frac{z}{2}+\frac{\tau}{2}-\frac{1}{2},\ \frac{z}{2}-\frac{\tau}{2}+\frac{1}{2},\ 0\Big) \\
&= \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \frac{z}{2}+\frac{\tau}{2}+\frac{1}{2},\ \frac{z}{2}-\frac{\tau}{2}-\frac{1}{2},\ 0\Big) \\
\widetilde{\psi}^{(2)[m]\,*}(\tau,z) &:= \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\ \frac{z}{2}+\frac{\tau}{2},\ \frac{z}{2}-\frac{\tau}{2},\ 0\Big)
\end{aligned}$$

The function $\widetilde{\psi}^{(1)[m]\,*}(\tau,z)$ satisfies the follwing modular transformation properties.

Lemma 8.1. *Let $m\in\mathbf{N}$, then*

1) $$\widetilde{\psi}^{(1)[m]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = (-1)^m\,\tau\, e^{\frac{\pi i m}{2\tau}z^2}\, e^{-\frac{\pi i m}{2\tau}}\, q^{-\frac{m}{4}}\,\widetilde{\psi}^{(1)[m]\,*}(\tau,z)$$

2) $\quad \widetilde{\psi}^{(1)[m]\,*}(\tau+1,z) \;=\; \widetilde{\psi}^{(2)[m]\,*}(\tau,z)$

Proof. These are obtained easily from Lemma 2.1 in [17] as follows.

$$1)\quad \widetilde{\psi}^{(1)[m]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \widetilde{\Phi}^{[m,0]\,*}\Big(-\frac{1}{\tau},\frac{z}{2\tau}-\frac{1}{2\tau}+\frac{1}{2},\frac{z}{2\tau}+\frac{1}{2\tau}-\frac{1}{2},0\Big)$$

$$= \widetilde{\Phi}^{[m,0]\,*}\Big(-\frac{1}{\tau},\frac{\frac{z}{2}-\frac{1}{2}+\frac{\tau}{2}}{\tau},\frac{\frac{z}{2}+\frac{1}{2}-\frac{\tau}{2}}{\tau},0\Big)$$

$$= \tau\, e^{\frac{2\pi i m}{\tau}(\frac{z}{2}-\frac{1}{2}+\frac{\tau}{2})(\frac{z}{2}+\frac{1}{2}-\frac{\tau}{2})}\widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\frac{z}{2}-\frac{1}{2}+\frac{\tau}{2},\frac{z}{2}+\frac{1}{2}-\frac{\tau}{2},0\Big)$$

$$= (-1)^m\,\tau\, e^{\frac{\pi i m}{2\tau}z^2}e^{-\frac{\pi i m}{2\tau}}q^{-\frac{m}{4}}\widetilde{\psi}^{(1)[m]\,*}(\tau,z)\,, \qquad \text{proving 1).}$$

$$2)\quad \widetilde{\psi}^{(1)[m]\,*}(\tau+1,z) = \widetilde{\Phi}^{[m,0]\,*}\Big(\tau+1,\,\frac{z}{2}+\frac{\tau+1}{2}-\frac{1}{2},\,\frac{z}{2}-\frac{\tau+1}{2}+\frac{1}{2},\,0\Big)$$

$$= \widetilde{\Phi}^{[m,0]\,*}\Big(\tau+1,\,\frac{z}{2}+\frac{\tau}{2},\,\frac{z}{2}-\frac{\tau}{2},\,0\Big)$$

$$= \widetilde{\Phi}^{[m,0]\,*}\Big(\tau,\,\frac{z}{2}+\frac{\tau}{2},\,\frac{z}{2}-\frac{\tau}{2},\,0\Big) = \widetilde{\psi}^{(2)[m]\,*}(\tau,z)\,, \qquad \text{proving 2).}$$

□

8.2 $\Xi^{(i)[m,p]\,*}(\tau,z)$ and $\Upsilon^{(i)[m,p]\,*}(\tau,z)$

For $m\in\mathbf{N}$ and $p\in\mathbf{Z}$ such that $0\le p\le 2m$ and $i\in\{1,2,3\}$, we define functions $\Xi^{(i)[m,p]\,*}(\tau,z)$ and $\Upsilon^{(i)[m,p]\,*}(\tau,z)$ as follows:

$$\Xi^{(1)[m,p]\,*}(\tau,z) \;:=\; (-1)^p\,q^{-\frac{m}{4}}\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)\cdot\widetilde{\psi}^{(1)[m]\,*}(\tau,z) \tag{8.1a}$$

$$\Xi^{(2)[m,p]\,*}(\tau,z) \;:=\; q^{-\frac{m}{4}}\,\theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)\cdot\widetilde{\psi}^{(2)[m]\,*}(\tau,z) \tag{8.1b}$$

and

$$\Upsilon^{(1)[m,p]\,*}(\tau,z) \;:=\; \eta(\tau)^3\cdot\frac{\theta_{p,m+\frac{1}{2}}(\tau,z)\,+\,\theta_{-p,m+\frac{1}{2}}(\tau,z)}{\theta_{0,\frac{1}{2}}(\tau,z)} \tag{8.2a}$$

$$\Upsilon^{(2)[m,p]\,*}(\tau,z) \;:=\; \eta(\tau)^3\cdot\frac{\theta^{(-)}_{p,m+\frac{1}{2}}(\tau,z)\,+\,\theta^{(-)}_{-p,m+\frac{1}{2}}(\tau,z)}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)} \tag{8.2b}$$

To compute modular transformation of these functions, we use the following formulas which are obtained easily from Lemmas 1.3 and 1.4 in [17].

Note 8.1. *Let $m\in\mathbf{Z}_{\ge 0}$ and $p\in\mathbf{Z}$. Then*

$$1)\quad \text{(i)}\ \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}\Big(-\frac{1}{\tau},0\Big) = -\,i\,(-1)^{m+p}\,\frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}}\sum_{p'=0}^{2m}(-1)^{p'}e^{-\frac{2\pi i}{2m+1}pp'}\theta^{(-)}_{p'-m-\frac{1}{2},m+\frac{1}{2}}(\tau,0)$$

$$\text{(ii)}\ \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau+1,\ z) \ = \ (-1)^p\, e^{\frac{\pi i}{2}(m+\frac{1}{2})}\, e^{\frac{\pi i}{2m+1}p^2}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau, z)$$

2) (i)
$$\big[\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}\big]\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big)$$
$$= \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i}{2\tau}(m+\frac{1}{2})z^2} \sum_{p'\in\mathbf{Z}/(2m+1)\mathbf{Z}} e^{\frac{2\pi i}{2m+1}pp'}\, \big[\theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}}\big](\tau, z)$$
$$= \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i}{2\tau}(m+\frac{1}{2})z^2} \sum_{p'\in\mathbf{Z}/(2m+1)\mathbf{Z}} e^{-\frac{2\pi i}{2m+1}pp'}\, \big[\theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}}\big](\tau, z)$$

$$\text{(ii)}\ \big[\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}\big](\tau+1, z) \ = \ e^{\frac{\pi i}{2m+1}p^2}\, \big[\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}\big](\tau, z)$$

Then modular transformation properties of $\Xi^{(1)[m,p]\,*}$ and $\Upsilon^{(1)[m,p]\,*}$ are given by the following formulas:

Lemma 8.2. *Let $m \in \mathbf{N}$ and $p \in \mathbf{Z}$. Then*

1) $$\Xi^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) \ = \ \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i m}{2\tau}z^2} \sum_{p'=0}^{2m} e^{-\frac{2\pi i}{2m+1}pp'}\, \Xi^{(1)[m,p']\,*}(\tau, z)$$

2) $$\Xi^{(1)[m,p]\,*}(\tau+1, z) \ = \ e^{\frac{\pi i}{4}}\, e^{\frac{\pi i}{2m+1}p^2}\, \Xi^{(2)[m,p]\,*}(\tau, z)$$

Proof. 1) $\Xi^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big)$

$$= (-1)^p e^{-2\pi i\frac{m}{4}(-\frac{1}{\tau})}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}\Big(-\frac{1}{\tau}, 0\Big)\, \widetilde{\psi}^{(1)[m]\,*}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big)$$
$$= (-1)^p\, e^{\frac{\pi i m}{2\tau}} \times \Big\{-i\,(-1)^{m+p}\, \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} \sum_{p'=0}^{2m}(-1)^{p'} e^{-\frac{2\pi i}{2m+1}pp'}\theta^{(-)}_{p'-m-\frac{1}{2},m+\frac{1}{2}}(\tau, 0)\Big\}$$
$$\times\ (-1)^m\, \tau\, e^{\frac{\pi i m}{2\tau}z^2}\, e^{-\frac{\pi i m}{2\tau}}\, q^{-\frac{m}{4}}\, \widetilde{\psi}^{(1)[m]\,*}(\tau, z)$$
$$= \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i m}{2\tau}z^2} \sum_{p'=0}^{2m} e^{-\frac{2\pi i}{2m+1}pp'}\, \underbrace{(-1)^{p'}\, q^{-\frac{m}{4}}\, \theta^{(-)}_{p'-m-\frac{1}{2},m+\frac{1}{2}}(\tau, 0)\, \widetilde{\psi}^{(1)[m]\,*}(\tau, z)}_{\Xi^{(1)[m,p']\,*}(\tau, z)}$$

proving 1).

2) $$\Xi^{(1)[m,p]\,*}(\tau+1, z) \ = \ (-1)^p\, e^{-\frac{\pi i m}{2}(\tau+1)}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau+1, 0)\, \widetilde{\psi}^{(1)[m]\,*}(\tau+1, z)$$
$$= (-1)^p\, q^{-\frac{m}{4}}\, e^{-\frac{\pi i m}{2}} \times \Big\{(-1)^p\, e^{\frac{\pi i}{2}(m+\frac{1}{2})}\, e^{\frac{\pi i}{2m+1}p^2}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau, z)\Big\}\, \widetilde{\psi}^{(2)[m]\,*}(\tau, z)$$
$$= e^{\frac{\pi i}{4}}\, e^{\frac{\pi i}{2m+1}p^2}\, \underbrace{q^{-\frac{m}{4}}\, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau, z)\, \widetilde{\psi}^{(2)[m]\,*}(\tau, z)}_{\Xi^{(2)[m,p]\,*}(\tau, z)}, \qquad \text{proving 2).}$$

□

Lemma 8.3. *Let $m \in \mathbf{N}$ and $p \in \mathbf{Z}$. Then*

$$1)\ \Upsilon^{(1)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) \;=\; \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i m}{2\tau}z^2} \sum_{p'=0}^{2m} e^{-\frac{2\pi i}{2m+1}pp'}\, \Upsilon^{(1)[m,p']\,*}(\tau,z)$$

$$2)\ \Upsilon^{(1)[m,p]\,*}(\tau+1,z) \;=\; e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}}\, \Upsilon^{(2)[m,p]\,*}(\tau,z)$$

Proof. 1) $\Upsilon^{(1)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \eta(-\frac{1}{\tau})^3 \cdot \dfrac{\theta_{p,m+\frac{1}{2}}(-\frac{1}{\tau},\frac{z}{\tau}) \,+\, \theta_{-p,m+\frac{1}{2}}(-\frac{1}{\tau},\frac{z}{\tau})}{\theta_{0,\frac{1}{2}}(-\frac{1}{\tau},\,\frac{z}{\tau})}$

$$= (-i\tau)^{\frac{3}{2}}\eta(\tau)^3\, \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i}{2\tau}(m+\frac{1}{2})z^2} \sum_{p'\in\mathbf{Z}(2m+1)\mathbf{Z}} e^{-\frac{2\pi i}{2m+1}pp'}\, \frac{\big[\theta_{p',m+\frac{1}{2}}+\theta_{-p',m+\frac{1}{2}}\big](\tau,z)}{(-i\tau)^{\frac{1}{2}}\, e^{\frac{\pi i z^2}{4\tau}}\, \theta_{0,\frac{1}{2}}(\tau,z)}$$

$$= \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}\, e^{\frac{\pi i m}{2\tau}z^2} \sum_{p'\in\mathbf{Z}(2m+1)\mathbf{Z}} e^{-\frac{2\pi i}{2m+1}pp'} \underbrace{\Bigg\{\eta(\tau)^3\, \frac{\big[\theta_{p',m+\frac{1}{2}}+\theta_{-p',m+\frac{1}{2}}\big](\tau,z)}{\theta_{0,\frac{1}{2}}(\tau,z)}\Bigg\}}_{\substack{||\\ \Upsilon^{(1)[m,p']\,*}(\tau,z)}}$$

proving 1).

$$2)\ \Upsilon^{(1)[m,p]\,*}(\tau+1,z) = \eta(\tau+1)^3\, \frac{[\theta_{p,m+\frac{1}{2}}+\theta_{-p,m+\frac{1}{2}}](\tau+1,z)}{\theta_{0,\frac{1}{2}}(\tau+1,z)}$$

$$= [e^{\frac{\pi i}{12}}\eta(\tau)]^3\, \frac{e^{\frac{\pi i}{2m+1}p^2}\,[\theta^{(-)}_{p,m+\frac{1}{2}}+\theta^{(-)}_{-p,m+\frac{1}{2}}](\tau,z)}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}$$

$$= e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}} \underbrace{\eta(\tau)^3\, \frac{[\theta^{(-)}_{p,m+\frac{1}{2}}-\theta^{(-)}_{-p,m+\frac{1}{2}}](\tau,z)}{\theta^{(-)}_{0,\frac{1}{2}}(\tau,z)}}_{\substack{||\\ \Upsilon^{(2)[m,p]\,*}(\tau,z)}}, \qquad \text{proving 2).}$$

□

9 $G^{(i)[m,p]\,*}(\tau,z)$ and $g_k^{(i)[m,p]\,*}(\tau)$

9.1 $G^{(i)[m,p]\,*}(\tau,z)$

For $m \in \mathbf{N}$ and $p \in \mathbf{Z}$ such that $0 \le p \le 2m$ and $i \in \{1,2\}$, we put

$$G^{(i)[m,p]\,*}(\tau,z) \;:=\; \Xi^{(i)[m,p]\,*}(\tau,z) - \Upsilon^{(i)[m,p]\,*}(\tau,z) \tag{9.1a}$$

Then, by Proposition 7.1, $G^{(i)[m,p]\,*}(\tau,z)$ can be written in the following form:

$$G^{(i)[m,p]\,*}(\tau,z) \;=\; \sum_{\substack{k\in\mathbf{Z}\\ 0\le k\le m}} g_k^{(i)[m,p]\,*}(\tau)\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$= \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} g_k^{(i)[m,p]\,*}(\tau)\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \,+\, 2\,g_0^{(i)[m,p]\,*}(\tau)\theta_{0,m}(\tau,z)$$

$$+\, 2\,g_m^{(i)[m,p]\,*}(\tau)\theta_{m,m}(\tau,z) \tag{9.1b}$$

The modular transformation properties of $G^{(1)[m,p]\,*}(\tau,z)$ are obtained immediately from (9.1a) and Lemma 8.2 and Lemma 8.3 as follows:

Proposition 9.1. *Let $m\in\mathbf{N}$ and $p\in\mathbf{Z}_{\geq 0}$ such that $0\leq p\leq 2m$. Then*

1) $$G^{(1)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}e^{\frac{\pi i m}{2\tau}z^2}\sum_{p'=0}^{2m} e^{-\frac{2\pi i}{2m+1}pp'}G^{(1)[m,p']\,*}(\tau,z) \tag{9.2a}$$

2) $$G^{(1)[m,p]\,*}(\tau+1,z) \;=\; e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}}\,G^{(2)[m,p]\,*}(\tau,z) \tag{9.2b}$$

The formula (9.2a) is rewritten as follows:

Proposition 9.2. *Let $m\in\mathbf{N}$ and $\ell\in\mathbf{Z}_{\geq 0}$ such that $0\leq \ell\leq 2m$. Then*

$$\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell}\,G^{(1)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) \;=\; (-i\tau)^{\frac{3}{2}}\sqrt{2m+1}\,e^{\frac{\pi i m}{2\tau}z^2}\,G^{(1)[m,\ell]\,*}(\tau,z) \tag{9.3}$$

Proof. Applying $\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell}$ to (9.2a), we have

$$\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell}\,G^{(1)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)$$

$$= \sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell}\,\frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}e^{\frac{\pi i m}{2\tau}z^2}\sum_{p'=0}^{2m} e^{-\frac{2\pi i}{2m+1}pp'}\,G^{(1)[m,p']\,*}(\tau,z)$$

$$= \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}}\,e^{\frac{\pi i m}{2\tau}z^2}\sum_{p'=0}^{2m}\underbrace{\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell}e^{-\frac{2\pi i}{2m+1}pp'}}_{\substack{||\\(2m+1)\,\delta_{p',\ell}}}\,G^{(1)[m,p']\,*}(\tau,z)$$

$$= (-i\tau)^{\frac{3}{2}}\sqrt{2m+1}\,e^{\frac{\pi i m}{2\tau}z^2}\,G^{(1)[m,\ell]\,*}(\tau,z)\,,$$ proving Proposition 9.2. □

9.2 Modular transformation of $g_k^{(i)[m,p]\,*}(\tau)$

To compute modular transformation of $g_k^{(1)[m,p]\,*}(\tau)$, we use the following formulas which are obtained easily from Lemmas 1,3 and 1.4 in [17].

Note 9.1. *Let $m\in\mathbf{N}$ and $k\in\mathbf{Z}$. Then*

1) (i) $$\big[\theta_{k,m}+\theta_{-k,m}\big]\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}}\, e^{\frac{\pi i m}{2\tau}z^2} \sum_{j\in\mathbf{Z}/2m\mathbf{Z}} \cos\frac{\pi jk}{m}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$$

$$= (-i\tau)^{\frac{1}{2}}\sqrt{\frac{2}{m}}\, e^{\frac{\pi i m}{2\tau}z^2}$$

$$\times \Bigg\{ \sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} \cos\frac{\pi jk}{m}\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + (-1)^k\,\theta_{m,m}(\tau,z)\Bigg\}$$

(ii) $$\theta_{0,m}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}}\, e^{\frac{\pi i m}{2\tau}z^2}\Bigg\{ \sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + \theta_{m,m}(\tau,z)\Bigg\}$$

(iii) $$\theta_{m,m}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) =$$

$$\frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}}\, e^{\frac{\pi i m}{2\tau}z^2}\Bigg\{ \sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} (-1)^j\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + (-1)^m\,\theta_{m,m}(\tau,z)\Bigg\}$$

2) (i) $$\big[\theta_{k,m}+\theta_{-k,m}\big](\tau+1,z) = e^{\frac{\pi i}{2m}k^2}\,\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

(ii) $$\theta_{0,m}(\tau+1,z) = \theta_{0,m}(\tau,z)$$

(iii) $$\theta_{m,m}(\tau+1,z) = e^{\frac{\pi i m}{2}}\,\theta_{m,m}(\tau,z)$$

The relation between modular transformation properties of $G^{(i)[m,p]\,*}(\tau,z)$ and those of $g_k^{(i)[m,p]\,*}(\tau)$, for $i\in\{1,2\}$, is obtained by using the above formulas as follows :

Lemma 9.1. *Let $m\in\mathbf{N}$ and $p\in\mathbf{Z}$ and $i\in\{1,2\}$ Then*

1) $$G^{(i)[m,p]\,*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = (-i\tau)^{\frac{1}{2}}\sqrt{\frac{2}{m}}\, e^{\frac{\pi i m}{2\tau}z^2}\Bigg($$

$$\sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} g_k^{(i)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}}\cos\frac{\pi jk}{m}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + (-1)^k\,\theta_{m,m}(\tau,z)\Bigg\}$$

$$+\ g_0^{(i)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + \theta_{m,m}(\tau,z)\Bigg\}$$

$$+\ g_m^{(i)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}}(-1)^j\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + (-1)^m\,\theta_{m,m}(\tau,z)\Bigg\}\Bigg) \tag{9.4a}$$

$$2) \quad G^{(i)[m,p]*}(\tau+1,z) \;=\; \sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} e^{\frac{\pi i}{2m}k^2} g_k^{(i)[m,p]*}(\tau+1)\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+\;\; 2\, g_0^{(i)[m,p]*}(\tau+1)\,\theta_{0,m}(\tau,z) \;\;+\;\; 2\, e^{\frac{\pi i m}{2}}\, g_m^{(i)[m,p]*}(\tau+1)\,\theta_{m,m}(\tau,z) \tag{9.4b}$$

Proof. By (9.1b) and Note 9.1, we have

$$G^{(i)[m,p]*}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) \;=\; \sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} g_k^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\big[\theta_{k,m}+\theta_{-k,m}\big]\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)$$

$$+\;2\, g_0^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\,\theta_{0,m}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) \;+\; 2\, g_m^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\,\theta_{m,m}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)$$

$$=\;\; (-i\tau)^{\frac{1}{2}}\sqrt{\frac{2}{m}}\, e^{\frac{\pi i m}{2\tau}z^2}\Bigg($$

$$\sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} g_k^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} \cos\frac{\pi jk}{m}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; (-1)^k\,\theta_{m,m}(\tau,z)\Bigg\}$$

$$+\;\; g_0^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; \theta_{m,m}(\tau,z)\Bigg\}$$

$$+\;\; g_m^{(i)[m,p]*}\Big(-\frac{1}{\tau}\Big)\Bigg\{\sum_{\substack{j\in\mathbf{Z}\\ 0<j<m}} (-1)^j\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; (-1)^m\,\theta_{m,m}(\tau,z)\Bigg\}\Bigg)$$

and

$$G^{(i)[m,p]*}(\tau+1,z) \;=\; \sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} g_k^{(i)[m,p]*}(\tau+1)\;\underbrace{\big[\theta_{k,m}+\theta_{-k,m}\big](\tau+1,z)}_{\substack{||\\ e^{\frac{\pi i}{2m}k^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)}}$$

$$+\;\; 2\, g_0^{(i)[m,p]*}(\tau+1)\;\underbrace{\theta_{0,m}(\tau+1,z)}_{\substack{||\\ \theta_{0,m}(\tau,z)}} + 2\, g_m^{(i)[m,p]*}(\tau+1)\;\underbrace{\theta_{m,m}(\tau+1,z)}_{\substack{||\\ e^{\frac{\pi i m}{2}}\theta_{m,m}(\tau,z)}}$$

$$=\;\; \sum_{\substack{k\in\mathbf{Z}\\ 0<k<m}} e^{\frac{\pi i}{2m}k^2} g_k^{(i)[m,p]*}(\tau+1)\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z)$$

$$+\;\; 2\, g_0^{(i)[m,p]*}(\tau+1)\,\theta_{0,m}(\tau,z) \;\;+\;\; 2\, e^{\frac{\pi i m}{2}}\, g_m^{(i)[m,p]*}(\tau+1)\,\theta_{m,m}(\tau,z)$$

proving Lemma 9.1. ∎

Proposition 9.3. *Let $m \in \mathbf{N}$ and $p, k \in \mathbf{Z}$ such that $0 \le p \le 2m$ and $0 \le k \le m$. Then*

1) *S-transformation of $g_k^{(1)[m,p]\,*}(\tau)$ are as follows:*

$$\text{(i)} \quad \underset{(0<j<m)}{g_j^{(1)[m,p]\,*}}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} e^{\frac{2\pi i}{2m+1}pp'} \cos\frac{\pi jk}{m}\, g_k^{(1)[m,p']\,*}(\tau)$$

$$+ \frac{-i\tau}{\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{(1)[m,p]\,*}(\tau) + (-1)^j\, g_m^{(1)[m,p']\,*}(\tau) \Big\}$$

$$\text{(ii)} \quad g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{2\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} e^{\frac{2\pi i}{2m+1}pp'}\, g_k^{(1)[m,p']\,*}(\tau)$$

$$+ \frac{-i\tau}{2\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{(1)[m,p']\,*}(\tau) + g_m^{(1)[m,p']\,*}(\tau) \Big\}$$

$$\text{(iii)} \quad g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{2\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} (-1)^k\, e^{\frac{2\pi i}{2m+1}pp'}\, g_k^{(1)[m,p']\,*}(\tau)$$

$$+ \frac{-i\tau}{2\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{(1)[m,p']\,*}(\tau) + (-1)^m\, g_m^{(1)[m,p']\,*}(\tau) \Big\}$$

2) *T-transformation of $g_k^{(1)[m,p]\,*}(\tau)$ are as follows:*

$$\text{(i)} \quad \underset{(0<j<m)}{g_j^{(1)[m,p]\,*}}(\tau+1) = e^{\frac{\pi i}{2m+1}p^2 + \frac{\pi i}{4} - \frac{\pi i}{2m}j^2}\, g_j^{(2)[m,p]\,*}(\tau)$$

$$\text{(ii)} \quad g_0^{(1)[m,p]\,*}(\tau+1) = e^{\frac{\pi i}{2m+1}p^2 + \frac{\pi i}{4}}\, g_0^{(2)[m,p]\,*}(\tau)$$

$$\text{(iii)} \quad g_m^{(1)[m,p]\,*}(\tau+1) = e^{\frac{\pi i}{2m+1}p^2 + \frac{\pi i}{4} - \frac{\pi i m}{2}}\, g_m^{(2)[m,p]\,*}(\tau)$$

Proof. Substituting (9.1b) and (9.4a) into (9.3), we have

$$\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} (-i\tau)^{\frac{1}{2}} \sqrt{\frac{2}{m}}\, e^{\frac{\pi i m}{2\tau}z^2} \Bigg($$

$$\sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} g_k^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \Big\{ \sum_{\substack{j \in \mathbf{Z} \\ 0<j<m}} \cos\frac{\pi jk}{m} \big[\theta_{j,m} + \theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + (-1)^k\, \theta_{m,m}(\tau,z) \Big\}$$

$$+ g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \Big\{ \sum_{\substack{j \in \mathbf{Z} \\ 0<j<m}} \big[\theta_{j,m} + \theta_{-j,m}\big](\tau,z) + \theta_{0,m}(\tau,z) + \theta_{m,m}(\tau,z) \Big\}$$

$$+ \; g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)\Big\{ \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} (-1)^j \, \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; (-1)^m\,\theta_{m,m}(\tau,z)\Big\}\Big)$$

$$= \quad (-i\tau)^{\frac{3}{2}}\sqrt{2m+1}\;\; e^{\frac{\pi i m}{2\tau}z^2}\left\{ \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} g_j^{(1)[m,\ell]\,*}(\tau)\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)\right.$$

$$\left. + \;\; 2\,g_0^{(1)[m,\ell]\,*}(\tau)\,\theta_{0,m}(\tau,z) \;\; + \;\; 2\,g_m^{(1)[m,\ell]\,*}(\tau)\,\theta_{m,m}(\tau,z)\right\}$$

namely

$$(-i\tau)^{-1}\sqrt{\frac{2}{m(2m+1)}}\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} g_k^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$\times\;\Big\{ \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} \cos\frac{\pi jk}{m}\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; (-1)^k\,\theta_{m,m}(\tau,z)\Big\}$$

$$+ \;\; (-i\tau)^{-1}\sqrt{\frac{2}{m(2m+1)}}\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$\times\;\Big\{ \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; \theta_{m,m}(\tau,z)\Big\}$$

$$+ \;\; (-i\tau)^{-1}\sqrt{\frac{2}{m(2m+1)}}\sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$\times\;\Big\{ \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} (-1)^j\,\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) \;+\; \theta_{0,m}(\tau,z) \;+\; (-1)^m\,\theta_{m,m}(\tau,z)\Big\}$$

$$= \sum_{\substack{j\in\mathbf{Z}\\0<j<m}} g_j^{(1)[m,\ell]\,*}(\tau)\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z) + 2\,g_0^{(1)[m,\ell]\,*}(\tau)\,\theta_{0,m}(\tau,z) + 2\,g_m^{(1)[m,\ell]\,*}(\tau)\,\theta_{m,m}(\tau,z)$$

Comparing the coefficients of $\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$, $\theta_{0,m}(\tau,z)$ and $\theta_{m,m}(\tau,z)$ in this equation, we have

$$\underset{(0<j<m)}{g_j^{(1)[m,\ell]\,*}(\tau)} \;\; = \;\; \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}}\sum_{p=0}^{2m} \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} e^{\frac{2\pi i}{2m+1}p\ell}\,\cos\frac{\pi jk}{m}\,g_k^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$+ \ \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} \Big\{ g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \ + \ (-1)^j \, g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \Big\}$$

$$2\, g_0^{(1)[m,\ell]\,*}(\tau) \ = \ \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} e^{\frac{2\pi i}{2m+1}p\ell} \, g_k^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$+ \ \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} \Big\{ g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \ + \ g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \Big\}$$

$$2\, g_m^{(1)[m,\ell]\,*}(\tau) \ = \ \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} \sum_{\substack{k \in \mathbf{Z} \\ 0<k<m}} (-1)^k \, e^{\frac{2\pi i}{2m+1}p\ell} \, g_k^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big)$$

$$+ \ \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{2m+1}p\ell} \Big\{ g_0^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \ + \ (-1)^m \, g_m^{(1)[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \Big\}$$

Then, replacing τ with $-\frac{1}{\tau}$ in these equations, we obtain the formulas in the claim 1).

2) Substituting (9.1b) and (9.4b) into (9.2b), we have

$$\sum_{\substack{j \in \mathbf{Z} \\ 0<j<m}} e^{\frac{\pi i}{2m}j^2} g_j^{(1)[m,p]\,*}(\tau+1) \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$$

$$+ \ 2\, g_0^{(1)[m,p]\,*}(\tau+1)\,\theta_{0,m}(\tau,z) \ + \ 2\, e^{\frac{\pi i m}{2}} \, g_m^{(1)[m,p]\,*}(\tau+1)\,\theta_{m,m}(\tau,z)$$

$$= \ e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}} \Big\{ \sum_{\substack{j \in \mathbf{Z} \\ 0<j<m}} g_j^{(2)[m,p]\,*}(\tau) \big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$$

$$+ \ 2\, g_0^{(2)[m,p]\,*}(\tau)\,\theta_{0,m}(\tau,z) \ + \ 2\, g_m^{(2)[m,p]\,*}(\tau)\,\theta_{m,m}(\tau,z) \Big\}$$

Comparing the coefficients of $\big[\theta_{j,m}+\theta_{-j,m}\big](\tau,z)$, $\theta_{0,m}(\tau,z)$ and $\theta_{m,m}(\tau,z)$ in this equation, we have

$$\begin{cases} e^{\frac{\pi i}{2m}j^2} \underset{(0<j<m)}{g_j^{(1)[m,p]\,*}(\tau+1)} &= \ e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}} \, g_j^{(2)[m,p]\,*}(\tau) \\ g_0^{(1)[m,p]\,*}(\tau+1) &= \ e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}} \, g_0^{(2)[m,p]\,*}(\tau) \\ e^{\frac{\pi i m}{2}} \, g_m^{(1)[m,p]\,*}(\tau+1) &= \ e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}} \, g_m^{(2)[m,p]\,*}(\tau) \end{cases}$$

namely

$$\begin{cases} g_{\substack{j \\ (0<j<m)}}^{(1)[m,p]\,*}(\tau+1) &=& e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}-\frac{\pi i}{2m}j^2}\, g_j^{(2)[m,p]\,*}(\tau) \\ g_0^{(1)[m,p]\,*}(\tau+1) &=& e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}}\, g_0^{(2)[m,p]\,*}(\tau) \\ g_m^{(1)[m,p]\,*}(\tau+1) &=& e^{\frac{\pi i}{2m+1}p^2+\frac{\pi i}{4}-\frac{\pi i m}{2}}\, g_m^{(2)[m,p]\,*}(\tau) \end{cases}$$

proving the claim 2). □

9.3 Explicit formula for $g_k^{(i)[m,p]\,*}(\tau)$

The explicit formulas for $G^{(i)[m,p]\,*}(\tau,z)\quad (i\in\{1,2\})$ are obtained from Proposition 7.1 and the formulas (8.1a), (8.1b), (8.2a), (8.2b) and (9.1a) as follows:

$$\begin{aligned} G^{(1)[m,p]\,*}(\tau,z) \;&= \\ &- \Bigg[\sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r<j}} - \sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ j\le r<0}}\Bigg] \sum_{\substack{k\in\mathbf{Z} \\ 0<k<m}} (-1)^{j-\frac{1}{2}+k}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \\ &\qquad\qquad \times\; q^{-\frac{1}{4m}\,[2mr+k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\ &- \Bigg[\sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r\le j}} - \sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ j<r<0}}\Bigg] \sum_{\substack{k\in\mathbf{Z} \\ 0\le k\le m}} (-1)^{j-\frac{1}{2}+k}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \\ &\qquad\qquad \times\; q^{-\frac{1}{4m}\,[2mr-k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\ &- \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z} \\ -p<r\le p}} \sum_{\substack{k\in\mathbf{Z} \\ 0\le k<m}} (-1)^k q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \\ &+ 2\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z} \\ 0\le r\le p-1}} q^{-m(r+\frac{1}{2})^2}\,\theta_{0,m}(\tau,z) \\ &- (-1)^m\,\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{\substack{r\in\mathbf{Z} \\ -p\le r\le p}} q^{-mr^2}\,\theta_{m,m}(\tau,z) \end{aligned}$$

$$\begin{aligned} G^{(2)[m,p]\,*}(\tau,z) \;&= \\ &- (-1)^p\Bigg[\sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0\le r<j}} - \sum_{\substack{j,r\in\frac{1}{2}\mathbf{Z}_{\rm odd} \\ j\le r<0}}\Bigg] \sum_{\substack{k\in\mathbf{Z} \\ 0<k<m}} (-1)^{j-\frac{1}{2}}\, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} \\ &\qquad\qquad \times\; q^{-\frac{1}{4m}\,[2mr+k+2mp]^2}\big[\theta_{k,m}+\theta_{-k,m}\big](\tau,z) \end{aligned}$$

$$- \; (-1)^p \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k \le m}} (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2}$$

$$\times \; q^{-\frac{1}{4m}[2mr-k+2mp]^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

$$- \; (-1)^p \, \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbf{Z} \\ -p < r \le p}} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le k < m}} q^{-\frac{1}{4m}\big(m(2r-1)+k\big)^2} \big[\theta_{k,m} + \theta_{-k,m}\big](\tau, z)$$

$$+ \; 2\,(-1)^p \, \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-m(r+\frac{1}{2})^2} \, \theta_{0,m}(\tau, z)$$

$$- \; (-1)^p \, \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbf{Z} \\ -p \le r \le p}} q^{-mr^2} \, \theta_{m,m}(\tau, z)$$

Then the explicit formulas for $g_k^{(i)[m,p]\,*}(\tau)$ follow immediately from (9.1b) and the above formulas as follows:

Proposition 9.4. *Let $m \in \mathbf{N}$ and $p, k \in \mathbf{Z}$ such that $0 \le p \le 2m$ and $0 \le k \le m$. Then $g_k^{(i)[m,p]\,*}(\tau) \quad (i \in \{1,2\})$ are as follows:*

1) (i) $\underset{(0<k<m)}{g_k^{(1)[m,p]\,*}(\tau)}$

$$= \; - \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r < j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j \le r < 0}} \Bigg] (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}[2mr+k+2mp]^2}$$

$$- \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 \le r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] (-1)^{j-\frac{1}{2}+k} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}[2mr-k+2mp]^2}$$

$$- \; \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbf{Z} \\ -p < r \le p}} (-1)^k q^{-\frac{1}{4m}\big(m(2r-1)+k\big)^2}$$

(ii) $2\,g_0^{(1)[m,p]\,*}(\tau)$

$$= \; -2 \Bigg[\sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ 0 < r \le j}} - \sum_{\substack{j,r \in \frac{1}{2}\mathbf{Z}_{\rm odd} \\ j < r < 0}} \Bigg] (-1)^{j-\frac{1}{2}} \, q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2 - m(r+p)^2}$$

$$- 2\, \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbf{Z} \\ 0 \le r \le p-1}} q^{-m(r+\frac{1}{2})^2}$$

(iii) $2\, g_m^{(1)[m,p]\,*}(\tau)$

$$= \ -2\,(-1)^m \Bigg[\sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0<r\le j}} - \sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ j<r<0}} \Bigg] (-1)^{j-\frac12}\, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2 - m(r+p-\frac12)^2}$$
$$- \ (-1)^m\, \theta^{(-)}_{2mp+m+\frac12, m+\frac12}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p\le r\le p}} q^{-mr^2}$$

2) (i) $\underset{(0<k<m)}{g_k^{(2)[m,p]\,*}(\tau)}$

$$= \ - \ (-1)^p \Bigg[\sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0\le r<j}} - \sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ j\le r<0}} \Bigg] (-1)^{j-\frac12}\, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2}\, q^{-\frac{1}{4m}\,[2mr+k+2mp]^2}$$
$$- \ (-1)^p \Bigg[\sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0\le r\le j}} - \sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ j<r<0}} \Bigg] (-1)^{j-\frac12}\, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2}\, q^{-\frac{1}{4m}\,[2mr-k+2mp]^2}$$
$$- \ (-1)^p\, \theta^{(-)}_{2mp+m+\frac12, m+\frac12}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p<r\le p}} q^{-\frac{1}{4m}\left(m(2r-1)+k\right)^2}$$

(ii) $2\, g_0^{(2)[m,p]\,*}(\tau)$

$$= \ -2\,(-1)^p \Bigg[\sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0\le r\le j}} - \sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ j<r<0}} \Bigg] (-1)^{j-\frac12}\, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2 - m(r+p)^2}$$
$$-2\,(-1)^p\, \theta^{(-)}_{2mp+m+\frac12, m+\frac12}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ 0\le r\le p-1}} q^{-m(r+\frac12)^2}$$

(iii) $2\, g_m^{(2)[m,p]\,*}(\tau)$

$$= \ -2\,(-1)^p \Bigg[\sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ 0\le r\le j}} - \sum_{\substack{j, r \in \frac12 \mathbf{Z}_{\rm odd} \\ j<r<0}} \Bigg] (-1)^{j-\frac12}\, q^{(m+\frac12)(j+\frac{2pm}{2m+1})^2 - m(r+p-\frac12)^2}$$
$$- \ (-1)^p\, \theta^{(-)}_{2mp+m+\frac12, m+\frac12}(\tau,0) \sum_{\substack{r \in \mathbf{Z} \\ -p\le r\le p}} q^{-mr^2}$$

10 Indefinite modular forms $g_j^{[m,p]\,*}(\tau)$

By Proposition 9.4 we observe that the following formula

$$g_j^{(2)[m,p]\,*}(\tau) \;=\; (-1)^{j+p}\, g_j^{(1)[m,p]\,*}(\tau) \tag{10.1}$$

holds for all $m \in \mathbf{N}$ and $p,\, j \in \mathbf{Z}$ such that $0 \le p \le 2m$ and $0 \le j \le m$. Then, simplifying the notation, we define the functions $g_j^{[m,p]\,*}(\tau)$ by

$$g_j^{[m,p]\,*}(\tau) \;=\; g_j^{(1)[m,p]\,*}(\tau) \tag{10.2}$$

Then the modular transformation formulas for these functions $g_j^{[m,p]\,*}(\tau)$ are obtained from Proposition 9.4 as follows:

Proposition 10.1. *Let $m \in \mathbf{N}$ and $p, j \in \mathbf{Z}$ such that $0 \le p \le 2m$ and $0 \le j \le m$. Then*

1) (i)

$$\underset{(0<j<m)}{g_j^{[m,p]\,*}\Big(-\frac{1}{\tau}\Big)} \;=\; \frac{-i\tau}{\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} e^{\frac{2\pi i}{2m+1}pp'} \cos\frac{\pi jk}{m}\, g_k^{[m,p']\,*}(\tau)$$
$$+\; \frac{-i\tau}{\sqrt{m(m+\frac{1}{2})}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{[m,p]\,*}(\tau) \,+\, (-1)^j\, g_m^{[m,p']\,*}(\tau)\Big\}$$

(ii)

$$g_0^{[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \;=\; \frac{-i\tau}{\sqrt{2m(2m+1)}} \sum_{p'=0}^{2m} \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} e^{\frac{2\pi i}{2m+1}pp'}\, g_k^{[m,p']\,*}(\tau)$$
$$+\; \frac{-i\tau}{\sqrt{2m(2m+1)}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{[m,p']\,*}(\tau) \,+\, g_m^{[m,p']\,*}(\tau)\Big\}$$

(iii)

$$g_m^{[m,p]\,*}\Big(-\frac{1}{\tau}\Big) \;=\; \frac{-i\tau}{\sqrt{2m(2m+1)}} \sum_{p'=0}^{2m} \sum_{\substack{k\in\mathbf{Z}\\0<k<m}} (-1)^k\, e^{\frac{2\pi i}{2m+1}pp'}\, g_k^{[m,p']\,*}(\tau)$$
$$+\; \frac{-i\tau}{\sqrt{2m(2m+1)}} \sum_{p'=0}^{2m} e^{\frac{2\pi i}{2m+1}pp'} \Big\{ g_0^{[m,p']\,*}(\tau) \,+\, (-1)^m\, g_m^{[m,p']\,*}(\tau)\Big\}$$

2)

$$\underset{(0\le j\le m)}{g_j^{[m,p]\,*}(\tau+1)} \;=\; e^{\frac{\pi i}{2m+1}(p+\frac{2m+1}{2})^2 - \frac{\pi i}{2m}(j+m)^2}\, g_j^{[m,p]\,*}(\tau)$$

11 An example $\sim$ the case $m = 1$

The functions which take place in the case $m = 1$ are $g_k^{[1,p]\,*}(\tau)$ $(p = 0, 1, 2;\ k = 0, 1)$ and they are, by Proposition 9.4, as follows:

$$\begin{aligned} 2\,g_0^{[1,p]\,*}(\tau) &= -2\Bigg[\sum_{\substack{j,\,r\,\in\,\frac12\mathbf{Z}_{\rm odd}\\ 0<r\le j}} - \sum_{\substack{j,\,r\,\in\,\frac12\mathbf{Z}_{\rm odd}\\ j<r<0}}\Bigg](-1)^{j-\frac12}\, q^{\frac32(j+\frac{2p}{3})^2-(r+p)^2} \\ &\quad -2\,\theta^{(-)}_{2p+\frac32,\frac32}(\tau,0) \sum_{\substack{r\,\in\,\mathbf{Z}\\ 0\le r\le p-1}} q^{-(r+\frac12)^2} \end{aligned} \tag{11.1a}$$

$$\begin{aligned} 2\,g_1^{[1,p]\,*}(\tau) &= 2\Bigg[\sum_{\substack{j,\,r\,\in\,\frac12\mathbf{Z}_{\rm odd}\\ 0<r\le j}} - \sum_{\substack{j,\,r\,\in\,\frac12\mathbf{Z}_{\rm odd}\\ j<r<0}}\Bigg](-1)^{j-\frac12}\, q^{\frac32(j+\frac{2p}{3})^2-(r+p-\frac12)^2} \\ &\quad +\,\theta^{(-)}_{2p+\frac32,\frac32}(\tau,0) \sum_{\substack{r\,\in\,\mathbf{Z}\\ -(p-1)\le r\le p-1}} q^{-r^2} \end{aligned} \tag{11.1b}$$

Putting $\left\{\begin{array}{lcl} j-\frac12 &=& j' \\ r-\frac12 &=& r' \end{array}\right.$, the above formulas are rewritten as follows:

$$\begin{aligned} g_0^{[1,p]\,*}(\tau) &= -\Bigg[\sum_{\substack{j',\,r'\,\in\,\mathbf{Z}\\ 0\le r'\le j'}} - \sum_{\substack{j',\,r'\,\in\,\mathbf{Z}\\ j'<r'<0}}\Bigg](-1)^{j'}\, q^{\frac32(j'+\frac12+\frac{2p}{3})^2-(r'+\frac12+p)^2} \\ &\quad -\,\theta^{(-)}_{2p+\frac32,\frac32}(\tau,0) \sum_{\substack{r\,\in\,\mathbf{Z}\\ 0\le r\le p-1}} q^{-(r+\frac12)^2} \end{aligned} \tag{11.2a}$$

$$\begin{aligned} g_1^{[1,p]\,*}(\tau) &= \Bigg[\sum_{\substack{j',\,r'\,\in\,\mathbf{Z}\\ 0\le r'\le j'}} - \sum_{\substack{j',\,r'\,\in\,\mathbf{Z}\\ j'<r'<0}}\Bigg](-1)^{j'}\, q^{\frac32(j'+\frac12+\frac{2p}{3})^2-(r'+p)^2} \\ &\quad +\,\frac12\,\theta^{(-)}_{2p+\frac32,\frac32}(\tau,0) \sum_{\substack{r\,\in\,\mathbf{Z}\\ -(p-1)\le r\le p-1}} q^{-r^2} \end{aligned} \tag{11.2b}$$

These functions are written explicitly as follows:

Note 11.1.

1) $g_0^{[1,0]\,*}(\tau) = -\Bigg[\sum\limits_{\substack{j,\,r\,\in\,\mathbf{Z}\\ 0\le r\le j}} - \sum\limits_{\substack{j,\,r\,\in\,\mathbf{Z}\\ j<r<0}}\Bigg](-1)^j\, q^{\frac32(j+\frac12)^2-(r+\frac12)^2} \;=\; -q^{\frac18}+\cdots$

2) $g_1^{[1,0]\,*}(\tau) = \Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r\le j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{2})^2-r^2} \quad = \quad q^{\frac{3}{8}}+\cdots$

3) $$g_0^{[1,1]\,*}(\tau) = g_0^{[1,2]\,*}(\tau) = \Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r<j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j\le r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{6})^2-(r+\frac{1}{2})^2} = q^{\frac{19}{24}}+\cdots$$

4) $$g_1^{[1,1]\,*}(\tau) = g_1^{[1,2]\,*}(\tau) = -\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r<j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j\le r<0}}\Bigg](-1)^j q^{\frac{3}{2}(j+\frac{1}{6})^2-r^2} + \frac{1}{2}\,\theta^{(-)}_{\frac{1}{2},\frac{3}{2}}(\tau,0) = -\frac{1}{2}\,q^{\frac{1}{24}}+\cdots$$

The S-transformation of these functions is computed by Proposition 9.3 as follows:

$$g_0^{[1,p]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\sum_{p'=0}^{2} e^{\frac{2\pi i}{3}pp'}\Big\{g_0^{[1,p']\,*}(\tau) + g_1^{[1,p']\,*}(\tau)\Big\}$$

$$g_1^{[1,p]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\sum_{p'=0}^{2} e^{\frac{2\pi i}{3}pp'}\Big\{g_0^{[1,p']\,*}(\tau) - g_1^{[1,p']\,*}(\tau)\Big\}$$

Since $g_k^{[1,1]\,*}(\tau) = g_k^{[1,2]\,*}(\tau) \quad (k=0,1)$ by Note 11.1, these formulas are written explicitly as follows:

Note 11.2.

1) $g_0^{[1,0]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\Big\{g_0^{[1,0]\,*}(\tau) + g_1^{[1,0]\,*}(\tau) + 2\,g_0^{[1,1]\,*}(\tau) + 2\,g_1^{[1,1]\,*}(\tau)\Big\}$

2) $g_1^{[1,0]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\Big\{g_0^{[1,0]\,*}(\tau) - g_1^{[1,0]\,*}(\tau) + 2\,g_0^{[1,1]\,*}(\tau) - 2\,g_1^{[1,1]\,*}(\tau)\Big\}$

3) $g_0^{[1,1]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\Big\{g_0^{[1,0]\,*}(\tau) + g_1^{[1,0]\,*}(\tau) - g_0^{[1,1]\,*}(\tau) - g_1^{[1,1]\,*}(\tau)\Big\}$

4) $g_1^{[1,1]\,*}\Big(-\frac{1}{\tau}\Big) = \frac{-i\tau}{\sqrt{6}}\Big\{g_0^{[1,0]\,*}(\tau) - g_1^{[1,0]\,*}(\tau) - g_0^{[1,1]\,*}(\tau) + g_1^{[1,1]\,*}(\tau)\Big\}$

From Notes 11.1 and 11.2, we have the following:

Lemma 11.1. *Define the functions $f_i(\tau) \quad (i=0,1,2,3)$ by*

$$\left\{\begin{array}{lcl} f_0(\tau) & := & \dfrac{-\,2\,g_1^{[1,1]\,*}(\tau)}{\eta(\tau)^2} \\[2ex] f_3(\tau) & := & \dfrac{2\,g_0^{[1,1]\,*}(\tau)}{\eta(\tau)^2} \end{array}\right. \qquad \left\{\begin{array}{lcl} f_1(\tau) & := & \dfrac{-\,g_0^{[1,0]\,*}(\tau)}{\eta(\tau)^2} \\[2ex] f_2(\tau) & := & \dfrac{g_1^{[1,0]\,*}(\tau)}{\eta(\tau)^2} \end{array}\right. \tag{11.3}$$

Then,

1) *these functions $f_i(\tau)$ satisfy the following S-transformation properties:*

$$\begin{array}{rrcl}
(0) & f_0\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{f_0(\tau) + 2\,f_1(\tau) + 2\,f_2(\tau) + f_3(\tau)\Big\} \\
(\mathrm{i}) & f_1\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{f_0(\tau) + f_1(\tau) - f_2(\tau) - f_3(\tau)\Big\} \\
(\mathrm{ii}) & f_2\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{f_0(\tau) - f_1(\tau) - f_2(\tau) + f_3(\tau)\Big\} \\
(\mathrm{iii}) & f_3\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{f_0(\tau) - 2\,f_1(\tau) + 2\,f_2(\tau) - f_3(\tau)\Big\}
\end{array}$$

2) *the leading terms of $f_i(\tau)$ are as follows* :

$$\left\{\begin{array}{lcrl}
f_0(\tau) & = & q^{-\frac{1}{24}} & + \cdots \\
f_1(\tau) & = & q^{\frac{1}{24}} & + \cdots \\
f_2(\tau) & = & q^{\frac{7}{24}} & + \cdots \\
f_3(\tau) & = & 2\,q^{\frac{17}{24}} & + \cdots
\end{array}\right.$$

Next we consider the Jacobi's theta function

$$\theta_{j,3}(\tau,z) \;=\; \sum_{n\,\in\,\mathbf{Z}} q^{3(n+\frac{j}{6})^2}\, e^{6\pi i(n+\frac{j}{6})z}$$

The S-transformation of $\theta_{j,3}(\tau,0)$ is computed by using Lemmas 1.2 and 1.3 in [17] as follows:

$$\begin{array}{rrcl}
(\mathrm{i}) & \theta_{0,3}\Big(-\dfrac{1}{\tau},0\Big) & = & \dfrac{(-i\tau)^{\frac{1}{2}}}{\sqrt{6}}\Big\{\theta_{0,3}(\tau,0) + 2\,\theta_{1,3}(\tau,0) + 2\,\theta_{2,3}(\tau,0) + \theta_{3,3}(\tau,0)\Big\} \\
(\mathrm{ii}) & \theta_{1,3}\Big(-\dfrac{1}{\tau},0\Big) & = & \dfrac{(-i\tau)^{\frac{1}{2}}}{\sqrt{6}}\Big\{\theta_{0,3}(\tau,0) + \theta_{1,3}(\tau,0) - \theta_{2,3}(\tau,0) - \theta_{3,3}(\tau,0)\Big\} \\
(\mathrm{iii}) & \theta_{2,3}\Big(-\dfrac{1}{\tau},0\Big) & = & \dfrac{(-i\tau)^{\frac{1}{2}}}{\sqrt{6}}\Big\{\theta_{0,3}(\tau,0) - \theta_{1,3}(\tau,0) - \theta_{2,3}(\tau,0) + \theta_{3,3}(\tau,0)\Big\} \\
(\mathrm{iv}) & \theta_{3,3}\Big(-\dfrac{1}{\tau},0\Big) & = & \dfrac{(-i\tau)^{\frac{1}{2}}}{\sqrt{6}}\Big\{\theta_{0,3}(\tau,0) - 2\,\theta_{1,3}(\tau,0) + 2\,\theta_{2,3}(\tau,0) - \theta_{3,3}(\tau,0)\Big\}
\end{array}$$

And the leading terms of $\theta_{j,3}(\tau,0)$ are

$$\left\{\begin{array}{lcrl}
\theta_{0,3}(\tau,0) & = & q^{0} & + \cdots \\
\theta_{1,3}(\tau,0) & = & q^{\frac{1}{12}} & + \cdots \\
\theta_{2,3}(\tau,0) & = & q^{\frac{1}{3}} & + \cdots \\
\theta_{3,3}(\tau,0) & = & 2\,q^{\frac{3}{4}} & + \cdots
\end{array}\right.$$

Then putting

$$h_j(\tau) \;:=\; \frac{\theta_{j,3}(\tau,0)}{\eta(\tau)} \qquad (j=0,1,2,3) \tag{11.4}$$

we have

Lemma 11.2. 1) *Functions $h_j(\tau)$ satisfy the following S-transformation properties:*

$$\begin{array}{rrcl}
(0) & h_0\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{h_0(\tau)\,+\,2\,h_1(\tau)\,+\,2\,h_2(\tau)\,+\,h_3(\tau)\Big\} \\
(\mathrm{i}) & h_1\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{h_0(\tau)\,+\,h_1(\tau)\,-\,h_2(\tau)\,-\,h_3(\tau)\Big\} \\
(\mathrm{ii}) & h_2\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{h_0(\tau)\,-\,h_1(\tau)\,-\,h_2(\tau)\,+\,h_3(\tau)\Big\} \\
(\mathrm{iii}) & h_3\Big(-\dfrac{1}{\tau}\Big) & = & \dfrac{1}{\sqrt{6}}\Big\{h_0(\tau)\,-\,2\,h_1(\tau)\,+\,2\,h_2(\tau)\,-\,h_3(\tau)\Big\}
\end{array}$$

2) *the leading terms of* $h_j(\tau)$ *are as follows* :
$$\left\{\begin{array}{rcrl}
h_0(\tau) & = & q^{-\frac{1}{24}} & + \cdots \\
h_1(\tau) & = & q^{\frac{1}{24}} & + \cdots \\
h_2(\tau) & = & q^{\frac{7}{24}} & + \cdots \\
h_3(\tau) & = & 2\,q^{\frac{17}{24}} & + \cdots
\end{array}\right.$$

From these formulas, we obtain the following :

Proposition 11.1.

(i) $$\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r\le j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{2})^2-(r+\frac{1}{2})^2} \;=\; \eta(\tau)\,\theta_{1,3}(\tau,0)$$

(ii) $$\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r\le j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j<r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{2})^2-r^2} \;=\; \eta(\tau)\,\theta_{2,3}(\tau,0)$$

(iii) $$\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r<j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j\le r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{6})^2-(r+\frac{1}{2})^2} \;=\; \frac{1}{2}\,\eta(\tau)\,\theta_{3,3}(\tau,0)$$

(iv) $$\Bigg[\sum_{\substack{j,r\in\mathbf{Z}\\ 0\le r<j}} - \sum_{\substack{j,r\in\mathbf{Z}\\ j\le r<0}}\Bigg](-1)^j\, q^{\frac{3}{2}(j+\frac{1}{6})^2-r^2} \;=\; \frac{1}{2}\Big\{\eta(\tau)\,\theta_{0,3}(\tau,0)\,+\,\theta^{(-)}_{\frac{1}{2},\frac{3}{2}}(\tau,0)\Big\}$$

Proof. By Lemmas 11.1 and 11.2, we see that the functions $\{f_i(\tau)\}_{i=0,1,2,3}$ and $\{h_i(\tau)\}_{i=0,1,2,3}$ satisfy the same S-transformation properties and have the same polar parts. Then, by Lemma 4.8 in [14], we have
$$f_i(\tau)\,=\,h_i(\tau) \qquad \text{for all } i,$$
namely
$$\left\{\begin{array}{rcl}
2\,g_0^{[1,1]\,*}(\tau) & = & \eta(\tau)\,\theta_{3,3}(\tau,0) \\
-\,2\,g_1^{[1,1]\,*}(\tau) & = & \eta(\tau)\,\theta_{0,3}(\tau,0)
\end{array}\right. \qquad
\left\{\begin{array}{rcl}
-\,g_0^{[1,0]\,*}(\tau) & = & \eta(\tau)\,\theta_{1,3}(\tau,0) \\
g_1^{[1,0]\,*}(\tau) & = & \eta(\tau)\,\theta_{2,3}(\tau,0)
\end{array}\right.$$
by (11.3) and (11.4). Rewriting $g_k^{[1,p]\,*}(\tau)$ by using Note 11.1, we obtain the formulas in Proposition 11.1. □

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