POSITIVE SOLUTIONS FOR THE FRACTIONAL p-LAPLACIAN VIA MIXED TOPOLOGICAL AND VARIATIONAL METHODS

ANTONIO IANNIZZOTTO

ABSTRACT. We study a nonlinear, nonlocal Dirichlet problem driven by the degenerate fractional p-Laplacian via a combination of topological methods (degree theory for operators of monotone type) and variational methods (critical point theory). We assume local conditions ensuring the existence of sub- and supersolutions. So we prove existence of two positive solutions, in both the coercive and noncoercive cases.

Version of March 1, 2023

1. Introduction

In the study of nonlinear elliptic partial differential equations, two main approaches have been followed in the past decades: in the *variational* approach, solutions of the examined problem are seen as critical points of an energy functional and detected via minimization or min-max schemes; in the *topological* approach, solutions are seen as zeros of a nonlinear operator and are found via fixed point theorems or degree theory. The topological approach is more general, as it allows to deal with gradient-depending terms, but it usually provides less precise information about the number of solutions. Classical references on the variational and topological methods for nonlinear equations are [22] and [8], respectively.

When quasilinear equations are considered (think for instance of the p-Laplacian operator), both variational and topological methods are affected by technical difficulties: no Hilbertian structure is available, energy functionals fail to be twice differentiable, so the Leray-Schauder degree for nonlinear operators does not apply. Thus, the p-Laplace equation has been the benchmark for developing both a critical point theory for C^1 -functionals on Banach spaces, and an effective degree theory for operators of monotone type mapping a Banach space into its dual, as well as useful combinations of the two methods, as in the very interesting paper [19]. See [18] for a comprehensive account on such techniques.

In the last decade, beside nonlinearity, nonlocality has come into play as a major feature, with the increasing interest in fractional order elliptic operators and the inherent new difficulties, which are mainly related to regularity, maximum and comparison principles, and the boundary behavior of solutions. The semilinear case, dealing with the fractional Laplacian or variants of it, is by now well established, see for instance [17] for the variational approach. In the quasilinear case, namely for the fractional p-Laplacian, a purely variational approach for existence results was proposed in [10], based on Morse theory and the spectral theory of [16]. Much has been achieved since then and, despite missing a full organic theory, we have several partial results providing a basic toolbox for the treatment of equations, at least in the degenerate case, i.e., when the summability exponent is p > 2 (examples are found in [5] for the variational method and in [6] for the topological method).

In this paper, in some sense a companion work of [6], we aim at combining the variational and topological approaches, relying on recent regularity and comparison results, in order to prove multiplicity of positive solutions for a fractional p-Laplace equations under mild assumptions on the reaction. Precisely, we study the following fractional order nonlinear equation with Dirichlet condition:

(1.1)
$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

²⁰¹⁰ Mathematics Subject Classification. 35R11, 47H11, 35A15.

Key words and phrases. Fractional p-Laplacian, Degree theory, Multiple solutions.

Here $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with $C^{1,1}$ boundary, $p \geq 2$, $s \in (0,1)$ s.t. N > ps, the leading operator is the fractional p-Laplacian, defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough and all $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_{\varepsilon}^c(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N + ps}} \, dy,$$

which for p=2 reduces to the fractional Laplacian and for $s\to 1^-$ converges to the classical p-Laplacian (up to multiplicative constants). Finally, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory mapping subject to several growth conditions. We focus on asymptotically (p-1)-linear reactions both at infinity and at zero, i.e., we assume that the quotient

$$t \mapsto \frac{f(\cdot, t)}{t^{p-1}}$$

is bounded (uniformly in Ω) as $t \to \infty$, 0, respectively. The present work differs from [6], mainly because we do not assume a qualitatively different behavior of the quotient above at $\pm \infty$ and at 0 (jumping reaction). So we distinguish two cases, roughly speaking:

- (a) in the coercive case, the limits of the quotient above lie below the principal eigenvalue of the operator $(-\Delta)_p^s$ with nonresonance on a positively measured subset of Ω ;
- (b) in the noncoercive case, the same limits lie above the principal eigenvalue.

Note that in general we allow for the reaction to have exactly the same behavior at infinity and at zero, unlike in many similar works (i.e., we exclude the so-called concave-convex reactions). On the other hand, we shall assume some local conditions ensuring the existence of sub- and supersolutions, namely, in case (a) we require that the reaction lies above a power t^{q-1} (q > p) near the origin, and in case (b) we require that it is nonpositive at some point b > 0. Under such hypotheses, in both cases we prove that problem (1.1) admits at least two positive solutions, the first being obtained as a local minimizer of the energy functional, and the second being detected through a degree theoretic argument (see Theorems 4.3 and 5.2 below). We are inspired by similar results from [19] for the local case (s = 1). We remark that our results are new even in the semilinear framework p = 2.

The structure of the paper is the following: in Section 2 we recall the degree theory for operators of class $(S)_+$; in Section 3 we collect some general results on the fractional p-Laplacian and the related Dirichlet problem; in Section 4 we prove our multiplicity result for the coercive case (a); and in Section 5 we deal with the noncoercive case (b).

Notation. Throughout the paper, for any $A \subset \mathbb{R}^N$ we shall set $A^c = \mathbb{R}^N \setminus A$. For any two measurable functions $u, v : \Omega \to \mathbb{R}$, $u \leqslant v$ in Ω will mean that $u(x) \leqslant v(x)$ for a.e. $x \in \Omega$ (and similar expressions). The positive (resp., negative) part of u is denoted u^+ (resp., u^-). Every function u defined in Ω will be identified with its 0-extension to \mathbb{R}^N . If X is an ordered Banach space, then X_+ will denote its non-negative order cone. For all $r \in [1, \infty]$, $\|\cdot\|_r$ denotes the standard norm of $L^r(\Omega)$ (or $L^r(\mathbb{R}^N)$), which will be clear from the context). Moreover, C will denote a positive constant (whose value may change case by case).

2. Degree theory for $(S)_+$ -maps

For the reader's convenience, we recall here some basic notions and properties of Browder's topological degree for $(S)_+$ -maps, introduced in [2] (we follow the general exposition of [18, Section 4.3]).

Let X be a separable, reflexive Banach space, X^* be its dual space, and $U \subseteq X$. An operator $A: U \to X^*$ is an $(S)_+$ -map if, for any sequence (u_n) in X, $u_n \to u$ in X and

$$\limsup_{n} \langle A(u_n), u_n - u \rangle \leqslant 0$$

imply $u_n \to u$ (strongly) in X. Also, A is demicontinuous if it is strong to weak* continuous. Note that, if A is a demicontinuous $(S)_+$ -map and $B: U \to X^*$ is a completely continuous map, then A+B is a demicontinuous $(S)_+$ -map as well. In particular, by classical functional analysis (Troyanski's renorming theorem) there exists a $(S)_+$ -homeomorphism $\mathcal{F}: X \to X^*$ (duality map) s.t. for all $u \in X$

$$\|\mathcal{F}(u)\|^2 = \|u\|^2 = \langle \mathcal{F}(u), u \rangle.$$

Now consider a triple (A, U, u^*) where $U \subset X$ is a bounded open set, $A : \overline{U} \to X^*$ is a demicontinuous $(S)_+$ -map, and $u^* \in X^* \setminus A(\partial U)$. By separability, we can perform a Galerkin type approximation of X by

means of an increasing sequence (X_n) of finite-dimensional subspaces. For all $n \in \mathbb{N}$ we set $U_n = U \cap X_n$ and define $A_n : \overline{U}_n \to X_n^*$ by setting for all $u \in \overline{U}_n$, $v \in X_n$

$$\langle A_n(u), v \rangle = \langle A(u), v \rangle$$

(for simplicity, we use the same notation for the duality pairings between X_n^* and X_n , and between X^* and X, respectively). By [18, Proposition 4.38] Brouwer's degree of the triple (A_n, U_n, u^*) eventually stabilizes as $n \to \infty$, so we can define for some $n \in \mathbb{N}$ big enough

$$\deg_{(S)_{+}}(A, U, u^{*}) = \deg(A_{n}, U_{n}, u^{*}).$$

The integer-valued map $\deg_{(S_+)}$ inherits the main properties of Brouwer's degree:

Proposition 2.1. [18, Theorem 4.42] Let $U \subset X$ be a bounded open set, $A : \overline{U} \to X^*$ be a demicontinuous $(S)_+$ -map, $u^* \notin A(\partial U)$. Then:

- (i) (normalization) if $u^* \in \mathcal{F}(U)$, then $\deg_{(S)_+}(\mathcal{F}, U, u^*) = 1$;
- (ii) (domain additivity) if $U = U_1 \cup U_2$, with $U_1, U_2 \subset X$ nonempty open sets s.t. $U_1 \cap U_2 = \emptyset$ and $u^* \notin A(\partial U_1 \cup \partial U_2)$, then

$$\deg_{(S)_{\perp}}(A, U, u^*) = \deg_{(S)_{\perp}}(A, U_1, u^*) + \deg_{(S)_{\perp}}(A, U_2, u^*);$$

(iii) (excision) if $C \subset \overline{U}$ is closed s.t. $u^* \notin A(C)$, then

$$\deg_{(S)_+}(A, U \setminus C, u^*) = \deg_{(S)_+}(A, U, u^*);$$

(iv) (homotopy invariance) if $h:[0,1]\times \overline{U}\to X^*$ is a $(S)_+$ -homotopy s.t. $u^*\notin h(t,\partial U)$ for all $t\in[0,1]$, then the function

$$t \mapsto \deg_{(S)_+}(h(t,\cdot), U, u^*)$$

is constant in [0,1];

- (v) (solution) if $\deg_{(S)_+}(A, U, u^*) \neq 0$, then there exists $u \in U$ s.t. $A(u) = u^*$;
- (vi) (boundary dependence) if $B: \overline{U} \to X^*$ is a demicontinuous $(S)_+$ -map s.t. A(u) = B(u) for all $u \in \partial U$, then

$$\deg_{(S)_+}(A,U,u^*) = \deg_{(S)_+}(B,U,u^*).$$

For our purposes, the most important property is (iv). We recall that a $(S)_+$ -homotopy is a map $h: [0,1] \times \overline{U} \to X^*$ s.t. $t_n \to t$ in [0,1], $u_n \rightharpoonup u$ in X, and

$$\lim\sup_{n} \langle h(t_n, u_n), u_n - u \rangle \leqslant 0$$

imply $u_n \to u$ in X and $h(t_n, u_n) \rightharpoonup h(t, u)$ in X^* . For instance, if $A, B : \overline{U} \to X^*$ are demicontinuous $(S)_+$ -maps, then by [18, Proposition 4.41] we can define a $(S)_+$ -homotopy as follows:

$$h(t, u) = (1 - t)A(u) + tB(u).$$

The bridge between variational and topological methods is represented by the case of a functional $\Phi \in C^1(X)$, whose Gâteaux derivative $\Phi': X \to X^*$ is a (demi-) continuous $(S)_+$ -map (potential operator). We set

$$K(\Phi) = \{ u \in X : \Phi'(u) = 0 \}.$$

In such cases, the degree of Φ' in some special sets is related to the local and asymptotic behavior of Φ , respectively, as first proved in [21] (for the Leray-Schauder degree):

Proposition 2.2. [18, Corollaries 4.46, 4.49] Let $\Phi \in C^1(X)$ be a functional s.t. $\Phi' : X \to X^*$ is a continuous $(S)_+$ -map:

(i) if $u_0 \in X$ is a local minimizer and an isolated critical point of Φ , then for all $\rho > 0$ small enough

$$\deg_{(S)_+}(\Phi', B_{\rho}(u_0), 0) = 1;$$

(ii) if Φ is coercive and $K(\Phi)$ is bounded, then for all R>0 big enough

$$\deg_{(S)_{\perp}}(\Phi', B_R(0), 0) = 1.$$

3. The Dirichlet problem for the fractional p-Laplacian

Here we recall some basic features of the existing theory about problem (1.1). First, for all open $\Omega \subseteq \mathbb{R}^N$ and all measurable $u: \Omega \to \mathbb{R}$ we define the Gagliardo seminorm

$$[u]_{s,p,\Omega} = \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \right)^{\frac{1}{p}}.$$

We introduce the fractional Sobolev spaces (see [4] for details)

$$W^{s,p}(\Omega) = \{ u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty \},\,$$

$$\widetilde{W}^{s,p}(\Omega) = \left\{ u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : u \in W^{s,p}(\Omega') \text{ for some } \Omega' \ni \Omega \text{ and } \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}} \, dx < \infty \right\},$$

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c \right\}.$$

Clearly $W_0^{s,p}(\Omega) \subset \widetilde{W}^{s,p}(\Omega)$, and conversely for all $u \in \widetilde{W}^{s,p}(\Omega)$ s.t. u = 0 in Ω^c we have $u \in W_0^{s,p}(\Omega)$, see [5, Lemma 2.1]. If $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ -smooth boundary $\partial\Omega$, then $W_0^{s,p}(\Omega)$ is a uniformly convex, separable Banach space with norm

$$||u|| = [u]_{s,p,\mathbb{R}^N},$$

whose dual space is denoted $W^{-s,p'}(\Omega)$. Also, the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1, p_s^*]$ and compact for all $r \in [1, p_s^*]$, where the fractional critical exponent is defined by

$$p_s^* = \frac{Np}{N - ps}.$$

By [13, Lemma 2.3], we can extend the definition of the fractional p-Laplacian to a wider class of functions. For all $u \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \in W_0^{s,p}(\Omega)$ set

$$\langle (-\Delta)_p^s u, \varphi \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dx dy.$$

Then we have $(-\Delta)_p^s u \in W^{-s,p'}(\Omega)$, and the definition above agrees with the one given in Section 1 for u smooth enough (for instance, $u \in C_c^{\infty}(\Omega)$). The restricted operator $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a continuous $(S)_+$ -map (in fact, the duality map of $W_0^{s,p}(\Omega)$ is $u \mapsto (-\Delta)_p^s u/\|u\|^{p-2}$), as well as the gradient of the C^1 -functional

$$u \mapsto \frac{\|u\|^p}{p}.$$

We recall a useful formula, holding for any $u \in W_0^{s,p}(\Omega)$:

$$||u^{\pm}||^p \leqslant \langle (-\Delta)_p^s u, \pm u^{\pm} \rangle.$$

Also, such operator is strictly (T)-monotone:

Proposition 3.1. [16, proof of Lemma 9] Let $u, v \in \widetilde{W}^{s,p}(\Omega)$ s.t. $(u-v)^+ \in W_0^{s,p}(\Omega)$ satisfy

$$\langle (-\Delta)_p^s u - (-\Delta)_p^s v, (u-v)^+ \rangle \leqslant 0,$$

then $u \leq v$ in Ω .

Now let us consider problem (1.1), under the following basic hypothesis:

 \mathbf{H}_0 $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $c_0 > 0$, $r \in (1, p_s^*)$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|f(x,t)| \le c_0(1+|t|^{r-1}).$$

We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a (weak) supersolution of (1.1) if for all $\varphi \in W_0^{s,p}(\Omega)_+$

$$\langle (-\Delta)_p^s u, \varphi \rangle \geqslant \int_{\Omega} f(x, u) \varphi dx,$$

and similarly we define a (weak) subsolution. A (weak) solution of (1.1) is both a super- and a subsolution, i.e., a function $u \in W_0^{s,p}(\Omega)$ s.t. for all $\varphi \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s u, \varphi \rangle = \int_{\Omega} f(x, u) \varphi \, dx,$$

The definitions of super-, subsolutions and solutions for other Dirichlet problems will be analogous. We have the following a priori bound:

Proposition 3.2. [3, Theorem 3.3] Let \mathbf{H}_0 hold, $u \in W_0^{s,p}(\Omega)$ be a solution of (1.1). Then, $u \in L^{\infty}(\Omega)$ with $||u||_{\infty} \leq C$, for some C = C(||u||) > 0.

In fractional regularity theory, the following weighted Hölder spaces play a major role. Set for all $x \in \mathbb{R}^N$

$$d_{\Omega}(x) = \operatorname{dist}(x, \Omega^c),$$

and for all $\alpha \in (0,1)$

$$C_s^{\alpha}(\overline{\Omega}) = \Big\{ u \in C^0(\overline{\Omega}) : \frac{u}{\operatorname{d}_{\Omega}^s} \text{ has a α-H\"{o}lder continuous extension to $\overline{\Omega}$} \Big\},$$

which is a Banach space endowed with the norm

$$||u||_{\alpha,s} = \left|\left|\frac{u}{\mathrm{d}_{\Omega}^{s}}\right|\right|_{\infty} + \sup_{x \neq y} \frac{|u(x)/\mathrm{d}_{\Omega}^{s}(x) - u(y)/\mathrm{d}_{\Omega}^{s}(y)|}{|x - y|^{\alpha}}.$$

For $\alpha = 0$, the space $C_s^0(\overline{\Omega})$ is defined similarly, with α -Hölder continuous replaced by continuous and the corresponding norm

$$||u||_{0,s} = \left|\left|\frac{u}{\mathrm{d}_{\Omega}^s}\right|\right|_{\infty}.$$

In addition, the interior of the positive order cone of $C_s^0(\overline{\Omega})$ is

$$\operatorname{int}(C_s^0(\overline{\Omega})_+) = \left\{ u \in C_s^0(\overline{\Omega}) : \inf_{\Omega} \frac{u}{\mathrm{d}_{\Omega}^s} > 0 \right\}.$$

By Proposition 3.2 and [14, Theorem 1.1] we have the following global regularity result:

Proposition 3.3. Let \mathbf{H}_0 hold, $u \in W_0^{s,p}(\Omega)$ be a solution of (1.1). Then, $u \in C_s^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$ (independent of u).

While Proposition 3.1 above can be regarded as a weak comparison principle, strong maximum and comparison principles can be stated as follows:

Proposition 3.4. [12, Theorems 2.6, 2.7] Let $g \in C^0(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$:

(i) if $u \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$\begin{cases} (-\Delta)_p^s u + g(u) \geqslant g(0) & \text{in } \Omega \\ u \geqslant 0 & \text{in } \mathbb{R}^N \end{cases}$$

and $u \neq 0$, then

$$\inf_{\Omega} \frac{u}{\mathrm{d}_{\Omega}^s} > 0;$$

(ii) if $u \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\overline{\Omega})$, $v \in W_0^{s,p}(\Omega) \cap C^0(\overline{\Omega})$, C > 0 satisfy

$$\begin{cases} (-\Delta)_p^s \, v + g(v) \leqslant (-\Delta)_p^s \, u + g(u) \leqslant C & \text{in } \Omega \\ 0 < v \leqslant u & \text{in } \Omega \\ 0 = v \leqslant u & \text{in } \Omega^c \end{cases}$$

and $u \neq v$, then

$$\inf_{\Omega} \frac{u - v}{\mathrm{d}_{\Omega}^s} > 0.$$

Referring to [9] for details, we consider the following nonlinear weighted eigenvalue problem with weight function $m \in L^{\infty}(\Omega)$:

(3.1)
$$\begin{cases} (-\Delta)_p^s u = \lambda m(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

The following result summarizes the main properties of the principal eigenvalue of (3.1):

Proposition 3.5. [9, Propositions 3.3, 4.2] Let $m \in L^{\infty}(\Omega)$ be s.t. $m^+ \neq 0$. Then, the smallest eigenvalue of (3.1) is

$$\lambda_1(m) = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\int_{\Omega} m(x) |u|^p dx} > 0.$$

In addition:

- (i) $\lambda_1(m)$ is attained at a unique positive, normalized eigenfunction $e_1(m) \in \operatorname{int}(C^0_{\mathfrak{o}}(\overline{\Omega})_+)$, while any nonprincipal eigenfunction changes sign in Ω ;
- (ii) for all $\tilde{m} \in L^{\infty}(\Omega)$ s.t. $m \leq \tilde{m}$ in Ω and $m \neq \tilde{m}$, we have

$$\lambda_1(m) > \lambda_1(\tilde{m}).$$

In particular, we will write $\lambda_1 = \lambda_1(1)$, $e_1 = e_1(1)$. We recall two technical results, related to the eigenvalue λ_1 . The first is of variational nature:

Proposition 3.6. [11, Lemma 2.7] Let $\xi \in L^{\infty}(\Omega)$ be s.t. $\xi \leqslant \lambda_1$ in Ω , and $\xi \neq \lambda_1$. Then, there exists $\sigma > 0$ s.t. for all $u \in W_0^{s,p}(\Omega)$

$$||u||^p - \int_{\Omega} \xi(x)|u|^p dx \geqslant \sigma ||u||^p.$$

The second is a nonlocal version of the antimaximum principle of [7]:

Proposition 3.7. [6, Lemma 3.9] Let $m, \beta \in L^{\infty}(\Omega)_+ \setminus \{0\}, \lambda \geqslant \lambda_1(m), u \in W_0^{s,p}(\Omega)$ solve

$$\begin{cases} (-\Delta)_p^s u = \lambda m(x) |u|^{p-2} u + \beta(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Then, $u^- \neq 0$.

Now we focus on the variational formulation of problem (1.1). First, set for all $(x,t) \in \Omega \times \mathbb{R}$

$$F(x,t) = \int_0^t f(x,\tau) \, d\tau.$$

Then, set for all $u \in W_0^{s,p}(\Omega)$

$$\Phi(u) = \frac{\|u\|^p}{p} - \int_{\Omega} F(x, u) \, dx.$$

By virtue of \mathbf{H}_0 , it is easily seen that $\Phi \in C^1(W_0^{s,p}(\Omega))$. For all $u \in W_0^{s,p}(\Omega)$ we have

$$\Phi'(u) = (-\Delta)_p^s u - N_f(u),$$

where $N_f:W^{s,p}_0(\Omega)\to W^{-s,p'}(\Omega)$ is the completely continuous Nemitskii operator defined for all $u,\varphi\in$ $W_0^{s,p}(\Omega)$ by

$$\langle N_f(u), \varphi \rangle = \int_{\Omega} f(x, u) \varphi \, dx.$$

So, as seen in Section 2, $\Phi': W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a (demi)-continuous $(S)_+$ -map. In addition, Φ is sequentially weakly l.s.c. and satisfies a bounded (PS)-condition, i.e., whenever (u_n) is a bounded sequence in $W_0^{s,p}(\Omega)$ s.t. $|\Phi(u_n)| \leq C$ for all $n \in \mathbb{N}$ and $\Phi'(u_n) \to 0$ in $W^{-s,p'}(\Omega)$, then up to a subsequence $u_n \to u$ in $W_0^{s,p}(\Omega)$.

Finally, we recall the equivalence between Sobolev and Hölder local minimizers of Φ :

Proposition 3.8. [15, Theorem 1.1] Let \mathbf{H}_0 hold, $u \in W_0^{s,p}(\Omega)$. Then, the following are equivalent:

- (i) there exists $\sigma > 0$ s.t. $\Phi(u+v) \geqslant \Phi(u)$ for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$, $||v||_{0,s} \leqslant \sigma$; (ii) there exists $\rho > 0$ s.t. $\Phi(u+v) \geqslant \Phi(u)$ for all $v \in W_0^{s,p}(\Omega)$, $||v|| \leqslant \rho$.

Remark 3.9. All results of this section also hold in the singular case $p \in (1,2)$, but the regularity result Proposition 3.3, which has only been proved for the degenerate case $p \ge 2$ so far, and consequently Proposition 3.8.

4. Coercive case

In this section we deal with the case when the limits of the quotient

$$t \mapsto \frac{f(x,t)}{t^{p-1}}$$

both at zero and infinity lie below the principal eigenvalue λ_1 without resonance (in fact we will assume a slightly more general condition). This case is called *coercive*, since the energy functional corresponding to problem (1.1) tends to infinity as $||u|| \to \infty$. In order to detect a positive subsolution, we will make use of an auxiliary Dirichlet problem for the *p-fractional Lane-Emden equation*:

(4.1)
$$\begin{cases} (-\Delta)_p^s v = |v|^{q-2}v & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c. \end{cases}$$

Here $q \in (p, p_s^*)$, and recalling that $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ we set

$$c_q = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\|u\|}{\|u\|_q} > 0.$$

The following technical result is a nonlocal version of [20, Theorem 1]:

Lemma 4.1. Let $q \in (p, p_s^*)$. Then problem (4.1) has at least one solution $v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$ s.t.

$$||v_q||^p = ||v_q||_q^q = c_q^{\frac{pq}{q-p}}.$$

Proof. Set

$$S_q = \{ v \in W_0^{s,p}(\Omega) : ||v||_q^q = c_q^{\frac{pq}{q-p}} \}.$$

By the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, it is easily seen that \mathcal{S}_q is sequentially weakly closed as a subset of $W_0^{s,p}(\Omega)$. Also for all $v \in \mathcal{S}_q$ we have

By definition of c_q , there exists a sequence (u_n) in $W_0^{s,p}(\Omega) \setminus \{0\}$ s.t.

$$\frac{\|u_n\|}{\|u_n\|_q} \to c_q.$$

By replacing if necessary u_n with $|u_n|$, we may assume $u_n \ge 0$ in Ω , for all $n \in \mathbb{N}$. Set

$$v_n = c_q^{\frac{p}{q-p}} \frac{u_n}{\|u_n\|_q} \in \mathcal{S}_q,$$

then $||v_n||^p \to c_q^{\frac{pq}{q-p}}$. In particular, (v_n) is bounded in $W_0^{s,p}(\Omega)$. Passing if necessary to a subsequence, we have $v_n \to v_q$ in $W_0^{s,p}(\Omega)$, $v_n \to v_q$ in $L^q(\Omega)$, and $v_n(x) \to v_q(x)$ for a.e. $x \in \Omega$. So $v_q \in \mathcal{S}_q$ and $v_q \geqslant 0$ in Ω . Also,

$$||v_q||^p \leqslant \liminf_n ||v_n||^p = c_q^{\frac{pq}{q-p}},$$

which along with (4.2) gives

$$||v_q||^p = ||v_q||_q^q = c_q^{\frac{pq}{q-p}}.$$

Thus, v_q is a minimizer of the functional $v \mapsto ||v||^p$ restricted the C^1 -manifold \mathcal{S}_q . By Lagrange's multipliers rule, there exists $\mu \in \mathbb{R}$ s.t. for all $\varphi \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s v_q, \varphi \rangle = \mu \int_{\Omega} v_q^{q-1} \varphi \, dx.$$

Testing the relation above with $v_q \in W_0^{s,p}(\Omega)$ we get

$$||v_q||^p = \mu ||v_q||_q^q,$$

which implies $\mu = 1$. So v_q is a non-negative weak solution of (4.1), in the sense of Section 3. Clearly the reaction

$$(x,t)\mapsto |t|^{q-2}t$$

satisfies \mathbf{H}_0 , so by Proposition 3.3 we have $v_q \in C_s^{\alpha}(\overline{\Omega})$. Now apply Proposition 3.4 (i) (with $g(t) = -|t|^{q-2}t$) to find $v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Our hypotheses on the reaction f, in the present case, are the following:

 $\mathbf{H}_1 \ f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

(i) for all M > 0 there exists $a_M \in L^{\infty}(\Omega)_+$ s.t. for a.e. $x \in \Omega$ and all $t \in [0, M]$

$$0 \leqslant f(x,t) \leqslant a_M(x);$$

(ii) there exists $\theta \in L^{\infty}(\Omega)_+$ s.t. $\theta \leqslant \lambda_1$ in Ω , $\theta \neq \lambda_1$, and uniformly for a.e. $x \in \Omega$

$$\limsup_{t \to \infty} \frac{f(x,t)}{t^{p-1}} \leqslant \theta(x);$$

(iii) there exists $\eta \in L^{\infty}(\Omega)_+$ s.t. $\eta \leqslant \lambda_1$ in $\Omega, \eta \neq \lambda_1$, and uniformly for a.e. $x \in \Omega$

$$\limsup_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} \leqslant \eta(x);$$

(iv) there exists $q\in(p,p_s^*)$ s.t. for a.e. $x\in\Omega$ and all t>0

$$f(x,t) > \min\{t, \|v_q\|_{\infty}\}^{q-1}.$$

Hypotheses \mathbf{H}_1 above only concern the behavior of $f(x,\cdot)$ in the positive semiaxis, but since we are seeking positive solutions, we can use \mathbf{H}_1 (iii) and set for all $x \in \Omega$, $t \in \mathbb{R}^-$

$$f(x,t) = 0.$$

Clearly \mathbf{H}_1 imply \mathbf{H}_0 , hence all results of Section 3 apply. Hypotheses \mathbf{H}_1 (ii) (iii) conjure a (p-1)-sublinear growth at infinity and a (p-1)-superlinear growth at the origin, with the principal eigenvalue $\lambda_1 > 0$ (defined in Proposition 3.5) as a threshold slope. Also, $v_q \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ in \mathbf{H}_1 (iv) is defined by Lemma 4.1.

Example 4.2. Assume that, for convenient Ω , p, q we have $\lambda_1 > ||v_q||_{\infty}^{q-p}$. Then we can find $\lambda \in (0, \lambda_1)$, $\mu > 1$ s.t. $\lambda > \mu ||v_q||_{\infty}^{q-p}$. So set for all $t \in \mathbb{R}$

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \mu t^{q-1} & \text{if } 0 < t \leq ||v_q||_{\infty} \\ \lambda t^{p-1} + \mu ||v_q||_{\infty}^{q-1} - \lambda ||v_q||_{\infty}^{p-1} & \text{if } t > ||v_q||_{\infty}. \end{cases}$$

Elementary calculations show that $f \in C^0(\mathbb{R})$ satisfies \mathbf{H}_1 .

Our multiplicity result for the coercive case is the following, extending [18, Theorem 11.13] to the nonlocal framework (see also the first part of [19, Theorem 1]):

Theorem 4.3. Let \mathbf{H}_1 hold. Then, problem (1.1) has at least two solutions $u_1, u_2 \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Proof. By \mathbf{H}_1 (and the extension of f to the negative semiaxis) we can define the primitive $F: \Omega \times \mathbb{R} \to \mathbb{R}$ and the functional $\Phi \in C^1(W_0^{s,p}(\Omega))$ as in Section 3.

Our first claim is that 0 is a local minimizer of Φ . Let $\sigma > 0$ be as in Proposition 3.6 (with $\xi = \eta$ from \mathbf{H}_1 (iii). Fix $\varepsilon \in (0, \sigma \lambda_1)$, then by \mathbf{H}_1 (iii) and de l'Hôpital's rule there exists $\delta > 0$ s.t. for a.e. $x \in \Omega$ and all $t \in [0, \delta]$

$$F(x,t) \leqslant \frac{\eta(x) + \varepsilon}{p} t^p.$$

By Proposition 3.6 and the variational characterization of λ_1 , for all $u \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $||u||_{\infty} \leq \delta$ we have

$$\Phi(u) \geqslant \frac{\|u\|^p}{p} - \int_{\Omega} \frac{\eta(x) + \varepsilon}{p} (u^+)^p dx$$
$$\geqslant \frac{1}{p} \Big(\|u\|^p - \int_{\Omega} \eta(x) |u|^p dx \Big) - \frac{\varepsilon}{p} \|u\|_p^p$$
$$\geqslant \Big(\sigma - \frac{\varepsilon}{\lambda_1} \Big) \frac{\|u\|^p}{p} \geqslant 0.$$

Since $\Phi(0) = 0$ and $C_s^0(\overline{\Omega}) \hookrightarrow L^{\infty}(\Omega)$, we see that 0 is a local minimizer in $C_s^0(\overline{\Omega})$ for Φ . By Proposition 3.8, it is such as well in $W_0^{s,p}(\Omega)$, as claimed.

Now let $v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$ be as in Lemma 4.1. By \mathbf{H}_1 (iv) we have for all $\varphi \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$

$$\langle (-\Delta)_p^s v_q, \varphi \rangle = \int_{\Omega} v_q^{q-1} \varphi \, dx < \int_{\Omega} f(x, v_q) \varphi \, dx,$$

so v_q is a (strict) subsolution of (1.1). Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\hat{f}(x,t) = f(x, \max\{t, v_q(x)\}), \ \hat{F}(x,t) = \int_0^t \hat{f}(x,\tau) d\tau,$$

so $\hat{f}: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies \mathbf{H}_0 . Then set for all $u \in W_0^{s,p}(\Omega)$

$$\hat{\Phi}(u) = \frac{\|u\|^p}{p} - \int_{\Omega} \hat{F}(x, u) \, dx.$$

As in Section 3 we see that $\hat{\Phi} \in C^1(W_0^{s,p}(\Omega))$ and is sequentially weakly l.s.c. Moreover, $\hat{\Phi}$ is coercive. To see this, let $\sigma > 0$ be as in Proposition 3.6 (this time with $\xi = \theta$ from \mathbf{H}_1 (ii)). Fix $\varepsilon \in (0, \sigma \lambda_1)$, then by \mathbf{H}_1 (ii) there exists $M > \|v_q\|_{\infty}$ s.t. for a.e. $x \in \Omega$ and all $t \geq M$

$$f(x,t) \leqslant (\theta(x) + \varepsilon)t^{p-1}$$
.

Further, by \mathbf{H}_1 (i) we can find $a_M \in L^{\infty}(\Omega)_+$ s.t. for a.e. $x \in \Omega$ and all $t \geq M$

$$\hat{F}(x,t) = \int_0^{v_q} f(x,v_q) d\tau + \int_{v_q}^M f(x,\tau) d\tau + \int_M^t f(x,\tau) d\tau$$

$$\leq a_M(x)M + \frac{\theta(x) + \varepsilon}{p} (t^p - M^p).$$

Using again \mathbf{H}_1 (i) we find C > 0 s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$\hat{F}(x,t) \leqslant \frac{\theta(x) + \varepsilon}{p} (t^+)^p + C.$$

So, by Proposition 3.6 and the variational characterization of λ_1 , we have for all $u \in W_0^{s,p}(\Omega)$

$$\hat{\Phi}(u) \geqslant \frac{\|u\|^p}{p} - \int_{\Omega} \left(\frac{\theta(x) + \varepsilon}{p} (u^+)^p + C\right) dx$$

$$\geqslant \frac{1}{p} \left(\|u\|^p - \int_{\Omega} \theta(x) |u|^p dx\right) - \frac{\varepsilon}{p} \|u\|_p^p - C$$

$$\geqslant \left(\sigma - \frac{\varepsilon}{\lambda_1}\right) \frac{\|u\|^p}{p} - C,$$

and the latter tends to ∞ as $||u|| \to \infty$. Thus, there exists $u_1 \in W_0^{s,p}(\Omega)$ s.t.

(4.3)
$$\hat{\Phi}(u_1) = \inf_{u \in W_0^{s,p}(\Omega)} \hat{\Phi}(u).$$

In particular, we have weakly in Ω

$$(4.4) (-\Delta)_p^s u_1 = \hat{f}(x, u_1).$$

Testing (4.1) and (4.4) with $(v_q - u_1)^+ \in W_0^{s,p}(\Omega)$, and using \mathbf{H}_1 (iv), we have

$$\langle (-\Delta)_p^s v_q - (-\Delta)_p^s u_1, (v_q - u_1)^+ \rangle = \int_{\{u_1 < v_q\}} \left(v_q^{q-1} - f(x, v_q) \right) (v_q - u_1) \, dx \leqslant 0.$$

By Proposition 3.1 we have $u_1 \geqslant v_q$ in Ω . By construction, we may replace \hat{f} with f in (4.4) and see that u_1 solves (1.1). By Proposition 3.3, then, we have $u_1 \in C_s^{\alpha}(\overline{\Omega})$. By \mathbf{H}_1 (iv) we have weakly in Ω

$$(-\Delta)_p^s v_q = v_q^{q-1} \leqslant f(x, u_1) = (-\Delta)_p^s u_1.$$

Besides, since v_q is a strict subsolution of (1.1), we have $u_1 \neq v_q$. So Proposition 3.4 (ii) (with g(t) = 0) implies

$$u_1 - v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+),$$

in particular $u_1 \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$. Now set

$$V = \{ v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega}) : v - v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+) \}.$$

Since $u_1 - v_q \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$, there exists $\sigma > 0$ s.t. for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $||v - u_1||_{0,s} \leqslant \sigma$ we have $v \in V$. By (4.3) we have for all $v \in V$

$$\Phi(v) = \hat{\Phi}(v) \geqslant \hat{\Phi}(u_1) = \Phi(u_1),$$

hence u_1 is a local minimizer of Φ in $C_s^0(\overline{\Omega})$. By Proposition 3.8, it is as well a local minimizer of Φ in $W_0^{s,p}(\Omega)$.

There remains to prove that Φ has a further critical point, beside 0 and u_1 . With this aim in mind, we distinguish two cases:

- (a) If either 0 or u_1 is not an isolated critical point of Φ , then clearly Φ has infinitely many critical points.
- (b) If both 0 and u_1 are isolated critical points of Φ , then in particular they are *strict* local minimizers. We then apply Proposition 2.2 (i) and find for all $\rho > 0$ small enough

(4.5)
$$\deg_{(S)_{+}}(\Phi', B_{\rho}(0), 0) = \deg_{(S)_{+}}(\Phi', B_{\rho}(u_{1}), 0) = 1.$$

Also, arguing as we did above with $\hat{\Phi}$, we see that Φ is coercive in $W_0^{s,p}(\Omega)$. So, by Proposition 2.2 (ii) we have for all R > 0 big enough

(4.6)
$$\deg_{(S)_{+}}(\Phi', B_{R}(0), 0) = 1.$$

We choose $0 < \rho < R$ in the relations above so that

$$\overline{B}_{\rho}(0) \cap \overline{B}_{\rho}(u_1) = \emptyset, \ \overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_1) \subset B_R(0).$$

Using (4.5) (4.6) and Proposition 2.1 (ii) (domain additivity) twice, we get

$$1 = \deg_{(S)_{+}}(\Phi', B_{R}(0), 0)$$

$$= \deg_{(S)_{+}}(\Phi', B_{\rho}(0), 0) + \deg_{(S)_{+}}(\Phi', B_{\rho}(u_{1}), 0) + \deg_{(S)_{+}}(\Phi', B_{R}(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_{1})), 0)$$

$$= 2 + \deg_{(S)_{+}}(\Phi', B_{R}(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_{1})), 0),$$

which rephrases as

$$\deg_{(S)_+}(\Phi', B_R(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_1)), 0) = -1.$$

Then, by Proposition 2.1 (v) (solution), there exists $u_2 \in B_R(0) \setminus (\overline{B}_o(0) \cup \overline{B}_o(u_1))$ s.t.

$$\Phi'(u_2) = 0.$$

In either case, we end up with two nontrivial critical points $u_1, u_2 \in K(\Phi) \setminus \{0\}$. We already know that $u_1 \in \text{int}(C_s^0(\overline{\Omega})_+)$ solves (1.1).

So, let us consider u_2 . By Proposition 3.3 we see that $u_2 \in C_s^{\alpha}(\overline{\Omega})$ solves (1.1). Testing with $-u_2^- \in W_0^{s,p}(\Omega)$ and using \mathbf{H}_1 (i) we have

$$||u_2^-||^p \leqslant \langle (-\Delta)_p^s u_2, -u_2^- \rangle$$

= $\int_{\{u_2 < 0\}} f(x, u_2) u_2 dx \leqslant 0,$

hence $u_2 \ge 0$ in Ω . Now apply Proposition 3.4 (i) (with g(t) = 0) and recall that $u_2 \ne 0$, to conclude that $u_2 \in \text{int}(C_s^0(\overline{\Omega})_+)$, which ends the proof.

Remark 4.4. In the proof of Theorem 4.3, the degree-theoretical argument used to detect a second positive solution can be replaced by an equivalent variational one, based on the mountain pass theorem (see [18, Theorem 5.40]). Also, we note that under hypotheses symmetric to \mathbf{H}_1 on the negative semiaxis, we can prove existence of two negative solutions.

5. Noncoercive case

In this section we consider the more delicate case in which the limits of

$$t \mapsto \frac{f(x,t)}{t^{p-1}},$$

both for $t \to \infty, 0^+$ lie above the principal eigenvalue λ_1 . Such asymptotic behavior prevents both coercivity of the energy functional and the existence of a local minimum at 0. So, in the present case the use of degree theory is more meaningful.

Our hypotheses on the reaction f are the following:

 $\mathbf{H}_2 \ f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

(i) for all M > 0 there exists $a_M \in L^{\infty}(\Omega)_+$ s.t. for a.e. $x \in \Omega$ and all $t \in [0, M]$

$$|f(x,t)| \leqslant a_M(x);$$

(ii) there exist $\theta_1, \theta_2 \in L^{\infty}(\Omega)_+$ s.t. $\theta_1 \geqslant \lambda_1$ in $\Omega, \theta_1 \neq \lambda_1$, and uniformly for a.e. $x \in \Omega$

$$\theta_1(x) \leqslant \liminf_{t \to \infty} \frac{f(x,t)}{t^{p-1}} \leqslant \limsup_{t \to \infty} \frac{f(x,t)}{t^{p-1}} \leqslant \theta_2(x);$$

(iii) there exist $\eta_1, \eta_2 \in L^{\infty}(\Omega)_+$ s.t. $\eta_1 \geqslant \lambda_1$ in $\Omega, \eta_1 \neq \lambda_1$, and uniformly for a.e. $x \in \Omega$

$$\eta_1(x) \leqslant \liminf_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} \leqslant \limsup_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} \leqslant \eta_2(x);$$

(iv) there exist b, K > 0 s.t. uniformly for a.e. $x \in \Omega$

$$\limsup_{t \to b^{-}} \frac{f(x,t)}{(b-t)^{p-1}} \leqslant K.$$

As in Section 4, by \mathbf{H}_2 (iii) we may set for all $x \in \Omega$, $t \leq 0$

$$f(x,t) = 0.$$

So \mathbf{H}_2 imply \mathbf{H}_0 . By \mathbf{H}_2 (ii) (iii), $f(x,\cdot)$ has precisely a (p-1)-linear growth both at 0 and at ∞ , with limits slopes above the principal eigenvalue λ_1 (Proposition 3.5) without resonance. Also, hypothesis \mathbf{H}_2 (iv) implies that the constant b is a supersolution of (1.1).

Example 5.1. Define an autonomous reaction $f \in C^0(\mathbb{R})$ by setting for all $t \ge 0$

$$f(t) = \mu t^{p-1} - \gamma \arctan(t^{q-1}),$$

with q > p, $\mu > \lambda_1$, and $\gamma > 0$. Elementary calculus shows that f satisfies \mathbf{H}_2 (i) – (iii). Moreover, if γ is big enough (depending on μ , p, and q), then f becomes negative at some b > 0, hence \mathbf{H}_2 (iv) is satisfied as well.

Our multiplicity theorem for the noncoercive case, extending [18, Theorem 11.15] to the nonlocal framework (see also the second part of [19, Theorem 1]), is the following:

Theorem 5.2. Let \mathbf{H}_2 hold. Then, problem (1.1) has at least two solutions $u_1, u_2 \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Proof. Since \mathbf{H}_0 holds, we can define $\Phi \in C^1(W_0^{s,p}(\Omega))$ as in Section 3. First, we note that by \mathbf{H}_2 (ii) we have f(x,0)=0 for a.e. $x\in\Omega$, so 0 is a critical point of Φ .

As anticipated above, the constant function $b \in \widetilde{W}^{s,p}(\Omega)$ is a (strict) supersolution of (1.1). Indeed, by \mathbf{H}_2 (iv) we have $f(x,b) \leq 0$ for a.e. $x \in \Omega$, hence for all $\varphi \in W_0^{s,p}(\Omega)_+$

$$\langle (-\Delta)_p^s b, \varphi \rangle = 0 \geqslant \int_{\Omega} f(x, b) \varphi \, dx.$$

Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\check{f}(x,t) = f(x,\min\{t,b\}), \ \check{F}(x,t) = \int_0^t \check{f}(x,\tau) d\tau.$$

Then, $\check{f}: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies \mathbf{H}_0 . So set for all $u \in W^{s,p}_0(\Omega)$

$$\check{\Phi}(u) = \frac{\|u\|^p}{p} - \int_{\Omega} \check{F}(x, u) \, dx.$$

As seen in Section 3, $\check{\Phi} \in C^1(W_0^{s,p}(\Omega))$ is sequentially weakly l.s.c. In addition, such functional is coercive. Indeed, by \mathbf{H}_2 (i) and the definition of \check{f} we have for a.e. $x \in \Omega$ and all $t \geqslant b$

$$\check{F}(x,t) = \int_0^b f(x,\tau) d\tau + \int_b^t f(x,b) d\tau \leqslant a_b(x)b$$

(recall that $f(\cdot, b) \leq 0$ in Ω), which implies for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$\check{F}(x,t) \leqslant C.$$

So, by the continuous embedding $W^{s,p}_0(\Omega) \hookrightarrow L^1(\Omega)$ we have for all $u \in W^{s,p}_0(\Omega)$

$$\check{\Phi}(u) \geqslant \frac{\|u\|^p}{p} - C,$$

and the latter tends to ∞ as $||u|| \to \infty$. Thus, there exists $u_1 \in W_0^{s,p}(\Omega)$ s.t.

(5.1)
$$\check{\Phi}(u_1) = \inf_{u \in W_0^{s,p}(\Omega)} \check{\Phi}(u).$$

By (5.1) we have weakly in Ω

$$(5.2) (-\Delta)_p^s u_1 = \check{f}(x, u_1).$$

Since \check{f} satisfies \mathbf{H}_0 , by Proposition 3.3 we have $u_1 \in C_s^{\alpha}(\overline{\Omega})$. Recalling that $f(\cdot,t) = 0$ in Ω for all $t \leq 0$, and testing (5.2) with $-u_1^- \in W_0^{s,p}(\Omega)$ we have

$$||u_1^-||^p \leqslant \langle (-\Delta)_p^s u_1, -u_1^- \rangle$$

= $\int_{\{u_1 < 0\}} f(x, u_1) u_1 dx = 0,$

hence $u_1 \geqslant 0$ in Ω . On the other hand, by \mathbf{H}_2 (iv) we have $f(\cdot, b) \leqslant 0$ in Ω . So, testing (5.2) with $(u_1 - b)^+ \in W_0^{s,p}(\Omega)$ we have

$$\langle (-\Delta)_p^s u_1 - (-\Delta)_p^s b, (u_1 - b)^+ \rangle = \int_{\Omega} \check{f}(x, u_1)(u_1 - b)^+ dx$$
$$= \int_{\{u_1 > b\}} f(x, b)(u_1 - b) dx \le 0,$$

hence $u_1 \leq b$ in Ω . This already implies that we can replace \check{f} with f in (5.2) and see that $u_1 \in C_s^{\alpha}(\overline{\Omega})$ solves (1.1) (see Proposition 3.3). Note that, so far, $u_1 = 0$ may occur. Set

$$V = \{ v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega}) : v < b \text{ in } \Omega \}.$$

We aim at proving that V is a neighborhood of u_1 in the $C_s^0(\overline{\Omega})$ -topology, distinguishing two cases:

- (a) If $u_1 = 0$, then clearly there exists $\sigma > 0$ s.t. for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $||v||_{0,s} \leqslant \sigma$ we have v < b in Ω , hence $v \in V$.
- (b) If $u_1 \neq 0$, then we have $u_1 > 0$ in Ω . To see this, fix $\varepsilon \in (0, \lambda_1)$. By \mathbf{H}_2 (iii) there exists $\delta \in (0, b)$ s.t. for a.e. $x \in \Omega$ and all $t \in [0, \delta]$

$$f(x,t) > \varepsilon t^{p-1}$$
.

Besides, by \mathbf{H}_2 (i) we have for a.e. $x \in \Omega$ and all $t \in [\delta, b]$

$$f(x,t) \geqslant -a_b(x) \geqslant -\frac{\|a_b\|_{\infty}}{\delta^{p-1}} t^{p-1}.$$

Recalling the construction of \check{f} , we find C>0 s.t. for a.e. $x\in\Omega$ and all $t\geqslant0$

$$\check{f}(x,t) \geqslant -Ct^{p-1}$$
.

By (5.2) we have weakly in Ω

$$(-\Delta)_p^s u_1 + C u_1^{p-1} \geqslant 0,$$

while $u_1 \ge 0$ in Ω and $u_1 \ne 0$. By Proposition 3.4 (i) (with $g(t) = C(t^+)^{p-1}$) we have

$$\inf_{\Omega} \frac{u_1}{\mathrm{d}_{\Omega}^s} > 0.$$

In particular, we have $u_1 \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$. Now, let us compare u_1 to b. By \mathbf{H}_2 (iv), for all $\varepsilon > 0$ there exists $\delta \in (0, b)$ s.t. for a.e. $x \in \Omega$ and all $t \in [\delta, b]$

$$f(x,t) \leqslant (K+\varepsilon)(b-t)^{p-1}$$
.

Besides, by \mathbf{H}_2 (i) we have for a.e. $x \in \Omega$ and all $t \in [0, \delta]$

$$f(x,t) \leqslant a_{\delta}(x) \leqslant \frac{\|a_{\delta}\|_{\infty}}{(b-\delta)^{p-1}} (b-t)^{p-1}.$$

So we can find C > 0 s.t. for a.e. $x \in \Omega$ and all $t \in [0, b]$

$$\check{f}(x,t) \leqslant C(b-t)^{p-1}.$$

Now recall that $0 < u_1 \le b$ in Ω and (5.2) holds, so we have weakly in Ω

$$(-\Delta)_p^s u_1 - C(b-u_1)^{p-1} \leqslant 0 = (-\Delta)_p^s b - C(b-b)^{p-1}$$

By Proposition 3.4 (ii) (with $g(t) = -C((b-t)^+)^{p-1}$) we have

$$\inf_{\Omega} \frac{b - u_1}{\mathrm{d}_{\Omega}^s} > 0.$$

So, we can find $\sigma > 0$ s.t. for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $||v - u_1||_{0,s} \leqslant \sigma$ we have $v \in V$.

By (5.1) and the estimates above, for all $v \in V$ we have

$$\Phi(v) = \check{\Phi}(v) \geqslant \check{\Phi}(u_1) = \Phi(u_1).$$

So, u_1 is a local minimizer of Φ in $C_s^0(\overline{\Omega})$. By Proposition 3.8, it is as well a local minimizer of Φ in $W_0^{s,p}(\Omega)$. We seek now another critical point of Φ , using degree theory. First, note that by \mathbf{H}_2 (ii) we can find M, C > 0 s.t. for a.e. $x \in \Omega$ and all $t \geqslant M$

$$|f(x,t)| \leqslant Ct^{p-1}$$
.

Also, by \mathbf{H}_2 (i) there exists $a_M \in L^{\infty}(\Omega)_+$ s.t. for a.e. $x \in \Omega$ and all $t \in [0, M]$

$$|f(x,t)| \leqslant a_M(x).$$

Recalling that $f(\cdot,t)=0$ in Ω for all $t\leq 0$, we have for a.e. $x\in \Omega$ and all $t\in \mathbb{R}$

$$|f(x,t)| \le C(1+|t|^{p-1}),$$

i.e., f satisfies \mathbf{H}_0 (with r=p). Define the completely continuous map $N_f: W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ as in Section 3. So, $\Phi': W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a continuous $(S)_+$ -map. Our first claim is that, for all R>0 big enough,

(5.4)
$$\deg_{(S)_+}(\Phi', B_R(0), 0) = 0.$$

To see this, first let $\theta_1, \theta_2 \in L^{\infty}(\Omega)_+$ be as in \mathbf{H}_2 (ii), then set for all $u, \varphi \in W_0^{s,p}(\Omega)$

$$\langle K_{\infty}(u), \varphi \rangle = \int_{\Omega} \frac{\theta_1(x) + \theta_2(x)}{2} (u^+)^{p-1} \varphi \, dx.$$

So, $K_{\infty}: W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a completely continuous map. We define a continuous $(S)_+$ -homotopy $h_{\infty}: [0,1] \times W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ by setting for all $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$

$$h_{\infty}(t,u) = (-\Delta)_{n}^{s} u - (1-t)N_{f}(u) - tK_{\infty}(u).$$

We prove next that for all R > 0 big enough, $t \in [0,1]$, and ||u|| = R

$$(5.5) h_{\infty}(t, u) \neq 0.$$

Arguing by contradiction, assume that there exist sequences (t_n) in [0,1] and (u_n) in $W_0^{s,p}(\Omega)$, s.t. $||u_n|| \to \infty$ and for all $n \in \mathbb{N}$

$$h_{\infty}(t_n, u_n) = 0.$$

Passing if necessary to a subsequence, we have $t_n \to t$ for some $t \in [0, 1]$. Also, testing the relation above with $-u_n^- \in W_0^{s,p}(\Omega)$ and recalling that $f(\cdot, t) = 0$ in Ω for all $t \leq 0$, we have for all $n \in \mathbb{N}$

$$||u_n^-||^p \leqslant \langle (-\Delta)_p^s u_n, -u_n^- \rangle$$

$$= (1 - t_n) \int_{\{u_n < 0\}} f(x, u_n) u_n \, dx + t_n \int_{\{u_n < 0\}} \frac{\theta_1(x) + \theta_2(x)}{2} (u_n^+)^{p-1} u_n \, dx = 0,$$

so $u_n \ge 0$ in Ω . Clearly we may assume $u_n \ne 0$, so set

$$v_n = \frac{u_n}{\|u_n\|} \in W_0^{s,p}(\Omega)_+ \setminus \{0\}.$$

For all $n \in \mathbb{N}$ we have weakly in Ω

$$(5.6) (-\Delta)_p^s v_n = (1 - t_n) \frac{f(x, u_n)}{\|u_n\|^{p-1}} + t_n \frac{\theta_1(x) + \theta_2(x)}{2} v_n^{p-1}.$$

Clearly $||v_n|| = 1$, so up to a subsequence we have $v_n \to v$ in $W_0^{s,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$, and $v_n(x) \to v(x)$ for a.e. $x \in \Omega$ with $L^p(\Omega)$ -dominated convergence. Hence $v \ge 0$ in Ω . Testing (5.6) with $(v_n - v) \in W_0^{s,p}(\Omega)$, using (5.3) and Hölder's inequality, we get for all $n \in \mathbb{N}$

$$\langle (-\Delta)_p^s v_n, v_n - v \rangle = (1 - t_n) \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|_{p-1}} (v_n - v) \, dx + t_n \int_{\Omega} \frac{\theta_1(x) + \theta_2(x)}{2} v_n^{p-1} (v_n - v) \, dx$$

$$\leq C \frac{1 + \|u_n\|_p^{p-1}}{\|u_n\|_{p-1}} \|v_n - v\|_p + C \|v_n\|_p^{p-1} \|v_n - v\|_p \leq C \|v_n - v\|_p,$$

and the latter tends to 0 as $n \to \infty$. By the $(S)_+$ -property of $(-\Delta)_p^s$, we have $v_n \to v$ in $W_0^{s,p}(\Omega)$, in particular ||v|| = 1. By (5.3) we have for all $n \in \mathbb{N}$

$$\int_{\Omega} \left| \frac{f(x, u_n)}{\|u_n\|^{p-1}} \right|^{p'} dx \leqslant C \frac{1 + \|u_n\|_p^p}{\|u_n\|^p} \leqslant C.$$

By reflexivity, we can find $g_{\infty} \in L^{p'}(\Omega)$ s.t. up to a subsequence we have in $L^{p'}(\Omega)$

$$\frac{f(\cdot, u_n)}{\|u_n\|^{p-1}} \rightharpoonup g_{\infty}.$$

Fix $\varepsilon > 0$ and set for all $n \in \mathbb{N}$

$$\Omega_n^{\varepsilon} = \left\{ x \in \Omega : u_n(x) > 0, \ \theta_1(x) - \varepsilon \leqslant \frac{f(x, u_n(x))}{u_n(x)^{p-1}} \leqslant \theta_2(x) + \varepsilon \right\}.$$

Also set

$$\Omega^+ = \{ x \in \Omega : v(x) > 0 \}.$$

Then we have $u_n(x) = ||u_n||v_n(x) \to \infty$ for a.e. $x \in \Omega^+$, so by \mathbf{H}_2 (ii) we have $\chi_{\Omega_n^{\varepsilon}} \to 1$ in Ω^+ , with dominated convergence. So we have in $L^{p'}(\Omega^+)$

(5.7)
$$\chi_{\Omega_n^{\varepsilon}} \frac{f(\cdot, u_n)}{\|u_n\|^{p-1}} \rightharpoonup g_{\infty}.$$

Besides, by definition of Ω_n^{ε} , for all $n \in \mathbb{N}$ we have in Ω^+

$$\chi_{\Omega_n^{\varepsilon}}(\theta_1 - \varepsilon)v_n^{p-1} \leqslant \chi_{\Omega_n^{\varepsilon}} \frac{f(\cdot, u_n)}{\|u_n\|^{p-1}} \leqslant \chi_{\Omega_n^{\varepsilon}}(\theta_2 + \varepsilon)v_n^{p-1}.$$

By Mazur's theorem (see [1, Corollary 3.8]), up to a convex combination we have strong convergence in (5.7), so we can pass to the limit as $n \to \infty$ in the inequalities above and find in Ω^+

$$(\theta_1 - \varepsilon)v^{p-1} \leqslant g_{\infty} \leqslant (\theta_2 + \varepsilon)v^{p-1}.$$

Let $\varepsilon \to 0^+$ to find in Ω^+

$$\theta_1 v^{p-1} \leqslant g_\infty \leqslant \theta_2 v^{p-1}$$
.

Besides, in $\Omega \setminus \Omega^+$ we have $v_n \to 0$. By (5.3) and the definition of v_n , for all $n \in \mathbb{N}$ we have in $\Omega \setminus \Omega^+$

$$\left| \frac{f(x, u_n)}{\|u_n\|^{p-1}} \right| \leqslant C \frac{1 + u_n^{p-1}}{u_n^{p-1}} v_n^{p-1} \leqslant C v_n^{p-1},$$

and the latter tends to 0 as $n \to \infty$. So we may set $g_{\infty} = 0$ in $\Omega \setminus \Omega^+$. In conclusion, there exists $\theta \in L^{\infty}(\Omega)$ s.t. in Ω

$$\theta_1 \leqslant \theta \leqslant \theta_2, \ g_{\infty} = \theta v^{p-1}.$$

Besides, we have in $L^{p'}(\Omega)$

$$\frac{\theta_1 + \theta_2}{2} v_n^{p-1} \to \frac{\theta_1 + \theta_2}{2} v^{p-1}.$$

Define now

$$m_{\infty} = (1-t)\theta + t\frac{\theta_1 + \theta_2}{2} \in L^{\infty}(\Omega).$$

Passing to the limit in (5.6) as $n \to \infty$, we have weakly in Ω

$$(-\Delta)_p^s v = m_{\infty}(x)v^{p-1}.$$

So $v \neq 0$ is an eigenfunction of problem (3.1) with weight $m = m_{\infty}$, associated to the eigenvalue 1. By \mathbf{H}_2 (ii) and the computation above we have $m_{\infty} \geqslant \lambda_1$ in Ω , $m_{\infty} \neq \lambda_1$, so by Proposition 3.5 (ii) we have

$$\lambda_1(m_\infty) < \lambda_1(\lambda_1) = 1.$$

Thus, v is a non-principal eigenfunction, which along with Proposition 3.5 (i) implies that v is nodal, a contradiction. So (5.5) is proved.

Using (5.5) and Proposition 2.1 (iv) (homotopy invariance), we see that for all R > 0 big enough

(5.8)
$$\deg_{(S)_{+}}(\Phi', B_{R}(0), 0) = \deg_{(S)_{+}}((-\Delta)_{p}^{s} - K_{\infty}, B_{R}(0), 0).$$

In order to evaluate the latter quantity, we shall define a new homotopy. Let $w \in C_c^{\infty}(\Omega)_+ \setminus \{0\}$ be identified with an element of $W^{-s,p'}(\Omega)$, and set for all $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$

$$\tilde{h}_{\infty}(t,u) = (-\Delta)_p^s u - K_{\infty}(u) - tw.$$

Clearly $\tilde{h}_{\infty}: [0,1] \times W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a continuous $(S)_+$ -homotopy. Moreover, for all $t \in [0,1]$ and all $u \in W_0^{s,p}(\Omega) \setminus \{0\}$ we have

$$\tilde{h}_{\infty}(t,u) \neq 0.$$

Arguing by contradiction, let \tilde{h}_{∞} vanish at some $t \in [0,1], u \in W_0^{s,p}(\Omega) \setminus \{0\}$. Then we have weakly in Ω

(5.10)
$$(-\Delta)_p^s u = \frac{\theta_1(x) + \theta_2(x)}{2} (u^+)^{p-1} + tw(x).$$

Testing (5.10) with $-u^- \in W_0^{s,p}(\Omega)$ we have

$$||u^-||^p \leqslant \langle (-\Delta)_p^s u, -u^- \rangle$$

$$= \int_{\{u < 0\}} \frac{\theta_1(x) + \theta_2(x)}{2} (u^+)^{p-1} u \, dx + t \int_{\{u < 0\}} w(x) u \, dx \leqslant 0,$$

so $u \geqslant 0$ in Ω . Now set

$$m = \frac{\theta_1 + \theta_2}{2} \in L^{\infty}(\Omega),$$

and note that by \mathbf{H}_2 (ii) $m \ge \lambda_1$ in Ω , with $m \ne \lambda_1$. By Proposition 3.5 (ii) we have

$$\lambda_1(m) < \lambda_1(\lambda_1) = 1.$$

Now we distinguish two cases:

- (a) If t = 0, then by (5.10) we see that u is an eigenfunction of problem (3.5) with weight m, associated to the non-principal eigenvalue 1, hence u must be nodal, a contradiction.
- (b) If $t \in (0,1]$, then we apply Proposition 3.7 with m as above, $\lambda = 1$, and $\beta = tw$, and by (5.10) we deduce that $u^- \neq 0$, a contradiction again.

So (5.9) is proved. By Proposition 2.1 (iv) (homotopy invariance) we have for all R > 0

(5.11)
$$\deg_{(S)_{+}}((-\Delta)_{p}^{s} - K_{\infty}, B_{R}(0), 0) = \deg_{(S)_{+}}((-\Delta)_{p}^{s} - K_{\infty} - w, B_{R}(0), 0).$$

In addition, by (5.9) and Proposition 2.1 (v) (solution) we have for all R > 0

$$\deg_{(S)_+}((-\Delta)_n^s - K_{\infty} - w, B_R(0), 0) = 0.$$

Concatenating (5.8), (5.11), and the above relation we achieve the proof of (5.4).

Now we study the behavior of Φ' near the origin, where for all $\rho > 0$ small enough we have

(5.12)
$$\deg_{(S)_{+}}(\Phi', B_{\rho}(0), 0) = 0.$$

Due to the similarity of hypotheses \mathbf{H}_2 (ii) (iii), we may prove (5.12) with an analogous argument to that used for (5.4), so we omit the details. Set for all $u, \varphi \in W_0^{s,p}(\Omega)$

$$\langle K_0(u), \varphi \rangle = \int_{\Omega} \frac{\eta_1(x) + \eta_2(x)}{2} (u^+)^{p-1} \varphi \, dx,$$

with $\eta_1, \eta_2 \in L^{\infty}(\Omega)_+$ as in \mathbf{H}_2 (iii). Also, set for all $(t, u) \in [0, 1] \times W_0^{s,p}(\Omega)$

$$h_0(t, u) = (-\Delta)_p^s u - (1 - t)N_f(u) - tK_0(u).$$

Then, $h_0: [0,1] \times W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a continuous $(S)_+$ -homotopy s.t. or all $t \in [0,1]$ and all ||u|| > 0 small enough

$$h_0(t,u) \neq 0.$$

By homotopy invariance we have for all $\rho > 0$ small enough

$$\deg_{(S)_+}(\Phi', B_{\rho}(0), 0) = \deg_{(S)_+}((-\Delta)_p^s - K_0, B_{\rho}(0), 0).$$

In addition, fix as before $w \in C_c^{\infty}(\Omega)_+ \setminus \{0\}$ and set for all $(t, u) \in [0, 1] \times W_0^{s, p}(\Omega)$

$$\tilde{h}_0(t, u) = (-\Delta)_p^s u - K_0(u) - tw.$$

So we have for all $t \in [0,1]$ and all $u \in W_0^{s,p}(\Omega) \setminus \{0\}$

$$\tilde{h}_0(t,u) \neq 0.$$

By the homotopy invariance and solution properties, then, for all $\rho > 0$

$$\deg_{(S)_{+}}((-\Delta)_{p}^{s} - K_{0}, B_{\rho}(0), 0) = \deg_{(S)_{+}}((-\Delta)_{p}^{s} - K_{0} - w, B_{\rho}(0), 0) = 0.$$

Concatenating the relations above, we get (5.12).

Finally, we go back to the solution $u_1 \in W_0^{s,p}(\Omega)_+$ detected in (5.1), recalling that it is a local minimizer of Φ . Once again we distinguish two cases:

- (a) If u_1 is not an isolated critical point of Φ , then clearly there exist infinitely many nontrivial critical points of such functional.
- (b) If u_1 is an isolated critical point of Φ , then it is in particular a *strict* local minimizer. So, by Proposition 2.2 (i) we have for all $\rho > 0$ small enough

(5.13)
$$\deg_{(S)_+}(\Phi', B_{\rho}(u_1), 0) = 1.$$

Comparing (5.12) and (5.13) we see that $u_1 \neq 0$. Also, fixing $\rho > 0$ small enough and R > 0 big enough we may have

$$\overline{B}_{\rho}(0) \cap \overline{B}_{\rho}(u_1) = \emptyset, \ \overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_1) \subset B_R(0),$$

as well as (5.4) (5.12) (5.13). Applying twice Proposition 2.1 (ii) (domain additivity) we get

$$0 = \deg_{(S)_{+}}(\Phi', B_{R}(0), 0)$$

$$= \deg_{(S)_{+}}(\Phi', B_{\rho}(0), 0) + \deg_{(S)_{+}}(\Phi', B_{\rho}(u_{1}), 0) + \deg_{(S)_{+}}(\Phi', B_{R}(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_{1})), 0)$$

$$= 1 + \deg_{(S)_{+}}(\Phi', B_{R}(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_{1})), 0),$$

that is,

$$\deg_{(S)_+}(\Phi', B_R(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_1)), 0) = -1.$$

By Proposition 2.1 (v) (solution), there exists $u_2 \in B_R(0) \setminus (\overline{B}_{\rho}(0) \cup \overline{B}_{\rho}(u_1))$ s.t.

$$\Phi'(u_2) = 0.$$

In either case, there exist (at least) two nontrivial critical points $u_1, u_2 \in K(\Phi) \setminus \{0\}$. Choose $i \in \{1, 2\}$ and argue as before. By Proposition 3.3 we have $u \in C_s^{\alpha}(\overline{\Omega})$ satisfies weakly in Ω

$$(-\Delta)_p^s u_i = f(x, u_i).$$

Testing with $-u_i \in W_0^{s,p}(\Omega)$ we have as above

$$||u_i^-||^p \leqslant \langle (-\Delta)_p^s u_i, -u_i^- \rangle$$

= $\int_{\{u_i < 0\}} f(x, u_i) u_i dx = 0,$

hence $u_i \ge 0$ in Ω . In addition, by \mathbf{H}_2 (i) (iii) we have for a.e. $x \in \Omega$ and all $t \in [0, ||u_i||_{\infty}]$

$$f(x,t) \geqslant -Ct^{p-1}$$
.

So we have weakly in Ω

$$(-\Delta)_p^s u_i + C u_i^{p-1} \geqslant 0.$$

Applying Proposition 3.4 (i) (with $g(t) = (t^+)^{p-1}$), we deduce that $u_i \in \operatorname{int}(C_s^0(\overline{\Omega}))$, which concludes the proof.

Remark 5.3. Under hypotheses symmetric to \mathbf{H}_2 , we can prove that (1.1) admits two negative solutions.

Acknowledgement. The author is a member of GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica 'Francesco Severi'). He is partially supported by the research projects: Evolutive and Stationary Partial Differential Equations with a Focus on Biomathematics (Fondazione di Sardegna 2019), Analysis of PDEs in connection with real phenomena (CUP F73C22001130007, Fondazione di Sardegna 2021), Nonlinear Differential Problems via Variational, Topological and Set-valued Methods (PRIN-2017AYM8XW). The authors wishes to thank the anonymous Referee for her/his careful reading of the work and precious suggestions.

References

- [1] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New York (2011).
- [2] F.E. Browder, Degree of mapping for nonlinear operators of monotone type, Proc. Nat. Acad. Sci. U.S.A. 80 (1983) 1771–1773.
- [3] W. CHEN, S. MOSCONI, M. SQUASSINA, Nonlocal problems with critical Hardy nonlinearity, J. Funct. Anal. 275 (2018) 3065–3114.
- [4] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
- [5] S. Frassu, A. Iannizzotto, Five solutions for the fractional p-Laplacian with noncoercive energy, Nonlinear Differ. Equ. Appl. 29 (2022) art. 43.
- [6] S. Frassu, A. Iannizzotto, Multiple solutions for the fractional p-Laplacian with jumping reactions, J. Fixed Point Theory Appl. 25 (2023) art. 25.
- [7] T. GODOY, J.P. GOSSEZ, S. PACZKA, On the antimaximum principle for the p-Laplacian with indefinite weight, Nonlinear Anal. 51 (2002) 449–467.
- [8] A. Granas, J. Dugundji, Fixed point theory, Springer, New York (2003).
- [9] A. IANNIZZOTTO, Monotonicity of eigenvalues of the fractional p-Laplacian with singular weights, Topol. Methods Nonlinear Anal. (2023) DOI: 10.12775/TMNA.2022.024.
- [10] A. IANNIZZOTTO, S. LIU, K. PERERA, M. SQUASSINA, Existence results for fractional p-Laplacian problems via Morse theory, Adv. Calc. Var. 9 (2016) 101–125.
- [11] A. IANNIZZOTTO, R. LIVREA, Four solutions for fractional p-Laplacian equations with asymmetric reactions, Mediterr. J. Math. 18 (2021) art. 220.
- [12] A. IANNIZZOTTO, S. MOSCONI, N.S. PAPAGEORGIOU, On the logistic equation for the fractional p-Laplacian, Math. Nachr. (2023) DOI: 10.1002/mana.202100025.
- [13] A. IANNIZZOTTO, S. MOSCONI, M. SQUASSINA, Global Hölder regularity for the fractional p-Laplacian, Rev. Mat. Iberoam. 32 (2016) 1353–1392.
- [14] A. IANNIZZOTTO, S. MOSCONI, M. SQUASSINA, Fine boundary regularity for the degenerate fractional p-Laplacian, J. Funct. Anal. 279 (2020) art. 108659.
- [15] A. IANNIZZOTTO, S. MOSCONI, M. SQUASSINA, Sobolev versus Hölder minimizers for the degenerate fractional p-Laplacian, Nonlinear Anal. 191 (2020) art. 111635.
- [16] E. LINDGREN, P. LINDQVIST, Fractional eigenvalues, Calc. Var. Partial Differential Equations 49 (2014) 795–826.
- [17] G. MOLICA BISCI, V.D. RĂDULESCU, R. SERVADEI, Variational methods for nonlocal fractional problems, Cambridge University Press, Cambridge (2016).

- [18] D. MOTREANU, V. V. MOTREANU, N. S. PAPAGEORGIOU, Topological and variational methods with applications to nonlinear boundary value problems, Springer, New York (2014).
- [19] D. MOTREANU, V. V. MOTREANU, N. S. PAPAGEORGIOU, A degree theoretic approach for multiple solutions of constant sign for nonlinear elliptic equations, Manuscripta Math. 124 (2007) 507–531.
- [20] M. Ôtani, Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations, J. Funct. Anal. 76 (1988) 140–159.
- [21] P.H. RABINOWITZ, A note on topological degree for potential operators, J. Math. Anal. Appl. 51 (1975) 483–492.
- [22] M. Struwe, Variational methods, Springer, Berlin (2008).

(A. Iannizzotto) DIPARTIMENTO DI MATEMATICA E INFORMATICA

Università di Cagliari

VIA OSPEDALE 72, 09124 CAGLIARI, ITALY

 $Email\ address: \verb| antonio.iannizzotto@unica.it| \\$