Dilations of commuting C_0 -semigroups with bounded generators and the von Neumann polynomial inequality

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ABSTRACT. Consider d commuting C_0 -semigroups (or equivalently: d-parameter C_0 semigroups) over a Hilbert space for $d \in \mathbb{N}$. In the literature (cf. [31, 27, 28, 24, 18, 26]), conditions are provided to classify the existence of unitary and regular unitary dilations. Some of these conditions require inspecting values of the semigroups, some provide only sufficient conditions, and others involve verifying sophisticated properties of the generators. By focussing on semigroups with bounded generators, we establish a simple and natural condition on the generators, viz. complete dissipativity, which naturally extends the basic notion of the dissipativity of the generators. Using examples of non-doubly commuting semigroups, this property can be shown to be strictly stronger than dissipativity. As the first main result, we demonstrate that complete dissipativity completely characterises the existence of regular unitary dilations, and extend this to the case of arbitrarily many commuting C_0 -semigroups. We furthermore show that all multi-parameter C_0 -semigroups (with bounded generators) admit a weaker notion of regular unitary dilations, and provide simple sufficient norm criteria for complete dissipativity. The paper concludes with an application to the von Neumann polynomial inequality problem, which we formulate for the semigroup setting and solve negatively for all $d \ge 2$.

1. INTRODUCTION

Dynamical systems can often be described by evolving contractive operators over Hilbert or Banach spaces. Characterising the possibility of embedding these into larger systems described by surjective isometries began in 1953 with the research of Sz.-Nagy, *et al.* in [30], in which the unitary (power) dilation of contractions and of 1-parameter contractive C_0 -semigroups over Hilbert spaces is presented. In 1955, Stinespring [29] introduced dilation to the non-commutative setting of Banach and C^* -algebras, thus opening the way for results for more sophisticated dynamical systems. For an overview of this development, see *e.g.* [3, 25]. With the backdrop of these theoretical frameworks, intensive research has been conducted over the decades to yield concrete results for families of operators (see *e.g.* [1]), classes of semigroups (see *e.g.* [27, 28, 24, 18, 26]), and dynamical systems over C^* - and W^* -algebras (see *e.g.* [32, 8, 7, 16, 17]).

In this paper we focus on the semigroup setting. Consider d commuting C_0 -semigroup T_1, T_2, \ldots, T_d over a Hilbert space \mathcal{H} for some $d \in \mathbb{N}$. We say that T_1, T_2, \ldots, T_d have a simultaneous unitary dilation if there is a Hilbert space \mathcal{H}' , a bounded operator (necessarily an isometry) $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$, and d commuting unitary C_0 -semigroups U_1, U_2, \ldots, U_d over \mathcal{H}' (which can be uniquely extended to commuting unitary representations of $(\mathbb{R}, +, 0)$), such that $\prod_{i=1}^d T(t_i) = r^*(\prod_{i=1}^d U(t_i))r$ holds for all $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d$. We call this a simultaneous regular unitary dilation if the stronger condition $(\prod_{i=1}^d T(t_i^-))^*(\prod_{i=1}^d T(t_i^+)) = r^*(\prod_{i=1}^d U(t_i))r$ holds for all $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}^d$, where $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$ denote the positive and negative parts of t for any $t \in \mathbb{R}$. One reason to consider regular unitary dilations, is that they are more directly related to the notion of 'positive definite functions', which in turn characterise group representation of locally compact Hausdorff groups (cf. [31, Theorem I.7.1 b)], [23, Notes, p. 52], and [9, §3.3]).

In [31, Theorem I.8.1], [27], [28, Theorem 2], and [24, Theorem 2.3], it was proved that T_1, T_2, \ldots, T_d have a simultaneous unitary dilation if they are contractive and $d \in \{1, 2\}$. In [24, Theorem 3.2] a general condition on T_1, T_2, \ldots, T_d which we shall refer to as *Brehmer positivity* (see Definition 2.3) was found for the existence of simultaneous regular unitary dilations. However, to verify this condition one needs to consider values of the semigroups. In [18, Theorem 2.2 and Theorem 3.1] Le Merdy fully classified the existence of a *weaker notion* of simultaneous

²⁰²⁰ Mathematics Subject Classification. 47A13, 47A20, 47D03, 47D06.

Key words and phrases. Semigroups of operators, bounded semigroups, dilations, infinitesimal generator.

unitary dilations, and applied this to commuting families of bounded analytic C_0 -semigroups. More recently, Shamovich and Vinnivok established in [26] sufficient conditions on generators for the existence of simultaneous unitary dilations. These conditions are quite sophisticated and involve proving the existence of embeddings of the generators of the marginals.

Note that there is a natural correspondence between d commuting C_0 -semigroups, T_1, T_2, \ldots, T_d and d-parameter C_0 -semigroups, T, i.e. SOT-continuous morphisms between the algebraic structures $(\mathbb{R}^d_{\geq 0}, +, \mathbf{0})$ and $(\mathfrak{L}(\mathcal{H}), \circ, \mathbf{I})$. This correspondence is realised via the constructions $T(\mathbf{t}) = \prod_{i=1}^{d} T_i(t_i)$ for all $\mathbf{t} \in \mathbb{R}^d_{\geq 0}$ and the co-ordinate maps $T_i(t) = T(0, 0, \ldots, t, \ldots, 0)$ for $t \in \mathbb{R}_{\geq 0}$ and $i \in \{1, 2, \ldots, d\}$. In this way, the T_i may be viewed as the marginal semigroups (or simply: the marginals) of T. It is well known that their generators A_1, A_2, \ldots, A_d commute (even if they are unbounded). See for example [5, Proposition 1.1.8–9]. It is also straightforward to see that T is contractive/unitary/isometric if and only if each of the T_i are. Hence one may interchangeably refer to commuting families of d contractive/unitary/isometric C_0 -semigroups. For convenience, we shall primarily use the multi-parameter presentation throughout this paper.

In our research, we focus primarily on multi-parameter C_0 -semigroups with bounded generators. In §2 we introduce the special conditions of *complete dissipativity* and *complete super dissipativity* on the generators of T (see Definition 2.8). In §3 we develop algebraic identities involving these notions. In §4 the main classification result is proved:

Theorem 1.1 (Classification via complete dissipativity). Let $d \in \mathbb{N}$ and T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators. Then the following statements are equivalent.

- (1) The semigroup T has a regular unitary dilation.
- (2) The generators of T are completely dissipative.
- (3) There is a net $(T^{(\alpha)})_{\alpha \in \mathcal{I}}$ consisting of regularly unitarily dilatable *d*-parameter C_0 semigroups over \mathcal{H} , such that $(T^{(\alpha)})_{\alpha \in \mathcal{I}} \longrightarrow T$ uniformly in norm on compact subsets of $\mathbb{R}^d_{\geq 0}$, i.e. $\sup_{\mathbf{t} \in L} \|T^{(\alpha)}(\mathbf{t}) T(\mathbf{t})\| \xrightarrow{\alpha} 0$ for all compact $L \subseteq \mathbb{R}^d_{\geq 0}$.
- (4) There is a net $(T^{(\alpha)})_{\alpha \in \mathcal{I}}$ consisting of regularly unitarily dilatable d-parameter C_0 semigroups over \mathcal{H} , such that $(T^{(\alpha)})_{\alpha \in \mathcal{I}} \longrightarrow T$ uniformly in the SOT-topology on compact subsets of $\mathbb{R}^d_{\geq 0}$, i.e. $\sup_{\mathbf{t} \in L} ||(T^{(\alpha)}(\mathbf{t}) T(\mathbf{t}))\xi|| \xrightarrow{\alpha} 0$ for all $\xi \in \mathcal{H}$ and all compact $L \subseteq \mathbb{R}^d_{\geq 0}$.

As a consequence of this classification, we further show that all multi-parameter C_0 semigroups with bounded generators admit weaker notions of regular unitary dilations.

Corollary 1.2 (Regular unitary dilation up to exponential equivalence). Let $d \in \mathbb{N}$ and T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators. Then for some $\boldsymbol{\omega} \in \mathbb{R}^d_{\geq 0}$, the modified d-parameter C_0 -semigroup $\tilde{T} := (e^{-\langle \mathbf{t}, \boldsymbol{\omega} \rangle} T(\mathbf{t}))_{\mathbf{t} \in \mathbb{R}^d_{\geq 0}}$ has a regular unitary dilation.

In the remainder of §4 we extend Theorem 1.1 to arbitrarily many commuting C_0 semigroups (see Corollary 4.5). We further explore the set of $\omega \in \mathbb{R}^d$ for which Corollary 1.2
holds and provide simple norm conditions sufficient for the existence of regular unitary dilations:

Theorem 1.3 (Sufficient norm conditions for regular unitary dilations). Let $d \in \mathbb{N}$ and T a (necessarily contractive) d-parameter C_0 -semigroup over a Hilbert space \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . If the generators satisfy $\frac{\|A_i + \omega_i \cdot \mathbf{I}\|}{\omega_i} \leq 2^{1/d} - 1$ for all $i \in \{1, 2, \ldots, d\}$ and some $\boldsymbol{\omega} = (\omega_i)_{i=1}^d \in \mathbb{R}^d_{>0}$, then T has a regular unitary dilation.

In §5, we investigate complete (super) dissipativity for concrete classes of semigroups. In the case of normal semigroups, the notions coincide with the more basic properties of dissipativity and negative spectral bounds respectively. And by working with a naturally definable class of non-doubly commuting generators, we demonstrate that our notions are in general not equivalent to these properties.

The paper concludes in §6 with a non-trivial application of complete dissipativity to the von Neumann inequality problem. We provide a natural generalisation of the problem to our context, defining the regular polynomial bounds and the regular von Neumann polynomial inequality problem for multi-parameter C_0 -semigroups (see Definition 6.2). We then establish a second characterisation of regular unitary dilations:

Theorem 1.4 (Classification via polynomial bounds). Let $d \in \mathbb{N}$ and T be a *d*-parameter C_0 -semigroup over a Hilbert space \mathcal{H} with bounded generators. Then the following are equivalent:

- (1) The semigroup T has a regular unitary dilation.
- (2) T satisfies regular polynomial bounds.
- (3) T satisfies regular polynomial bounds in a neighbourhood of **0**.

Using this, we negatively solve the regular von Neumann polynomial inequality problem for multi-parameter C_0 -semigroups:

Corollary 1.5 Let \mathcal{H} be a Hilbert space with dim $(\mathcal{H}) \ge 2$ and let $d \in \mathbb{N}$ with $d \ge 2$. Then there exist d-parameter contractive C_0 -semigroups with bounded generators which have strictly negative spectral bounds,^a for which regular polynomial bounds fail.

2. Definitions

In this paper we fix the following notation. Let $d \in \mathbb{N}$. Let $A, A_1, A_2, \ldots, A_d \in \mathfrak{L}(\mathcal{H})$ be any bounded operators, let $C, C_1, C_2, K \subseteq \{1, 2, \ldots, d\}$, and let $\pi = (\pi(i))_i$ be a (possibly empty) finite sequence of indices from $\{1, 2, \ldots, d\}$.

- We write $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0\}$, $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$, and $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$. The set \mathbb{T} denotes the unit circle in the complex plane $\{z \in \mathbb{C} \mid |z| = 1\}$.
- In any algebraic context empty sums and products shall always be taken to be the additive and multiplicative identities respectively.
- Depending on the context I shall denote the identity operator on a space and 0 denotes the zero operator on a space or between spaces.
- In the context of \mathbb{R}^d let $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^d$ and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$, and let $\mathbf{e}_i = (0, 0, \dots, \underbrace{1}_i, \dots, 0)$ denote the canonical unit vectors for $i \in \{1, 2, \dots, d\}$. Also set $\mathbf{e}_C := \sum_{i \in C} \mathbf{e}_i$. For $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}^d$ let $\mathbf{t}^+ := (t_i^+)_{i=1}^d = (\max\{t_i, 0\})_{i=1}^d \in \mathbb{R}^d_{\geq 0}$ and $\mathbf{t}^- := (t_i^-)_{i=1}^d = (\max\{-t_i, 0\})_{i=1}^d \in \mathbb{R}^d_{\geq 0}$ denote the positive and negative parts respectively. We further define the support $\operatorname{supp}(\mathbf{t}) := \{i \in \{1, 2, \dots, d\} \mid t_i \neq 0\}$ as usual.
- For $\boldsymbol{\omega} = (\omega_i)_{i=1}^d \in \mathbb{R}^d$ we shall denote $\omega_K := \prod_{i \in K} \omega_i$.
- $\Re e A \coloneqq \frac{1}{2}(A + A^*)$ and $\Im m A \coloneqq \frac{1}{2i}(A A^*)$ denote the self-adjoint operators, referred to as the real and imaginary parts of the operator A.
- We adopt the notation $A(\pi) := \prod_{i=1}^{|\pi|} A_{\pi(i)}$ and $A(C) := \prod_{i \in C} A_i$ where the order of the indices can be taken to be ascending. If these operators commute, then the order of multiplication is irrelevant.
- We use the notation A_i^- for $-A_i$ for each *i* and define $A^-(\pi)$ and $A^-(C)$ as above.
- We write $(C_1, C_2) \in Part(K)$ to denote that $\{C_1, C_2\}$ is a partition of K. We shall frequently compute sums over partitions of products. If $a_1, a_2, \ldots, a_d \in \mathcal{A}$ are commuting elements of some algebra \mathcal{A} with unit, then using binomial expansion one obtains

^aThe spectral bound of a linear operator $A : \text{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is given by $\sup\{\Re e \mid \lambda \in \sigma(A)\}$ (cf. [6, Definition 1.12]).

 $\sum_{(C_1,C_2)\in \operatorname{Part}(K)} \prod_{i\in C_1} a_i = \sum_{C\subseteq K} \prod_{i\in K} \begin{cases} a_i & : i\in C\\ 1 & : i\notin C \end{cases} = \prod_{i\in K} (1+a_i).$ This observation shall be repeatedly used in computations.

To avoid confusion throughout this paper we shall consistently call self-adjoint elements $a \in \mathcal{A}$ of a unital C^* -algebra \mathcal{A} positive (in symbols $a \ge \mathbf{0}$) if $a = b^*b$ for some $b \in \mathcal{A}$, which holds if and only if $\sigma(a) \subseteq \mathbb{R}_{\ge 0}$. We call a strictly positive if $a - c\mathbf{I} \ge \mathbf{0}$ for some $c \in \mathbb{R}_{>0}$. For self-adjoint bounded operators $A \in \mathfrak{L}(\mathcal{H})$ one equivalently has that A is positive if and only if $\langle A\xi, \xi \rangle \ge 0$ for all $\xi \in \mathcal{H}$ and strictly positive if and only if $\langle A\xi, \xi \rangle \ge c \|\xi\|^2$ for all $\xi \in \mathcal{H}$ and some $c \in \mathbb{R}_{>0}$.

2.1 Notions of dilation. As alluded to in the introduction, there is a natural correspondence between commuting families of (contractive/isometric/unitary) C_0 -semigroups and multiparameter (contractive/isometric/unitary) C_0 -semigroups. Working with the multi-parameter presentation, we provide definitions of (regular) unitary dilations corresponding to those in the introduction.^c

Definition 2.1 A unitary dilation of T is a tuple (U, \mathcal{H}', r) , where U is a d-parameter unitary C_0 -semigroup over a Hilbert space \mathcal{H}' and $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$ (necessarily isometric), such that $T(\mathbf{t}) = r^* U(\mathbf{t})r$ holds for all $\mathbf{t} \in \mathbb{R}^d_{\geq 0}$.

Note that any *d*-parameter unitary C_0 -semigroup U can always be (uniquely) extended to an SOT-continuous representation of $(\mathbb{R}^d, +, \mathbf{0})$ via the definition $U(\mathbf{t}) = U(\mathbf{t}^-)^* U(\mathbf{t}^+)$ for all $\mathbf{t} \in \mathbb{R}^d$. By capturing this behaviour, one obtains the following stronger notion of dilation:

Definition 2.2 A regular unitary dilation of T is a tuple (U, \mathcal{H}', r) , where U is an SOTcontinuous unitary representation of $(\mathbb{R}^d, +, \mathbf{0})$ over a Hilbert space \mathcal{H}' and $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$ (necessarily isometric), such that $T(\mathbf{t}^-)^*T(\mathbf{t}^+) = r^*U(\mathbf{t})r$ holds for all $\mathbf{t} \in \mathbb{R}^d$.

Clearly, the existence of a regular unitary dilation implies the existence of a unitary dilation. This implication is strict (*cf.* 5.3 and Remark 5.5).

Definition 2.3 The self-adjoint *Brehmer operators* associated to T shall be defined via

$$B_{T,K}(t) := \sum_{C \subseteq K} (-1)^{|C|} T(t\mathbf{e}_C)^* T(t\mathbf{e}_C)$$

for all $K \subseteq \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_{\geq 0}$. We say that T satisfies the Brehmer positivity criterion, if for sufficiently small $t \in \mathbb{R}_{>0}$ it holds that $B_{T,K}(t) \geq \mathbf{0}$ for all $K \subseteq \{1, 2, \ldots, d\}$.

Theorem 2.4 (Ptak, 1985). Let $d \in \mathbb{N}$ and T be a *d*-parameter C_0 -semigroup over \mathcal{H} . Then the following statements are equivalent:

- (1) T has a regular unitary dilation.
- (2) T satisfies the Brehmer positivity criterion.
- (3) $B_{T,K}(t) \ge \mathbf{0}$ for all $K \subseteq \{1, 2, \dots, d\}$ and all $t \in \mathbb{R}_{>0}$.

A proof of this can be found in [24, Theorem 3.2] and is based on the more general characterisation of regular unitary dilations of operator-valued functions defined on topological groups provided in [31, Theorem I.7.1], which in turn is rooted in the correspondence between 'positive definite functions' and group representations (*cf.* [9, Proposition 3.14, Theorem 3.20, and Proposition 3.35]). Note that the Brehmer positivity criterion in Ptak's paper is more weakly defined in terms of a sequence of *t*-values converging to 0. However, in the proof it is stated that the existence of a regular unitary dilation implies that $B_{T,K}(t) \ge \mathbf{0}$ for all $t \in \mathbb{R}_{>0}$ (*cf.* (a) \Rightarrow (b) in [24, Theorem 3.2]). Thus the above formulation is equivalent to Ptak's.

^bThe first property is also called *positive semi-definite*.

^cSince a multi-parameter C_0 -semigroup is a single object as opposed to a collection, we drop the term 'simultaneous'.

In general, it can be difficult to verify this condition, as we need to consider the values of the semigroup at all points (or at least at points close to 0). For semigroups with bounded generators, however, Theorem 1.1 establishes a simpler condition purely in terms of the generators.

Note that in the discrete case, the existence of simultaneous regular unitary dilations for families of commuting contractions is equivalent to a similarly defined notion of Brehmer positivity (*cf.* [31, Theorem I.9.1]). In [4] the authors inspect slightly weaker conditions and construct simultaneous regular *isometric* dilations for families of commuting contractions satisfying this.

In addition to these definitions there are weaker variations of dilation that one may consider. These all revolve around dilations of certain natural modifications. In the discrete setting, *i.e.* for tuples of commuting bounded operators over \mathcal{H} , there is the notion of ρ -dilations (see *e.g.* [31, §I.11]). In the continuous setting of multi-parameter C_0 -semigroups one has the following:

Definition 2.5 Say that T has a *(regular) unitary dilation up to similarity* if it can be written as $T = s \tilde{T}(\cdot) s^{-1} := (s \tilde{T}(\mathbf{t}) s^{-1})_{\mathbf{t} \in \mathbb{R}^d_{\geq 0}}$ for some invertible $s \in \mathfrak{L}(\mathcal{H})$ and some *d*-parameter C_0 semigroup \tilde{T} over \mathcal{H} , where \tilde{T} has a (regular) unitary dilation.

Definition 2.6 Say that T has a (regular) unitary dilation up to exponential equivalence if it can be written as $T = e^{\langle \cdot, \omega \rangle} \tilde{T}(\cdot) := (e^{\langle \mathbf{t}, \omega \rangle} \tilde{T}(\mathbf{t}))_{\mathbf{t} \in \mathbb{R}^d_{\geq 0}}$ for some $\omega \in \mathbb{R}^d$ and some d-parameter C_0 -semigroup \tilde{T} over \mathcal{H} , where \tilde{T} has a (regular) unitary dilation.

In [18, Theorem 2.2] Le Merdy provides a characterisation of C_0 -semigroups which are similar to C_0 -semigroups that have a unitary dilation. The characterisation involves demonstrating the complete boundedness of a certain functional calculus map induced by the resolvents of the generators of the marginal semigroups. In [18, Theorem 3.1] *d*-parameter contractive C_0 semigroups with bounded analytic marginal semigroups are shown to satisfy this criterion.

By focussing on the smaller class of multi-parameter C_0 -semigroups with bounded generators, we obtain a considerably simpler condition. This condition is not fulfilled by all multiparameter C_0 -semigroups, as shall be demonstrated in §5.3. Nonetheless in a similar vein to [18], we demonstrate in Corollary 1.2 that regular unitary dilatability up to exponential equivalence always holds.

2.2 Dissipativity conditions on the generators. It turns out that we can capture the existence of regular unitary dilations in terms of a stronger but natural notion of dissipativity. Recall that an operator $A \in \mathfrak{L}(\mathcal{H})$ is called *dissipative* if $\mathfrak{Re}\langle A\xi, \xi \rangle \leq 0$ for all $\xi \in \mathcal{H}$.^d

Remark 2.7 Let $A \in \mathfrak{L}(\mathcal{H})$. Since the spectrum $\sigma(A)$ is bounded, the resolvent set satisfies $\rho(A) \cap \mathbb{R}_{>0} \neq \emptyset$. The Lumer-Phillips form of the Hille-Yosida theorem (see [10, Theorem 3.3]) thus yields that A is the generator of a contractive C_0 -semigroup if and only if A is dissipative.

It shall be convenient to furthermore call $A \in \mathfrak{L}(\mathcal{H})$ super dissipative if $\mathfrak{Re}\langle A\xi, \xi \rangle \leq -\omega \|\xi\|^2$ for all $\xi \in \mathcal{H}$ and some $\omega \in \mathbb{R}_{>0}$. Now, as an alternative formulation of these terms, observe that A is dissipative if and only if $\mathfrak{Re} A \leq \mathbf{0}$ and A is super dissipative if and only if $\mathfrak{Re} A \leq -\omega \mathbf{I}$ for some $\omega \in \mathbb{R}_{>0}$. Based on these observations, we introduce the following generalised notions:

Definition 2.8 The self-adjoint *dissipation operators* associated to the generators A_1, A_2, \ldots, A_d of T shall be defined by

$$S_{T,K} := 2^{-|K|} \sum_{(C_1, C_2) \in \operatorname{Part}(K)} A^-(C_1)^* A^-(C_2)$$

for all $K \subseteq \{1, 2, \ldots, d\}$. For $k \in \mathbb{N}_0$ we shall refer to $S_{T,K}$ for $K \subseteq \{1, 2, \ldots, d\}$ with |K| = k as the k^{th} order dissipation operators. Setting

$$\beta_T := \min_{K \subseteq \{1,2,\dots,d\}} \sigma(S_{T,K}),$$

^dDissipativity is typically defined for unbounded operators (*cf.* [34, VI.5 (8)]), but we shall not require this level of generality.

we say that the generators of T are completely dissipative if $\beta_T \ge 0$ (equivalently: $S_{T,K} \ge \mathbf{0}$ for each $K \subseteq \{1, 2, \ldots, d\}$) and completely super dissipative if $\beta_T > 0$.

Observe that for $K = \emptyset$ one has $S_{T,\emptyset} = 2^{-0}A^{-}(\emptyset)^*A^{-}(\emptyset) = \mathbf{I}$, since $\{\emptyset, \emptyset\}$ is the only partition of K. And for $K = \{\alpha\}$ for some $\alpha \in \{1, 2, \ldots, d\}$ one has

$$S_{T,\{\alpha\}} = 2^{-1} (A^{-}(\{\alpha\})^* A^{-}(\emptyset) + A^{-}(\emptyset)^* A^{-}(\{\alpha\})) = 2^{-1} ((A_{\alpha}^{-})^* \cdot \mathbf{I} + \mathbf{I} \cdot A_{\alpha}^{-}) = -\Re e A_{\alpha},$$

since $\{\{\alpha\}, \emptyset\}$ is the only partition of K. So if T has completely dissipative generators, then for each $\alpha \in \{1, 2, ..., d\}$ $0 \leq \beta_T \leq \min(\sigma(-\Re e A_\alpha)) = -\max(\sigma(\Re e A_\alpha))$, which, by the correspondence between numerical ranges and spectra for self-adjoint operators (*cf.* [12, Theorem 1.2.1–4]), implies $\Re e\langle A_\alpha \xi, \xi \rangle = \langle (\Re e A_\alpha)\xi, \xi \rangle \leq 0$ for all $\xi \in \mathcal{H}$, *i.e.* A_α is dissipative. In a similar way, we see that the complete super dissipativity of the generators of T implies that each of the generators must be super dissipative. Furthermore, if the generators are normal, then super dissipativity is equivalent to strict negativity of the spectral bounds.

Hence complete dissipativity extends the notion of dissipativity, and complete super dissipativity extends the notion of super dissipativity as well as, in the case of normal generators, strict negativity of the spectral bounds. In §5.3 it shall be shown that complete (super) dissipativity is a strictly stronger property than (super) dissipativity.

3. Algebraic identities

We now lay out some usual algebraic identities for the computation of dissipation operators. To start, we observe the following basic recursive relations. For $t \in \mathbb{R}_{\geq 0}$, $K \subseteq \{1, 2, \ldots, d\}$, and $\alpha \in \{1, 2, \ldots, d\} \setminus K$ one has

$$B_{T,\emptyset}(t) = (-1)^0 T(\mathbf{0})^* T(\mathbf{0}) = 1 \cdot \mathbf{I}^* \mathbf{I} = \mathbf{I} \text{ and}$$

$$B_{T,K\cup\{\alpha\}}(t) = \sum_{C\subseteq K} (-1)^{|C\cup\{\alpha\}|} T(t\mathbf{e}_{C\cup\{\alpha\}})^* T(t\mathbf{e}_{C\cup\{\alpha\}})$$

$$+ \sum_{C\subseteq K} (-1)^{|C|} T(t\mathbf{e}_{C})^* T(t\mathbf{e}_{C})$$

$$= \sum_{C\subseteq K} (-1)^{|C|+1} T_{\alpha}(t)^* T(t\mathbf{e}_{C})^* T(t\mathbf{e}_{C}) T_{\alpha}$$

$$+ \sum_{C\subseteq K} (-1)^{|C|} T(t\mathbf{e}_{C})^* T(t\mathbf{e}_{C})$$

$$= B_{T,K}(t) - T_{\alpha}(t)^* B_{T,K}(t) T_{\alpha}(t)$$
(3.1)

as well as

$$S_{T,\emptyset} = 2^{-0}A^{-}(\emptyset)^{*}A^{-}(\emptyset) = \mathbf{I}^{*}\mathbf{I} = \mathbf{I}, \text{ and}$$

$$S_{T,K\cup\{\alpha\}} = 2^{-|K\cup\{\alpha\}|} \sum_{(C_{1},C_{2})\in\operatorname{Part}(K\cup\{\alpha\})} A^{-}(C_{1})^{*}A^{-}(C_{2})$$

$$= 2^{-|K|-1} \sum_{(C_{1},C_{2})\in\operatorname{Part}(K)} A^{-}(C_{1}\cup\{\alpha\})^{*}A^{-}(C_{2})$$

$$+ 2^{-|K|} \sum_{(C_{1},C_{2})\in\operatorname{Part}(K)} A^{-}(C_{1})^{*}A^{-}(C_{2}\cup\{\alpha\})$$

$$= \frac{1}{2}2^{-|K|} \sum_{(C_{1},C_{2})\in\operatorname{Part}(K)} A^{-}(C_{1})^{*}A^{-}(C_{2})A_{\alpha}^{-}$$

$$= -\frac{1}{2}(A_{\alpha}^{*}S_{T,K} + S_{T,K}A_{\alpha})$$

$$= \frac{1}{2}\{-\Re e A_{\alpha}, S_{T,K}\} + i\frac{1}{2}[\Im m A_{\alpha}, S_{T,K}], \qquad (3.2)$$

where $[\cdot, \cdot]$ und $\{\cdot, \cdot\}$ denote the algebraic *commutator* and *anti-commutator* operations.^e Note that the above recursive relations for $B_{T,\cdot}(\cdot)$ also hold for semigroups with unbounded generators.

3.1 **Dissipation operators under shifts.** Shifting the generators by constant multiples of the identity lead to self-similarities in the dissipation operators. In this subsection we consider an arbitrary *d*-parameter C_0 -semigroup T with marginals T_1, T_2, \ldots, T_d which have bounded generators A_1, A_2, \ldots, A_d on which we place no further assumptions. For arbitrary $\boldsymbol{\omega} \in \mathbb{R}^d$ we define the 1-dimensional continuous representation $E_{\boldsymbol{\omega}} := (e^{\langle t, \boldsymbol{\omega} \rangle})_{t \in \mathbb{R}^d}$ of the group $(\mathbb{R}^d, +, \mathbf{0})$ and define the modification

$$E_{\boldsymbol{\omega}} \cdot T \quad := \quad (e^{-\langle \mathbf{t}, \ \boldsymbol{\omega} \rangle} T(\mathbf{t}))_{\mathbf{t} \in \mathbb{R}^{d}_{\geq 0}},$$

which is clearly a *d*-parameter C_0 -semigroup over \mathcal{H} whose marginals have bounded generators $\tilde{A}_i = A_i - \omega_i \mathbf{I}$ for each $i \in \{1, 2, \dots, d\}$.

^eFor any algebra \mathcal{A} , which is at least a \mathbb{Z} -module, and for any $a, b \in \mathcal{A}$ one defines [a, b] := ab - ba and $\{a, b\} := ab + ba$.

Definition 3.1 Call the *d*-parameter C_0 -semigroup $E_{\omega} \cdot T$ the ω -shift of T.

Proposition 3.2 (Self-similarity under shifts). Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . Further consider the $\boldsymbol{\omega}$ -shift $E_{\boldsymbol{\omega}} \cdot T$ of T for some $\boldsymbol{\omega} \in \mathbb{R}^d$. Then

$$S_{E_{\omega}\cdot T,K} = \sum_{K'\subseteq K} \omega_{K\setminus K'} S_{T,K'}$$
(3.3)

holds for all $K \subseteq \{1, 2, \ldots, d\}$.

Proof. We prove this by induction over the size of K and using the recursion laid out in (3.2). For $K = \emptyset$, the left- and right-hand sides of (3.3) are both **I** and thus equal. Let $K \subseteq \{1, 2, \ldots, d\}$ and $\alpha \in \{1, 2, \ldots, d\} \setminus K$. Assume that (3.3) holds for K. Then

$$\begin{split} S_{E_{\boldsymbol{\omega}}\cdot T,K\cup\{\alpha\}} &\stackrel{(3.2)}{=} & -\frac{1}{2}(\tilde{A}_{\alpha}S_{E_{\boldsymbol{\omega}}\cdot T,K} + S_{E_{\boldsymbol{\omega}}\cdot T,K}\tilde{A}_{\alpha}^{*}) \\ &= & -\frac{1}{2}((A_{\alpha} - \omega_{\alpha}\mathbf{I})S_{E_{\boldsymbol{\omega}}\cdot T,K} + S_{E_{\boldsymbol{\omega}}\cdot T,K}(A_{\alpha}^{*} - \omega_{\alpha}\mathbf{I})) \\ &\stackrel{\text{ind.}}{=} & -\frac{1}{2}\sum_{K'\subseteq K} \omega_{K\setminus K'} \cdot ((A_{\alpha} - \omega_{\alpha}\mathbf{I})S_{T,K'} + S_{T,K'}(A_{\alpha}^{*} - \omega_{\alpha}\mathbf{I})) \\ &= & \sum_{K'\subseteq K} \omega_{K\setminus K'}\omega_{\alpha}S_{T,K'} - \frac{1}{2}\sum_{K'\subseteq K} \omega_{K\setminus K'} \cdot (A_{\alpha}S_{T,K'} + S_{T,K'}A_{\alpha}^{*}) \\ &\stackrel{(3.2)}{=} & \sum_{K'\subseteq K} \omega_{K\setminus K'}\omega_{\alpha}S_{T,K'} + \sum_{K'\subseteq K} \omega_{K\setminus K'}S_{T,K'\cup\{\alpha\}} \\ &= & \sum_{K'\subseteq K} \omega_{(K\cup\{\alpha\})\setminus K'}S_{T,K'} + \sum_{K'\subseteq K} \omega_{(K\cup\{\alpha\})\setminus (K'\cup\{\alpha\})}S_{T,K'\cup\{\alpha\}} \\ &= & \sum_{K'\subseteq K\cup\{\alpha\}} \omega_{(K\cup\{\alpha\})\setminus K'}S_{T,K'}, \end{split}$$

i.e. (3.3) holds for $K \cup \{\alpha\}$. Hence the claim holds by induction.

3.2 Dissipation operators under inversions. This subsection is independent of the rest of the paper, and provides a connection to inverse infinitesimal generators, which are of relatively recent interest in semigroup theory in particular in the unbounded case (*cf.* [11, 35]).

Let $P \subseteq \{1, 2, ..., d\}$. We shall assume that the generators $A_1, A_2, ..., A_d$ of T are bounded, and that A_i is invertible for each $i \in P$. We then define $A_{P,i} := A_i^{-1}$ for $i \in P$ and $A_{P,i} := A_i$ for $i \in \{1, 2, ..., d\} \setminus P$. Since the A_i commute, one clearly has that the $A_{P,i}$ commute. We can thus define the *d*-parameter C_0 -semigroup

$$T_P := (e^{\sum_{i=1}^d t_i A_{P,i}})_{\mathbf{t} \in \mathbb{R}^d_{\geq 0}}$$

over \mathcal{H} , which has $A_{P,1}, A_{P,2}, \ldots, A_{P,d}$ as its generators.

Definition 3.3 Call T_P the *P*-inversion associated to the semigroup *T*.

Observe that for $K \subseteq \{1, 2, \dots, d\}$

$$\begin{aligned} & A^{-}(K \cap P)^{*}S_{T_{P},K}A^{-}(K \cap P) \\ &= 2^{-|K|}\sum_{(C_{1},C_{2})\in\operatorname{Part}(K)}A^{-}(K \cap P)^{*}\Big(\prod_{i\in C_{1}}A_{P,i}^{-}\Big)^{*}\Big(\prod_{i\in C_{2}}A_{P,i}^{-}\Big)A^{-}(K \cap P) \\ &= 2^{-|K|}\sum_{(C_{1},C_{2})\in\operatorname{Part}(K)}\Big(\prod_{i\in C_{1}\setminus P}A_{i}^{-}\prod_{i\in C_{1}\cap P}(A_{i}^{-})^{-1}\prod_{i\in K\cap P}A_{i}^{-}\Big)^{*} \\ &\quad \cdot\Big(\prod_{i\in C_{2}\setminus P}A_{i}^{-}\prod_{i\in C_{2}\cap P}(A_{i}^{-})^{-1}\prod_{i\in K\cap P}A_{i}^{-}\Big) \\ &= 2^{-|K|}\sum_{(C_{1},C_{2})\in\operatorname{Part}(K)}\Big(\prod_{i\in C_{1}\setminus P}A_{i}^{-}\prod_{i\in (K\setminus C_{1})\cap P}A_{i}^{-}\Big)^{*} \cdot\Big(\prod_{i\in C_{2}\setminus P}A_{i}^{-}\prod_{i\in (K\setminus C_{2})\cap P}A_{i}^{-}\Big) \end{aligned}$$

$$= 2^{-|K|} \sum_{\substack{(C_1,C_2)\in \operatorname{Part}(K)\\ (C_1,C_2)\in \operatorname{Part}(K)}} \left(\prod_{i\in C_1\setminus P} A_i^- \prod_{i\in C_2\cap P} A_i^-\right)^* \cdot \left(\prod_{i\in C_2\setminus P} A_i^- \prod_{i\in C_1\cap P} A_i^-\right)$$
$$= 2^{-|K|} \sum_{\substack{(C_1,C_2)\in \operatorname{Part}(K)\\ (\tilde{C}_1,\tilde{C}_2)\in \operatorname{Part}(K)}} A^-((C_1)^*A^-(\tilde{C}_2) \stackrel{\text{Defn}}{=} S_{T,K},$$

where equality between the final lines hold, since $(C_1, C_2) \mapsto ((C_1 \setminus P) \cup (C_2 \cap P), (C_2 \setminus P) \cup (C_1 \cap P))$ clearly constitutes a bijection on the set of partitions of K.

Due to the above relation and the invertibility of the generators, if $\beta_{T_P} \ge 0$, it follows that

for all $\xi \in \mathcal{H}$ and $K \subseteq \{1, 2, \dots, d\}$. Hence we have shown

$$\beta_{T_P} \ge 0 \Longrightarrow \beta_T \ge \frac{\beta_{T_P}}{\max_{K \subseteq \{1, 2, \dots, d\}} \|A^- (K \cap P)^{-1}\|^2} = \frac{\beta_{T_P}}{\max_{K \subseteq P} \|A^- (K)^{-1}\|^2} \ge 0$$
(3.4)

and, since $(T_P)_P = T$ clearly holds, it follows analogously that

$$\beta_T \ge 0 \Longrightarrow \beta_{T_P} \ge \frac{\beta_{(T_P)_P}}{\max_{K \subseteq P} \|A_P^-(K)^{-1}\|^2} = \frac{\beta_T}{\max_{K \subseteq P} \|A^-(K)\|^2} \ge 0.$$
(3.5)

The implications and inequalities in (3.4) and (3.5) yield $\beta_T \ge 0$ if and only if $\beta_{T_P} \ge 0$, as well as $\beta_T > 0$ if and only if $\beta_{T_P} > 0$. Putting this together yields

Proposition 3.4 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . Further let $P \subseteq \{1, 2, \ldots, d\}$ and assume that A_i is invertible for $i \in P$. Then T is completely dissipative (resp. completely super dissipative) if and only if T_P is.

4. Main results

Our goal is to relate regular unitary dilatability to the notions of complete (super) dissipativity. To achieve this, we first establish relations between the Brehmer and dissipation operators.

Proposition 4.1 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . Then for all $t \in \mathbb{R}_{\geq 0}$ and $K \subseteq \{1, 2, \ldots, d\}$

$$B_{T,K}(t) = 2^{|K|} t^{|K|} S_{T,K} + (-1)^{|K|} t^{|K|+1} \Delta_{T,K}(t), \qquad (4.1)$$

where $\Delta_{T,K}$ is an operator-valued function which is norm bounded on compact subsets of $\mathbb{R}_{\geq 0}$.

Proof. Let $K \subseteq \{1, 2, ..., d\}$ and $t \in \mathbb{R}_{\geq 0}$ be arbitrary. Relying on the boundedness of the generators we can expand the expression defining the Brehmer operator $B_{T,K}(t)$ in terms of an absolute convergent power series:

$$B_{T,K}(t) = \sum_{C \subseteq K} (-1)^{|C|} T(t\mathbf{e}_{C})^{*} T(t\mathbf{e}_{C})$$

$$= \sum_{C \subseteq K} (-1)^{|C|} \left(e^{t \sum_{i \in C} A_{i}} \right)^{*} e^{t \sum_{i \in C} A_{i}}$$

$$= \sum_{k,l \in \mathbb{N}_{0}} \frac{t^{k+l}}{k!l!} \sum_{C \subseteq K} (-1)^{|C|} \left(\sum_{i \in C} A_{i}^{*} \right)^{k} \left(\sum_{i \in C} A_{i} \right)^{l}$$

$$= \sum_{k,l \in \mathbb{N}_{0}} \frac{t^{k+l}}{k!l!} \sum_{C \subseteq K} \sum_{\substack{(\pi_{1}, \pi_{2}) \in K^{k} \times K^{l}:\\ \operatorname{ran}(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) \subseteq C}} (-1)^{|C|} \cdot \underbrace{\left(\prod_{i=1}^{k} A_{\pi_{1}(i)} \right)^{*} \prod_{j=1}^{l} A_{\pi_{2}(j)}}_{=A(\pi_{1})^{*}A(\pi_{2})}$$

$$= \sum_{k,l\in\mathbb{N}_{0}} \frac{t^{k+l}}{k!l!} \sum_{(\pi_{1},\pi_{2})\in K^{k}\times K^{l}} \underbrace{\sum_{\substack{C\subseteq K:\\ C\supseteq\operatorname{ran}(\pi_{1})\cup\operatorname{ran}(\pi_{2})}}_{(*)} (-1)^{|C|} A(\pi_{1})^{*} A(\pi_{2}).$$

We can simplify (*) using the binomial expansion. Observe for any $P \subseteq K$ that $\sum_{C \subseteq K: C \supseteq P} (-1)^{|C|} = \sum_{C' \subseteq K \setminus P} (-1)^{|C' \cup P|} = (-1)^{|P|} \sum_{C' \subseteq K \setminus P} (-1)^{|C'|}$, which equals $(-1)^{|P|} \cdot (1 + (-1))^{|K \setminus P|} = 0$ if $P \neq K$ and otherwise $(-1)^{|K|}$ if P = K. Thus the sum (*) vanishes except when $\operatorname{ran}(\pi_1) \cup \operatorname{ran}(\pi_2) = K$, in which case it equals $(-1)^{|K|}$. Applying this to the above computation thus yields:

$$B_{T,K}(t) = (-1)^{|K|} \sum_{\substack{k,l \in \mathbb{N}_{0} \\ \operatorname{ran}(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) = K}} \sum_{\substack{t,l \in \mathbb{N}_{0}: \\ k+l \in \mathbb{N}_{0}: \\ nn(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) = K \\ = K + (-1)^{|K|} t^{|K|+1} \sum_{\substack{k,l \in \mathbb{N}_{0}: \\ k+l = n \\ ran(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) \in K^{k} \times K^{l}: \\ k+l = n \\ ran(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) = K \\ k+l = n \\ nn(\pi_{1}) \cup \operatorname{ran}(\pi_{2}) = K \\ =: \Delta_{T,K}} \underbrace{\frac{t^{n-|K|-1}}{k!l!} A(\pi_{1})^{*} A(\pi_{2})}_{=:\Delta_{T,K}}$$

To complete the proof, it remains to simplify the first sum in the last expression, and to show that $\Delta_{T,K}$ is norm-bounded on compact subsets of $\mathbb{R}_{\geq 0}$.

Observe that for $k, l \in \mathbb{N}_0$ with k + l = |K| a pair of sequences $(\pi_1, \pi_2) \in K^k \times K^l$ satisfy $\operatorname{ran}(\pi_1) \cup \operatorname{ran}(\pi_2) = K$ if and only if π_1, π_2 are injective functions mapping to the respective parts of a partition (C_1, C_2) of K with $|C_1| = k$ and $|C_2| = l$. In this case, by commutativity of the generators, one has that $A^-(\pi_1) = A^-(C_1)$ and $A^-(\pi_2) = A^-(C_2)$. Conversely, for any partition (C_1, C_2) of K setting $k = |C_1|$ and $l = |C_2|$ there exist k! sequences $\pi_1 \in K^k$ with $\operatorname{ran}(\pi_1) = C_1$ and l! sequences $\pi_2 \in K^l$ with $\operatorname{ran}(\pi_2) = C_2$. Hence $B^0_{T,K}$ simplifies to

$$B_{T,K}^{0} = \sum_{\substack{k,l \in \mathbb{N}_{0}:\\ k+l = |K|}} \frac{1}{k!l!} \sum_{\substack{(C_{1},C_{2}) \in \operatorname{Part}(K):\\ |C_{1}| = k, |C_{2}| = l}} k!l! \cdot A^{-}(C_{1})^{*}A^{-}(C_{2})$$
$$= \sum_{\substack{(C_{1},C_{2}) \in \operatorname{Part}(K)\\ = 2^{|K|}S_{T,K},}} A^{-}(C_{1})^{*}A^{-}(C_{2})$$

and thus when inserted into the above computation, one sees that (4.1) holds.

For the second term, using the boundedness of the generators, one may fix $M := \max_{i \in \{1,2,\dots,d\}} ||A_i|| \in \mathbb{R}_{\geq 0}$ and obtain the following norm bound on $\Delta_{T,K}(t)$:

$$\begin{aligned} \|\Delta_{T,K}(t)\| &\leq \sum_{n=|K|+1}^{\infty} \frac{t^{n-|K|-1}}{n!} \sum_{k=0}^{n} \sum_{(\pi,\pi')\in K^{k}\times K^{n-k}} \frac{n!}{k!(n-k)!} M^{n} \\ &= \sum_{n=|K|+1}^{\infty} \frac{t^{n-|K|-1}}{n!} \sum_{k=0}^{n} \binom{n}{k} |K|^{k} |K|^{n-k} M^{n} \end{aligned}$$

$$= \sum_{n=|K|+1}^{\infty} \frac{t^{n-|K|-1}}{n!} 2^n |K|^n M^n,$$

whereby the expression on the right constitutes a power series with infinite converge radius. It follows that $\|\Delta_{T,K}(\cdot)\|$ is bounded on compact subsets of $\mathbb{R}_{\geq 0}$.

Lemma 4.2 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . If the generators of T are completely super dissipative, then T has a regular unitary dilation.

Proof. By Theorem 2.4 it is necessary and sufficient to show that T satisfies the Brehmer positivity criterion. By Proposition 4.1 one has the connection between the Brehmer and dissipation operators: $B_{T,K}(t) = 2^{|K|}t^{|K|}S_{T,K} + (-1)^{|K|}t^{|K|+1}\Delta_{T,K}(t)$ for each $K \subseteq \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_{\geq 0}$, where $\Delta_{T,K}$ is norm-bounded on compact subsets of $\mathbb{R}_{\geq 0}$. Set $C := \sup\{1, \|\Delta_{T,K}(t)\| \mid t \in [0, 1], K \subseteq \{1, 2, \ldots, d\}\}$, which is finite and positive.

Since the generators of T are completely super dissipative, then $\beta_T > 0$. Set $t^* := \min\{1, \frac{\beta_T}{2C}\} \in \mathbb{R}_{>0}$. Applying the spectral theory of self-adjoint operators yields $\langle S_{T,K}\xi, \xi \rangle \ge \min(\sigma(S_{T,K})) \|\xi\|^2 \ge \beta_T \|\xi\|^2$ for all $\xi \in \mathcal{H}$. For $t \in (0, t^*)$ one thus obtains

for all $\xi \in \mathcal{H}$ and all $K \subseteq \{1, 2, \dots, d\}$. So $B_{T,K}(t) \ge \mathbf{0}$ for all $t \in (0, t^*)$ and all $K \subseteq \{1, 2, \dots, d\}$. Hence T satisfies the Brehmer positivity criterion.

Complete super dissipativity shall prove to be a strictly stronger condition than the existence of regular unitary dilations (see §5.2). The correct condition turns out to be complete dissipativity. To prove this, we shall make use of approximations via semigroups with completely super dissipative generators. And to enable this, we use the following lemma to establish sufficient conditions for shifts to guarantee this property.

Lemma 4.3 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . If $\beta_T < 0$, then the ω -shift $E_{\omega} \cdot T$ of T is completely super dissipative for all $\omega \in \mathbb{R}^d_{>0}$, provided

$$\prod_{i=1}^{d} (1 + \omega_i^{-1}) < 1 + \frac{1}{|\beta_T|}$$
(4.2)

holds. And if $\beta_T \ge 0$, then $E_{\omega} \cdot T$ is completely super dissipative for all $\omega \in \mathbb{R}^d_{>0}$.

Proof. Let $\omega \in \mathbb{R}^d_{>0}$ be arbitrary. Note that $\omega_K = \prod_{i \in K} \omega_i > 0$ and $S_{T,K} \ge \beta_T \mathbf{I} \ge -\beta_T^- \mathbf{I}$ for all $K \subseteq \{1, 2, \ldots, d\}$. Applying (3.3) from Proposition 3.2 to T yields

$$S_{E_{\boldsymbol{\omega}}\cdot T,K} = \sum_{K'\subseteq K} \omega_{K\setminus K'}^{-1} S_{T,K'}$$

$$= \omega_{K} \mathbf{I} + \omega_{K} \sum_{\substack{K'\subseteq K:\\K'\neq\emptyset}} \omega_{K'}^{-1} S_{T,K'}$$

$$\geqslant \omega_{K} \mathbf{I} - \omega_{K} \sum_{\substack{K'\subseteq K:\\K'\neq\emptyset}} \beta_{T}^{-} \omega_{K'}^{-1} \cdot \mathbf{I}$$

$$= \omega_{K} \cdot \left(1 - \beta_{T}^{-} \cdot \left(\sum_{\substack{K'\subseteq K\\K'\subseteq K}} \omega_{K'}^{-1} - 1\right)\right) \mathbf{I}$$

$$= \omega_{K} \cdot \left(1 - \beta_{T}^{-} \cdot \left(\prod_{i\in K} (1 + \omega_{i}^{-1}) - 1\right)\right) \mathbf{I}$$

$$\geq \omega_{K} \cdot \underbrace{\left(1 - \beta_{T}^{-} \cdot \left(\prod_{i=1}^{d} (1 + \omega_{i}^{-1}) - 1\right)\right)}_{=:\gamma} \mathbf{I}$$

for each $K \subseteq \{1, 2, \ldots, d\}$.

Now if $\beta_T \ge 0$, then $\beta_T^- = 0$ and hence $\gamma = 1 - 0 > 0$. Otherwise $\beta_T < 0$ and one has $\beta_T^- = |\beta_T| > 0$. In this case, since (4.2) is assumed, one has $\gamma > 1 - 1 = 0$. Since $S_{E_{\omega} \cdot T,K} \ge \omega_K \gamma \mathbf{I}$ for $K \subseteq \{1, 2, \dots, d\}$, one thus obtains $\beta_{E_{\omega} \cdot T} \ge \min_{K \subseteq \{1, 2, \dots, d\}} \omega_K \gamma > 0$, whence $E_{\omega} \cdot T$ is completely super dissipative.

4.1 Classification of regular unitary dilatability. We can now prove Theorem 1.1.

Proof (of Theorem 1.1). As per usual, we let T_1, T_2, \ldots, T_d denote the marginals of T and A_1, A_2, \ldots, A_d their respective generators.

(1) \Rightarrow (2): By Proposition 4.1 one has $B_{T,K}(t) = 2^{|K|}t^{|K|}S_{T,K} + (-1)^{|K|}t^{|K|+1}\Delta_{T,K}(t)$ where $\Delta_{T,K}$ is norm-bounded on compact subsets of $\mathbb{R}_{\geq 0}$ for each $K \subseteq \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_{\geq 0}$. Thus $C := \sup\{1, \|\Delta_{T,K}(t)\| \mid t \in [0, 1], K \subseteq \{1, 2, \ldots, d\}\}$ is a finite, positive real.

We shall prove the implication by contraposition. So, suppose that $\beta_T < 0$. By definition there must exist some $\tilde{K} \subseteq \{1, 2, \ldots, d\}$ such that $\min(\sigma(S_{T,\tilde{K}})) = \beta_T < 0$. By the correspondence between the numerical range of self-adjoint operators and their spectra (cf. [12, Theorem 1.2.1-4]), it holds that $\beta_T = \inf\{\langle S_{T,\tilde{K}}\xi, \xi\rangle \mid \xi \in \mathcal{H}, \|\xi\| = 1\}$. Set $t^* := \min\{1, -\frac{2^{|\tilde{K}|}\beta_T}{2C}\} \in \mathbb{R}_{>0}$. For $t \in (0, t^*)$ one computes

$$\begin{split} \langle B_{T,\tilde{K}}(t)\xi, \ \xi \rangle &= t^{|K|} \langle 2^{|K|} S_{T,\tilde{K}}\xi, \ \xi \rangle + (-1)^{|K|} t^{|K|+1} \langle \Delta_{T,\tilde{K}}(t)\xi, \ \xi \rangle \\ &\leqslant t^{|\tilde{K}|} 2^{|\tilde{K}|} \langle S_{T,\tilde{K}}\xi, \ \xi \rangle + t^{|\tilde{K}|+1} \| \Delta_{T,\tilde{K}}(t) \| \| \xi \|^2 \\ &\leqslant t^{|\tilde{K}|} \left(2^{|\tilde{K}|} \langle S_{T,\tilde{K}}\xi, \ \xi \rangle + Ct \cdot 1^2 \right) \quad (\text{since } t \in (0, \ t^*) \subseteq [0, \ 1]) \\ &\leqslant t^{|\tilde{K}|} \left(2^{|\tilde{K}|} \langle S_{T,\tilde{K}}\xi, \ \xi \rangle - \frac{2^{|\tilde{K}|}\beta_T}{2} \right) \quad (\text{since } t < t^* \leqslant -\frac{2^{\tilde{K}}\beta_T}{2C}) \\ &= (2t)^{|\tilde{K}|} (\langle S_{T,\tilde{K}}\xi, \ \xi \rangle - \frac{\beta_T}{2}) \end{split}$$

for all $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Taking infima over such vectors ξ thus yields

$$\inf_{\xi} \langle B_{T,\tilde{K}}(t)\xi, \xi \rangle \leqslant (2t)^{|\tilde{K}|} (\inf_{\xi} \langle S_{T,\tilde{K}}\xi, \xi \rangle - \frac{\beta_T}{2}) = (2t)^{|\tilde{K}|} (\beta_T - \frac{\beta_T}{2}) < 0,$$

whence there must exist some $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ such that $\langle B_{T,\tilde{K}}(t)\xi, \xi\rangle < 0$. In particular, $B_{T,\tilde{K}}(t)$ is not a positive operator. Since this is the case for all $t \in (0, t^*)$, the Brehmer positivity criterion fails and by Theorem 2.4 T does not have a regular unitary dilation.

(2) \Rightarrow (3): Directly order $\mathbb{R}_{>0}^d$ via $\omega' \geq \omega :\Leftrightarrow \forall i \in \{1, 2, \dots, d\} : \omega'_i \leq \omega_i$ for each $\omega, \omega' \in \mathbb{R}_{>0}^d$, and consider the net $(T^{(\omega)} := E_{\omega} \cdot T)_{\omega \in \mathbb{R}_{>0}^d}$. Since T has completely dissipative generators, by Lemma 4.3, this net consists of d-parameter C_0 -semigroups with completely super dissipative generators, and further by Lemma 4.2, these semigroups have regular unitary dilations.

We now demonstrate uniform norm-convergence on compact subsets of $\mathbb{R}^d_{\geq 0}$. To this end fix an arbitrary compact subset $L \subseteq \mathbb{R}^d_{\geq 0}$. By compactness, $L \subseteq \prod_{i=1}^d [0, C]$ for some $C \in \mathbb{R}_{\geq 0}$. Since SOT-continuous 1-parameter C_0 -semigroups are norm bounded on compact subsets, there exists some $M \in \mathbb{R}_{\geq 0}$ such that $||T_i(t)|| \leq M$ for all $t \in [0, C]$ and all $i \in \{1, 2, \ldots, d\}$. Putting this together yields

$$\sup_{\mathbf{t}\in L} \|T^{(\boldsymbol{\omega})}(\mathbf{t}) - T(\mathbf{t})\| = \sup_{\mathbf{t}\in L} |e^{-\langle \mathbf{t}, \boldsymbol{\omega} \rangle} - 1| \|T(\mathbf{t})\| \leq M^d \cdot (1 - e^{-C\sum_{i=1}^d \omega_i})$$

for each $\boldsymbol{\omega} \in \mathbb{R}^d_{\geq 0}$. The right- and thus the left-hand side of the above clearly converge to 0 as $\boldsymbol{\omega} \longrightarrow \mathbf{0}$. So $T^{(\boldsymbol{\omega})} \xrightarrow{} T$ uniformly in norm on compact subsets of $\mathbb{R}^d_{\geq 0}$. Hence (3) holds.

 $(3) \Rightarrow (4)$: This trivially holds.

(4) \Rightarrow (1): By Theorem 2.4, it is necessary and sufficient to show that $B_{T,K}(t) \ge \mathbf{0}$ for all $K \subseteq \{1, 2, \ldots, d\}$ and all $t \in \mathbb{R}_{>0}$. So fix arbitrary $K \subseteq \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_{>0}$. Since each of the $T^{(\alpha)}$ are regularly unitarily dilatable, by Theorem 2.4, one has that $B_{T^{(\alpha)},K}(t) \ge \mathbf{0}$ for each $\alpha \in \mathcal{I}$. Now, for $\xi \in \mathcal{H}$ one has

$$\begin{aligned} |\langle (B_{T^{(\alpha)},K}(t) - B_{T,K}(t))\xi, \xi\rangle| &= \left| \sum_{\substack{C \subseteq K \\ C \subseteq K}} (-1)^{|C|} (\|T^{(\alpha)}(t\mathbf{e}_{C})\xi\|^{2} - \|T(t\mathbf{e}_{C})\xi\|^{2}) \right| \\ &\leqslant \sum_{\substack{C \subseteq K \\ C \subseteq K}} |\|T^{(\alpha)}(t\mathbf{e}_{C})\xi\|^{2} - \|T(t\mathbf{e}_{C})\xi\|^{2}|, \end{aligned}$$

which clearly converges to 0, since by assumption $(T^{(\alpha)})_{\alpha} \longrightarrow T$ uniformly in the SOT-topology on compact subsets of $\mathbb{R}^d_{\geq 0}$ (e.g. $\{t\mathbf{e}_C\}$). Since $B_{T^{(\alpha)},K}(t) \geq \mathbf{0}$ for all $\alpha \in \mathcal{I}$, the above computation implies $\langle B_{T,K}(t)\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$, *i.e.* $B_{T,K}(t) \geq \mathbf{0}$.

Having established Theorem 1.1, we may now prove Corollary 1.2.

Proof (of Corollary 1.2). By Lemma 4.3 we can find $\boldsymbol{\omega} \in \mathbb{R}^d_{>0}$ such that the $\boldsymbol{\omega}$ -shift $E_{\boldsymbol{\omega}} \cdot T$ of T has completely (super!) dissipative generators. By Theorem 1.1 it follows that $\tilde{T} = E_{\boldsymbol{\omega}} \cdot T$ has a regular unitary dilation.

Remark 4.4 Observe that, if we let (4') denote the same statement as (4) in Theorem 1.1, except with *uniform* SOT-convergence on compact subsets of $\mathbb{R}^d_{\geq 0}$ replaced by the weaker condition of pointwise SOT-convergence, then the above proof makes clear that (4) \Rightarrow (4') \Rightarrow (1). So the equivalences in the theorem also hold with (4) replaced by the weaker statement (4'). Observe further that the statement and proof of (4) \Rightarrow (1) do not require T to have bounded generators. It would be interesting to know whether the concept of complete dissipativity can be well-defined without the assumption of bounded generators, and whether (1) \Rightarrow (2) and (2) \Rightarrow (4) in the theorem remain true.

4.2 Infinitely many commuting C_0 -semigroups. We now extend the notion of complete dissipativity and the classification result to arbitrarily many commuting C_0 -semigroups.

Let I be an arbitrary non-empty set and $(T_i)_{i \in I}$ be commuting C_0 -semigroups over a Hilbert space \mathcal{H} , with bounded generators $(A_i)_{i \in I} \subseteq \mathfrak{L}(\mathcal{H})$. For finite subsets $K \subseteq I$ and $t \in \mathbb{R}_{\geq 0}$ we define

$$B_{(T_i)_{i\in I},K}(t) := \sum_{\substack{C\subseteq K \\ C_1,K}} (-1)^{|C|} \Big(\prod_{i\in C} T_i(t)\Big)^* \Big(\prod_{i\in C} T_i(t)\Big), \text{ and}$$
$$S_{(T_i)_{i\in I},K} := 2^{-|K|} \sum_{\substack{(C_1,C_2)\in \operatorname{Part}(K) \\ (C_1,C_2)\in \operatorname{Part}(K)}} \Big(\prod_{i\in C_1} A_i^-\Big)^* \Big(\prod_{i\in C_2} A_i^-\Big),$$

which clearly coincide with the *Brehmer* and *dissipation operators* in Definition 2.3 and Definition 2.8 respectively in the finite case of $I = \{1, 2, ..., d\}$ for some $d \in \mathbb{N}$. We shall say that the generators of $(T_i)_{i \in I}$ are *completely dissipative* if $S_{(T_i)_{i \in I}, K} \ge \mathbf{0}$ for all finite subsets $K \subseteq I$. Again, this coincides with Definition 2.8 in the finite case.

We say that $(T_i)_{i\in I}$ has a simultaneous regular unitary dilation, if there exist commuting unitary C_0 -semigroups $(U_i)_{i\in I}$ over some Hilbert space \mathcal{H}' and $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$ (necessarily isometric), such that

$$\left(\prod_{i\in\operatorname{supp}(\mathbf{t}^{-})}T_i(t_i^{-})\right)^*\left(\prod_{i\in\operatorname{supp}(\mathbf{t}^{+})}T_i(t_i^{+})\right) = r^*\left(\prod_{i\in K}U_i(t_i)\right)r$$

for all $\mathbf{t} \in \mathbb{R}^K$ and finite $K \subseteq \{1, 2, \dots, d\}$, where we extend the unitary semigroups to representations of $(\mathbb{R}, +, 0)$ in the usual way. This clearly coincides with the definition of a regular unitary dilation in the finite case (*cf.* Definition 2.2 and the introduction).

Using these definitions we obtain the following classification as a simple consequence of the classification result in the finite case:

Corollary 4.5 (General classification of regular unitary dilatability). Let I be an arbitrary non-empty set and $(T_i)_{i \in I}$ be C_0 -semigroups over a Hilbert space \mathcal{H} with bounded generators $(A_i)_{i \in I} \subseteq \mathfrak{L}(\mathcal{H})$. Then $(T_i)_{i \in I}$ has a regular unitary dilation if and only if the generators of $(T_i)_{i \in I}$ are completely dissipative.

This proof relies on the fact that Theorem 2.4 is formulated for arbitrarily large families of commuting C_0 -semigroups in [24].

Proof (of Corollary 4.5). For the 'only if'-direction, suppose that $(T_i)_{i\in I}$ has regular unitary dilation. Then using this dilation, clearly every finite subset $K \subseteq I$ has a regular unitary dilation, and thus by the classification theorem (Theorem 1.1), $(T_i)_{i\in K}$ has completely dissipative generators. In particular, $S_{(T_i)_{i\in I},K} = S_{(T_i)_{i\in K},K} \ge \mathbf{0}$. Since this holds for all finite $K \subseteq I$, by definition, the generators of $(T_i)_{i\in I}$ are completely dissipative.

Towards the 'if'-direction, suppose that the generators of $(T_i)_{i\in I}$ are completely dissipative. Then clearly for every finite subset $K \subseteq I$ the generators of $(T_i)_{i\in K}$ are completely dissipative, and thus by the classification theorem (Theorem 1.1) $(T_i)_{i\in K}$ has a regular unitary dilation. By Theorem 2.4 it follows that the Brehmer operators associated to $(T_i)_{i\in K}$ are positive for all $t \in \mathbb{R}_{>0}$. Thus $B_{(T_i)_{i\in I},K}(t) = B_{(T_i)_{i\in K},K}(t) \ge \mathbf{0}$ for all $t \in \mathbb{R}_{>0}$. So $B_{(T_i)_{i\in I},K}(t) \ge \mathbf{0}$ for all $t \in \mathbb{R}_{>0}$ and for all finite $K \subseteq I$. *I.e.*, the commuting system $(T_i)_{i\in I}$ of C_0 -semigroups satisfies the Brehmer positivity criterion. By [24, Theorem 3.2] it follows that $(T_i)_{i\in I}$ has a regular unitary dilation.

4.3 Sufficient norm conditions. As it is not always so easy to check the spectral values of (the dissipation) operators, we wish to provide simple norm-conditions in terms on the generators, which guarantee complete (super) dissipativity. We shall show that it suffices to control the relative norm-bounds of the generators. To this end we introduce the following definition.

Definition 4.6 Let $A \in \mathfrak{L}(\mathcal{H})$ and $r \in \mathbb{R}_{\geq 0}$. Say that A is r-circular (resp. strictly r-circular) if $\frac{\|A-\omega\mathbf{I}\|}{\omega} \leq r$ (resp. $\frac{\|A-\omega\mathbf{I}\|}{\omega} < r$) for some $\omega \in \mathbb{R}_{>0}$. If r = 1 we shall say (strictly) circular instead of (strictly) 1-circular.

These notions are visualised in Figure 1 in the case of normal operators.

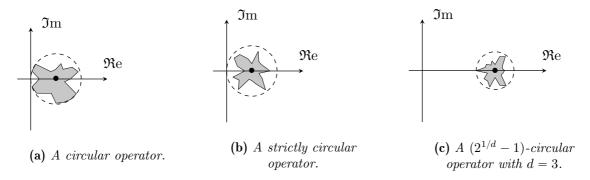


FIGURE 1. Visual examples of the spectra of normal operators under the definition of (strict) r-circularity, depicted as shaded regions of the complex plane contained in a disc around ω .

Proposition 4.7 If each of the dissipation operators associated to the generators of T are (strictly) circular, then the generators of T are completely (super) dissipative.

Proof. First observe that if a self-adjoint operator $S \in \mathfrak{L}(\mathcal{H})$ is (strictly) circular, then for some $\omega \in \mathbb{R}_{>0}$ one has $S \ge \omega \mathbf{I} - \|S - \omega \mathbf{I}\| \mathbf{I} = \delta \mathbf{I}$ where $\delta = \omega \cdot (1 - \frac{\|S - \omega \mathbf{I}\|}{\omega}) \ge 0$ (resp. $\delta > 0$). Thus since each $S_{T,K}$ is (strictly) circular, one has that $S_{T,K} \ge \delta_K \mathbf{I}$ for some $\delta_K \in \mathbb{R}_{\geq 0}$ (resp. $\delta_K \in \mathbb{R}_{>0}$) for each $K \subseteq \{1, 2, \ldots, d\}$. It follows that $\beta_T \ge \min\{\delta_K \mid K \subseteq \{1, 2, \ldots, d\}\} \ge 0$ (resp. $\beta_T > 0$) whence by definition the generators of T are completely (super) dissipative. \Box

Relying on the self-similarity of dissipation operators under shifts the following result provides sufficient means to guarantee (strict) circularity.

Proposition 4.8 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . Fix $\boldsymbol{\omega} \in (\mathbb{R} \setminus \{0\})^d$ and let $\tilde{T} := E_{-\boldsymbol{\omega}} \cdot T$ be the $(-\boldsymbol{\omega})$ -shift of T. Then

$$\frac{\|S_{T,K} - \omega_K \mathbf{I}\|}{|\omega_K|} \leqslant \sum_{\substack{K' \subseteq K:\\K' \neq \emptyset}} \frac{\|S_{\bar{T},K'}\|}{|\omega_{K'}|} \leqslant \prod_{i \in K} \left(1 + \frac{\|A_i + \omega_i \mathbf{I}\|}{|\omega_i|}\right) - 1$$
(4.3)

holds for all $K \subseteq \{1, 2, \ldots, d\}$.

Proof. We shall apply Proposition 3.2 to \tilde{T} and $\boldsymbol{\omega}$. Observe that $T = E_{\boldsymbol{\omega}} \cdot \tilde{T}$ and \tilde{T} has generators $\tilde{A}_i := A_i + \omega_i \mathbf{I}$ for each $i \in \{1, 2, \dots, d\}$. Let $K \subseteq \{1, 2, \dots, d\}$. Since $S_{\tilde{T}, \emptyset} = \mathbf{I}$, applying (3.3) to \tilde{T} and $\boldsymbol{\omega}$ yields

$$\omega_K^{-1} \cdot (S_{T,K} - \omega_K \mathbf{I}) = \omega_K^{-1} \cdot (S_{E_{\boldsymbol{\omega}} \cdot \tilde{T},K} - \omega_K \mathbf{I}) = \sum_{\substack{K' \subseteq K:\\K' \neq \emptyset}} \omega_{K'}^{-1} S_{\tilde{T},K'}$$

which implies the first inequality. Now, since $S_{\tilde{T},K'}$ is simply an average of products of the operators $\{(\tilde{A}_i^-)^*, \tilde{A}_i^- \mid i \in K'\}$ with each index from K' occurring exactly once in each product, one has $\|S_{\tilde{T},K'}\| \leq \prod_{i \in K'} \|\tilde{A}_i^-\| = \prod_{i \in K'} \|A_i + \omega_i \mathbf{I}\|$ for each $K' \subseteq K$, whence

$$\sum_{\substack{K' \subseteq K: \\ K' \neq \emptyset}} \frac{\|S_{\tilde{T},K'}\|}{|\omega_{K'}|} \leqslant \sum_{\substack{K' \subseteq K: \\ K' \neq \emptyset}} \prod_{i \in K'} \frac{\|A_i + \omega_i \mathbf{I}\|}{|\omega_i|} = \prod_{i \in K} \left(1 + \frac{\|A_i + \omega_i \mathbf{I}\|}{|\omega_i|}\right) - 1,$$

he second inequality.

which proves the second inequality.

Thus, if the generators are (strictly) r-circular, then the dissipation operators are at least (strictly) $((1+r)^d - 1)$ -circular. In this way, we may prove Theorem 1.3.

Proof (of Theorem 1.3). By Theorem 1.1 it suffices to show that the generators of T are completely dissipative. And by Proposition 4.7 it suffices in turn to show that the dissipation operators associated to T are circular. Using $\boldsymbol{\omega} \in \mathbb{R}^d_{>0}$ as in the assumption we now consider the ω -shift $E_{\omega} \cdot T$ of T. By Proposition 4.8, in order to obtain the circularity of T, it suffices to show that the right-hand expression in (4.3) is bounded by 1 for each $K \subseteq \{1, 2, \ldots, d\}$. Applying the inequality in the assumption yields $\prod_{i \in K} (1 + \frac{\|A_i + \omega_i \mathbf{I}\|}{\omega_i}) - 1 \leq \prod_{i=1}^d (1 + \frac{\|A_i + \omega_i \mathbf{I}\|}{\omega_i}) - 1 \leq (1 + 2^{1/d} - 1)^d - 1 = 1$ for all $K \subseteq \{1, 2, \ldots, d\}$. This completes the proof. \Box

Remark 4.9 Theorem 1.3 further enables us to obtain an alternative proof of Corollary 1.2. Let A_1, A_2, \ldots, A_d be the generators of T. Choose any $\boldsymbol{\omega} \in \mathbb{R}^d_{>0} \cap \prod_{i=1}^d \left[\frac{\|A_i\|}{2^{1/d}-1}, \infty\right)$ and set $\tilde{T} := E_{\boldsymbol{\omega}} \cdot T$. Then \tilde{T} is a *d*-parameter C_0 -semigroup with generators $\tilde{A}_i = A_i - \omega_i \cdot \mathbf{I}$ for all $i \in \{1, 2, \dots, d\}$. By construction one thus has $\frac{\|\tilde{A}_i + \omega_i \cdot \mathbf{I}\|}{\omega_i} = \frac{\|A_i\|}{\omega_i} \leq 2^{1/d} - 1$ for all $i \in \{1, 2, \dots, d\}$. Thus $\boldsymbol{\omega}$ witnesses that \tilde{T} satisfies the conditions in Theorem 1.3, which says that \tilde{T} possesses a regular unitary dilation.

Finally, observe that, since the spectral radius of a bounded operator over a Hilbert space is bounded by its norm, the norm requirement in Theorem 1.3 clearly implies that the generators of T have strictly negative spectral bounds, provided $d \ge 2$. However, as shall be shown in §5.3, this assumption does not suffice for the existence of regular unitary dilations.

4.4 **Regular exponents.** In Corollary 1.2 we showed that all multi-parameter C_0 -semigroups with bounded generators can be modified (shifted) to regularly unitarily dilatable semigroups. In this section we provide some basic facts about the set of such shifts.

Definition 4.10 Let T be a d-parameter C_0 -semigroup over \mathcal{H} . The regular exponents of T is the set $\operatorname{reg}(T) \subseteq \mathbb{R}^d$ of all $\omega \in \mathbb{R}^d$ for which the ω -shift $E_\omega \cdot T$ of T has a regular unitary dilation.

In other words, $\operatorname{reg}(T)$ contains all information about how to modify the dynamical system T so that it can be embedded via regular unitary dilations into reversible processes. Note that by Theorem 1.1 $\operatorname{reg}(T)$ can be equivalently formulated as the set of all $\omega \in \mathbb{R}^d$ such that the d generators of $E_{\omega} \cdot T$ are completely dissipative.

In order to study reg(T), we first show that $\mathbb{R}^d \ni \boldsymbol{\omega} \mapsto \beta_{E_{\boldsymbol{\omega}} \cdot T} \in \mathbb{R}$ is continuous.

Proposition 4.11 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators A_1, A_2, \ldots, A_d . Then

$$\prod_{i=1}^{d} \|A_i - \omega_i\| - \prod_{i=1}^{d} (|\omega_i| + \|A_i - \omega_i\|) \le \beta_{E_{\omega} \cdot T} - \beta_T \le \prod_{i=1}^{d} (|\omega_i| + \|A_i\|) - \prod_{i=1}^{d} \|A_i\|$$
(4.4)

for all $\boldsymbol{\omega} \in \mathbb{R}^d$. In particular, $\boldsymbol{\omega} \mapsto \beta_{E_{\boldsymbol{\omega}} \cdot T}$ is continuous.

Proof. Let $K \subseteq \{1, 2, ..., d\}$. By (3.3) in Proposition 3.2 one has $S_{E_{\omega} \cdot T, K} = S_{T,K} + \sum_{K' \subsetneq K} \omega_{K \setminus K'} S_{T,K'}$. Thus

$$S_{T,K} = S_{E_{\boldsymbol{\omega}}\cdot T,K} - \sum_{K' \subsetneq K} \omega_{K \setminus K'} S_{T,K'}$$

$$\geqslant \beta_{E_{\boldsymbol{\omega}}\cdot T} \mathbf{I} - \sum_{K' \subsetneq K} |\omega_{K \setminus K'}| ||S_{T,K'}|| \mathbf{I}$$

$$\stackrel{(*)}{\geqslant} \beta_{E_{\boldsymbol{\omega}}\cdot T} \mathbf{I} - \sum_{K' \subsetneq K} |\omega_{K \setminus K'}| \prod_{i \in K'} ||A_i|| \mathbf{I}$$

$$\geqslant \beta_{E_{\boldsymbol{\omega}}\cdot T} \mathbf{I} - \sum_{K' \subsetneq \{1,2,\dots,d\}} |\omega_{K \setminus K'}| \prod_{i \in K'} ||A_i|| \mathbf{I}$$

$$= \beta_{E_{\boldsymbol{\omega}}\cdot T} \mathbf{I} - \left(\prod_{i=1}^{d} (||A_i|| + |\omega_i|) - \prod_{i=1}^{d} ||A_i||\right) \mathbf{I}$$

where (*) holds, since each $S_{T,K'}$ is an average of products of $\{A_i^-, (A_i^-)^* \mid i \in K'\}$ with each index occurring exactly once in each product, and the final equality holds by the binomial expansion. Since this inequality holds for all $K \subseteq \{1, 2, \ldots, d\}$, it follows that

$$\beta_{E_{\omega} \cdot T} \leq \beta_T + \prod_{i=1}^d (\|A_i\| + |\omega_i|) - \prod_{i=1}^d \|A_i\|,$$

and hence the right-hand inequality in (4.4) holds.

Let $\tilde{T} := E_{\omega} \cdot T$. From the generality of the right-hand inequality in (4.4), observing that \tilde{T} has generators $\tilde{A}_i = A_i - \omega_i$ for each $i \in \{1, 2, \ldots, d\}$ and $E_{-\omega} \cdot \tilde{T} = T$, one readily obtains the left-hand inequality in (4.4).

From (4.4) it is clear that $|\beta_{E_{\omega}:T} - \beta_{T}| \longrightarrow 0$ for $\mathbb{R}^{d} \ni \omega \longrightarrow 0$. Letting $\omega \in \mathbb{R}^{d}$ be arbitrary and $\tilde{T} := E_{\omega} \cdot T$, if we consider $\mathbb{R}^{d} \ni \omega' \longrightarrow \omega$, then $\mathbb{R}^{d} \ni \omega' - \omega \longrightarrow 0$ and hence $\beta_{E_{\omega'}:T} = \beta_{E_{\omega'}:T} \longrightarrow \beta_{\tilde{T}} = \beta_{E_{\omega}:T}$. Thus the map is continuous in all points.

Proposition 4.12 Let T be a d-parameter C_0 -semigroup over \mathcal{H} with bounded generators. Then the following hold:

- (i) The set reg(T) is non-empty and closed.
- (ii) The inclusion $\operatorname{reg}(T) \supseteq \omega + \mathbb{R}^d_{\geq 0}$ holds for each $\omega \in \operatorname{reg}(T)$.
- (iii) For each $\boldsymbol{\omega} \in \mathbb{R}^d$ and $i \in \{1, 2, ..., d\}$ there exists $r \in \mathbb{R}$, such that $\boldsymbol{\omega} t\mathbf{e}_i \notin \operatorname{reg}(T)$ for all $t \in (r, \infty)$.
- (iv) The interior int(reg(T)) is non-empty and consists of all $\omega \in \mathbb{R}^d$ for which $E_{\omega} \cdot T$ has completely super dissipative generators.
- (v) The boundary $\partial \operatorname{reg}(T)$ consists of all shifts $\boldsymbol{\omega} \in \mathbb{R}^d$ for which $\beta_{\boldsymbol{\omega} \cdot T} = 0$.
- (vi) The set $\operatorname{reg}(T)$ is completely determined by the boundary via $\operatorname{reg}(T) = \partial \operatorname{reg}(T) + \mathbb{R}^d_{\geq 0}$.

Proof. (i): By Corollary 1.2 reg(T) is non-empty. Since reg(T) = { $\omega \in \mathbb{R}^d \mid \beta_{E_{\omega} \cdot T} \ge 0$ } and $\omega \mapsto \beta_{E_{\omega} \cdot T}$ is continuous, it follows that reg(T) is closed.

(ii): Fix an arbitrary $\boldsymbol{\omega} \in \operatorname{reg}(T)$. By closedness, it suffices to show that $\operatorname{reg}(T) \supseteq \boldsymbol{\omega} + \mathbb{R}^d_{>0}$. Since $\tilde{T} \coloneqq E_{\boldsymbol{\omega}} \cdot T$ has completely dissipative generators, by Lemma 4.3 $E_{\boldsymbol{\omega}'+\boldsymbol{\omega}} \cdot T = E_{\boldsymbol{\omega}'} \cdot \tilde{T}$ has completely (super) dissipative generators for $\boldsymbol{\omega}' \in \mathbb{R}^d_{>0}$. Thus $\operatorname{reg}(T) \supseteq \boldsymbol{\omega} + \mathbb{R}^d_{>0}$.

(iii): Let $i \in \{1, 2, ..., d\}$ and $\boldsymbol{\omega} \in \mathbb{R}^d$. Fix some $\lambda \in \sigma(\mathfrak{Re}(A_i - \omega_i \mathbf{I})) \subseteq \mathbb{R}$ and set $r := -\lambda$. For each $t \in (r, \infty)$ one has $\sigma(\mathfrak{Re}(A_i - (\omega_i - t)\mathbf{I})) = \sigma(\mathfrak{Re}(A_i - \omega_i)) + t \ni -r + t$ and hence $\sigma(\mathfrak{Re}A_i - (\omega_i - t)\mathbf{I}) \cap \mathbb{R}_{>0} \neq \emptyset$. It follows that $\mathfrak{Re}(A_i - (\omega_i - t)\mathbf{I})$ is non-dissipative and thus by the Lumer-Phillips form of the Hille-Yosida theorem (*cf.* Remark 2.7) the 1-parameter C_0 -semigroup, \tilde{T}_i , induced by the generator $A_i - (\omega_i - t)\mathbf{I}$ is not contractive. Thus $E_{\boldsymbol{\omega}-t\mathbf{e}_i} \cdot T$, which has \tilde{T}_i has its *i*th marginal, is not contractive and thus cannot have a regular unitary dilation. Hence $\boldsymbol{\omega} - t\mathbf{e}_i \notin \operatorname{reg}(T)$ for all $t \in (r, \infty)$.

(iv): The non-emptiness of the interior clearly follows the non-emptiness of reg(T). and (ii). By continuity $\{\omega \in \mathbb{R}^d \mid \beta_{E_\omega \cdot T} > 0\}$ is clearly open and contained in reg(T). Conversely, consider an arbitrary point ω in the interior of reg(T). Then $\prod_{i=1}^d (\omega_i - 2r, \omega_r + 2r) \subseteq \operatorname{reg}(T)$ for some $r \in \mathbb{R}_{>0}$. So $\omega' := (\omega_i - r)_{i=1}^d \in \operatorname{reg}(T)$. So the generators of $\tilde{T} := E_{\omega'} \cdot T$ are completely dissipative. Since $\omega'' := \omega - \omega' \in \mathbb{R}_{>0}^d$, by Lemma 4.3 $E_\omega \cdot T = E_{\omega''} \cdot \tilde{T}$ has completely super dissipative generators. Hence the interior $\operatorname{int}(\operatorname{reg}(T))$ coincides with $\{\omega \in \mathbb{R}^d \mid \beta_{E_\omega \cdot T} > 0\}$.

(v): By (i) and (iv) it follows that $\partial \operatorname{reg}(T) = \overline{\operatorname{reg}(T)} \setminus \operatorname{int}(\operatorname{reg}(T)) = \operatorname{reg}(T) \setminus \operatorname{int}(\operatorname{reg}(T)) = \{\omega \in \mathbb{R}^d \mid \beta_{E_{\omega} \cdot T} \ge 0 \text{ and } \beta_{E_{\omega} \cdot T} \ge 0\} = \{\omega \in \mathbb{R}^d \mid \beta_{E_{\omega} \cdot T} = 0\}.$

(vi): The \supseteq -inclusion follows from (i) and (ii). Towards the \subseteq -inclusion, let $\boldsymbol{\omega} \in \operatorname{reg}(T)$ be arbitrary. Repeated applications of (iii) yields a sufficiently large $t_0 \in \mathbb{R}_{>0}$, such that $\boldsymbol{\omega} - t_0 \mathbf{1} \notin \operatorname{reg}(T)$. Thus $\beta_{E_{\boldsymbol{\omega}} \cdot T} \ge 0$ and $\beta_{E_{\boldsymbol{\omega} - t_0} \mathbf{1} \cdot T} < 0$. Since the map $t \in \mathbb{R} \mapsto \beta_{E_{\boldsymbol{\omega} - t_1} \cdot T}$ is continuous, by the intermediate value theorem there exists some $t^* \in [0, t_0]$, such that $\beta_{E_{\boldsymbol{\omega} - t_1} \cdot T} = 0$. By (v), $\boldsymbol{\omega} - t^* \mathbf{1} \in \partial \operatorname{reg}(T)$, and thus $\boldsymbol{\omega} = (\boldsymbol{\omega} - t^* \mathbf{1}) + t^* \mathbf{1} \in \partial \operatorname{reg}(T) + \mathbb{R}_{\ge 0}^d$. \Box

5. Examples

Throughout this section we work with *d*-parameter C_0 -semigroups T over a Hilbert space \mathcal{H} whose marginals T_1, T_2, \ldots, T_d have bounded generators A_1, A_2, \ldots, A_d respectively.

This section has three goals. Working within well understood cases of d = 1 as well as when $d \ge 1$ and the marginals of T doubly commute, we first show that the equivalence $(1) \Leftrightarrow (2)$ in the classification theorem (Theorem 1.1) holds without appealing to it. Within the case of doubly commuting (resp. normal) marginal semigroups we demonstrate secondly that the notion of complete dissipativity (resp. complete super dissipativity) corresponds to dissipativity (resp. strictly negative spectral bounds). Finally, working outside these cases, we shall apply the classification theorem and demonstrate that the notions of complete (super) dissipativity are not equivalent to the mentioned properties, and in the process produce concrete examples of d-parameter contractive C_0 -semigroups which are not regularly unitarily dilatable.

5.1 **One-parameter contractive semigroups.** Consider the case d = 1. Applying the definitions yields $B_{T,\emptyset}(t) = S_{T,\emptyset} = \mathbf{I}$ and

$$\begin{array}{rcl} B_{T,\{1\}}(t) &=& \mathbf{I} - T_1(t)^* \mathbf{I} T_1(t) = \mathbf{I} - T_1(t)^* T_1(t) \text{ and} \\ S_{T,\{1\}} &=& -\frac{1}{2} \{ \Re e \, A_1, \, \mathbf{I} \} + \imath \frac{1}{2} [\Im m \, A_1, \, \mathbf{I}] = - \, \Re e \, A_1 \end{array}$$

for all $t \in \mathbb{R}_{\geq 0}$.

If T is contractive, then clearly $B_{T,\emptyset}(t) \ge \mathbf{0}$ and $B_{T,\{1\}}(t) \ge \mathbf{0}$ for all $t \in \mathbb{R}_{\ge 0}$, whence T satisfies the Brehmer positivity criterion and thus by Theorem 2.4 has a regular unitary dilation. Conversely, if T has a regular unitary dilation, it is necessarily contractive. Hence T has a regular unitary dilation if and only if it is contractive.

The above expressions also demonstrate that the generator of T is completely dissipative if and only if $\Re e A_1 \leq 0$, *i.e.* if and only if A_1 is dissipative, which, by the Lumer-Phillips form of the Hille-Yosida theorem (*cf.* Remark 2.7) in turn holds if and only if T is contractive. We also see that the generator of T is completely super dissipative if and only if $\Re e A_1 \leq -\beta \mathbf{I}$ for some $\beta > 0$, which holds if and only if T is super dissipative.

Hence for d = 1 the existence of a regular unitary dilation is equivalent to T being contractive, which in turn is equivalent to the generator of T being completely dissipative. And one also has that complete super dissipativity and super dissipativity coincide.

5.2 Doubly commuting generators. Recall that operators $(X_i)_i \subseteq \mathfrak{L}(\mathcal{H})$ are said to doubly commute if X_i commutes with X_j and X_j^* for all i, j with $i \neq j$. Suppose that the marginals T_1, T_2, \ldots, T_d are doubly commuting semigroups, *i.e.* $(T_1(t_1), T_2(t_2), \ldots, T_d(t_d))$ doubly commute for all $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d$. Then it is easy to see that the generators (A_1, A_2, \ldots, A_d) doubly commute (recall that we also assume here that the A_i are bounded). The converse also clearly holds. Observe further, that double commutativity trivially holds in the case of d = 1. Furthermore, by Fuglede's theorem (*cf.* [21, Proposition 4.4.12]), if the commuting marginal semigroups are normal (or equivalently: the commuting generators are normal), then they doubly commute (or equivalently: the generators doubly commute).

Under the assumption of doubly commuting marginals, since for each $K \subseteq \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_{\geq 0}$ the Brehmer operator $B_{T,K}(t)$ is an algebraic expression over $\{T_i(t), T_i(t)^* \mid i \in K\}$ and the dissipation operator $S_{T,K}$ is an algebraic expression over $\{A_i, A_i^* \mid i \in K\}$, one has that $(T_\alpha(t), B_{T,K}(t))$ doubly commute and similarly $(A_\alpha, S_{T,K})$ doubly commute for all $\alpha \in \{1, 2, \ldots, d\} \setminus K$. Applying the recursive expressions in (3.1) and (3.2), one thus obtains $B_{T,\emptyset}(t) = S_{T,\emptyset} = \mathbf{I}$ and

$$B_{T,K\cup\{\alpha\}}(t) = B_{T,K}(t) - T_{\alpha}(t)^{*}B_{T,K}(t)T_{\alpha}(t) = B_{T,K}(t) \cdot (\mathbf{I} - T_{\alpha}(t)^{*}T_{\alpha}(t)), \text{ and} S_{T,K\cup\{\alpha\}}(t) = -\frac{1}{2}(A_{\alpha}S_{T,K} + S_{T,K}A_{\alpha}^{*}) = S_{T,K} \cdot (-\frac{1}{2}(A_{\alpha} + A_{\alpha}^{*}))$$

for all $K \subseteq \{1, 2, \dots, d\}$, $\alpha \in \{1, 2, \dots, d\} \setminus K$, and $t \in \mathbb{R}_{\geq 0}$. By induction one thus obtains

$$B_{T,K}(t) = \prod_{i \in K} (\mathbf{I} - T_i(t)^* T_i(t)) \text{ and } S_{T,K} = \prod_{i \in K} (-\Re e A_i)$$
(5.1)

for all $K \subseteq \{1, 2, \dots, d\}$ and $t \in \mathbb{R}_{\geq 0}$. Using these expressions yields the following results.

Proposition 5.1 If the marginals of T doubly commute (e.g. if d = 1), then the following are equivalent:

- (1) T has a regular unitary dilation.
- (2) T is contractive.
- (3) The generators of T are completely dissipative.
- (4) The generators of T are dissipative.

Proof. The implication $(1) \Rightarrow (2)$ is clear. And by the discussion in §2.2, we already know that complete dissipativity clearly generalises dissipativity, *i.e.* $(3) \Rightarrow (4)$ holds.

(2) \Leftrightarrow (4): By the correspondence in the introduction, the *d*-parameter C_0 -semigroup T is contractive if and only if each of the marginals T_i is contractive, which, by the Lumer-Phillips form of the Hille-Yosida theorem (*cf.* Remark 2.7), in turn holds if and only if each of the generators A_i is dissipative.

(2) \Rightarrow (1): If T is contractive, or equivalently (by the correspondence mentioned in the introduction) each T_i is contractive, then $\mathbf{I} - T_i(t)^*T_i(t) \ge \mathbf{0}$ for all $i \in \{1, 2, ..., d\}$. Since by the double commutativity assumption $\{\mathbf{I} - T_i(t)^*T_i(t) \mid i \in \{1, 2, ..., d\}\}$ is a set of commuting positive operators, it follows that any product of these is positive.^f By (5.1) it follows that $B_{T,K}(t) \ge \mathbf{0}$ for all $t \in \mathbb{R}_{\ge 0}$ and all $K \subseteq \{1, 2, ..., d\}$. So T satisfies the Brehmer positivity criterion and by Theorem 2.4 it has a regular unitary dilation.

 $(4) \Rightarrow (3)$: If each of the A_i are dissipative, then $-\Re e A_i \ge \mathbf{0}$ for all $i \in \{1, 2, \ldots, d\}$. Since by the double commutativity assumption $\{-\Re e A_i \mid i \in \{1, 2, \ldots, d\}\}$ is a set of commuting positive operators, it follows that any product of these is positive.^f By (5.1) it follows that $S_{T,K} \ge \mathbf{0}$ for all $K \subseteq \{1, 2, \ldots, d\}$. So the generators of T are completely dissipative. \Box

Proposition 5.2 If the marginals of T are normal, then the following are equivalent:

- (1) The generators of T have strictly negative spectral bounds.
- (2) The generators of T are completely super dissipative.
- (3) The generators of T are super dissipative.

Note that the equivalence of (2) and (3) holds under the weaker assumption of double commutativity. In particular, if d = 1, (2) \Leftrightarrow (3) holds without any assumptions.

Proof. First note that normality of the commuting marginal semigroups implies that the marginals double commute (see above).

(1) \Leftrightarrow (3): Let $i \in \{1, 2, ..., d\}$. The generator, A_i , of T_i has a strictly negative spectral bound, *i.e.* $\sigma(A_i) \subseteq \{z \in \mathbb{C} \mid \Re e \ z \leq -\omega\}$ for some $\omega \in \mathbb{R}_{>0}$, if and only if (by the assumption of normality) $\Re e A_i \leq -\omega I$ for some $\omega \in \mathbb{R}_{>0}$, which in turn holds if and only if A_i is super dissipative (cf. §2.2).

 $(2) \Rightarrow (3)$: Complete super dissipativity clearly generalises super dissipativity (cf. §2.2).

(3) \Rightarrow (2): If the generators of T are super dissipative, then for some $\boldsymbol{\omega} \in \mathbb{R}_{>0}^d$ we have $-\Re e A_i \geq \omega_i \mathbf{I}$ for each $i \in \{1, 2, \ldots, d\}$. Since by double commutativity $\{-\Re e A_i \mid i \in \{1, 2, \ldots, d\}\}$ is a set of commuting strictly positive operators, any product of these is strictly positive.^f By (5.1) it follows that $S_{T,K} > 0$ for each $K \subseteq \{1, 2, \ldots, d\}$. (More precisely, $S_{T,K} \geq \omega_K \mathbf{I}$ for each $K \subseteq \{1, 2, \ldots, d\}$.) It follows that $\beta_T > 0$ and the generators of T are completely super dissipative.

Observe that $(1) \Leftrightarrow (3)$ in Proposition 5.1 agrees with $(1) \Leftrightarrow (2)$ in the classification theorem (Theorem 1.1). Moreover, Proposition 5.1 and Proposition 5.2 demonstrate under the assumption of double commutativity the equivalence of complete dissipativity of the generators of T and the dissipativity of the individual generators, as well as the equivalence of complete super dissipativity, super dissipativity, and (under the assumption of normality) strictly negative spectral bounds.

Further note that if T is a d-parameter unitary C_0 -semigroup, then its generators are purely imaginary, *i.e.* $\Re e A_i = \mathbf{0}$ for each $i \in \{1, 2, ..., d\}$. By (5.1) it follows that $S_{T,K} = \mathbf{0}$ for all non-empty $K \subseteq \{1, 2, ..., d\}$, whence $\beta_T = 0$. So the generators are completely dissipative but not completely super dissipative. Hence the latter property is strictly stronger than the former.

5.3 Non-doubly commuting generators. Suppose that T is contractive and has bounded generators, which furthermore have strictly negative spectral bounds. Then, if the marginals of T doubly commute, by Proposition 5.1 T is completely dissipative and thus by the classification theorem (Theorem 1.1) has a regular unitary dilation. We shall show that the first three assumptions do not together suffice to yield regular unitary dilatability. In order to achieve this, we clearly need to consider non-doubly commuting semigroups. A typical place to look is upper (or lower) triangular matrices.

Proposition 5.3 Let \mathcal{H} be a Hilbert space with dim $(\mathcal{H}) \ge 2$, and let $d \in \mathbb{N}$ with $d \ge 2$. Then there exists a *d*-parameter contractive C_0 -semigroup T over \mathcal{H} with bounded generators which have strictly negative spectral bounds, such that T has no regular unitary dilation.

^fLet \mathcal{A} be a unital C^* -algebra and $a_1, a_2, \ldots, a_n \in \mathcal{A}$ be commuting positive elements. Then the elements the C^* -subalgebra generated by these elements is commutative. In particular $\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}$ are commutative positive elements of \mathcal{A} , whence $\prod_{i=1}^n a_i = \prod_{i=1}^n (\sqrt{a_i^*}\sqrt{a_i}) = \prod_{i=1}^n \sqrt{a_i^*} \prod_{j=1}^n \sqrt{a_j} = (\prod_{i=1}^n \sqrt{a_i})^* \prod_{j=1}^n \sqrt{a_j}$, which is positive. Note that positive elements are strictly positive if and only if they are invertible (*cf.* [21, Proposition 3.2.12]). Since the product of invertible (resp. commuting positive) elements is invertible (resp. positive), it follows that the product of commuting strictly positive.

Proof. We first construct a space of operators over \mathcal{H} , from which we shall pick our generators. Since dim $(\mathcal{H}) \ge 2$ one can find orthonormal closed subspaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$ with $0 < \dim(\mathcal{H}_2) \le \dim(\mathcal{H}_1)$ and $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$. Consider the following collection of operators

$$\mathcal{G} = \bigcup_{c \in \mathbb{C}^{\times}} \underbrace{\{ c \cdot \mathbf{I} + \begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mid D \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_1) \}}_{=:\mathcal{G}_c}$$

and observe for $D, D_1, D_2 \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_1)$ and $E \in \mathfrak{L}(\mathcal{H}_2)$ the following algebraic relations:

$$\begin{pmatrix} \mathbf{0} \ D_1 \\ \mathbf{0} \ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} \ D_2 \\ \mathbf{0} \ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \ D_1 \\ \mathbf{0} \ \mathbf{0} \end{pmatrix}^* \begin{pmatrix} \mathbf{0} \ D_2 \\ \mathbf{0} \ \mathbf{0} \end{pmatrix}^* = \begin{pmatrix} \mathbf{0} \ \mathbf{0} \\ \mathbf{0} \ E \end{pmatrix} \begin{pmatrix} \mathbf{0} \ D \\ \mathbf{0} \ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \ D \\ \mathbf{0} \ \mathbf{0} \end{pmatrix}^* \begin{pmatrix} \mathbf{0} \ \mathbf{0} \\ \mathbf{0} \ E \end{pmatrix} = \mathbf{0}$$
(5.2)
which one can derive

from which one can derive

$$\begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}^* = \mathbf{I} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ D^* & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix} = \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_1 + D_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}^* \cdot \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix} = \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_2 \\ D_1^* & D_1^* D_2 \end{pmatrix}, \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I} + \begin{pmatrix} \mathbf{0} & D_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}^* = \mathbf{I} + \begin{pmatrix} D_2 D_1^* & D_2 \\ D_1^* & \mathbf{0} \end{pmatrix},$$
(5.3)

and thus

$$\left(\mathbf{I} + \begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) \cdot \left(\mathbf{I} + \begin{pmatrix} \mathbf{0} & -D \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) = \mathbf{I} + \begin{pmatrix} \mathbf{0} & D + (-D) \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \mathbf{I}.$$
(5.4)

By (5.3) and (5.4), \mathcal{G} forms a commutative group under multiplication and the elements in \mathcal{G} in general are not normal nor do they doubly commute.

Properties of the generators: The spectra of the elements are straightforward to compute: For $c \in \mathbb{C}^{\times}$ and $A := c \cdot \mathbf{I} + \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \in \mathcal{G}_c$ and $\lambda \in \mathbb{C} \setminus \{c\}$ one clearly has $\lambda \cdot \mathbf{I} - A \in \mathcal{G}_{\lambda-c} \subseteq \mathcal{G}$, and thus $\lambda \cdot \mathbf{I} - A$ is invertible. So $\sigma(A) \subseteq \{c\}$. Since the spectrum of bounded operators is non-empty, it follows that $\sigma(A) = \{c\}$ for all $A \in \mathcal{G}_c$ and all $c \in \mathbb{C}^{\times}$. In particular, $\sigma(A) = \{-1\} \subseteq \{z \in \mathbb{C} \mid \Re e z \leq -1\}$, for all $A \in \mathcal{G}_{-1}$, whence each operator in \mathcal{G}_{-1} generates a C_0 -semigroup whose generator has a strictly negative spectral bound.

For
$$A := -1 \cdot \mathbf{I} + \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \in \mathcal{G}_{-1}$$
 one has

$$\mathfrak{Re}\langle A(x \oplus y), \ x \oplus y \rangle = -\|x \oplus y\|^2 + \mathfrak{Re}\langle Dy, \ x \rangle$$

$$\leqslant -\|x\|^2 - \|y\|^2 + \|D\| \|x\| \|y\|$$

$$= -(\|x\| - \|y\|)^2 - (2 - \|D\|) \|x\| \|y\|$$

for $x \in \mathcal{H}_1$ und $y \in \mathcal{H}_2$. Thus A is dissipative, provided $||D|| \leq 2$.

This establishes a broad class of possibilities for generating our *d*-parameter C_0 -semigroups. Let $A_1, A_2, \ldots, A_d \in \mathcal{G}_{-1}$ with $A_i = -\mathbf{I} + \begin{pmatrix} 0 & -2D_i \\ 0 & 0 \end{pmatrix}$ for some contractive $D_i \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_1)$ and let *T* be the *d*-parameter C_0 -semigroup whose marginals have the commuting operators A_1, A_2, \ldots, A_d as generators. By the above, each A_i is bounded, dissipative, and has a strictly negative spectral bound. By dissipativity and the Lumer-Phillips form of the Hille-Yosida theorem (*cf.* Remark 2.7), the marginals of *T* and thus *T* itself are contractive. By applying the algebraic relations in (5.3) one can readily verify that *T* is explicitly given by:

$$T(\mathbf{t}) = e^{-\sum_{i=1}^{d} t_i} \cdot \left(\mathbf{I} + \sum_{i=1}^{d} t_i (\mathbf{I} + A_i) \right)$$
(5.5)

for each $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}^d_{\geq 0}$.

Complete dissipativity: To prove that T has no regular unitary dilation, by the classification theorem (Theorem 1.1) it suffices to show that the generators of T are not completely dissipative. To assist with the computation of the dissipation operators associated to the generators of T, define the following aggregates

$$\langle D\rangle_K:=\sum_{i\in K}D_i,\quad \langle |D|^2\rangle_K:=\sum_{i\in K}D_i^*D_i$$

for each $K \subseteq \{1, 2, \ldots, d\}$. We now claim that the dissipativity operators are given by

$$S_{T,K} = \left(\mathbf{I} + \begin{pmatrix} \mathbf{0} \langle D \rangle_{K} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right)^{*} \left(\mathbf{I} - \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} \langle |D|^{2} \rangle_{K} \\ \mathbf{0} & \langle D \rangle_{K} \end{pmatrix}}_{=:V_{K}}\right) \underbrace{\left(\mathbf{I} + \begin{pmatrix} \mathbf{0} \langle D \rangle_{K} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right)}_{=:C_{K}}$$

$$= C_{K}^{*}C_{K} - C_{K}^{*}V_{K}C_{K}$$

$$= C_{K}^{*}C_{K} - V_{K}$$
(5.6)

holds for all $K \subseteq \{1, 2, ..., d\}$. (Note that the final equality holds by applying the identities in (5.2) to $\langle D \rangle_K$ and $\langle |D|^2 \rangle_K$.) We prove this by induction over the size of K using the algebraic relations in (5.3) and the recursions in (3.2). For $K = \emptyset$ one has $C_{\emptyset}^* C_{\emptyset} - V_{\emptyset} =$ $(\mathbf{I} + \mathbf{0})^* (\mathbf{I} + \mathbf{0}) - \mathbf{0} = \mathbf{I} = S_{T,\emptyset}$, so (5.6) holds. And if (5.6) holds for some $K \subseteq \{1, 2, ..., d\}$, then for $\alpha \in \{1, 2, ..., d\} \setminus K$

$$S_{T,K\cup\{\alpha\}} \stackrel{(3.2)}{=} \frac{1}{2}(-A_{\alpha}^{*})S_{T,K} + \frac{1}{2}S_{T,K}(-A_{\alpha})$$

$$= \frac{1}{2}\left(\mathbf{I} + 2\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}\right)S_{T,K} + \frac{1}{2}S_{T,K}\left(\mathbf{I} + 2\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}\right)$$

$$\stackrel{\text{ind.}}{=} (C_{K}^{*}C_{K} - V_{K}) + \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}(C_{K}^{*}C_{K} - V_{K}) + (C_{K}^{*}C_{K} - V_{K})\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}$$

$$\stackrel{(*)}{=} (C_{K}^{*}C_{K} - V_{K}) + \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}C_{K} - \mathbf{0} \end{pmatrix} + \begin{pmatrix} C_{K}^{*}\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix} - \mathbf{0} \end{pmatrix}$$

$$= (C_{K}^{*}C_{K} - V_{K}) + \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}C_{K} + C_{K}^{*}\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}$$

$$= (C_{K} + \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix})^{*} \begin{pmatrix} C_{K} + \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix} - V_{K} - \begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}^{*}\begin{pmatrix} 0 & D_{\alpha} \\ 0 & 0 \end{pmatrix}$$

$$= C_{K\cup\{\alpha\}}^{*}C_{K\cup\{\alpha\}} - \begin{pmatrix} V_{K} + \begin{pmatrix} 0 & 0 \\ 0 & D_{\alpha}^{*}D_{\alpha} \end{pmatrix} \end{pmatrix}$$

where the cancellations in (*) hold by applying (5.2) to $\langle D \rangle_K$, D_{α} , and $\langle |D|^2 \rangle_K$. This computation shows that (5.6) holds for $K \cup \{\alpha\}$. It follows by induction that (5.6) holds for all $K \subseteq \{1, 2, \ldots, d\}$.

Let $K \subseteq \{1, 2, ..., d\}$ be arbitrary. Observe that $C_K \in \mathcal{G}_1$, and thus by the above discussions, C_K is invertible. By (5.6) one has $S_{T,K} = C_K^*(\mathbf{I} - V_K)C_K$. Since $S_{T,K}$ and $\mathbf{I} - V_K$ are self-adjoint and C_K is invertible, it follows that the signs of the minimal spectral values of $S_{T,K}$ and $\mathbf{I} - V_K$ coincide.^g To compute the latter observe that $\sigma(\mathbf{I} - V_K) = 1 - \sigma(V_K) = 1 - \{0\} \cup \sigma(\langle |D|^2 \rangle_K) = \{1\} \cup (1 - \sigma(\langle |D|^2 \rangle))$, whence by the positivity of $\langle |D|^2 \rangle_K = \sum_{i \in K} D_i^* D_i$ the spectral theorem yields

$$\min \sigma(\mathbf{I} - V_K) = 1 - \max(\sigma(\langle |D|^2 \rangle_K)) = 1 - \|\langle |D|^2 \rangle_K\| = 1 - \left\| \sum_{i \in K} D_i^* D_i \right\|$$

For each $i \in \{1, 2, ..., d\}$ we now choose $D_i = \alpha V_i$ where V_i is an isometry and $\alpha \in \mathbb{C}$ with $|\alpha| \in (\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d-1}}) \subseteq [0, 1]$. This can be achieved in the above construction, as we simply required that each $D_i \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_1)$ be contractive and since $0 < \dim(\mathcal{H}_2) \leq \dim(\mathcal{H}_1)$. For $K \subseteq \{1, 2, ..., d\}$ our choice yields $\|\sum_{i \in K} D_i^* D_i\| = \|\sum_{i \in K} |\alpha|^2 \mathbf{I}\| = |\alpha|^2 |K|$ and thus by the above computation $\operatorname{sgn}(\min(\sigma(S_{T,K}))) = \operatorname{sgn}(\min(\sigma(\mathbf{I}-V_K))) = \operatorname{sgn}(1-|\alpha|^2|K|)$. By our choice of α we have that $\operatorname{sgn}(\min(\sigma(S_{T,K}))) = +1$ for $K \subsetneq \{1, 2, ..., d\}$ and $\operatorname{sgn}(\min(\sigma(S_{T,K}))) = -1$ for $K = \{1, 2, ..., d\}$. In particular, $\beta_T = \min_{K \subseteq \{1, 2, ..., d\}} \min(\sigma(S_{T,K})) < 0$.

For any such construction, one thus has that T satisfies all the assumptions and $\beta_T < 0$. That is, the generators of T are not completely dissipative, and hence by Theorem 1.1 T has no regular unitary dilation.

Remark 5.4 The above construction satisfied $\min(\sigma(S_{T,K})) \ge 0$ for |K| < d and $\min(\sigma(S_{T,K})) < 0$ for |K| = d. This shows that, to determine whether a *d*-parameter C_0 -semigroup *T* has completely dissipative generators, the higher order dissipation operators are not redundant.

Remark 5.5 It is well known that all 2-parameter contractive C_0 -semigroups have unitary dilations (*cf.* [27], [28, Theorem 2], and [24, Theorem 2.3]). Thus Proposition 5.3 confirms that

^gLet \mathcal{A} be a unital C^* -algebra, $a \in \mathcal{A}$ be self-adjoint, and $c \in \mathcal{A}$ be invertible. Set $\lambda := \min(\sigma(a)), \lambda' := \min(\sigma(c^*ac)), r := \min(\sigma(c^*c)) > 0$, and $r' := \min(\sigma((c^{-1})^*c^{-1})) > 0$. Then $c^*ac \ge c^* \cdot \lambda \mathbf{I} \cdot c = \lambda c^*c \ge \lambda r \mathbf{I}$ so $\lambda' = \min(\sigma(c^*ac)) \ge \lambda r$. Since $a = (c^{-1})^*(c^*ac)c^{-1}$, one similarly obtains $\lambda \ge \lambda' r'$. Since r, r' > 0, it follows that $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\lambda')$.

the existence of regular unitary dilations is a strictly stronger condition than the existence of unitary dilations.

6. Application to the von Neumann polynomial inequality

We conclude this paper with an application of the complete dissipativity condition introduced in this paper. When studying the aspects of multiple operators, it is natural to consider algebraic combinations and their bounds.

Definition 6.1 Let $S_1, S_2, \ldots, S_d \in \mathfrak{L}(\mathcal{H})$ be commuting operators. We define the map $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}] \ni p \mapsto p(S_1, S_2, \ldots, S_d) \in \mathfrak{L}(\mathcal{H})$ as the unique linear map satisfying $p(S_1, S_2, \ldots, S_d) := (\prod_{i \in \text{supp}(\mathbf{n}^-)} S_i^{-n_i})^* (\prod_{i \in \text{supp}(\mathbf{n}^+)} S_i^{n_i})$ for all monomials of the form $p = \prod_{i=1}^d X_i^{n_i}$ where $\mathbf{n} \in \mathbb{Z}^d$. We refer to this map as the *regular polynomial evaluation*.

It is easy to see that the commutativity of the operators guarantees that regular polynomial evaluation is well-defined and linear. Note that, unless the operators are normal (which by Fuglede's theorem implies double commutativity), regular polynomial evaluation is not multiplicative. We may also consider $\mathbb{C}[X_1, X_2, \ldots, X_d]$ as a subalgebra of the ring $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}]$. The restriction of the above map to $\mathbb{C}[X_1, X_2, \ldots, X_d]$ yields the usual *polynomial evaluation* for tuples of operators.

Now in [33], [31, Proposition I.8.3 and Notes, p. 54], [23, Theorem 1.2], and [25, Theorem 1.1] the following bound is proved for commuting *d*-tuples (S_1, S_2, \ldots, S_d) of contractions and $p \in \mathbb{C}[X_1, X_2, \ldots, X_d]$:

$$||p(S_1, S_2, \dots, S_d)|| \leq \sup_{\boldsymbol{\lambda} \in \mathbb{T}^d} |p(\lambda_1, \lambda_2, \dots, \lambda_d)|,$$

provided $d \in \{1, 2\}$. In [20, pp. 488–489] an example of d = 3 commuting contractions is provided, which satisfies the above bounds, but admits no *simultaneous unitary power dilation.*^h The proper correspondence between polynomial bounds and unitary dilations can be found in [23, Corollary 4.9] (see also Corollary 4.13 in this book). There it is shown (in particular for $d \ge 3$) that a simultaneous unitary power dilation exists if and only if a strengthened version of the above inequalities hold with \mathbb{C} -valued polynomials replaced by $n \times n$ matrices $(p_{ij})_{ij}$ with polynomial entries for all $n \in \mathbb{N}$ and norms/absolute values replaced by appropriate matrix norms.

Further interesting connections can be found in recent literature. For example, in [2] the authors define the notion of *Siegel-dissipativity* (which bears no relation to complete dissipativity), and show that this condition suffices for a certain variation of the above bounds. And in [14, 15] finite dimensional dilations for operator systems and polynomial bounds for finite dimensional commuting matrices are investigated.

Moving to the continuous setting of multi-parameter (contractive) C_0 -semigroups, we can define a similar problem.

Definition 6.2 Let T be a d-parameter C_0 -semigroup over \mathcal{H} . Say that T satisfies polynomial bounds if

$$\|p(T_1(t_1), T_2(t_2), \dots, T_d(t_d))\| \leq \sup_{\boldsymbol{\lambda} \in \mathbb{T}^d} |p(\lambda_1, \lambda_2, \dots, \lambda_d)|$$
(6.1)

for all polynomials $p \in \mathbb{C}[X_1, X_2, \ldots, X_d]$ and all $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}^d_{\geq 0}$. We say T satisfies regular polynomial bounds, if the above holds for all $p \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}]$ and all $\mathbf{t} \in \mathbb{R}^d_{\geq 0}$. We say that T satisfies regular polynomial bounds in a neighbourhood of $\mathbf{0}$, if for some neighbourhood $W \subseteq \mathbb{R}^d_{\geq 0}$ of $\mathbf{0}$, the above holds for all $p \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}]$ and all $\mathbf{t} \in W$.

^hA tuple $(S_1, S_2, \ldots, S_d) \in \mathfrak{L}(\mathcal{H})$ of commuting operators, is said to have a *simultaneous unitary power dilation*, if a Hilbert space \mathcal{H}' exists, as well as *d* commuting unitaries $(U_1, U_2, \ldots, U_d) \in \mathfrak{L}(\mathcal{H}')$ and $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$ (necessarily isometric), such that $\prod_{i=1}^d S_i^{n_i} = r^*(\prod_{i=1}^d U_i^{n_i})r$ for all $\mathbf{n} = (n_i)_{i=1}^d \in \mathbb{N}_0^d$.

The (regular) von Neumann polynomial inequality problem for d-parameter C_0 -semigroups shall refer to determining whether T satisfies (regular) polynomial bounds for all d-parameter contractive C_0 -semigroups T over a Hilbert space \mathcal{H} .

Clearly, satisfaction of regular polynomial bounds implies satisfaction of polynomial bounds. Simple examples of the stronger condition are as follows:

Proposition 6.3 Every *d*-parameter unitary C_0 -semigroup U over \mathcal{H} satisfies regular polynomial bounds.

Proof. Let $p \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}]$ and $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d$ be arbitrary and fixed. Since $\{U_i(t_i) \mid i \in \{1, 2, \ldots, d\}\}$ are doubly commuting unitary operators, one can apply the spectral mapping theorem for commutative C^* -algebras to simultaneously diagonalise these to multiplication operators over a semi-finite measure space (see [19, Theorem 1.3.6], [22, 3.3.1 and 3.4.1], and [13]). That is, one can find a semi-finite measure space (X, μ) , measurable \mathbb{R} -valued functions $\theta_1, \theta_2, \ldots, \theta_d \in L^{\infty}(X, \mu)$, and a unitary operator $u \in \mathfrak{L}(\mathcal{H}, L^2(X, \mu))$, such that $U_i(t_i) = u^* M_{e^{i\theta_i(\cdot)}u}$ for all $i \in \{1, 2, \ldots, d\}$. One can readily verify that $p(U_1(t_1), U_2(t_2), \ldots, U_d(t_d)) = u^* M_{p(e^{i\theta_1(\cdot)}, e^{i\theta_2(\cdot)}, \ldots, e^{i\theta_d(\cdot)})}u$ and thus

$$\begin{aligned} \|p(U_1(t_1), U_2(t_2), \dots, U_d(t_d))\| &= \|M_{p(e^{i\theta_1(\cdot)}, e^{i\theta_2(\cdot)}, \dots e^{i\theta_d(\cdot)})}\|_{L^2(X, \mu)} \\ &= \|p(e^{i\theta_1(\cdot)}, e^{i\theta_2(\cdot)}, \dots e^{i\theta_d(\cdot)})\|_{L^\infty(X, \mu)} \\ &\leqslant \sup_{\lambda \in \mathbb{T}^d} |p(\lambda_1, \lambda_2, \dots, \lambda_d)|. \end{aligned}$$

Thus U satisfies regular polynomial bounds.

Proposition 6.4 Let T be a d-parameter C_0 -semigroup over \mathcal{H} . If T has a regular unitary dilation, then it satisfies regular polynomial bounds.

Proof. For any monomial $p = \prod_{i=1}^{d} X_i^{n_i}$ with $\mathbf{n} \in \mathbb{Z}^d$, and for any $\mathbf{t} \in \mathbb{R}^d_{\geq 0}$, one computes

$$p(T_1(t_1), T_2(t_2), \dots, T_d(t_d)) = (\prod_{i \in \text{supp}(\mathbf{n}^-)} T_i(t_i)^{-n_i})^* (\prod_{i \in \text{supp}(\mathbf{n}^+)} T_i(t_i)^{n_i})$$

$$= (\prod_{i \in \text{supp}(\mathbf{n}^-)} T_i(-n_i t_i))^* (\prod_{i \in \text{supp}(\mathbf{n}^+)} T_i(n_i t_i))$$

$$= T(\mathbf{n}^- \odot \mathbf{t})^* T(\mathbf{n}^+ \odot \mathbf{t})$$

$$= T((\mathbf{n} \odot \mathbf{t})^-)^* T((\mathbf{n} \odot \mathbf{t})^+),$$
(6.2)

where we define $\odot : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ via $\mathbf{s} \odot \mathbf{s}' := (s_i s'_i)_{i=1}^d$ for $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^d$.

By assumption, T has a regular unitary dilation (U, \mathcal{H}', r) . Applying the definitions yields

$$p(T_{1}(t_{1}), T_{2}(t_{2}), \dots, T_{d}(t_{d})) = T((\mathbf{n} \odot \mathbf{t})^{-})^{*}T((\mathbf{n} \odot \mathbf{t})^{+})$$

= $r^{*}U(\mathbf{n} \odot \mathbf{t})r$
= $r^{*}U((\mathbf{n} \odot \mathbf{t})^{-})^{*}U((\mathbf{n} \odot \mathbf{t})^{+})r$
= $r^{*}p(U_{1}(t_{1}), U_{2}(t_{2}), \dots, U_{d}(t_{d}))r$,

whereby the final equality holds by applying (6.2) to U instead of T. Since the polynomial map is linear and the monomials linearly span $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_d, X_d^{-1}]$, the above computation implies that

$$p(T_1(t_1), T_2(t_2), \dots, T_d(t_d)) = r^* p(U_1(t_1), U_2(t_2), \dots, U_d(t_d))r$$

for all $p \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_d, X_d^{-1}]$ and all $\mathbf{t} \in \mathbb{R}^d_{\ge 0}$.

Since U satisfies regular polynomial bounds (see Proposition 6.3) and $r \in \mathfrak{L}(\mathcal{H}, \mathcal{H}')$ is an isometry and thus contractive, one obtains

$$\|p(T_1(t_1), T_2(t_2), \dots, T_d(t_d))\| \leq \|r^*\| \|p(U_1(t_1), U_2(t_2), \dots, U_d(t_d))\| \|r\| \\ \leq 1 \cdot \sup_{\lambda \in \mathbb{T}^d} |p(\lambda_1, \lambda_2, \dots, \lambda_d)| \cdot 1.$$

Since this holds for all polynomials $p \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_d, X_d^{-1}]$, it follows that T satisfies regular polynomial bounds.

Now, to prove $(3) \Rightarrow (1)$ of Theorem 1.4, observe that the dissipation operators involve expressions of the form $(\prod_{i \in C_1} A_i^-)^* (\prod_{j \in C_2} A_j^-)$ for $(C_1, C_2) \in Part(K)$ and $K \subseteq \{1, 2, \ldots, d\}$,

which, in terms of their indices, are fitting for the kinds of expressions that arise in regular polynomial evaluations. However, regular polynomial evaluation captures values of the semigroups and not their generators. Using appropriate approximation arguments, this issue can nonetheless be overcome. In this way, one may prove Theorem 1.4.

Proof (of Theorem 1.4). The direction $(1) \Rightarrow (2)$ has already been proved in Proposition 6.4 and $(2) \Rightarrow (3)$ holds trivially. To prove $(3) \Rightarrow (1)$, assume that T satisfies regular polynomial bounds in a neighbourhood $W \subseteq \mathbb{R}^d_{\geq 0}$ of **0**. By Theorem 1.1, to show that T has a regular unitary dilation, it is necessary and sufficient to show that the generators of T are completely dissipative. To this end fix an arbitrary $K \subseteq \{1, 2, \ldots, d\}$. We need to show that $S_{T,K} \geq \mathbf{0}$.

Now, for each $i \in K$, we know that $t_i^{-1}(\mathbf{I} - T_i(t_i)) \longrightarrow -A_i$ in norm for $\mathbb{R}_{>0} \ni t_i \longrightarrow 0$. Thus, defining for $\mathbf{t} = (t_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d$ the product $t_K := \prod_{i \in K} t_i$ and the operator

$$P(\mathbf{t}) := \sum_{(C_1, C_2) \in \operatorname{Part}(K)} \prod_{i \in C_1} (\mathbf{I} - T_i(t_i)^*) \cdot \prod_{j \in C_2} (\mathbf{I} - T_j(t_j))$$

one has the norm convergence $\frac{1}{2^{|K|}t_K}P(\mathbf{t}) \longrightarrow S_{T,K}$ for $\mathbb{R}^d_{>0} \ni \mathbf{t} \longrightarrow 0$. Since the subspace $\mathfrak{L}(\mathcal{H})_{\geq 0}$ of $\mathfrak{L}(\mathcal{H})$ consisting of positive operators is norm-closed, in order to prove that $S_{T,K} \geq \mathbf{0}$, it suffices to prove that $P(\mathbf{t}) \geq \mathbf{0}$ for all $\mathbf{t} \in \mathbb{R}^d_{>0}$ sufficiently close to $\mathbf{0}$.

So fix an arbitrary $\mathbf{t} \in W \cap \mathbb{R}^d_{>0}$. We shall show that $P(\mathbf{t}) \ge \mathbf{0}$. Since $P(\mathbf{t}) \in \mathfrak{L}(\mathcal{H})$ is clearly a self-adjoint operator, it suffices to prove that $\|\mathbf{I} - \alpha P(\mathbf{t})\| \le 1$ for sufficiently small $\alpha > 0$.ⁱ

Now, applying regular polynomial evaluation to

$$p := \sum_{(C_1, C_2) \in \text{Part}(K)} \prod_{i \in C_1} (1 - X_i^{-1}) \cdot \prod_{j \in C_2} (1 - X_j)$$

yields

$$p(T_1(t_1), T_2(t_2), \dots, T_d(t_d)) = P(\mathbf{t}),$$

and since T satisfies regular polynomial bounds in W, it follows that

$$\|\mathbf{I} - \alpha P(\mathbf{t})\| = \|(1 - \alpha p)(T_1(t_1), T_2(t_2), \dots, T_d(t_d))\| \leq \sup_{\boldsymbol{\lambda} \in \mathbb{T}^d} |1 - \alpha p(\lambda_1, \lambda_2, \dots, \lambda_d)|$$

for $\alpha > 0$. Thus, to prove that $P(\mathbf{t}) \ge \mathbf{0}$, it suffices to prove for sufficiently small $\alpha > 0$, that $|1 - \alpha p(\lambda_1, \lambda_2, \dots, \lambda_d)| \le 1$ for all $\boldsymbol{\lambda} \in \mathbb{T}^d$. This in turn holds if p, when viewed as a function $p : \mathbb{T}^d \to \mathbb{C}$, is bounded, \mathbb{R} -valued, and non-negative. Indeed, for each $\boldsymbol{\lambda} \in \mathbb{T}^d$

$$p(\lambda_1, \lambda_2, \dots, \lambda_d) = \sum_{\substack{(C_1, C_2) \in \operatorname{Part}(K) \ i \in C_1}} \prod_{i \in C_1} (1 - \lambda_i^{-1}) \cdot \prod_{j \in C_2} (1 - \lambda_j)$$
$$= \prod_{i \in K} \underbrace{\left((1 - \lambda_i^{-1}) + (1 - \lambda_i) \right)}_{=2 \Re_e(1 - \lambda_i) \in [0, 4]} \in [0, 4^{|K|}],$$

which completes the proof.

Remark 6.5 The use of the complete dissipativity condition was brought to bear in the above argument. If one did not work with this concept, one might attempt to prove that satisfaction of regular polynomial bounds implies the existence of a regular unitary dilation, by showing that the Brehmer positivity criterion holds. To achieve this, one could attempt to capture the Brehmer operators via regular polynomial evaluation and then again rely on polynomial bounds. However, the expressions occurring in the Brehmer operators involve monomials of the form $(\prod_{i\in C} T_i(t))^*(\prod_{i\in C} T_i(t))$ for $C \subseteq \{1, 2, \ldots, d\}$, which do not seem to be expressions that can be captured by regular polynomial evaluation.

Combining the examples in §5.3 with Theorem 1.4 we can prove Corollary 1.5, which negatively solves the regular von Neumann polynomial problem for multi-parameter C_0 -semigroups:

ⁱLet \mathcal{A} be a unital C^* -algebra and $a \in \mathcal{A}$ be self-adjoint. Let $\lambda_{\min} := \min(\sigma(a))$ and $\lambda_{\max} := \max(\sigma(a))$ and $\alpha^* := \frac{1}{\max\{|\lambda_{\min}|, |\lambda_{\max}|\}+1}$. For $\alpha \in (0, \alpha^*)$ one has $\min(\sigma(\mathbf{I} - \alpha a)) = 1 - \alpha \lambda_{\max}$ and $\max(\sigma(\mathbf{I} - \alpha a)) = 1 - \alpha \lambda_{\min}$, which are both positive by the choice of α^* . By the spectral theorem it follows that $\|\mathbf{I} - \alpha a\| = 1 - \alpha \lambda_{\min}$, which does not exceed 1 if and only if $\lambda_{\min} \ge 0$. Now, a is a positive element if and only if $\lambda_{\min} \ge 0$, which by the preceding argument holds if and only if $\|\mathbf{I} - \alpha a\| \le 1$ for all sufficiently small $\alpha \in \mathbb{R}_{>0}$.

Proof (of Corollary 1.5). By Proposition 5.3 there exist *d*-parameter contractive C_0 -semigroups whose generators are bounded and have strictly negative spectral bounds, and which are not completely dissipative. By the classification theorem (Theorem 1.1), these have no regular unitary dilations, and thus by Theorem 1.4 do not satisfy regular polynomial bounds.

Remark 6.6 It would be of interest to know whether the characterisation in Theorem 1.4 also holds without the assumption of bounded generators. Certainly the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold, as neither Proposition 6.4 nor Proposition 6.3 required the semigroups to have bounded generators. But our proof of the implication $(3) \Rightarrow (1)$ relied heavily on the complete dissipativity condition. The latter notion might need to be reformulated for the unbounded setting.

Acknowledgement. The author is grateful to Tanja Eisner for her patience and feedback, to Leonardo Goller and Rainer Nagel for useful discussions, and to the referee for their constructive feedback.

References

- [1] T. Andô. On a pair of commutative contractions. Acta Sci. Math. (Szeged), 24:88–90, 1963.
- [2] N. Arcozzi, N. Chalmoukis, A. Monguzzi, M. M. Peloso, and M. Salvatori. The Drury-Arveson space on the Siegel upper half-space and a von Neumann type inequality. *Integral Equations Operator Theory*, 93(6):Paper No. 59, 22, 2021.
- [3] W. Arveson. Dilation theory yesterday and today. In A glimpse at Hilbert space operators, volume 207 of Oper. Theory Adv. Appl., pages 99–123. Birkhäuser Verlag, Basel, 2010.
- [4] S. Barik and B. K. Das. Isometric dilations of commuting contractions and Brehmer positivity. Complex Anal. Oper. Theory, 16(5):Paper No. 69, 25, 2022.
- [5] P. L. Butzer and H. Berens. Semi-groups of operators and approximation. Die Grundlehren der mathematischen Wissenschaften, Band 145. Springer-Verlag New York, Inc., New York, 1967.
- K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations, volume 194 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.
- [7] F. Fagnola. Quantum stochastic differential equations and dilation of completely positive semigroups. In Open quantum systems. II, volume 1881 of Lecture Notes in Math., pages 183–220. Springer, Berlin, 2006.
- [8] F. Fagnola and S. J. Wills. Solving quantum stochastic differential equations with unbounded coefficients. J. Funct. Anal., 198(2):279–310, 2003.
- [9] G. B. Folland. A course in abstract harmonic analysis. Number 29 in Textbooks in mathematics. CRC Press, 2 edition, 2015.
- [10] J. A. Goldstein. Semigroups of linear operators & applications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985.
- [11] A. M. Gomilko, K. Zvart, and Y. Tomilov. On the inverse operator of the generator of a C₀-semigroup. Mat. Sb., 198(8):35–50, 2007.
- [12] K. E. Gustafson and D. K. M. Rao. Numerical range. Universitext. Springer-Verlag, New York, 1997.
- [13] M. Haase. The functional calculus approach to the spectral theorem. Indagationes Mathematicae, 31(6):1066–1098, 2020.
- [14] M. Hartz and M. Lupini. Dilation theory in finite dimensions and matrix convexity. Israel J. Math., 245(1):39– 73, 2021.
- [15] M. Hartz, S. Richter, and O. M. Shalit. Von Neumann's inequality for row contractive matrix tuples. Math. Z., 301(4):3877–3894, 2022.
- [16] M. Izumi. E₀-semigroups: around and beyond Arveson's work. J. Operator Theory, 68(2):335–363, 2012.
- [17] M. Laca and B. Li. Dilation theory for right LCM semigroup dynamical systems. J. Math. Anal. Appl., 505(2):Paper No. 125586, 37, 2022.
- [18] C. Le Merdy. On dilation theory for c_0 -semigroups on Hilbert space. Indiana Univ. Math. J., 45(4):945–959, 1996.
- [19] G. J. Murphy. C*-algebras and operator theory. Academic Press, Inc., Boston, MA, 1990.
- [20] S. Parrott. Unitary dilations for commuting contractions. Pacific J. Math., 34:481–490, 1970.
- [21] G. K. Pedersen. Analysis now, volume 118 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.
- [22] G. K. Pedersen. C*-algebras and their automorphism groups. Pure and Applied Mathematics (Amsterdam). Academic Press, London, 2018.
- [23] G. Pisier. Similarity problems and completely bounded maps, volume 1618 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, expanded edition, 2001.
- [24] M. Ptak. Unitary dilations of multi-parameter semi-groups of operators. Annales Polonici Mathematici, XLV:237-243, 1985.
- [25] O. M. Shalit. Dilation theory: a guided tour. In Operator theory, functional analysis and applications, volume 282 of Oper. Theory Adv. Appl., pages 551–623. Birkhäuser/Springer, Cham, 2021.

- [26] E. Shamovich and V. Vinnikov. Dilations of semigroups of contractions through vessels. Integral Equations Operator Theory, 87(1):45–80, 2017.
- [27] M. Słociński. Unitary dilation of two-parameter semi-groups of contractions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 22:1011–1014, 1974.
- [28] M. Słociński. Unitary dilation of two-parameter semi-groups of contractions II. Zeszyty Naukowe Uniwersytetu Jagiellońskiego, 23:191–194, 1982.
- [29] W. F. Stinespring. Positive functions on C*-algebras. Proc. Amer. Math. Soc., 6:211-216, 1955.
- [30] B. Sz.-Nagy. Sur les contractions de l'espace de Hilbert, volume 15. 1953.
- [31] B. Sz.-Nagy and C. Foias. Harmonic Analysis of Operators on Hilbert Space. North-Holland, 1970.
- [32] R. Tevzadze. Markov dilation of diffusion type processes and its application to the financial mathematics. *Georgian Math. J.*, 6(4):363–378, 1999.
- [33] J. von Neumann. Eine Spektraltheorie f
 ür allgemeine Operatoren eines unit
 ären Raumes. Math. Nachr., 4:258–281, 1951.
- [34] K. Yosida. Functional analysis, volume 123 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, sixth edition, 1980.
- [35] H. Zwart. Is A⁻¹ an infinitesimal generator? In Perspectives in operator theory, volume 75 of Banach Center Publ., pages 303–313. Polish Acad. Sci. Inst. Math., Warsaw, 2007.

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