ON A CLASS OF ROBUST NONCONVEX QUADRATIC OPTIMIZATION PROBLEMS

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ABSTRACT. Let us consider the following robust nonconvex quadratic optimization problem:

min $\frac{1}{2}x^{\top}Ax + a^{\top}x$ s.t. $\alpha \leq \frac{1}{2}x^{\top}(B_1 + \mu B_2)x + (b_1 + \delta b_2)^{\top}x \leq \beta, \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2],$ where A, B_1, B_2 are real symmetric matrices, $\mu_1, \mu_2, \delta_1, \delta_2, \alpha, \beta \in \mathbb{R}$ satisfying $\mu_1 \leq \mu_2, \delta_1 \leq \delta_2$ and $\alpha < \beta$. We establish the robust alternative result; the robust S-lemma and the robust optimality for the above nonconvex problem. Nonconvex quadratic programming under uncertainty and Robust optimization and S-lemma and Global optimality Primary: 90C20 and 90C30 and 90C26 and 90C46

1. INTRODUCTION AND BASIC NOTATION

Robust optimization arises as a deterministic approach when addressing an optimization problem under uncertainty data. This paper revisites the following robust optimization problem:

(1.1)
$$\min\left\{\frac{1}{2}x^{\top}Ax + a^{\top}x: \ \alpha \leq \frac{1}{2}x^{\top}Bx + b^{\top}x \leq \beta, \ \forall \ (B,b) \in \mathcal{B}_b\right\},$$

where $\mathcal{B}_b \doteq \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}$, with all the matrices being real symmetric, $a, b \in \mathbb{R}^n$ and $\alpha, \beta, \delta_1, \delta_2, \mu_1, \mu_2$ are given real numbers. Optimization problems that can be modeled by quadratic functions appear, for instance, in [2, 8, 9, 10, 12].

Problem (1.1) includes that examined in [7]:

(1.2)
$$\min\left\{\frac{1}{2}x^{\top}Ax + a^{\top}x : \frac{1}{2}x^{\top}Bx + b^{\top}x \le \beta, \ \forall \ (B,b) \in \mathcal{B}_b\right\}.$$

Theorem 5.1 in [7] provides a characterization of robust optimality for problem (1.2) under the convexity of the set

$$\Big\{ (x^\top H_0 x, x^\top H_1 x, x^\top H_2 x) : x \in \mathbb{R}^{n+1} \Big\},\$$

where

$$H_{0} = \begin{pmatrix} A & a \\ a^{\top} & 2\gamma \end{pmatrix}, \ H_{1} = \begin{pmatrix} B_{1} + \mu_{1}B_{2} & b_{1} + \delta_{1}b_{2} \\ (b_{1} + \delta_{1}b_{2})^{\top} & -2\beta \end{pmatrix}, \ H_{2} = \begin{pmatrix} B_{1} + \mu_{2}B_{2} & b_{1} + \delta_{2}b_{2} \\ (b_{1} + \delta_{2}b_{2})^{\top} & -2\beta \end{pmatrix}$$

with $\gamma = -f(\overline{x})$. Here, \overline{x} is a feasible point of the robust optimization problem (1.2), which is either to be supposed becoming an optimal solution, or to be optimal for deriving optimality conditions.

Certainly the presence of the matrices H_0, H_1, H_2 is because the authors in [7] homogenize problem (1.2) in order to apply the Dines convexity theorem ([3]) valid for quadratic forms. Finally, we realize there is a gap in the proof of Theorem 5.1 in [7], but we were unable to find a counterexample to such a result under their assumptions. This is discussed in detail after Theorem 6 in Section 4. Notice that the approach employed in [7] was also applied in [1].

We have to point out that problem (1.1) (and so problem (1.2)) was studied without homogenizing the problem thanks to the convexity result established in [4, Theorem 4.19] valid for inhomogeneous quadratic functions. This allows us to impose the convexity of a set being the image of \mathbb{R}^n via the inhomogeneous quadratic functions.

Associated to problem (1.1), purposes of the present paper are to establish an alternative robust result (Theorem 3), a robust S-lemma (Theorem 4) and a characterization of robust optimality (Theorem 5), for problem (1.1). Finally, we provide a counterexample (Example 7) to the argument employed in the proof of Theorem 5.1 in [7] related to problem (1.2).

Thus, the structure of the present paper is as follows. Section 2 establishes the convexity of images for quadratic mappings by applying the Ramana-Goldman criterion [11] (see also [5, Theorem 2.1]). The main results are presented in Section 3, and Section 4 revisites problem (1.2) discussed in [7].

2. A preliminary result: convexity of images

By S^n we denote the set of symmetric matrices of order $n \in \mathbb{N}$ with real entries; S^n_+ denotes the subset of S^n whose elements are positive semidefinite matrices, and we write $A \succeq 0$ if $A \in S^n_+$; and S^n_{++} stands for the matrices in S^n that are positive definite, and in this case we write $A \succ 0$ if $A \in S^n_{++}$.

It is our purpose to prove the convexity of images for quadratic mappings under the Ramana-Goldman criterion [11] (see also [5, Theorem 2.1]). To that end, we are given $A_i \in S^n$, $b_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$, we set

$$M_i = \begin{pmatrix} A_i & b_i \\ b_i^\top & 2c_i \end{pmatrix}, f_i(x) = x^\top A_i x + 2b_i^\top x, \ \overline{f}_i(x) = x^\top A_i x, \ x \in \mathbb{R}^n.$$

Furthermore, let us consider the function

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$$G(x,t) = (g_0(x,t), g_1(x,t), \dots, g_m(x,t)), \ (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where g_i is defined by $g_i(x,t) = \begin{pmatrix} x \\ t \end{pmatrix}^\top M_i \begin{pmatrix} x \\ t \end{pmatrix}$.

Lemma 1. Let $A_i \in S^n$, $c_i \in \mathbb{R}$, $b_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, m$ be as above. Set

$$(x) = (f_0(x), f_1(x), \dots, f_m(x)), \ \overline{F}(x) = (\overline{f}_0(x), \overline{f}_1(x), \dots, \overline{f}_m(x)).$$

If
$$F(\mathbb{R}^n)$$
 and $\overline{F}(\mathbb{R}^n)$ are convex then $G(\mathbb{R}^{n+1})$ is convex.

Proof. Set $\Lambda := F(\mathbb{R}^n)$ and $\overline{\Lambda} := \overline{F}(\mathbb{R}^n)$ and $\Omega := G(\mathbb{R}^{n+1})$.

As Λ is convex, by the convexity criterion due to Ramana-Goldman (see also [5, Theorem 2.1]), $\Lambda + \overline{\Lambda} = \Lambda$. We easily get that for $(x, t) \in \mathbb{R}^{n+1}$,

$$\begin{pmatrix} x \\ t \end{pmatrix}^{\top} M_i \begin{pmatrix} x \\ t \end{pmatrix} = x^{\top} A_i x + 2t b_i^{\top} x + 2c_i t^2, \ i = 0, 1, \dots, m.$$

By setting $\overline{\gamma} = 2(c_0, c_1, \dots, c_m)$, we obtain (2.3) $\Lambda + \overline{\gamma} \subseteq \Omega$. Let $z_1 = G(x_1, t_1)$, $z_2 = G(x_2, t_2)$ be any elements in Ω and let $\lambda \in [0, 1[$. We distinguish three cases. (i): $t_1 \neq 0$ and $t_2 \neq 0$. Then, $z_1 = t_1^2(F(x_1/t_1) + \overline{\gamma})$ and $z_2 = t_2^2(F(x_2/t_2) + \overline{\gamma})$.

The convexity of Λ implies that

$$\frac{\lambda}{\lambda t_1^2 + (1-\lambda)t_2^2} z_1 + \frac{1-\lambda}{\lambda t_1^2 + (1-\lambda)t_2^2} z_2$$

= $\frac{\lambda t_1^2}{\lambda t_1^2 + (1-\lambda)t_2^2} F(x_1/t_1) + \frac{(1-\lambda)t_2^2}{\lambda t_1^2 + (1-\lambda)t_2^2} F(x_2/t_2) + \overline{\gamma} \in \Lambda + \overline{\gamma}.$

Taking into account (2.3) and the fact the Ω is a cone, we obtain

$$\lambda z_1 + (1 - \lambda) z_2 \in \mathbb{R}_{++}(\Lambda + \overline{\gamma}) \subseteq \Omega.$$

(*ii*): $t_1 = t_2 = 0$. Then, $z_1, z_2 \in \overline{\Lambda}$, and because of the convexity of $\overline{\Lambda}$, we get

$$\lambda \left(z_1 + (1 - \lambda) z_2 \right) \in \overline{\Lambda} \subseteq \Omega.$$

(*iii*): $t_1 \neq 0$ and $t_2 = 0$. Then, since $\overline{\Lambda}$ is a cone,

$$\lambda z_1 + (1 - \lambda) z_2 \in \lambda t_1^2(\Lambda + \overline{\gamma}) + \lambda t_1^2 \overline{\Lambda} \subseteq \lambda t_1^2(\Lambda + \overline{\gamma}) \subseteq \Omega.$$

This completes the proof that Ω is convex.

Part (a) of the following result is exactly Theorem 2.3 (i) in [2], and (b) is a consequence of the previous lemma.

Corollary 2. Let the same hypotheses of Lemma 1 be satisfied. Let $\rho_i \in \mathbb{R}$, for i = 1, ..., m. If $n \ge m + 1$, $A_0 \in S_{++}^n$, $A_i = \rho_i A_0$ for i = 1, ..., m, then

- (a) $F(\mathbb{R}^n)$ and $\overline{F}(\mathbb{R}^n)$ are convex.
- (b) $G(\mathbb{R}^{n+1})$ is convex.

Proof. By assumption on A_i , we can apply [2, Theorem 2.3 (i)] to obtain the convexity of $F(\mathbb{R}^n)$ and $\overline{F}(\mathbb{R}^n)$. Then, (b) follows from Lemma 1. \Box

3. The main results

Denote the function:

$$f(x) \doteq \frac{1}{2}x^{\top}Ax + a^{\top}x$$

and let us define the following matrices in \mathcal{S}^{n+1} :

$$H_0 \doteq \begin{pmatrix} A & a \\ a^{\top} & 2\gamma \end{pmatrix}, \ W(\delta, \lambda) \doteq \begin{pmatrix} B_1 & b_1 + \delta b_2 \\ (b_1 + \delta b_2)^{\top} & -2\lambda \end{pmatrix}, \ W_2 \doteq \begin{pmatrix} B_2 & 0 \\ 0^{\top} & 0 \end{pmatrix},$$

and set

$$W_{1_{\beta}} = W(\delta_1, \beta); \ W_{2_{\beta}} = W(\delta_2, \beta); \ W_{1_{\alpha}} = W(\delta_1, \alpha); \ W_{2_{\alpha}} = W(\delta_2, \alpha).$$

The following set will play an important role in the following.

$$\Omega_W \doteq \left\{ \left(\frac{1}{2}y^\top H_0 y, \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_{1_\beta} + \mu W_2)y), \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_{2_\beta} + \mu W_2)y, \right. \right.$$

(3.4)

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$$-\min_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^{\top}(W_{1_{\alpha}}+\mu W_2)y, -\min_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^{\top}(W_{2_{\alpha}}+\mu W_2)y): y\in\mathbb{R}^{n+1}\Big\} + \operatorname{int}\mathbb{R}^5_+.$$

By Corollary 1 in [6],
$$\Omega_W$$
 is convex if the set

$$\Omega_\mu \doteq \left\{ \left(y^\top H_0 y, y^\top (W_{1_\beta} + \mu_1 W_2) y, y^\top (W_{2_\beta} + \mu_1 W_2) y, y^\top (W_{1_\beta} + \mu_2 W_2) y, y^\top (W_{2_\beta} + \mu_2 W_2) y, -y^\top (W_{1_\alpha} + \mu_1 W_2) y, -y^\top (W_{2_\alpha} + \mu_1 W_2) y, -y^\top (W_{1_\alpha} + \mu_2 W_2) y, -y^\top (W_{2_\alpha} + \mu_2 W_2) y, -y^\top (W_{2_\alpha} + \mu_2 W_2) y \right\} : y \in \mathbb{R}^{n+1} \right\} + \text{int } \mathbb{R}^9_+$$

is so.

Theorem 3. (A robust alternative result) Let $A, B_1, B_2 \in S^n$, $a, b_1, b_2 \in \mathbb{R}^n$ and $\gamma, \alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R}$, with $\mu_1 \leq \mu_2, \delta_1 \leq \delta_2$ and $\alpha < \beta$. Assume that Ω_W is convex. Then, exactly one of the two following assertions hold:

$$(a) \exists x \in \mathbb{R}^{n} : \frac{1}{2}x^{\top}Ax + a^{\top}x + \gamma < 0, \ \alpha < \frac{1}{2}x^{\top}(B_{1} + \mu B_{2})x + (b_{1} + \delta b_{2})^{\top}x < \beta, \ \forall \ \mu \in [\mu_{1}, \mu_{2}], \ \forall \ \delta \in [\delta_{1}, \delta_{2}].$$
$$(b) \ \exists (\lambda_{0}, \lambda_{1}, \lambda_{2}) \in \mathbb{R}^{3}_{+} \setminus \{0\}, \ \exists \ \mu_{\alpha}, \mu_{\beta} \in [\mu_{1}, \mu_{2}], \ \exists \ \delta_{\alpha}, \delta_{\beta} \in [\delta_{1}, \delta_{2}]: \ \forall \ x \in \mathbb{R}^{n}$$

$$\lambda_0 \left(\frac{1}{2} x^\top A x + a^\top x + \gamma \right) + \lambda_1 \left(\frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \beta \right) + \lambda_2 \left(\alpha - \left(\frac{1}{2} x^\top (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x \right) \right) \ge 0,$$

where $\mu_{\alpha} + \mu_{\beta} = \mu_1 + \mu_2$.

In addition, we observe that (b) may be written equivalently as

 $\lambda_0 A + \lambda_1 (B_1 + \mu_\beta B_2) - \lambda_2 (B_1 + \mu_\alpha B_2) \succeq 0$ and

$$\exists \overline{x} \in \mathbb{R}^n : \left(\lambda_0 A + \lambda_1 (B_1 + \mu_\beta B_2) - \lambda_2 (B_1 + \mu_\alpha B_2)\right) \overline{x} + \lambda_0 a + \lambda_1 (b_1 + \delta_\beta b_2) + \lambda_2 (b_1 + \delta_\alpha b_2) = 0.$$

Proof. It is obvious that both statements (a) and (b) cannot be fulfilled simultaneously. Thus, we must check that if (a) does not hold, (b) does.

Step 1: The homogenization system. If (a) does not hold, then there exists no $x \in \mathbb{R}^n$ such that for all $\mu \in [\mu_1, \mu_2]$ and all $\delta \in [\delta_1, \delta_2]$

$$\frac{1}{2}x^{\top}Ax + a^{\top}x + \gamma < 0, \quad \frac{1}{2}x^{\top}(B_1 + \mu B_2)x + (b_1 + \delta b_2)^{\top}x < \beta, \\ -\left(\frac{1}{2}x^{\top}(B_1 + \mu B_2)x + (b_1 + \delta b_2)^{\top}x\right) < -\alpha.$$

By setting $\mathcal{B}_b \doteq \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}$, the previous is equivalent to the nonexistence of $x \in \mathbb{R}^n$ such that

$$\frac{1}{2}x^{\top}Ax + a^{\top}x + \gamma < 0, \ \max\left\{\frac{1}{2}x^{\top}Bx + b^{\top}x - \beta : (B,b) \in \mathcal{B}_b\right\} < 0,$$
$$-\min\left\{\frac{1}{2}x^{\top}Bx + b^{\top}x - \alpha : (B,b) \in \mathcal{B}_b\right\} < 0.$$

We claim that the following homogeneous system in \mathbb{R}^{n+1} : (3.5)

$$\frac{1}{2}x^{\top}Ax + ta^{\top}x + t^{2}\gamma < 0, \ \max\left\{\frac{1}{2}x^{\top}Bx + tb^{\top}x - t^{2}\beta : (B,b) \in \mathcal{B}_{b}\right\} < 0,$$

(3.6)
$$-\min\left\{\frac{1}{2}x^{\top}Bx + tb^{\top}x - t^{2}\alpha : (B,b) \in \mathcal{B}_{b}\right\} < 0$$

has no solution. If, on the contrary, there was a solution $(\overline{x}, \overline{t}) \in \mathbb{R}^{n+1}$ such that

$$\frac{1}{2}\overline{x}^{\top}A\overline{x} + \overline{t}a^{\top}\overline{x} + \overline{t}^{2}\gamma < 0, \quad \max\left\{\frac{1}{2}\overline{x}^{\top}B\overline{x} + \overline{t}b^{\top}\overline{x} - \overline{t}^{2}\beta : (B,b) \in \mathcal{B}_{b}\right\} < 0, \\ -\min\left\{\frac{1}{2}\overline{x}^{\top}B\overline{x} + \overline{t}b^{\top}\overline{x} - \overline{t}^{2}\alpha : (B,b) \in \mathcal{B}_{b}\right\} < 0,$$

we immediately reach a contradiction in case $\overline{t} \neq 0$. So, suppose that $\overline{t} = 0$. Then, the system (3.5)-(3.6) reduces to

$$\frac{1}{2}\overline{x}^{\top}A\overline{x} < 0, \quad \max\left\{\frac{1}{2}\overline{x}^{\top}(B_1 + \mu B_2)\overline{x} : \mu \in [\mu_1, \mu_2]\right\} < 0,$$
$$-\min\left\{\frac{1}{2}\overline{x}^{\top}(B_1 + \mu B_2)\overline{x} : \mu \in [\mu_1, \mu_2]\right\} < 0,$$

which is impossible to hold. Thus, the claim is proved.

On the other hand, observe that for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, the minimum and maximum values in (3.5)-(3.6) are achieved in, at least, one of the extreme points of the rectangle $[\mu_1, \mu_2] \times [\delta_1, \delta_2]$, that is, in one of the elements $(B_1 + \mu_1 B_2, b_1 + \delta_1 b_2)$, $(B_1 + \mu_1 B_2, b_1 + \delta_2 b_2)$, $(B_1 + \mu_2 B_2, b_1 + \delta_1 b_2)$ or $(B_1 + \mu_2 B_2, b_1 + \delta_2 b_2)$. Then, the nonexistence of solution to the system (3.5)-(3.6) is equivalent to the nonexistence of $y \in \mathbb{R}^{n+1}$ solution to the system

$$\frac{1}{2}y^{\top}H_{0}y < 0$$

$$\max\left\{\frac{1}{2}y^{\top}(W_{1_{\beta}} + \mu W_{2})y : \mu \in [\mu_{1}, \mu_{2}]\right\} < 0,$$

$$\max\left\{\frac{1}{2}y^{\top}(W_{2_{\beta}} + \mu W_{2})y : \mu \in [\mu_{1}, \mu_{2}]\right\} < 0.$$

$$-\min\left\{\frac{1}{2}y^{\top}(W_{1_{\alpha}} + \mu W_{2})y : \mu \in [\mu_{1}, \mu_{2}]\right\} < 0.$$

$$-\min\left\{\frac{1}{2}y^{\top}(W_{2_{\alpha}} + \mu W_{2})y : \mu \in [\mu_{1}, \mu_{2}]\right\} < 0.$$

This means that $(0, 0, 0, 0, 0) \notin \Omega_W$.

Step 2: A first use of a separation result. Since Ω_W is convex, there exist $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^5_+ \setminus \{0\}$ such that for all $y \in \mathbb{R}^{n+1}$

$$\lambda_{0} \left(\frac{1}{2}y^{\top}H_{0}y\right) + \lambda_{1} \left(\max_{\mu \in [\mu_{1}, \mu_{2}]} \frac{1}{2}y^{\top}(W_{1_{\beta}} + \mu W_{2})y\right) + \lambda_{2} \left(\max_{\mu \in [\mu_{1}, \mu_{2}]} \frac{1}{2}y^{\top}(W_{2_{\beta}} + \mu W_{2})y\right) + \lambda_{3} \left(-\min_{\mu \in [\mu_{1}, \mu_{2}]} \frac{1}{2}y^{\top}(W_{1_{\alpha}} + \mu W_{2})y\right) + \lambda_{4} \left(-\min_{\mu \in [\mu_{1}, \mu_{2}]} \frac{1}{2}y^{\top}(W_{2_{\alpha}} + \mu W_{2})y\right) \geq 0.$$

Since each of the minimum or maximum values are achieved in either μ_1 or μ_2 , we obtain for all $y \in \mathbb{R}^{n+1}$,

$$\begin{split} \lambda_0 \left(\frac{1}{2} y^\top H_0 y \right) + \lambda_1 \left(\max \left\{ \frac{1}{2} y^\top (W_{1_\beta} + \mu_1 W_2) y, \frac{1}{2} y^\top (W_{1_\beta} + \mu_2 W_2) y \right\} \right) + \\ \lambda_2 \left(\max \left\{ \frac{1}{2} y^\top (W_{2_\beta} + \mu_1 W_2) y, \frac{1}{2} y^\top (W_{2_\beta} + \mu_2 W_2) y \right\} \right) + \\ \lambda_3 \left(- \min \left\{ \frac{1}{2} y^\top (W_{1_\alpha} + \mu_1 W_2) y, \frac{1}{2} y^\top (W_{1_\alpha} + \mu_2 W_2) y \right\} \right) + \\ \lambda_4 \left(- \min \left\{ \frac{1}{2} y^\top (W_{2_\alpha} + \mu_1 W_2) y, \frac{1}{2} y^\top (W_{2_\alpha} + \mu_2 W_2) y \right\} \right) \ge 0. \end{split}$$

Thus, there is no $y \in \mathbb{R}^{n+1}$ solution to the system:

$$\begin{split} &\frac{1}{2}y^{\top}\Big(\lambda_{0}H_{0}+\lambda_{1}(W_{1_{\beta}}+\mu_{1}W_{2})+\lambda_{2}(W_{2_{\beta}}+\mu_{1}W_{2})-\lambda_{3}(W_{1_{\beta}}+\mu_{2}W_{2})-\lambda_{4}(W_{2_{\beta}}+\mu_{2}W_{2})\Big)y<0;\\ &\frac{1}{2}y^{\top}\Big(\lambda_{0}H_{0}+\lambda_{1}(W_{1_{\beta}}+\mu_{2}W_{2})+\lambda_{2}(W_{2_{\beta}}+\mu_{2}W_{2})-\lambda_{3}(W_{1_{\beta}}+\mu_{1}W_{2})-\lambda_{4}(W_{2_{\beta}}+\mu_{1}W_{2})\Big)y<0.\\ &\text{This means that }(0,0)\notin\Omega_{2}, \text{ where} \end{split}$$

$$\Omega_{2} \doteq \left\{ \left(y^{\top} (\lambda_{0}H_{0} + \lambda_{1}H_{1,1_{\beta}} + \lambda_{2}H_{1,2_{\beta}} - \lambda_{3}H_{2,1_{\beta}} - \lambda_{4}H_{2,2_{\beta}}) y, \right. \\ y^{\top} (\lambda_{0}H_{0} + \lambda_{1}H_{2,1_{\beta}} + \lambda_{2}H_{2,2_{\beta}} - \lambda_{3}H_{1,1_{\beta}} - \lambda_{4}H_{1,2_{\beta}}) y \right\} : y \in \mathbb{R}^{n+1} \right\},$$

with

$$\begin{split} W_{1_{\beta}} &+ \mu_1 W_2 = H_{1,1_{\beta}}, \quad W_{1_{\beta}} + \mu_2 W_2 = H_{2,1_{\beta}}, \\ W_{2_{\beta}} &+ \mu_1 W_2 = H_{1,2_{\beta}}, \quad W_{2_{\beta}} + \mu_2 W_2 = H_{2,2_{\beta}}, \\ W_{1_{\alpha}} &+ \mu_1 W_2 = H_{1,1_{\alpha}}, \quad W_{1_{\alpha}} + \mu_2 W_2 = H_{2,1_{\alpha}}, \\ W_{2_{\alpha}} &+ \mu_1 W_2 = H_{1,2_{\alpha}}, \quad W_{2_{\alpha}} + \mu_2 W_2 = H_{2,2_{\alpha}}. \end{split}$$

Step 3: A second use of a separation result and conclusion. By the Dines theorem, Ω_2 is convex. Thus, there exists $(\xi_1, \xi_2) \in \mathbb{R}^2_+ \setminus \{0\}$ such that for all $y \in \mathbb{R}^{n+1}$,

$$\xi_{1}y^{\top} \Big(\lambda_{0}H_{0} + \lambda_{1}H_{1,1_{\beta}} + \lambda_{2}H_{1,2_{\beta}} - \lambda_{3}H_{2,1_{\beta}} - \lambda_{4}H_{2,2_{\beta}}\Big)y \\ + \xi_{2}y^{\top} \Big(\lambda_{0}H_{0} + \lambda_{1}H_{2,1_{\beta}} + \lambda_{2}H_{2,2_{\beta}} - \lambda_{3}H_{1,1_{\beta}} - \lambda_{4}H_{1,2_{\beta}}\Big)y \ge 0$$

In particular, for y = (x, 1) with $x \in \mathbb{R}^n$, and by setting

$$\begin{split} \overline{\lambda}_0 &= \lambda_0(\xi_1 + \xi_2), \ \overline{\lambda}_1 = (\lambda_1 + \lambda_2)(\xi_1 + \xi_2), \ \overline{\lambda}_2 = (\lambda_3 + \lambda_4)(\xi_1 + \xi_2), \\ \mu_\beta &= \frac{\xi_1 \mu_1 + \xi_2 \mu_2}{\xi_1 + \xi_2}, \ \mu_\alpha = \frac{\xi_1 \mu_2 + \xi_2 \mu_1}{\xi_1 + \xi_2}, \ \delta_\beta = \frac{\lambda_1 \delta_1 + \lambda_2 \delta_2}{\lambda_1 + \lambda_2}, \text{ and } \delta_\alpha = \frac{\lambda_3 \delta_2 + \lambda_4 \delta_1}{\lambda_3 + \lambda_4}, \\ \text{one gets } (\overline{\lambda}_0, \overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^3_+ \backslash \{0\}, \ \mu_\beta, \mu_\alpha \in [\mu_1, \mu_2], \ \delta_\beta, \delta_\alpha \in [\delta_1, \delta_2] \text{ and} \\ \overline{\lambda}_0 \Big(\frac{1}{2} x^\top A x + a^\top x + \gamma \Big) + \overline{\lambda}_1 \Big(\frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \beta \Big) + \\ \overline{\lambda}_2 \Big(\alpha - \Big(\frac{1}{2} x^\top (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x \Big) \Big) \geq 0 \quad \forall \ x \in \mathbb{R}^n, \\ \text{where } \mu_\alpha + \mu_\beta = \mu_1 + \mu_2. \text{ This proves that } (b) \text{ holds.} \qquad \Box \qquad \Box$$

Theorem 4. (A robust S-lemma) Let $A, B_1, B_2 \in S^n$, $a, b_1, b_2 \in \mathbb{R}^n$ and $\gamma, \alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R}$, with $\mu_1 \leq \mu_2, \delta_1 \leq \delta_2$ and $\alpha < \beta$. Assume that Ω_W is convex and that there exists $x_0 \in \mathbb{R}^n$ satisfying (3.7)

$$\alpha < \frac{1}{2} x_0^\top (B_1 + \mu B_2) x_0 + (b_1 + \delta b_2)^\top x_0 < \beta, \ \forall \ \mu \in [\mu_1, \mu_2], \forall \ \delta \in [\delta_1, \delta_2].$$

Then, the following two assertions are equivalent:

$$(a) \quad \alpha \leq \frac{1}{2} x^{\top} (B_1 + \mu B_2) x + (b_1 + \delta b_2)^{\top} x \leq \beta, \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2] \\\Rightarrow \frac{1}{2} x^{\top} A x + a^{\top} x + \gamma \geq 0.$$

$$(b) \quad \exists \ (\lambda_1, \lambda_2) \in \mathbb{R}^2_+, \exists \ \mu_{\alpha}, \mu_{\beta} \in [\mu_1, \mu_2], \exists \ \delta_{\alpha}, \delta_{\beta} \in [\delta_1, \delta_2] : \forall x \in \mathbb{R}^n \\\frac{1}{2} x^{\top} A x + a^{\top} x + \gamma + \lambda_1 \Big(\frac{1}{2} x^{\top} (B_1 + \mu_{\beta} B_2) x + (b_1 + \delta_{\beta} b_2)^{\top} x - \beta \Big) + \\\lambda_2 \Big(\alpha - \Big(\frac{1}{2} x^{\top} (B_1 + \mu_{\alpha} B_2) x + (b_1 + \delta_{\alpha} b_2)^{\top} x \Big) \Big) \geq 0,$$

where $\mu_{\alpha} + \mu_{\beta} = \mu_1 + \mu_2$.

Proof. Clearly $(b) \Rightarrow (a)$.

Assume now that (a) is satisfied. Then (a) in Theorem 3 does not hold. Thus (b) of the same theorem fulfills, but then λ_0 is strictly positive because of (3.7), which implies the desired result. \Box

We are now ready to establish a characterization of optimality for the problem (1.1).

Theorem 5. (Characterizing robust optimality) Let $A, B_1, B_2 \in S^n$, $a, b_1, b_2 \in \mathbb{R}^n$ and $\alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R}$, with $\mu_1 \leq \mu_2, \delta_1 \leq \delta_2$ and $\alpha < \beta$. Let \overline{x} be feasible for problem (1.1) and put $\gamma = -f(\overline{x})$. Assume that Ω_W is convex, and the Slater-type condition (3.7) holds. Then, \overline{x} is optimal if, and only if there exist $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+$, $\mu_\alpha, \mu_\beta \in [\mu_1, \mu_2]$, $\delta_\alpha, \delta_\beta \in [\delta_1, \delta_2]$ such that the following statements are satisfied:

$$(a) \left(A + \lambda_1(B_1 + \mu_{\beta}B_2) - \lambda_2(B_1 + \mu_{\alpha}B_2)\right)\overline{x} = -\left(a + \lambda_1(b_1 + \delta_{\beta}b_2) - \lambda_2(b_1 + \delta_{\alpha}b_2)\right);$$

$$(b) \lambda_1 \left(\frac{1}{2}\overline{x}^{\top}(B_1 + \mu_{\beta}B_2)\overline{x} + (b_1 + \delta_{\beta}b_2)^{\top}\overline{x} - \beta\right) = 0;$$

$$\lambda_2 \left(\alpha - \left(\frac{1}{2}\overline{x}^{\top}(B_1 + \mu_{\alpha}B_2)\overline{x} + (b_1 + \delta_{\alpha}b_2)^{\top}\overline{x}\right)\right) = 0;$$

$$(c) A + \lambda_1(B_1 + \mu_{\beta}B_2) - \lambda_2(B_1 + \mu_{\alpha}B_2) \succeq 0.$$

Proof. The necessary condition follows from Theorem 4 where γ is substituted by $-f(\overline{x})$. Indeed, if \overline{x} is optimal then (a) in Theorem 4 holds, which means that (b) of the same theorem is also satisfied. This finally implies the desired statements.

The sufficiency part is already standard since (c) implies the convexity of the function

$$h(x) \doteq \frac{1}{2}x^{\top}Ax + a^{\top}x - f(\overline{x}) + \lambda_1 \left(\frac{1}{2}x^{\top}(B_1 + \mu_{\beta}B_2)x + (b_1 + \delta_{\beta}b_2)^{\top}x - \beta\right) +$$

$$\lambda_2 \left(\alpha - \left(\frac{1}{2} x^\top (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x \right) \right),$$

and (a) and (b) allow us to prove that \overline{x} is in fact a solution to problem (1.1).

4. Revisiting the case $\alpha = -\infty$

We consider the problem:

(4.8)

$$\min \frac{1}{2} x^{\top} A x + a^{\top} x$$
s.t. $\frac{1}{2} x^{\top} (B_1 + \mu B_2) x + (b_1 + \delta b_2)^{\top} \leq \beta, \ \forall \ \mu \in [\mu_1, \mu_2], \forall \ \delta \in [\delta_1, \delta_2],$

where A, B₁, B₂ are real symmetric matrices, $\mu_1, \mu_2, \beta \in \mathbb{R}$ satisfying $\mu_1 < \mu_2, \delta_1 < \delta_2$. This problem was also discussed in [7], and actually this paper motivated our study.

By looking at carefully the proof of Theorem 3, we immediately realize that in case there is no lower bound in the inequality constraint, all the terms where α appears, actually dissapear: they are superfluous. Hence, the set Ω_W reduces to

Thus, by Corollary 1 in [6], Ω_W^β is convex if the set

$$\Omega_{\mu}^{\beta} \doteq \left\{ \left(y^{\top} H_0 y, y^{\top} (W_{1_{\beta}} + \mu_1 W_2) y, y^{\top} (W_{2_{\beta}} + \mu_1 W_2) y, y^{\top} (W_{1_{\beta}} + \mu_2 W_2) y, y^{\top} (W_{2_{\beta}} + \mu_2 W_2) y \right\} : y \in \mathbb{R}^{n+1} \right\} + \text{int } \mathbb{R}^5_+.$$

Theorem 6. Let the data be as described above. Let \overline{x} be feasible for problem (4.8) and put $\gamma = -f(\overline{x})$. Assume that Ω_W^β is convex, and the Slater-type condition: there exists $x_0 \in \mathbb{R}^n$ such that

$$(4.10) \quad \frac{1}{2} x_0^\top (B_1 + \mu B_2) x_0 + (b_1 + \delta b_2)^\top x_0 < \beta, \ \forall \ \mu \in [\mu_1, \mu_2], \ \forall \ \delta \in [\delta_1, \delta_2]$$

is satisfied. Then, \overline{x} is optimal if, and only if there exist $\lambda \ge 0$, $\mu \in [\mu_1, \mu_2]$, $\delta \in [\delta_1, \delta_2]$ such that the following statements are satisfied:

(a)
$$\left(A + \lambda(B_1 + \mu B_2)\right)\overline{x} = -\left(a + \lambda(b_1 + \delta b_2)\right);$$

(b) $\lambda \left(\frac{1}{2}\overline{x}^{\top}(B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^{\top}\overline{x} - \beta\right) = 0;$
(c) $A + \lambda(B_1 + \mu B_2) \succeq 0.$

We recall that

$$H_0 = \begin{pmatrix} A & a \\ a^\top & 2\gamma \end{pmatrix},$$

with $\gamma = -f(\overline{x})$ as in the previous theorem. Set $H_1 \doteq H_{1,1_\beta}, \ H_2 \doteq H_{2,2_\beta}, H_3 \doteq H_{1,2_\beta}, \ H_4 \doteq H_{2,1_\beta}.$ The authors in [7, Theorem 5.1] proved the same result expressed in Theorem 6 under the convexity of the set

(4.11)
$$\left\{ \left(y^{\top} H_0 y, y^{\top} H_1 y, y^{\top} H_2 y \right) : y \in \mathbb{R}^{n+1} \right\} + \text{int } \mathbb{R}^3_+;$$

whereas ours requires the convexity of

$$\Omega^{\beta}_{\mu} = \left\{ \left(y^{\top} H_0 y, y^{\top} H_1 y, y^{\top} H_2 y, y^{\top} H_3 y, y^{\top} H_4 y \right) : y \in \mathbb{R}^{n+1} \right\} + \text{int } \mathbb{R}^5_+.$$

We believe that there is a gap in the proof of Theorem 5.1 in [7]. More precisely, the authors assert in page 221 of the same paper that the nonexistence of solution to the system (notice that our β is $-\beta$ in [7]) (4.12)

$$\frac{1}{2}x^{\top}Ax + ta^{\top}x + t^{2}\gamma < 0, \ \max\left\{\frac{1}{2}x^{\top}Bx + tb^{\top}x - t^{2}\beta : (B,b) \in \mathcal{B}_{b}\right\} < 0,$$

implies that

(4.13)
$$\frac{1}{2}y^{\top}H_0y < 0$$
 and $\forall \mu \in [\mu_1, \mu_2], \quad \frac{1}{2}y^{\top}(W_1 + \mu W_2)y < 0,$

has no solution. This is not necessarily true as Example 7 below shows. We recall that

$$W_{1} = \begin{pmatrix} B_{1} & b_{1} + \frac{\delta_{1}\mu_{2} - \delta_{2}\mu_{1}}{\mu_{2} - \mu_{1}}b_{2} \\ (b_{1} + \frac{\delta_{1}\mu_{2} - \delta_{2}\mu_{1}}{\mu_{2} - \mu_{1}}b_{2})^{\top} & -2\beta \end{pmatrix} \text{ and } \\ W_{2} = \begin{pmatrix} B_{2} & \frac{\delta_{2} - \delta_{1}}{\mu_{2} - \mu_{1}}b_{2} \\ \frac{\delta_{2} - \delta_{1}}{\mu_{2} - \mu_{1}}b_{2}^{\top} & 0 \end{pmatrix}.$$

Example 7. Let $n \ge 5$. Taking $A = I_n$, $B_1 = B_2 = 2I_n$, $a = b_1 = b_2 = s := (1, 1, ..., 1) \in \mathbb{R}^n$, $\mu_1 = -1$, $\mu_2 = 1$, $\delta_1 = -1$, $\delta_2 = 1$ and $\beta = 1$, Problem (4.8) takes the form

$$\min \frac{1}{2} \|x\|^2 + \sum_{i=1}^n x_i = \frac{1}{2} \|x+s\|^2 - \frac{n}{2}$$

s.t. $g(x,\mu,\delta) := (1+\mu) \|x\|^2 + (1+\delta) \sum_{i=1}^n x_i - 1 \le 0 \ \forall \ \mu, \delta \in [-1,1].$

Let us define the functions

$$f(x) := \frac{1}{2} \|x\|^2 + \sum_{i=1}^n x_i = \frac{1}{2} \|x+s\|^2 - \frac{n}{2};$$
$$g(x,\mu,\delta) := (1+\mu) \|x\|^2 + (1+\delta) \sum_{i=1}^n x_i - 1.$$

Then, problem (4.14) is equivalent to

$$\min f(x)$$

(4.15)
$$s.t. ||x||^2 - \frac{1}{2} \le 0;$$

(4.16)
$$\sum_{i=1}^{n} x_i - \frac{1}{2} \le 0;$$

(4.17)
$$||x||^2 + \sum_{i=1}^n x_i - \frac{1}{2} \le 0.$$

Let C be the set of constraints, that is, those x satisfying (4.15)-(4.17). It is clear that C is convex and compact. Thus, the unique solution to problem (4.14) is the projection of -s on C, which is $\overline{x} = -\frac{1}{\sqrt{2n}}s$. Hence, \overline{x} is a robust solution to problem (4.14). In this case, $\gamma = -f(\overline{x}) = \sqrt{\frac{n}{2}} - \frac{1}{4}$. By identifying the matrices involved in Theorem 6, we get

$$H_{0} = \begin{pmatrix} I_{n} & s \\ s^{\top} & 2\gamma \end{pmatrix}; H_{1} = \begin{pmatrix} \overline{0} & 0 \\ 0^{\top} & -2 \end{pmatrix}; H_{2} = \begin{pmatrix} 4I_{n} & 2s \\ 2s^{\top} & -2 \end{pmatrix}; H_{3} = \begin{pmatrix} \overline{0} & 2s \\ 2s^{\top} & 2\beta \end{pmatrix}; \quad H_{4} = \begin{pmatrix} 4I_{n} & 0 \\ 0^{\top} & 2\beta \end{pmatrix}.$$

Since

(4.18)
$$-2H_0 + (-2-2\gamma)H_1 + H_2 = \begin{pmatrix} 2I_n & 0\\ 0^{\top} & 2 \end{pmatrix} \succ 0,$$

applying [10, Theorem 2.1] we have that

(4.19)
$$\left\{ \left(y^{\top} H_0 y, y^{\top} H_1 y, y^{\top} H_2 y \right) : y \in \mathbb{R}^{n+1} \right\}$$

is convex. This means, according to [7], that (4.14) is regular with respect to \overline{x} . Also, by taking $x_0 = 0$, we have $g(x_0, \mu, \delta) < 0$, for all $\mu, \delta \in [-1, 1]$. So, we have all the conditions of [7, Theorem 5.1] are satisfied. From the first part of the proof of [7, Theorem 5.1] we have that the following homogeneous system in \mathbb{R}^{n+1} , (4.12) has no solution.

Coming back to our example, we obtain

$$W_1 = \begin{pmatrix} 2I_n & s \\ s^{\top} & -2 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} 2I_n & s \\ s^{\top} & 0 \end{pmatrix}.$$

Then, the following homogeneous system in \mathbb{R}^{n+1} (see (4.13))

$$\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^{\top} H_0 \begin{pmatrix} x \\ t \end{pmatrix} < 0 \text{ and } \forall \mu \in [-1,1], \ \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^{\top} (W_1 + \mu W_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0,$$

becomes

(4.20)
$$\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^{\top} \begin{pmatrix} I_n & s \\ s^{\top} & 2\gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} < 0 \text{ and}$$

(4.21)
$$\forall \mu \in [-1,1], \ \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^{\top} \begin{pmatrix} 2(1+\mu)I_n & (1+\mu)s \\ (1+\mu)s^{\top} & -2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} < 0.$$

We will see that such a system admits a solution, contradicting the assertion made in page 221 of [7] about the nonexistence of solution to the same system. Indeed, for $(-s, 1) \in \mathbb{R}^{n+1}$, (4.20) reduces to

$$\frac{1}{2} \begin{pmatrix} -s \\ 1 \end{pmatrix}^{\top} \begin{pmatrix} I_n & s \\ s^{\top} & 2\gamma \end{pmatrix} \begin{pmatrix} -s \\ 1 \end{pmatrix} = \frac{1}{2} (-n+2\gamma) = \frac{1}{2} \begin{pmatrix} -n+\sqrt{\frac{n}{2}} - \frac{1}{4} \end{pmatrix} < 0;$$

whereas (4.21) becomes: for all $\mu \in [-1, 1]$,

$$\frac{1}{2} \begin{pmatrix} -s \\ 1 \end{pmatrix}^{\top} \begin{pmatrix} 2(1+\mu)I_n & (1+\mu)s \\ (1+\mu)s^{\top} & -2 \end{pmatrix} \begin{pmatrix} -s \\ 1 \end{pmatrix} = \frac{1}{2}(n(1+\mu)-(1+\mu)n-2) = -1 < 0.$$

This proves our claim.

Observe that taking in Corollary 2: m = 4, $A_0 = I_n$, $\rho_1 = \rho_3 = 4$, $\rho_2 = \rho_4 = 0$, $a_0 = s$, $a_1 = a_3 = 0$, $a_2 = a_4 = 2s$, $c_0 = \gamma$, $c_1 = c_2 = -1$, $c_3 = c_4 = \beta$ we obtain that

$$\left\{ \left(y^{\top} H_0 y, y^{\top} H_1 y, y^{\top} H_2 y, y^{\top} H_3 y, y^{\top} H_4 y \right) : y \in \mathbb{R}^{n+1} \right\} \subset \mathbb{R}^5$$

is a convex set, which is required in our Theorem 6, providing the characterization of robust optimality for our example. $\hfill \Box$

Clearly, the previous example only shows there is a gap in the proof of Theorem 5.1 in [7]. We were unable to construct a real counterexample to that result under the convexity either of the set given in (4.11) or in (4.19).

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