Word of low complexity without uniform frequencies

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Abstract: In this paper, we construct a uniformely recurrent infinite word of low complexity without uniform frequencies of letters. This shows the optimality of a bound of Boshernitzan, which gives a sufficient condition for a uniformly recurrent infinite word to admit uniform frequencies.

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1 Introduction

Let us consider an infinite word u over a finite alphabet. We can naturally associate to it a subshift. The goal of this paper is to describe some ergodic properties of this subshift. By Oxtoby theorem, we know that the subshift is uniquely ergodic if and only if, in u, each finite word has uniform frequency. Moreover the subshift is minimal if u is uniformly recurrent.

For a long time, people have tried to find some conditions on infinite words which imply one of these properties. M. Keane gave in [10] a uniformly recurrent infinite word with complexity 3n+1 (from 4-interval exchange map) which does not possess uniform frequencies. Later, Boshernitzan, in [1], proved that a uniformly recurrent infinite word admits uniform frequencies if either of the following sufficient conditions is satisfied:

$$\liminf \frac{\mathbf{p}(n)}{n} < 2 \text{ or } \limsup \frac{\mathbf{p}(n)}{n} < 3$$

where \mathbf{p} denotes the complexity function of u (see [3] chap. 4, by J. Cassaigne and F. Nicolas for more details on the subject).

The bound of the second sufficient condition of Boshernitzan being already optimal by Keane's result, our goal in his work is to establish the

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optimality of the bound of the first sufficient condition. And we succeed to construct a uniformly recurrent infinite word without uniform frequencies, the complexity function of which verifies $\liminf \frac{\mathbf{p}(n)}{n} = 2$.

This result relates some properties of the complexity function and the ergodic measures of the subshift. This type of question has been investigated in the last years. The goal is to bound the number of ergodic measures of the subshift in terms of the complexity function.

Boshernitzan was the first to look at it, see [1]. During his Phd. thesis T. Monteil, see [11] and [3] (chap. 3 by S. Ferenczi and Th. Monteil) has proved, the same result with different techniques: If $\limsup \frac{\mathbf{p}(n)}{n} = K \ge 2$, then the subshift has at most K - 2 ergodic measures. Since this time, some results have appeared in the same veine: V. Cyr and B. Kra have also obtained similar results, see [6, 7]. In the first paper, they prove that the bound of Boshernitzan is sharp. In the second one, they construct minimal subshifts with complexity function arbitrarily close to linear but having uncountably many ergodic measures. We can also cite M. Damron and D. Fickenscher [8] who obtained the bound $\frac{K+1}{2}$ under a condition on the bispecial words.

Nevertheless, it seems that our proof is of a different nature, with an explicit construction of the infinite word.

After the preliminaries (section 2) we construct an infinite word which is uniformly recurrent in section 3, then we show in section 4 that this word is without uniform frequencies of letters and to finish we study the complexity of this word in section 5 and give in section 6 the proof of the main statement of section 3.

2 Preliminaries

In all that follows we consider the alphabet $\mathcal{A} = \{0, 1\}$. Let us denote \mathcal{A}^* , the set of the finite words on alphabet \mathcal{A} , ε the empty word. For all u in \mathcal{A}^* , |u| denotes the length (the number of letters it contains) of the word u $(|\varepsilon| = 0)$ and for any letter x of \mathcal{A} , $|u|_x$ is the number of occurrences of the letter x in u. We call Parikh vector of a finite word u, the vector denoted by U and defined by $\binom{|u|_0}{|u|_1}$.

A finite word u of length n formed by repeating a single letter x is typically denoted x^n . We define the n-th power of a finite word w as being the concatenation of n copies of w; we denote it w^n . An infinite word is an infinite sequence of letters of \mathcal{A} . We denote \mathcal{A}^{ω} the set of infinite words on \mathcal{A} . We say a finite word v is a factor of u if there exist two words u_1 and u_2 on the alphabet \mathcal{A} such that $u = u_1 v u_2$; we also say that u contains v. The factor v is said prefix (resp. suffix) if u_1 (resp. u_2) is the empty word. For any word u, the set of factors of length n is denoted $\mathcal{L}_n(u)$. The set of all factors of u is simply denoted $\mathcal{L}(u)$.

Definition 2.1. Let u be an infinite word on the alphabet $\mathcal{A} = \{0, 1\}$. A factor v of u is said to be

- a right special factor if v0 and v1 are both factors of u, and a left special factor if 0v and 1v are both factors of u.
- a bispecial factor of u if v is simultaneously a right special factor and a left special factor of u.
- a strong bispecial factor of u if 0v0, 0v1, 1v0, 1v1 are factors of u and a weak bispecial factor if uniquely 0v0 and 1v1, or 0v1 and 1v0, are factors of u.
- an ordinary bispecial factor of u if v is a bispecial factor of u which is neither strong nor weak.

An infinite word u is said to be recurrent if any factor of u appears infinitely often. An infinite word u is uniformly recurrent if for all $n \in \mathbb{N}$, there exists N such that any factor of u of length N contains all the factors of u of length n.

Definition 2.2. Let u be an infinite word on an alphabet \mathcal{A} . The complexity function of u is a function counting the number of distinct factor of u of length n for any given n. It is denoted \mathbf{p} and so that:

$$\mathbf{p}(n) = \# \mathcal{L}_n(u).$$

Let us denote **s** and **b** the functions respectively called first difference and second difference of the complexity of u; they are defined as follows: $\mathbf{s}(n) = \mathbf{p}(n+1) - \mathbf{p}(n)$ and $\mathbf{b}(n) = \mathbf{s}(n+1) - \mathbf{s}(n)$.

On a binary alphabet the function **s** counts the number of special factors for a given length in u. Let us denote **m** the map from $\mathcal{L}(u)$ to $\{-1, 0, +1\}$ defined by

$$\forall v \in \mathcal{L}(u), \ \mathbf{m}(v) = \begin{cases} -1 & \text{if } v \text{ is weak bispecial} \\ +1 & \text{if } v \text{ is strong bispecial} \\ 0 & \text{otherwise} \end{cases}$$

The following formula was given by the first author in [5]:

$$\forall n \ge 0, \ \mathbf{s}(n) = 1 + \sum_{\substack{w \in \mathcal{L}(u) \\ |w| < n}} \mathbf{m}(w) = 1 + \sum_{\substack{w \text{ bispecial} \\ |w| < n}} \mathbf{m}(w).$$

This relation allows to compute the complexity $\mathbf{p}(n)$ provided when we are able to describe the set of strong and weak bispecial factors of the binary infinite word u.

Definition 2.3. Two bispecial factors v and w of an infinite word u on the alphabet $\{0, 1\}$ are said to have the same type if they are all strong, weak, or ordinary. In other words the bispecial v and w have the same if $\mathbf{m}(v) = \mathbf{m}(w)$.

Definition 2.4. ([3] chap. 7, by S. Ferinczi and T. Monteil) Let u be an infinite word on an alphabet \mathcal{A} .

- We say that u admits frequencies if for any factor w, and any sequence (u_n) of prefixes of u such that $\lim_{n\to\infty} = \infty$, then $\lim_{n\to\infty} \frac{|u_n|_w}{|u_n|}$ exists.
- We say that u admits uniform frequencies if for any factor w, and any sequence (u_n) of factors of u such that $\lim_{n\to\infty} = \infty$, then $\lim_{n\to\infty} \frac{|u_n|_w}{|u_n|}$ exists.

In [10], M. Keane gave an example of a uniformly recurrent infinite word with complexity 3n + 1 which does not possess uniform frequencies. Later, Boshernitzan [1] obtained the following results:

Theorem 2.1. Let u be an infinite word on an alphabet A. Then, u admits uniform frequencies if its complexity function verifies at least one of the following conditions:

•
$$\liminf \frac{\mathbf{p}(n)}{n} < 2,$$

•
$$\limsup \frac{\mathbf{p}(n)}{n} < 3$$

The example of Keane ensures that constant 3 is optimal in the second condition, *i.e.*, it cannot be replaced with a larger constant.

3 Construction of a class of uniformly recurrent words

Let (l_i) , (m_i) , (n_i) be three integer sequences which are strictly increasing and verify the following conditions:

- $l_i < m_i < n_i$,
- $\frac{m_i}{l_i}$ increases exponentially to $+\infty$,
- $\frac{n_i}{m_i}$ increases exponentially to $+\infty$.

Let us define in \mathcal{A}^* two sequences (u_i) and (v_i) in the following way: $u_0 = 0, v_0 = 1$ and for all $i \in \mathbb{N}, u_{i+1} = u_i^{m_i} v_i^{l_i}$ and $v_{i+1} = u_i^{m_i} v_i^{n_i}$. The sequence (u_i) converges towards an infinite word u.

For $i \ge 1$, consider the substitution σ_i defined by $\sigma_i(0) = 0^{m_i} 1^{l_i}$, $\sigma_i(1) = 0^{m_i} 1^{n_i}$. Then, we have

 $u_i = \sigma_0 \sigma_1 \sigma_2 \dots \sigma_{i-1}(0)$ and $v_i = \sigma_0 \sigma_1 \sigma_2 \dots \sigma_{i-1}(1)$.

Theorem 3.1. Any infinite word u so defined is uniformly recurrent.

The proof is given at the end of this paper.

4 The word u is without uniform frequencies

Lemma 4.1. For all $i \ge 1$ we have:

1.
$$\frac{|u_i|_0}{|u_i|} \ge \left(1 + \frac{l_0}{m_0}\right)^{-1} \prod_{j=1}^{i-1} \left(1 + \frac{l_j n_{j-1}}{m_j l_{j-1}}\right)^{-1}$$

2. $\frac{|v_i|_1}{|v_i|} \ge \prod_{j=0}^{i-1} \left(1 + \frac{m_j}{n_j}\right)^{-1}$.

Proof. • Lower bound on $\frac{|u_{i+1}|_0}{|u_{i+1}|}$.

Firstly, we have for all $i \ge 0$, $|u_i| \le |v_i|$ since $|u_0| = |v_0| = 1$ and u_i is a strict prefix of v_i for $i \ge 1$. Then $\frac{|v_i|}{|u_i|} = \frac{m_{i-1}|u_{i-1}| + n_{i-1}|v_{i-1}|}{m_{i-1}|u_{i-1}| + l_{i-1}|v_{i-1}|} \le \frac{n_{i-1}}{l_{i-1}}$ since $l_{i-1} < n_{i-1}$ for $i \ge 1$. As

$$|u_{i+1}|_0 = m_i |u_i|_0 + l_i |v_i|_0 \ge m_i |u_i|_0$$

 $\quad \text{and} \quad$

$$|u_{i+1}| = m_i |u_i| + l_i |v_i| = |u_i| \left(m_i + l_i \frac{|v_i|}{|u_i|} \right)$$

we deduce the following inequalities:

$$|u_{i+1}| \le |u_i| \left(m_i + l_i \frac{n_{i-1}}{l_{i-1}} \right)$$

and

$$\frac{|u_{i+1}|_0}{|u_{i+1}|} \ge \left(1 + \frac{l_i}{m_i} \frac{n_{i-1}}{l_{i-1}}\right)^{-1} \cdot \frac{|u_i|_0}{|u_i|}.$$

Thus

$$\frac{|u_i|_0}{|u_i|} \ge \frac{|u_1|_0}{|u_1|} \prod_{j=1}^{i-1} \left(1 + \frac{l_j n_{j-1}}{m_j l_{j-1}}\right)^{-1}$$

• Lower bound on $\frac{|v_{i+1}|_1}{|v_{i+1}|}$. We have

 $|v_{i+1}|_1 = m_i |u_i|_1 + n_i |v_i|_1 \ge n_i |v_i|_1$ and $|v_{i+1}| = m_i |u_i| + n_i |v_i| \le |v_i| (m_i + n_i)$ since $|u_i| \le |v_i|$. So

$$\frac{|v_{i+1}|_1}{|v_{i+1}|} \ge \frac{n_i}{m_i + n_i} \cdot \frac{|v_i|_1}{|v_i|}.$$

Hence

$$\frac{|v_i|_1}{|v_i|} \ge \prod_{j=0}^{i-1} \left(1 + \frac{m_i}{n_i}\right)^{-1}.$$

In the rest of the paper we need to fix

$$l_i = 2^{2 \cdot 2^i + 4}, m_i = 2^{8 \cdot 2^i}, \text{ and } n_i = 2^{10 \cdot 2^i}, \text{ for } i \ge 0.$$
 (*)

Then the inequalities of Lemma 4.1 become:

1.
$$\frac{|u_i|_0}{|u_i|} \ge \prod_{j=1}^i \frac{1}{1+2^{-2^j}}$$

2. $\frac{|v_i|_1}{|v_i|} \ge \prod_{j=1}^i \frac{1}{1+2^{-2^j}}$

So we get

Lemma 4.2.

$$\forall i \ge 1, \min\left(\frac{|u_i|_0}{|u_i|}, \frac{|v_i|_1}{|v_i|}\right) \ge \prod_{j=1}^i \frac{1}{1+2^{-2^j}}.$$

Then we have the following lemma:

Lemma 4.3.

$$\forall i \in \mathbb{N}, \ \frac{|u_i|_0}{|u_i|} + \frac{|v_i|_1}{|v_i|} \ge \frac{3}{2}.$$

Proof. • For i = 0, the inequality is evident.

• For $i \ge 1$, write: $P_i = \prod_{j=1}^i \frac{1}{1+2^{-2^j}}$. The sequence (P_i) is decreasing and satisfies the following induction formula: $P_{i+1} = \frac{1}{1+2^{-2^{i+1}}}P_i$. Let us show, by induction, that $\frac{4}{3}P_i = \frac{1}{1-2^{-2^{i+1}}}$. We have $\frac{4}{3}P_0 = \frac{1}{1-2^{-2}}$. Assuming that for some $i \ge 0$, $\frac{4}{3}P_i = \frac{1}{1-2^{-2^{i+1}}}$ it follows : $\frac{4}{3}P_{i+1} = \frac{4}{3}P_i \times \frac{1}{1+2^{-2^{i+1}}} = \frac{1}{1-2^{-2^{i+1}}} \times \frac{1}{1+2^{-2^{i+1}}} = \frac{1}{1-2^{-2^{i+2}}}$. So $P_i = \frac{3}{4} \times \frac{1}{1-2^{-2^{i+1}}}$.

Hence, with Lemma 4.2 we get

$$\frac{|u_i|_0}{|u_i|} + \frac{|v_i|_1}{|v_i|} \ge 2 \times \frac{3}{4} \times \frac{1}{1 - 2^{-2^{i+1}}} \ge \frac{3}{2}.$$

Lemma 4.4. The letters of the word u do not admit unform frequencies.

Proof. If the letters of u possessed uniform frequencies, then the frequencies of 0 and 1, respectively denoted $\mathbf{f}_u(0)$ and $\mathbf{f}_u(1)$, should verify $\mathbf{f}_u(0) = \lim_{i \to \infty} \frac{|u_i|_0}{|u_i|}$, $\mathbf{f}_u(1) = \lim_{i \to \infty} \frac{|v_i|_1}{|v_i|}$ and $\mathbf{f}_u(0) + \mathbf{f}_u(1) = 1$. That is contradictory with Lemma 4.3.

5 Complexity of u

To estimate the complexity of u we are going to observe its bispecial factors.

Notation 5.1. Let $h, i \in \mathbb{N}$. We denote $u_i^{(h)}$ the finite word $\sigma_h \sigma_{h+1} \sigma_{h+2} \dots \sigma_{h+i-1}(0)$ and $u^{(h)}$ the infinite word $\lim_{i\to\infty} u_i^{(h)}$. However $u_i^{(0)}$ and $u^{(0)}$ are simply denoted respectively u_i and u.

Definition 5.1. A factor of $u^{(h)}$ is said to be short if it does not contain 10 as a factor. A factor of $u^{(h)}$ which is not short is said to be long.

Lemma 5.1. (Synchronization lemma): Let w be a long factor of $u^{(h)}$. Then there exist $x, y \in \mathcal{A}$ and $v \in \mathcal{A}^*$ such that xvy is a factor of $u_i^{(h+1)}$ and $w = s\sigma_h(v) p$, where s is a non-empty suffix of $\sigma_h(x)$, and p is a non-empty prefix of $\sigma_h(y)$. Moreover, the triple (s, v, p) is unique.

Proof. Since w is long, it cannot occur inside the image of one letter. Any occurrence of w in u is therefore of the form $s\sigma_h(v)p$, so existence follows.

Uniqueness is consequence of the fact that 10 occurs in $u^{(h)}$ only at the border between to images of letters under σ_h .

Lemma 5.2. 1. The short and strong bispecial factors of $u^{(h)}$ are ε and 1^{l_h} .

2. The short and weak bispecial factors of $u^{(h)}$ are 0^{m_h-1} and 1^{n_h-1} .

Proof. Let us first observe that in $u^{(h)}$, the factor 01 is always preceded by 10^{m_h-1} . Therefore a bispecial factor containing 01 must also contain 10 and is long.

Then the short bispecial factors are all of the form 0^k or 1^k , $k \ge 0$. We see that ε is strong bispecial (extensions 00, 01, 10, 11); 0^k ($0 \le k < m_h - 1$) is ordinary bispecial (extensions 00^k0 , 00^k1 , 10^k0), as well as 1^k ($1 \le k < n_h - 1$, $k \ne l_h$); 1^{l_h} is strong bispecial (extensions $01^{l_h}0$, $01^{l_h}1$, $11^{l_h}0$, $11^{l_h}1$); 0^{m_h-1} is weak bispecial (extensions $00^{m_h-1}1$ and $10^{m_h-1}0$), as well as 1^{n_h-1} ; 0^{m_h} and 1^{n_h} are not special, and 0^k ($k > m_h$) and 1^k ($k > n_h$) are not factors.

Lemma 5.3. Let w be a factor of $u^{(h)}$. Then the following assertions are equivalent:

(1) w is a long bispecial factor of $u^{(h)}$.

(2) There exists a bispecial factor v of $u^{(h+1)}$ such that $w = \hat{\sigma}_h(v)$ where $\hat{\sigma}_h(v) = 1^{l_h} \sigma_h(v) 0^{m_h} 1^{l_h}$.

Moreover v and w have the same type and |v| < |w|.

Proof. First, let us observe this fact: If a finite word v is a factor of $u^{(h+1)}$ then $\widehat{\sigma}_h(v) = 1^{l_h} \sigma_h(v) 0^{m_h} 1^{l_h}$ is a factor of $u^{(h)}$. Now, let us consider a bispecial factor v of $u^{(h+1)}$. Therefore the words $\widehat{\sigma}_h(0v)$, $\widehat{\sigma}_h(1v)$, $\widehat{\sigma}_h(v0)$ and $\widehat{\sigma}_h(v1)$ are factors of $u^{(h)}$; moreover $0\widehat{\sigma}_h(v)$ and $1\widehat{\sigma}_h(v)$ are respectively suffix of the first two words whereas $\widehat{\sigma}_h(v)0$ and $\widehat{\sigma}_h(v)1$ are respectively prefix of the last two words. Hence, the word $w = \widehat{\sigma}_h(v)$ is bispecial in $u^{(h)}$, and $\mathbf{m}(w) \ge \mathbf{m}(v)$.

Conversely, let w be a long bispecial factor of $u^{(h)}$. Then, according to the synchronization lemma, we can write w uniquely in the form $s\sigma_h(v)p$ where s and p are respectively non-empty suffix and prefix of images of letters.

As 0w and 1w are factors of $u^{(h)}$, and $\sigma_h(v)p$ starts with 0, it follows that 0s0 and 1s0 are factors of $u^{(h)}$. This is only possible if $s = 1^{l_h}$ ($s = 1^k$ with $1 \le k < l_h$ or $l_h < k < n_h$ are excluded since $0s0 \notin L(u^{(h)})$; $s = 0^k 1^{l_h}$ with $1 \le k < m_h$ and $s = 0^k 1^{n_h}$ with $0 \le k < m_h$ are excluded since $1s0 \notin \mathcal{L}(u^{(h)})$; and $s = 0^{m_h} 1^{l_h}$ and $s = 0^{m_h} 1^{n_h}$ are excluded since $0s0 \notin \mathcal{L}(u^{(h)})$.

Similarly, 1p0 and 1p1 are factors of $u^{(h)}$, and this is only possible if $p = 0^{m_h} 1^{l_h}$. Therefore $w = \hat{\sigma}_h(v)$.

If w extends as awb with $a, b \in \mathcal{A}$, then v also extends as avb. Therefore $\mathbf{m}(v) \geq \mathbf{m}(w)$. It follows that $\mathbf{m}(v) = \mathbf{m}(w)$: v and w have the same type. Moreover, it is clear that |v| < |w|.

In fact, long bispecial factors of $u^{(h)}$ are the images by $\hat{\sigma}_h$ of the "less long" bispecial factors of $u^{(h+1)}$. Thus, step by step, any non-ordinary bispecial factor w of $u^{(h)}$ of given type, will be write in the following form $\hat{\sigma}_h \hat{\sigma}_{h+1} \dots \hat{\sigma}_{h+i-1}(v)$ where v is a short bispecial factor of $u^{(h+i)}$ with the same type.

We will call bispecial factors of rank i, $(i \ge 0)$ of $u^{(h)}$, and write $a_i^{(h)}$, $b_i^{(h)}$, $c_i^{(h)}$, $d_i^{(h)}$ the following words

$$a_i^{(h)} = \widehat{\sigma}_h \widehat{\sigma}_{h+1} \dots \widehat{\sigma}_{h+i-1}(\varepsilon), \ b_i^{(h)} = \widehat{\sigma}_h \widehat{\sigma}_{h+1} \dots \widehat{\sigma}_{h+i-1}(1^{l_{h+i}}),$$
$$c_i^{(h)} = \widehat{\sigma}_h \widehat{\sigma}_{h+1} \dots \widehat{\sigma}_{h+i-1}(0^{m_{h+i}-1}) \text{ and } d_i^{(h)} = \widehat{\sigma}_h \widehat{\sigma}_{h+1} \dots \widehat{\sigma}_{h+i-1}(1^{n_{h+i}-1}).$$

The short bispecial ε , 1^{l_h} , 0^{m_h-1} and 1^{n_h-1} of $u^{(h)}$ are the bispecial factors of rank 0, $a_0^{(h)}$, $b_0^{(h)}$, $c_0^{(h)}$, and $d_0^{(h)}$.

The non-ordinary bispecial factors of u are therefore $a_i = a_i^{(0)}, b_i = b_i^{(0)}, c_i = c_i^{(0)}, d_i = d_i^{(0)}$.

Definition 5.2. Let $v, w \in \mathcal{A}^*$ and V, W be their corresponding Parikh vectors. Let us say that V is less than W and write V < W when $|v|_a \leq |w|_a$ for all $a \in \mathcal{A}$ and |v| < |w|.

Proposition 5.1. Let v, w, v', w' be four words such that $v' = \hat{\sigma}_i(v)$ and $w' = \hat{\sigma}_i(w)$. Then

$$V < W \Longrightarrow V' < W'.$$

Proof. Assume that V < W. Then, $|v|_0 \le |w|_0$, $|v|_1 \le |w|_1$, and |v| < |w|. On the one hand, we have $|v'|_0 = m_i (|v|+1)$ and $|w'|_0 = m_i (|w|+1)$; hence $|v'|_0 < |w'|_0$. On the other hand, we have $|v'|_1 = l_i |v|_0 + n_i |v|_1 + 2l_i$ and $|w'|_1 = l_i |w|_0 + n_i |w|_1 + 2l_i$; so $|v'|_1 \le |w'|_1$. Finally, $|v'| = |v'|_0 + |v'|_1 < |w'|_0 + |w'|_1 = |w'|_1$.

Lemma 5.4. For all $i \ge 0$, let A_i , B_i , C_i , D_i be the Parikh vectors corresponding to the non-ordinary bispecial factors of u, a_i , b_i , c_i , d_i . Then, we have

$$\forall i \ge 1, \ D_{i-1} < B_i < C_i < A_{i+1} < D_i$$

Proof. Applying $\widehat{\sigma}_{i-1}$ on the words $b_0^{(i)}$, $c_0^{(i)}$, $\widehat{\sigma}_i\left(a_0^{(i+1)}\right) = 1^{l_i}0^{m_i}1^{l_i}$, and $d_0^{(i)}$ we get the following words

$$\begin{cases} d_0^{(i-1)} = 1^{n_{i-1}-1} \\ b_1^{(i-1)} = 1^{l_{i-1}} (0^{m_{i-1}} 1^{n_{i-1}})^{l_i} 0^{m_{i-1}} 1^{l_{i-1}} \\ c_1^{(i-1)} = 1^{l_{i-1}} (0^{m_{i-1}} 1^{l_{i-1}})^{m_i-1} 0^{m_{i-1}} 1^{l_{i-1}} \\ a_2^{(i-1)} = 1^{l_{i-1}} (0^{m_{i-1}} 1^{n_{i-1}})^{l_i} (0^{m_{i-1}} 1^{l_{i-1}})^{m_i} (0^{m_{i-1}} 1^{n_{i-1}})^{l_i} 0^{m_{i-1}} 1^{l_{i-1}} \\ d_1^{(i-1)} = 1^{l_{i-1}} (0^{m_{i-1}} 1^{n_{i-1}})^{n_i-1} 0^{m_{i-1}} 1^{l_{i-1}}. \end{cases}$$

The Parikh vectors corresponding to these words are:

$$\begin{cases}
D_0^{(i-1)} = \begin{pmatrix} 0\\ n_{i-1} - 1 \end{pmatrix} \\
B_1^{(i-1)} = \begin{pmatrix} m_{i-1} (l_i + 1)\\ n_{i-1} l_i + 2 l_{i-1} \end{pmatrix} \\
C_1^{(i-1)} = \begin{pmatrix} m_i m_{i-1}\\ l_{i-1} (m_i + 1) \end{pmatrix} \\
A_2^{(i-1)} = \begin{pmatrix} m_{i-1} (m_i + 2 l_i + 1)\\ l_{i-1} (m_i + 2) + 2 l_i n_{i-1} \end{pmatrix} \\
D_1^{(i-1)} = \begin{pmatrix} m_{i-1} n_i\\ n_{i-1} (n_i - 1) + 2 l_{i-1} \end{pmatrix}.
\end{cases}$$

From (*) we have

 $n_{i-1}l_i + l_{i-1} < l_{i-1}m_i, \ m_i + 2l_i + 1 < n_i, \ l_{i-1}m_i + 2n_{i-1}l_i < n_{i-1}(n_i - 1).$

It follows the inequalities:

$$D_0^{(i-1)} < B_1^{(i-1)} < C_1^{(i-1)} < A_2^{(i-1)} < D_1^{(i-1)}$$

Applying $\hat{\sigma}_{i-2}$ on the words $d_0^{(i-1)}$, $b_1^{(i-1)}$, $c_1^{(i-1)}$, $a_2^{(i-1)}$, and $d_1^{(i-1)}$ we get the words $d_1^{(i-2)}$, $b_2^{(i-2)}$, $c_2^{(i-2)}$, $a_3^{(i-2)}$, and $d_2^{(i-2)}$; By Proposition 5.1, it results the following inqualities:

$$D_1^{(i-2)} < B_2^{(i-2)} < C_2^{(i-2)} < A_3^{(i-2)} < D_2^{(i-2)}.$$

And so on, after the *i*-th iteration we get:

$$D_{i-1}^{(0)} < B_i^{(0)} < C_i^{(0)} < A_{i+1}^{(0)} < D_i^{(0)}.$$

Lemma 5.5.

$$\forall i \ge 0, |b_i| < |c_i| < |a_{i+1}| < |d_i| < |b_{i+1}|.$$

Proof. • For $i \ge 1$, the inequalities $|b_i| < |c_i| < |a_{i+1}| < |d_i| < |b_{i+1}|$ follows from Lemma 5.4

• For i = 0, recall that

$$|b_0| = l_0, |c_0| = m_0 - 1, |a_1| = 2l_0 + m_0, |d_0| = n_0 - 1 \text{ and } |b_1| = l_1 (m_0 + n_0) + m_0 + 2l_0.$$

So

$$|b_0| < |c_0| < |a_1| < |d_0| < |b_1|.$$

Lemma 5.6. The function s associated to the word u verifies:

$$\forall n \in \mathbb{N}, \ \mathbf{s}(n) = \begin{cases} 1 & \text{if } n = 0\\ 2 & \text{if } n \in \bigcup_{i \ge 0} \left(\left| |c_i|, |a_{i+1}| \right| \cup \left| |d_i|, |b_{i+1}| \right| \right) \cup \left| 0, |b_0| \right| \\ 3 & \text{if } n \in \bigcup_{i \ge 0} \left(\left| |b_i|, |c_i| \right| \cup \left| |a_{i+1}|, |d_i| \right| \right). \end{cases}$$

Proof. Let $n \in \mathbb{N}$. We know that a_i, b_i, c_i , and $d_i, i \ge 0$ are the only bispecial factors of u which are strong or weak. Hence, we have

$$\begin{split} \mathbf{s} \, (n) &= 1 + \sum_{\substack{w \text{ bispecial } \\ |w| < n \\ }} \mathbf{m} \, (w) \\ &= 1 + \# \left\{ i \geq 0 : |a_i| < n \right\} \\ &+ \# \left\{ i \geq 0 : |b_i| < n \right\} \\ &- \# \left\{ i \geq 0 : |c_i| < n \right\} \\ &- \# \left\{ i \geq 0 : |d_i| < n \right\} . \end{split}$$

Since for $m \in]0, |b_0|[$ there is not strong or weak bispecial factor of u with length m we have,

for
$$0 < n \le |b_0|$$
, $\mathbf{s}(n) = 1 + \sum_{\substack{w \text{ bispecial} \\ |w| \le n-1}} \mathbf{m}(w) = 1 + \mathbf{m}(\varepsilon) = 2.$

Suppose $n > |b_0|$. Then, there exists $i \in \mathbb{N}$ such that $n \in [|b_i|, |b_{i+1}|]$. Since the sequences $|a_i|, |b_i|, |c_i|$, and $|d_i|$ are increasing we are in one of the following cases:

•
$$n \in [|b_i|, |c_i|[$$
, then $\mathbf{s}(n) = 1 + (i+1) + (i+1) - (i) - (i) = 3$.
• $n \in [|c_i|, |a_{i+1}|[$, then $\mathbf{s}(n) = 1 + (i+1) + (i+1) - (i+1) - (i) = 2$.
• $n \in [|a_{i+1}|, |d_i|[$, then $\mathbf{s}(n) = 1 + (i+2) + (i+1) - (i+1) - (i) = 3$.
• $n \in [|d_i|, |b_{i+1}|[$, then $\mathbf{s}(n) = 1 + (i+2) + (i+1) - (i+1) - (i+1) = 2$.

Theorem 5.2. The complexity function \mathbf{p} of u verifies:

$$\forall n \ge 1, \ \mathbf{p}(n) \le 3n+1.$$

Proof. By Lemma 5.6, $s(n) \le 3$ for all $n \ge 0$. So, $\mathbf{p}(n) = \mathbf{p}(0) + \sum_{m=0}^{n-1} \mathbf{s}(m) \le \mathbf{p}(0) + 3(n) = 3n + 1$.

Proposition 5.3. Let v, w, v', w' be four finite words such that $v' = \hat{\sigma}_i(v)$ and $w' = \hat{\sigma}_i(w)$. Then for all $\lambda > 0$ we have:

$$W > \lambda \left[V + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \Longrightarrow W' > \lambda \left[V' + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Proof. Assume that $W > \lambda \left[V + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$. Since $|v'|_0 = m_i (|v|+1)$ and $|v'|_1 = l_i |v|_0 + n_i |v|_1 + 2l_i$ then:

$$V' = \begin{pmatrix} m_i & m_i \\ l_i & n_i \end{pmatrix} \begin{pmatrix} |v|_0 \\ |v|_1 \end{pmatrix} + \begin{pmatrix} m_i \\ 2l_i \end{pmatrix}.$$

In the same way, we write W' (it suffices to replace V with W in the previous formula). It follows,

$$W' - \lambda \left[V' + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} m_i & m_i \\ l_i & n_i \end{pmatrix} \begin{pmatrix} |w|_0 - \lambda |v|_0 \\ |w|_1 - \lambda |v|_1 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} m_i \\ 2l_i \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since

$$W > \lambda V + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
 and $(1 - \lambda) \begin{pmatrix} m_i \\ 2l_i \end{pmatrix} > -\lambda \begin{pmatrix} m_i \\ 2l_i \end{pmatrix}$

it follows that:

$$W' - \lambda \left[V' + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] > \lambda \left[\begin{pmatrix} m_i & m_i \\ l_i & n_i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} m_i + 1 \\ 2l_i + 1 \end{pmatrix} \right] = \lambda \begin{pmatrix} m_i - 1 \\ n_i - l_i - 1 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This proposition allows to prove the following lemma:

Lemma 5.7.

$$\forall i \ge 0, \ B_{i+1} > l_{i+1} \left[D_i + \begin{pmatrix} 1\\1 \end{pmatrix} \right]$$

Proof. Let us choose an integer $i \geq 1$. Then, we have $b_1^{(i)} = \widehat{\sigma}_i \left(b_0^{(i+1)} \right) = \overline{\sigma}_i \left(b_0^{(i+1)} \right)$ $1^{l_i} (0^{m_i} 1^{n_i})^{l_{i+1}} 0^{m_i} 1^{l_i}$ and $d_0^{(i)} = 1^{n_i - 1}$; the corresponding Parikh vectors are: $B_1^{(i)} = \begin{pmatrix} l_{i+1}m_i + m_i \\ l_{i+1}n_i + 2l_i \end{pmatrix}$ and $D_0^{(i)} = \begin{pmatrix} 0 \\ n_i - 1 \end{pmatrix}$. It follows the inequality: $B_1^{(i)} > l_{i+1} \left[D_0^{(i)} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$

By regressive induction on $j \leq i$, suppose that:

$$B_{i+1-j}^{(j)} > l_{i+1} \left[D_{i-j}^{(j)} + \begin{pmatrix} 1\\1 \end{pmatrix} \right]$$

where $B_{i+1-j}^{(j)}$ and $D_{i-j}^{(j)}$ are respectively Parikh vectors of the words $b_{i+1-j}^{(j)}$ and $d_{i-j}^{(j)}$. Thus, by Proposition 5.3,

$$B_{i+2-j}^{(j-1)} > l_{i+1} \left[D_{i-j+1}^{(j-1)} + \begin{pmatrix} 1\\1 \end{pmatrix} \right]$$

since $B_{i+2-j}^{(j-1)}$ and $D_{i-j+1}^{(j-1)}$ are respectively Parikh vectors of $b_{i+2-j}^{(j-1)} = \widehat{\sigma}_{j-1}\left(b_{i+1-j}^{(j)}\right)$ and $d_{i-j+1}^{(j-1)} = \widehat{\sigma}_{j-1} \left(d_{i-j}^{(j)} \right)$. So,

$$B_{i+1-j}^{(j)} > l_{i+1} \left[D_{i-j}^{(j)} + \begin{pmatrix} 1\\1 \end{pmatrix} \right], \ 0 \le j \le i.$$

In the inequality above, we find the lemma by making j = 0.

Theorem 5.4. The complexity function \mathbf{p} of u verifies $\liminf \frac{\mathbf{p}(n)}{n} = 2$

Proof. We have $\mathbf{s}(n) = 2$ for $|d_i| < n \le |b_{i+1}|$. So

$$\mathbf{p}(|b_{i+1}|) = \mathbf{p}(|d_i|) + 2(|b_{i+1}| - |d_i|).$$

By Lemma 5.6, we have $p(n) \leq 3n + 1$ and we deduce that:

$$\mathbf{p}(|b_{i+1}|) \le 2|b_{i+1}| + 1 + \frac{1}{l_{i+1}}|b_{i+1}|$$

since $B_{i+1} > l_{i+1} \left[D_i + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] > l_{i+1} D_i$. So $\frac{\mathbf{p}(|b_{i+1}|)}{|b_{i+1}|} \le 2 + \frac{1}{|b_{i+1}|} + \frac{1}{l_{i+1}} \text{ and } \lim_{i \to \infty} \frac{\mathbf{p}(|b_{i+1}|)}{|b_{i+1}|} = 2.$

Thus, $\liminf \frac{\mathbf{p}(n)}{n} = 2$, since $\mathbf{s}(n) \ge 2$ (for all $n \ge 1$) implies $\liminf \frac{\mathbf{p}(n)}{n} \ge 2$.

6 Proof of theorem 3.1

Now, with Notation 5.1 we are able to explain the proof of Theorem 3.1.

Proof. Let us show that for $i \ge 0$, there exists N_i such that any factor of u of length N_i contains the prefix u_i . Indeed, u does not contain 1^{n_0+1} .

• For i = 0 any factor of u of length $N_0 = n_0 + 1$ contains the prefix $0 = u_0$.

• For $i \ge 1$, any factor of $u^{(i)}$ of length $N_0^{(i)} = n_i + 1$ contains the prefix $0 = u_0^{(i)}$ of $u^{(i)}$. Thus, any factor of $u^{(i-1)}$ of length

$$N_1^{(i-1)} = (m_{i-1} + n_{i-1}) \left(N_0^{(i)} + 1 \right)$$

contains $\sigma_{i-1}(0) = u_1^{(i-1)}$.

By regressive induction on j, suppose that for $j \leq i-1$, there exists $N_{i-j}^{(j)}$ such that any factor of $u^{(j)}$ of length $N_{i-j}^{(j)}$ contains the word $u_{i-j}^{(j)}$. Then, any factor of $u^{(j-1)}$ of length

$$N_{i-j+1}^{(j-1)} = (m_{j-1} + n_{j-1}) \left(N_{i-j}^{(j)} + 1 \right)$$

contains $\sigma_{j-1}\left(u_{i-j}^{(j)}\right) = u_{i-j+1}^{(j-1)}$.

So, for $0 \leq j \leq i-1$, there exists $N_{i-j}^{(j)}$ such that any factor of $u^{(j)}$ of length $N_{i-j}^{(j)}$ contains the word $u_{i-j}^{(j)}$.

Consequently, letting $N_i = N_i^{(0)}$, it follows that any factor of $u = u^{(0)}$ of length N_i contains the word u_i . This completes the proof.

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