

Kernel entropy estimation for long memory linear processes with infinite variance

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Abstract

Let $X = \{X_n : n \in \mathbb{N}\}$ be a long memory linear process with innovations in the domain of attraction of an α -stable law ($0 < \alpha < 2$). Assume that the linear process X has a bounded probability density function $f(x)$. Then, under certain conditions, we consider the estimation of the quadratic functional $\int_{\mathbb{R}} f^2(x) dx$ by using the kernel estimator

$$T_n(h_n) = \frac{2}{n(n-1)h_n} \sum_{1 \leq j < i \leq n} K\left(\frac{X_i - X_j}{h_n}\right).$$

The simulation study for long memory linear processes with symmetric α -stable innovations is also given.

Keywords: Linear process; domain of attraction of stable law; kernel entropy estimation; quadratic functional, long memory.

Subject Classification: Primary 60F05, 62M10; Secondary 60G10, 62G05.

1 Introduction

Let $X = \{X_n : n \in \mathbb{N}\}$ be a linear process defined by

$$X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i}, \quad (1.1)$$

where the innovations ε_i are i.i.d. real-valued random variables belonging to the domain of attraction of an α -stable law ($0 < \alpha < 2$), the coefficients $a_0 = 1$, $a_i \sim c_0 i^{-\beta}$, $i = 1, 2, \dots$, and c_0 is a positive constant. Here $a_i \sim a'_i$ means that $a_i/a'_i \rightarrow 1$ as $i \rightarrow \infty$. By Kolmogorov three-series theorem, the linear process X in (1.1) converges almost surely if $\alpha\beta > 1$. Assume that the linear process X has a bounded probability density function $f(x)$. Then the study of quadratic functional $\int_{\mathbb{R}} f^2(x) dx$ will help us to get more information on entropies related to the linear process X , say, quadratic Rényi entropy $R(f) = -\ln(\int_{\mathbb{R}} f^2(x) dx)$ and Shannon entropy $S(f) = -\int_{\mathbb{R}} f(x) \ln f(x) dx$.

Entropy is widely applied in the fields of information theory, statistical classification, pattern recognition and so on since it is a measure of uncertainty in a probability distribution. In the literature, different estimators for the quadratic functional and entropies with independent data have

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been well studied. However, there are very few works on estimations of the quadratic functional and the corresponding entropies for dependent case. In [6], Källberg, Leonenko and Seleznev extended the U -statistics method to m -dependence sequence. They showed the rate optimality and asymptotic normality of the U -statistics estimator for multivariate sequence. In [1], Ahmad obtained the strong consistency of the quadratic functional by orthogonal series method for stationary time series with strong mixing condition. In [9], kernel entropy estimation for quadratic functional and related entropies of regular time series data under certain mild conditions were studied. Although the linear processes in [9] can have infinite variance, it can only deal with the short memory case, see Definition 2.1 and Example 3.2 in [9] for more details. To the best of our knowledge, general results for quadratic functional estimations and related entropies of long memory linear processes with infinite variance are still unknown.

In this paper, for the linear process $X = \{X_n : n \in \mathbb{N}\}$ defined in (1.1), we only focus on the case $1 < \alpha\beta < 2$. According to Definition 2.1 in [9], this corresponds to the long memory case. When innovations are symmetric α -stable random variables, one can also refer to [3] for the definition of such long memory linear processes. To estimate the quadratic functional $\int_{\mathbb{R}} f^2(x) dx$ of the linear process $X = \{X_n : n \in \mathbb{N}\}$ defined in (1.1), we shall apply the kernel method

$$T_n(h_n) = \frac{2}{n(n-1)h_n} \sum_{1 \leq j < i \leq n} K\left(\frac{X_i - X_j}{h_n}\right), \quad (1.2)$$

where the kernel K is a symmetric and bounded function with $\int_{\mathbb{R}} K(u) du = 1$ and $\int_{\mathbb{R}} u^2 |K(u)| du < \infty$. The bandwidth sequence h_n satisfies $0 < h_n \rightarrow 0$ as $n \rightarrow \infty$.

Throughout this paper, if not mentioned otherwise, the letter c with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. We use ι to denote the imaginary unit $\sqrt{-1}$. For a complex number z , we use \bar{z} and $|z|$ to denote its conjugate and modulus, respectively. For any integrable function $g(x)$, its Fourier transform is defined as $\hat{g}(u) = \int_{\mathbb{R}} e^{\iota xu} g(x) dx$. Moreover, we let $\phi(\lambda)$ be the characteristic function of linear process $X = \{X_n : n \in \mathbb{N}\}$, and $\phi_\varepsilon(\lambda)$ the characteristic function of innovations. That is, $\phi(\lambda) = \mathbb{E}[e^{\iota \lambda X_n}]$ and $\phi_\varepsilon(\lambda) = \mathbb{E}[e^{\iota \lambda \varepsilon_1}]$. For simplicity of notation, we always assume that the coefficients a_i in the definition of the linear process X are nonzero.

The paper has the following structure. The main results are given in Section 2. A simulation study is given in Section 3. Section 4 is devoted to the proofs of Theorems 2.1 and 2.2 based on the Fourier transform and the projection method.

2 Main Results

It is well known that the characteristic function of an α -stable law $S_\alpha(\sigma, \eta, \mu)$ has the form

$$e^{\iota \lambda \mu - \sigma^\alpha |\lambda|^\alpha (1 - \iota \eta \operatorname{sign}(\lambda) \omega(\lambda, \alpha))},$$

where $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \eta \leq 1$, $\mu \in \mathbb{R}$, and

$$\omega(\lambda, \alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{for } \alpha \neq 1 \\ 2\pi^{-1} \log |\lambda| & \text{for } \alpha = 1. \end{cases}$$

It is called symmetric (or $S\alpha S$) if $\eta = \mu = 0$, and standard if $\sigma = 1$. For more details on stable laws, we refer to [8]. Let $\xrightarrow{\mathcal{L}}$ denote the convergence in distribution. A random variable Y is said

to be in the domain of attraction of an α -stable law if there exist i.i.d. random variables Y_n with the same distribution as Y , real numbers A_n , and strictly positive numbers B_n such that

$$\frac{\sum_{i=1}^n Y_i - A_n}{B_n} \xrightarrow{\mathcal{L}} S_\alpha(\sigma, \eta, \mu)$$

as $n \rightarrow \infty$, see, for example, [5]. Since the innovations ε_i belong to the domain of attraction of an α -stable law, by Theorem 2.6.5 in [5], the characteristic function ϕ_ε of ε_1 satisfies

$$|\phi_\varepsilon(\lambda)| = e^{-c_\alpha |\lambda|^{\alpha L(\lambda)(1+o(1))}}$$

as $\lambda \rightarrow 0$, where c_α is a positive constant depending on α and $L(\lambda)$ is a slowly varying function as $\lambda \rightarrow 0$. Since the coefficients satisfy $a_0 = 1$, $a_i \sim c_0 i^{-\beta}$, it is easy to see that

$$\sum_{i=0}^{\infty} \sqrt{\text{Var}(e^{i\lambda a_i \varepsilon_1})} = \infty \quad \text{but} \quad \sum_{i=0}^{\infty} \text{Var}(e^{i\lambda a_i \varepsilon_1}) < \infty$$

for $1 < \alpha\beta < 2$. So, by Definition 2.1 in [9], the linear process $X = \{X_n : n \in \mathbb{N}\}$ defined in (1.1) has long memory. This is consistent with the definition of long memory linear processes when innovations are symmetric α -stable random variables, see [3].

Let $G(x)$ be the distribution function of ε_1 . We introduce the following assumptions on the innovation ε_1 .

(A1) There exist strictly positive constants c_1 and δ such that $|\phi_\varepsilon(\lambda)| \leq \frac{c_1}{1+|\lambda|^\delta}$ for all $\lambda \in \mathbb{R}$;

(A2) There exists non-negative constants c_- and c_+ with $c_- + c_+ > 0$ such that

$$\lim_{x \rightarrow -\infty} |x|^\alpha G(x) = c_- \quad \text{and} \quad \lim_{x \rightarrow +\infty} x^\alpha (1 - G(x)) = c_+;$$

(A3) $G(x)$ is twice differentiable with derivatives $G^{(j)}(x)$ ($j = 1, 2$) satisfying the following inequalities: for any $x, y \in \mathbb{R}$, $|x - y| \leq 1$, $j = 1, 2$

$$|G^{(j)}(x)| \leq C(1 + |x|)^{-\alpha}$$

and

$$|G^{(j)}(x) - G^{(j)}(y)| \leq C|x - y|(1 + |x|)^{-\alpha}.$$

The assumption **(A1)** and $a_i \sim c_0 i^{-\beta}$ imply that (i) the linear process X defined in (1.1) has a bounded probability density function $f(x)$; (ii) the characteristic function $\phi(\lambda)$ of X_n decays at any polynomial rate to 0 as $|\lambda| \rightarrow +\infty$. Moreover, **(A1)** implies that there exists $m \in \mathbb{N}$ such that $|\phi_\varepsilon(\lambda)|^m \leq \frac{c_2}{1+|\lambda|^4}$. Following the proof of Lemma 1 of [2] we can get that $f(x)$ is twice continuously differentiable and all its derivatives up to the second order are uniformly bounded, see also the **P1** in [4].

Assumptions **(A2)** and **(A3)** are only needed in Theorem 2.2 to obtain the desired limiting theorems. The assumption **(A2)** shows that ε_1 belongs to the domain of attraction of α -stable law. For more details on the domain of attraction of α -stable law, we refer to [5]. If ε_1 is a symmetric α -stable random variable, then its distribution function G satisfies all assumptions **(A1)**, **(A2)** and **(A3)**.

The following are main results of our paper.

Theorem 2.1 Assume that **(A1)** holds. Then, for any $\eta \in (0, \alpha - \frac{1}{\beta})$, there exist positive constants c_1 and c_2 depending on η such that

$$\left| \mathbb{E} T_n(h_n) - \int_{\mathbb{R}} f^2(x) dx \right| \leq c_1 \left(n^{1-(\alpha-\eta)\beta} + h_n^2 \right)$$

and

$$\mathbb{E} \left| T_n(h_n) - \mathbb{E} T_n(h_n) - \frac{1}{n} \sum_{i=1}^n Y_i \right| \leq c_2 \left(n^{1-(\alpha-\eta)\beta} + \frac{1}{\sqrt{n^3 h_n^2}} + \frac{1}{\sqrt{n^2 h_n}} + h_n^2 n^{\frac{1-(\alpha-\eta)\beta}{2}} \right),$$

where $Y_i = 2 \left(f(X_i) - \int_{\mathbb{R}} f^2(x) dx \right)$.

Theorem 2.2 Under the assumptions of Theorem 2.1 and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$,

(1) if **(A2)** and **(A3)** hold with $c_- = c_+$, $1 < \alpha < 2$, $\frac{1}{\alpha} < \beta < 1$ and $\lim_{n \rightarrow \infty} n^{\frac{(\alpha\beta-1)(2-\alpha)}{4\alpha} + \frac{\eta\beta}{4}} h_n = 0$ for $\eta \in (0, \frac{(\alpha\beta-1)(\alpha-1)}{\alpha\beta})$, then we have

$$n^{\beta-\frac{1}{\alpha}} [T_n(h_n) - \mathbb{E} T_n(h_n)] \xrightarrow{\mathcal{L}} \tilde{c}Z, \quad (2.1)$$

where Z is a standard $S\alpha S$ random variable and

$$\tilde{c} = 2c_0 \left(2c_+ \frac{\Gamma(2-\alpha) \cos(\alpha\pi/2)}{1-\alpha} \int_{-\infty}^1 \int_0^1 (t-s)_+^{-\beta} dt ds \right)^{1/\alpha} \int_{\mathbb{R}} f(x) df(x).$$

(2) if **(A2)** and **(A3)** hold with $c_- = c_+$, $1 < \alpha < 2$, $1 < \beta < \frac{2}{\alpha}$ and $\lim_{n \rightarrow \infty} n^{\frac{(\alpha\beta-1)(2-\alpha\beta)}{4\alpha\beta} + \frac{\eta\beta}{4}} h_n = 0$ for $\eta \in (0, \frac{(\alpha\beta-1)^2}{\alpha\beta^2})$, then we have

$$n^{1-\frac{1}{\alpha\beta}} [T_n(h_n) - \mathbb{E} T_n(h_n)] \xrightarrow{\mathcal{L}} c_+^{1/\alpha\beta} c_f^+ L^+ + c_-^{1/\alpha\beta} c_f^- L^-, \quad (2.2)$$

where L^- and L^+ are i.i.d. random variables with stable law $S_{\alpha\beta}(1, 1, 0)$ and

$$c_f^\pm = 2\tilde{\sigma} \int_0^\infty (f_\infty(\pm u) - f_\infty(0)) u^{-(1+1/\beta)} du$$

with $f_\infty(x) = \mathbb{E}[f(X_1 + x)]$ and $\tilde{\sigma} = \left\{ \frac{c_0^\alpha (\alpha\beta-1)}{\Gamma(2-\alpha\beta) |\cos(\pi\alpha\beta/2)| \beta^{\alpha\beta}} \right\}^{1/(\alpha\beta)}$.

(3) if **(A2)** holds, $0 < \alpha < 1$, $1 < \alpha\beta < 2$ and $\lim_{n \rightarrow \infty} n^{\frac{(\alpha\beta-1)(2-\alpha\beta)}{4\alpha\beta} + \frac{\eta\beta}{4}} h_n = 0$ for $\eta \in (0, \frac{(\alpha\beta-1)^2}{\alpha\beta^2})$, then we have

$$n^{1-\frac{1}{\alpha\beta}} [T_n(h_n) - \mathbb{E} T_n(h_n)] \xrightarrow{\mathcal{L}} c_+^{1/\alpha\beta} c_f^+ L^+ + c_-^{1/\alpha\beta} c_f^- L^-, \quad (2.3)$$

where c_f^\pm and L^\pm are defined in (2).

Remark 2.3 The bandwidth h_n in Theorem 2.2 can be chosen independent of α and β . Note that for any $x \in (1, 2)$, $g(x) = \frac{(x-1)(2-x)}{4x} \in (0, \frac{3-2\sqrt{2}}{4}]$ and $\frac{3-2\sqrt{2}}{4} < \frac{3-2.8}{4} = \frac{1}{20}$. Choose $\eta \in (0, \alpha - \frac{1}{\beta})$ small enough. We only need to require $n^{\frac{1}{20}} h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, $\mathbb{E} T_n(h_n)$ in (2.1), (2.2) and (2.3) can be replaced by $\int_{\mathbb{R}} f^2(x) dx$ if h_n satisfies $\lim_{n \rightarrow \infty} n^{\frac{\alpha\beta-1}{2\alpha}} h_n = 0$ in (2.1) and $\lim_{n \rightarrow \infty} n^{\frac{\alpha\beta-1}{2\alpha\beta}} h_n = 0$ in (2.2) and (2.3). Clearly, if $h_n = O(n^{-\frac{1}{4}})$ and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$, we can replace $\mathbb{E} T_n(h_n)$ in Theorem 2.2 by $\int_{\mathbb{R}} f^2(x) dx$.

3 Simulation

We carry out a simulation study to examine properties of the kernel entropy estimator for the linear process $X = \{X_n : n \in \mathbb{N}\}$ defined in (1.1). Here we assume that the innovation ε_1 follows the standard symmetric α -stable law $S_\alpha(1, 0, 0)$. Moreover, we take the usual normal kernel function $K(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, the bandwidth $h_n = n^{-\frac{1}{5}}$ and coefficients

$$a_i = \begin{cases} 1, & \text{for } i = 0 \\ i^{-\beta}, & \text{for } i \geq 1 \end{cases}.$$

Since the innovations ε_i are i.i.d. $S_\alpha(1, 0, 0)$ random variables with $0 < \alpha < 2$, we have $\phi_\varepsilon(u) = e^{-|u|^\alpha}$. Therefore, the characteristic function of linear process $X = \{X_n : n \in \mathbb{N}\}$ is written as

$$\phi(u) = \mathbb{E}[e^{i\lambda X_n}] = \prod_{i=0}^{\infty} \mathbb{E}[e^{i\lambda a_i \varepsilon_{n-i}}] = e^{-|u|^\alpha \sum_{i=0}^{\infty} |a_i|^\alpha}.$$

Now, by Plancherel theorem, the quadratic functional of the linear process $X = \{X_n : n \in \mathbb{N}\}$ can be obtained as

$$\int_{\mathbb{R}} f^2(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\phi(\lambda)|^2 d\lambda = \frac{1}{\pi\alpha \left(2 \sum_{i=0}^{\infty} |a_i|^\alpha\right)^{\frac{1}{\alpha}}} \Gamma\left(\frac{1}{\alpha}\right).$$

We perform the simulation study by using the software MATLAB. Here, the sample sizes are $n = 1000, 2000, 5000$ and we always simulate $N = 1000$ times. In the following two tables, Mean, Var and Mse stand for the sample means, the sample variances and the sample mean squared errors, respectively.

The true values of $\int_{\mathbb{R}} f^2(x) dx$ and simulation results are summarized in the following two tables. Table 1 is for $\alpha = 0.5$, while Table 2 is for $\alpha = 1.5$. From these tables, we observe that

- (i) As n increases, the estimated value of $\int_{\mathbb{R}} f^2(x) dx$ approaches the true value and the bias steadily decreases;
- (ii) The estimator performs pretty well if $\alpha\beta$ is close to 2.

Table 1: $\alpha = 0.5$

β	True value	n	1000	2000	5000
		$h_n = n^{-1/5}$	0.2513	0.2187	0.1821
2.5	0.0051	Mean	0.0073	0.0070	0.0065
		Var($\times 10^{-3}$)	0.0099	0.0074	0.0046
		Mse($\times 10^{-3}$)	0.0148	0.0111	0.0064
3.5	0.0181	Mean	0.0184	0.0185	0.0182
		Var($\times 10^{-3}$)	0.0147	0.0080	0.0037
		Mse($\times 10^{-3}$)	0.0148	0.0081	0.0037
3.9	0.0219	Mean	0.0219	0.0221	0.0219
		Var($\times 10^{-3}$)	0.0146	0.0068	0.0028
		Mse($\times 10^{-3}$)	0.0146	0.0068	0.0028

Table 2: $\alpha = 1.5$

β	True value	n $h_n = n^{-1/5}$	1000	2000	5000
0.9	0.0668	Mean	0.0738	0.0728	0.0705
		Var($\times 10^{-3}$)	0.1110	0.0713	0.0544
		Mse($\times 10^{-3}$)	0.1601	0.1076	0.0679
1.1	0.0840	Mean	0.0858	0.0857	0.0846
		Var($\times 10^{-3}$)	0.0777	0.0420	0.0247
		Mse($\times 10^{-3}$)	0.0807	0.0447	0.0251
1.3	0.0935	Mean	0.0939	0.0940	0.0935
		Var($\times 10^{-3}$)	0.0525	0.0268	0.0124
		Mse($\times 10^{-3}$)	0.0527	0.0270	0.0124

4 Proofs

In this section, we will prove Theorems 2.1 and 2.2. To begin with, we introduce the projection method and two lemmas. Lemma 4.1 is on the characteristic function of innovations ε_i and Lemma 4.2 gives the desired estimation of the covariance

$$\mathbb{E} \left[(e^{\iota\lambda X_i} - \phi(\lambda))(e^{-\iota\lambda X_j} - \phi(-\lambda)) \right].$$

For each $i \in \mathbb{Z}$, let \mathcal{F}_i be the σ -field generated by random variables $\{\varepsilon_k : k \leq i\}$. Given an integrable complex-valued random variable Z , we define the projection operator \mathcal{P}_i as

$$\mathcal{P}_i Z = \mathbb{E}[Z|\mathcal{F}_i] - \mathbb{E}[Z|\mathcal{F}_{i-1}]$$

for each $i \in \mathbb{Z}$. It is easy to see that $\mathbb{E}[\mathcal{P}_i Z \mathcal{P}_j W] = 0$ if $i \neq j$, $\mathbb{E}|Z|^2 < \infty$ and $\mathbb{E}|W|^2 < \infty$.

Lemma 4.1 *If ε is in the domain of attraction of an α -stable law with $\alpha \in (0, 2)$ and $\phi_\varepsilon(\lambda)$ is its characteristic function, then for any $\eta \in (0, \alpha)$, there exists a positive constant $c_{\alpha, \eta}$ such that*

$$\mathbb{E} |e^{\iota\lambda\varepsilon} - \phi_\varepsilon(\lambda)|^2 \leq c_{\alpha, \eta} (|\lambda|^{\alpha-\eta} \wedge 1).$$

Proof: By Theorem 2.6.5 in [5],

$$|\phi_\varepsilon(\lambda)|^2 = e^{-c_1 |\lambda|^\alpha L(\lambda)(1+o(1))}$$

as $\lambda \rightarrow 0$, where L is a slowly varying function as $\lambda \rightarrow 0$. Therefore,

$$\mathbb{E} |e^{\iota\lambda\varepsilon_1} - \phi_\varepsilon(\lambda)|^2 = 1 - |\phi_\varepsilon(\lambda)|^2 \leq c_{\alpha, \eta} (|\lambda|^{\alpha-\eta} \wedge 1),$$

where in the last inequality we used the fact that $|1 - e^{-x}| \leq x$ for $x \geq 0$ and $\lim_{\lambda \rightarrow 0} |\lambda|^\eta L(\lambda) = 0$ for any $\eta > 0$. \square

Lemma 4.2 *For any $1 \leq i \neq j \leq n$ and $\eta \in (0, \alpha - \frac{1}{\beta})$, there exists a positive constant c_η such that*

$$\left| \mathbb{E} \left[(e^{\iota\lambda X_i} - \phi(\lambda))(e^{-\iota\lambda X_j} - \phi(-\lambda)) \right] \right| \leq \frac{c_\eta}{1 + |\lambda|^4} |i - j|^{1-(\alpha-\eta)\beta}.$$

Proof: For any $i \geq 1$,

$$\begin{aligned} e^{\iota\lambda X_i} - \phi(\lambda) &= \sum_{k=-\infty}^i \mathcal{P}_k(e^{\iota\lambda X_i} - \phi(\lambda)) \\ &= \sum_{k=-\infty}^i \left(\prod_{\ell=0}^{i-k-1} \phi_\varepsilon(\lambda a_\ell) \right) \left(e^{\iota\lambda a_{i-k}\varepsilon_k} - \phi_\varepsilon(\lambda a_{i-k}) \right) e^{\iota\lambda \sum_{\ell=i-k+1}^{\infty} a_\ell \varepsilon_{i-\ell}}. \end{aligned} \quad (4.1)$$

It suffices to consider the case $i > j$. Using the decomposition (4.1), we can obtain that

$$\mathbb{E} \left[(e^{\iota\lambda X_i} - \phi(\lambda))(e^{-\iota\lambda X_j} - \phi(-\lambda)) \right] = \sum_{k=-\infty}^j \mathbf{I}_1 \times \mathbf{I}_2 \times \mathbf{I}_3,$$

where

$$\begin{aligned} \mathbf{I}_1 &= \prod_{\ell=0}^{i-k-1} \phi_\varepsilon(\lambda a_\ell) \prod_{\ell=0}^{j-k-1} \phi_\varepsilon(-\lambda a_\ell), \\ \mathbf{I}_2 &= \mathbb{E} \left[(e^{\iota\lambda a_{i-k}\varepsilon_k} - \phi_\varepsilon(\lambda a_{i-k}))(e^{-\iota\lambda a_{j-k}\varepsilon_k} - \phi_\varepsilon(-\lambda a_{j-k})) \right], \\ \mathbf{I}_3 &= \prod_{\ell=1}^{\infty} \phi_\varepsilon(\lambda(a_{i-k+\ell} - a_{j-k+\ell})) \end{aligned}$$

with the convention $\prod_{\ell=0}^{-1} \phi_\varepsilon(\lambda a_\ell) = 1$.

By Assumption **(A1)**, there exists $m_0 \in \mathbb{N}$ such that $\prod_{\ell=0}^{m_0} |\phi_\varepsilon(\lambda a_\ell)|$ is less than a constant multiple of $\frac{1}{1+|\lambda|^6}$. Hence, for the case $i - j > m_0$ or $k < j - m_0$, $|\mathbf{I}_1|$ is less than a constant multiple of $\frac{1}{1+|\lambda|^6}$. Moreover, for the case $1 \leq i - j \leq m_0$ and $j - m_0 \leq k \leq j$, $\lim_{n \rightarrow \infty} n^\beta a_n = c_0$ implies that there exist infinitely many $\ell (\geq 1)$ such that

$$a_{i-k+\ell} - a_{j-k+\ell} = a_{i-j+j-k+\ell} - a_{j-k+\ell} \neq 0.$$

So, by Assumption **(A1)**, $|\mathbf{I}_3|$ is less than a constant multiple of $\frac{1}{1+|\lambda|^6}$ in the case $1 \leq i - j \leq m_0$ and $j - m_0 \leq k \leq j$.

Therefore, using Cauchy-Schwartz inequality and Lemma 4.1 with $\eta \in (0, \alpha - \frac{1}{\beta})$,

$$\begin{aligned} & \left| \mathbb{E} \left[(e^{\iota\lambda X_i} - \phi(\lambda))(e^{-\iota\lambda X_j} - \phi(-\lambda)) \right] \right| \\ & \leq \frac{c_1}{1+|\lambda|^6} \sum_{k=-\infty}^{i \wedge j} \left| \mathbb{E} \left[(e^{\iota\lambda a_{i-k}\varepsilon_k} - \phi_\varepsilon(\lambda a_{i-k}))(e^{-\iota\lambda a_{j-k}\varepsilon_k} - \phi_\varepsilon(-\lambda a_{j-k})) \right] \right| \\ & \leq \frac{c_2}{1+|\lambda|^6} |\lambda|^{\alpha-\eta} \sum_{k=-\infty}^{i \wedge j} |a_{i-k}|^{\frac{\alpha-\eta}{2}} |a_{j-k}|^{\frac{\alpha-\eta}{2}} \\ & \leq \frac{c_3}{1+|\lambda|^4} \sum_{\ell=1}^{\infty} (|i-j| + \ell)^{-\frac{(\alpha-\eta)\beta}{2}} \ell^{-\frac{(\alpha-\eta)\beta}{2}} \\ & \leq \frac{c_4}{1+|\lambda|^4} |i-j|^{1-(\alpha-\eta)\beta} \int_0^\infty (1+x)^{-\frac{(\alpha-\eta)\beta}{2}} x^{-\frac{(\alpha-\eta)\beta}{2}} dx \\ & \leq \frac{c_5}{1+|\lambda|^4} |i-j|^{1-(\alpha-\eta)\beta}. \end{aligned}$$

This gives the desired estimate. \square

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1 The proof will be done in several steps.

Step 1. We give the estimation for $|\mathbb{E} T_n(h_n) - \int_{\mathbb{R}} f^2(x) dx|$. Using the Fourier inverse transform, we can obtain that

$$\begin{aligned} & \mathbb{E} T_n(h_n) - \int_{\mathbb{R}} f^2(x) dx \\ &= \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) \mathbb{E} [(e^{\iota \lambda X_i} - \phi(\lambda))(e^{-\iota \lambda X_j} - \phi(-\lambda))] d\lambda \\ & \quad + \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{K}(\lambda h_n) - \widehat{K}(0)) |\phi(\lambda)|^2 d\lambda \\ &=: \text{II}_1 + \text{II}_2. \end{aligned}$$

By Lemma 4.2 and the boundedness of \widehat{K} , we can obtain that

$$|\text{II}_1| \leq \frac{c_1}{n^2} \sum_{1 \leq j < i \leq n} |i-j|^{1-(\alpha-\eta)\beta} \leq c_2 n^{1-(\alpha-\eta)\beta}.$$

Since K is a symmetric and bounded function with $\int_{\mathbb{R}} K(u) du = 1$ and $\int_{\mathbb{R}} u^2 |K(u)| du < \infty$,

$$|\widehat{K}(\lambda h_n) - \widehat{K}(0)| = \left| \int_{\mathbb{R}} (e^{\iota \lambda h_n u} - 1) K(u) du \right| \leq |\lambda|^2 h_n^2 \int_{\mathbb{R}} u^2 |K(u)| du.$$

Then, by Assumption (A1), $|\text{II}_2|$ is less than a constant multiple of h_n^2 . Hence

$$\left| \mathbb{E} T_n(h_n) - \int_{\mathbb{R}} f^2(x) dx \right| \leq c_3 \left(n^{1-(\alpha-\eta)\beta} + h_n^2 \right).$$

Step 2. We give the decomposition for $T_n(h_n) - \mathbb{E} T_n(h_n)$. It is easy to see that

$$T_n(h_n) - \mathbb{E} T_n(h_n) = 2N_n + A_n - \mathbb{E} A_n + B_n - \mathbb{E} B_n, \quad (4.2)$$

where

$$N_n = \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n, i-j > m} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) (e^{\iota \lambda X_i} - \phi(\lambda)) \phi(-\lambda) d\lambda,$$

$$A_n = \frac{1}{n(n-1)h_n} \sum_{1 \leq j < i \leq n, i-j \leq m} K\left(\frac{X_i - X_j}{h_n}\right),$$

$$B_n = \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n, i-j > m} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) (e^{\iota \lambda X_i} - \phi(\lambda)) (e^{-\iota \lambda X_j} - \phi(-\lambda)) d\lambda$$

and the proper choice of the natural number m will be specified in **Step 4**.

Step 3. We estimate $\mathbb{E} |A_n - \mathbb{E} A_n|^2$. For $1 \leq j < i \leq n$, we observe that

$$X_i - X_j = \sum_{\ell=0}^{\infty} a_{\ell}^{i,j} \epsilon_{i-\ell},$$

where

$$a_\ell^{i,j} = \begin{cases} a_\ell, & \text{for } 0 \leq \ell \leq i - j - 1, \\ a_\ell - a_{\ell-(i-j)}, & \text{for } \ell \geq i - j. \end{cases} \quad (4.3)$$

For $1 \leq j < i \leq n$ with $i - j \leq m$, it is easy to see that there exists $m_1 \in \mathbb{N}$ such that $m_1 \geq m$ and

$$\prod_{\ell=0}^{m_1} |\phi_\varepsilon(a_\ell^{i,j} \lambda)| \leq \frac{c_4}{1 + |\lambda|^4}. \quad (4.4)$$

Let

$$A_n^{i,j} = \frac{1}{n(n-1)h_n} \left(K\left(\frac{X_i - X_j}{h_n}\right) - \mathbb{E}K\left(\frac{X_i - X_j}{h_n}\right) \right).$$

Then

$$\begin{aligned} & \mathbb{E}|A_n - \mathbb{E}A_n|^2 \\ &= \sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| \leq 2m_1}} \mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}] + \sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| > 2m_1}} \mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}]. \end{aligned} \quad (4.5)$$

Boundedness of the kernel function K implies that

$$\sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| \leq 2m_1}} |\mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}]| \leq \frac{c_5}{n^3 h_n^2}. \quad (4.6)$$

To estimate

$$\sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| > 2m_1}} \mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}],$$

it suffices to consider the case $i_2 - i_1 > 2m_1$.

Applying the projection operator \mathcal{P}_k on the terms $A_n^{i,j}$, we see that

$$\begin{aligned} \mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}] &= \sum_{k=-\infty}^{i_1} \mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}] \\ &= \sum_{k=i_1-m_1}^{i_1} \mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}] + \sum_{k=-\infty}^{i_1-m_1-1} \mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}]. \end{aligned}$$

By Fourier inverse transform and the boundedness of \widehat{K} ,

$$\begin{aligned} & |\mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}]| \\ &= \left| \frac{1}{4\pi^2 n^2 (n-1)^2} \int_{\mathbb{R}^2} \widehat{K}(\lambda_1 h_n) \widehat{K}(\lambda_2 h_n) \right. \\ & \quad \times \mathbb{E}[\mathcal{P}_k(e^{\iota\lambda_1(X_{i_1} - X_{j_1})} - \mathbb{E}e^{\iota\lambda_1(X_{i_1} - X_{j_1})}) \mathcal{P}_k(e^{\iota\lambda_2(X_{i_2} - X_{j_2})} - \mathbb{E}e^{\iota\lambda_2(X_{i_2} - X_{j_2})})] d\lambda_1 d\lambda_2 \left. \right| \\ &\leq \frac{c_6}{n^4} \int_{\mathbb{R}^2} \left| \mathbb{E}[\mathcal{P}_k(e^{\iota\lambda_1(X_{i_1} - X_{j_1})} - \mathbb{E}e^{\iota\lambda_1(X_{i_1} - X_{j_1})}) \mathcal{P}_k(e^{\iota\lambda_2(X_{i_2} - X_{j_2})} - \mathbb{E}e^{\iota\lambda_2(X_{i_2} - X_{j_2})})] \right| d\lambda_1 d\lambda_2. \end{aligned} \quad (4.7)$$

In the case $i_1 - m_1 \leq k \leq i_1$, recall the choice of m_1 in (4.4) and $i_2 - i_1 > 2m_1$, we see that

$$\left| \mathbb{E}[\mathcal{P}_k(e^{\iota\lambda_1(X_{i_1} - X_{j_1})} - \mathbb{E}e^{\iota\lambda_1(X_{i_1} - X_{j_1})}) \mathcal{P}_k(e^{\iota\lambda_2(X_{i_2} - X_{j_2})} - \mathbb{E}e^{\iota\lambda_2(X_{i_2} - X_{j_2})})] \right|$$

in (4.7) is less than a constant multiple of

$$\left| \mathbb{E}[(e^{\iota\lambda_1 a_{i_1-k}^{i_1, j_1} \varepsilon_k} - \phi_\varepsilon(\lambda_1 a_{i_1-k}^{i_1, j_1})) (e^{\iota\lambda_1 a_{i_2-k}^{i_2, j_2} \varepsilon_k} - \phi_\varepsilon(\lambda_2 a_{i_2-k}^{i_2, j_2}))] \right| \frac{1}{1 + |\lambda_2|^4} \prod_{\ell=m_1+1}^{\infty} |\phi_\varepsilon(a_\ell^{i_1, j_1} \lambda_1 + a_\ell^{i_2, j_2} \lambda_2)|.$$

According to Assumption **(A1)** and $1 \leq i_\theta - j_\theta \leq m$ for $\theta = 1, 2$, it is easy to see that

$$\int_{\mathbb{R}} \prod_{\ell=m_1+1}^{\infty} |\phi_\varepsilon(a_\ell^{i_1, j_1} \lambda_1 + a_\ell^{i_2, j_2} \lambda_2)| d\lambda_1 \leq M,$$

where M a finite positive number independent of $\lambda_2, i_1, j_1, i_2, j_2$.

In the case $k \leq i_1 - m_1 - 1$, recall the choice of m_1 in (4.4) and $i_2 - i_1 > 2m_1$, we see that

$$\left| \mathbb{E}[\mathcal{P}_k(e^{\iota\lambda_1(X_{i_1} - X_{j_1})} - \mathbb{E}e^{\iota\lambda_1(X_{i_1} - X_{j_1})}) \mathcal{P}_k(e^{\iota\lambda_2(X_{i_2} - X_{j_2})} - \mathbb{E}e^{\iota\lambda_2(X_{i_2} - X_{j_2})})] \right|$$

in (4.7) is less than a constant multiple of

$$\left| \mathbb{E}[(e^{\iota\lambda_1 a_{i_1-k}^{i_1, j_1} \varepsilon_k} - \phi_\varepsilon(\lambda_1 a_{i_1-k}^{i_1, j_1})) (e^{\iota\lambda_2 a_{i_2-k}^{i_2, j_2} \varepsilon_k} - \phi_\varepsilon(\lambda_2 a_{i_2-k}^{i_2, j_2}))] \right| \frac{1}{1 + |\lambda_1|^4} \frac{1}{1 + |\lambda_2|^4}.$$

Therefore, by using Cauchy-Schwartz inequality, Lemma 4.1 and the definition of $a_\ell^{i, j}$ in (4.3), we can obtain that

$$\begin{aligned} \sum_{k=i_1-m_1}^{i_1} |\mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}]| &\leq \frac{c_7}{n^4} \sum_{k=i_1-m_1}^{i_1} \int_{\mathbb{R}^2} |a_{i_2-k}^{i_2, j_2} \lambda_2|^{\frac{\alpha-\eta}{2}} \frac{1}{1 + |\lambda_2|^4} \prod_{\ell=m_1+1}^{\infty} |\phi_\varepsilon(a_\ell^{i_1, j_1} \lambda_1 + a_\ell^{i_2, j_2} \lambda_2)| d\lambda_1 d\lambda_2 \\ &\leq \frac{c_8}{n^4} \sum_{k=i_1-m_1}^{i_1} \int_{\mathbb{R}} |a_{i_2-k}^{i_2, j_2} \lambda_2|^{\frac{\alpha-\eta}{2}} \frac{1}{1 + |\lambda_2|^4} d\lambda_2 \\ &\leq \frac{c_9}{n^4} \sum_{k=i_1-m_1}^{i_1} |a_{i_2-k}^{i_2, j_2}|^{\frac{\alpha-\eta}{2}} \\ &\leq \frac{c_{10}}{n^4} |i_2 - i_1|^{-\frac{(\alpha-\eta)\beta}{2}} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=-\infty}^{i_1-m_1-1} |\mathbb{E}[\mathcal{P}_k A_n^{i_1, j_1} \mathcal{P}_k A_n^{i_2, j_2}]| &\leq \frac{c_{11}}{n^4} \sum_{k=-\infty}^{i_1-m_1-1} \int_{\mathbb{R}^2} |a_{i_1-k}^{i_1, j_1} \lambda_1|^{\frac{\alpha-\eta}{2}} |a_{i_2-k}^{i_2, j_2} \lambda_2|^{\frac{\alpha-\eta}{2}} \frac{1}{1 + |\lambda_1|^4} \frac{1}{1 + |\lambda_2|^4} d\lambda_1 d\lambda_2 \\ &\leq \frac{c_{12}}{n^4} \sum_{k=-\infty}^{i_1-m_1-1} |a_{i_1-k}^{i_1, j_1}|^{\frac{\alpha-\eta}{2}} |a_{i_2-k}^{i_2, j_2}|^{\frac{\alpha-\eta}{2}} \\ &\leq \frac{c_{13}}{n^4} \sum_{k=-\infty}^{i_1-m_1-1} |a_{i_1-k}^{i_1, j_1}|^{\frac{\alpha-\eta}{2}} |a_{i_2-k}^{i_2, j_2}|^{\frac{\alpha-\eta}{2}} \\ &\leq \frac{c_{14}}{n^4} |i_2 - i_1|^{1-(\alpha-\eta)\beta}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| > 2m_1}} |\mathbb{E}[A_n^{i_1, j_1} A_n^{i_2, j_2}]| \\ &\leq \frac{c_{15}}{n^4} \sum_{\substack{1 \leq j_1 < i_1 \leq n, 1 \leq j_2 < i_2 \leq n; \\ i_1 - j_1 \leq m, i_2 - j_2 \leq m, |i_2 - i_1| > 2m_1}} (|i_2 - i_1|^{-\frac{(\alpha-\eta)\beta}{2}} + |i_2 - i_1|^{1-(\alpha-\eta)\beta}) \\ &\leq \frac{c_{16}}{n^{1+(\alpha-\eta)\beta}}. \end{aligned} \tag{4.8}$$

Combining (4.5), (4.6) and (4.8) gives

$$\mathbb{E}|A_n - \mathbb{E}A_n|^2 \leq c_{17} \left(\frac{1}{n^3 h_n^2} + \frac{1}{n^{1+(\alpha-\eta)\beta}} \right). \quad (4.9)$$

Step 4. We estimate $\mathbb{E}|B_n - \mathbb{E}B_n|$. For each $i \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, define

$$H(X_i)(\lambda) = e^{i\lambda X_i} - \phi(\lambda).$$

Then

$$B_n = B_{n,1} + B_{n,2} + B_{n,3}, \quad (4.10)$$

where

$$\begin{aligned} B_{n,1} &= \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n, i-j > m} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) \sum_{k=-\infty}^j \mathcal{P}_k H(X_i)(\lambda) \mathcal{P}_k H(X_j)(-\lambda) d\lambda \\ B_{n,2} &= \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n, i-j > m} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) \sum_{\substack{k \leq i, \ell \leq j, k \neq \ell \\ i-k \leq m_0, j-\ell \leq m_0}} \mathcal{P}_k H(X_i)(\lambda) \mathcal{P}_\ell H(X_j)(-\lambda) d\lambda \\ B_{n,3} &= \frac{1}{\pi n(n-1)} \sum_{1 \leq j < i \leq n, i-j > m} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) \sum_{\substack{k \leq i, \ell \leq j, k \neq \ell \\ i-k > m_0 \text{ or } j-\ell > m_0}} \mathcal{P}_k H(X_i)(\lambda) \mathcal{P}_\ell H(X_j)(-\lambda) d\lambda. \end{aligned}$$

Using similar arguments as in **Step 1.**, we can show that

$$\mathbb{E}|B_{n,1}| \leq c_{18} n^{1-(\alpha-\eta)\beta}. \quad (4.11)$$

Note that

$$\begin{aligned} &\mathbb{E}|B_{n,2}|^2 \\ &\leq \frac{c_{19}}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \int_{\mathbb{R}^2} |\widehat{K}(\lambda_1 h_n)| |\widehat{K}(\lambda_2 h_n)| \\ &\quad \times \left| \mathbb{E} \left[\mathcal{P}_{k_1} H(X_{i_1})(\lambda_1) \mathcal{P}_{\ell_1} H(X_{j_1})(-\lambda_1) \mathcal{P}_{k_2} H(X_{i_2})(\lambda_2) \mathcal{P}_{\ell_2} H(X_{j_2})(-\lambda_2) \right] \right| d\lambda_1 d\lambda_2. \end{aligned} \quad (4.12)$$

In the sequel, we will estimate the expectation

$$\mathbb{E} \left[\mathcal{P}_{k_1} H(X_{i_1})(\lambda_1) \mathcal{P}_{\ell_1} H(X_{j_1})(-\lambda_1) \mathcal{P}_{k_2} H(X_{i_2})(\lambda_2) \mathcal{P}_{\ell_2} H(X_{j_2})(-\lambda_2) \right] \quad (4.13)$$

and specify the choice of m . Assume that m is larger than $4m_0$. Then there are four possibilities for the orderings of i_1, j_1, i_2, j_2 :

$$(1) i_1 \geq i_2 > j_1 \geq j_2, \quad (2) i_1 \geq i_2 > j_2 \geq j_1, \quad (3) i_2 \geq i_1 > j_1 \geq j_2, \quad (4) i_2 \geq i_1 > j_2 \geq j_1.$$

By symmetry, it suffices to consider the first two cases. In the first case $i_1 \geq i_2 > j_1 \geq j_2$, the expectation (4.13) is equal to zero if $k_1 \neq k_2$. When $k_1 = k_2$, $0 \leq i_1 - k_1 \leq m_0$ and $0 \leq i_2 - k_2 \leq m_0$ imply $0 \leq i_1 - i_2 \leq m_0$. If $m > m_0 + m_2$, then there is a factor

$$\prod_{q=i_2-m_0-m_2}^{i_2-m_0-1} \phi_\varepsilon(a_{i_1-q}\lambda_1 + a_{i_2-q}\lambda_2)$$

in the expectation (4.13). By Assumption (A1), we can choose $m_2 \in \mathbb{N}$ independent of i_2 such that

$$\begin{aligned} \prod_{q=i_2-m_0-m_2}^{i_2-m_0-1} |\phi_\varepsilon(a_{i_1-q}\lambda_1 + a_{i_2-q}\lambda_2)| &\leq \prod_{p=m_0+1}^{m_0+m_2} \frac{c_{20}}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^\delta} \\ &\leq \sum_{p=m_0+1}^{m_0+m_2} \frac{c_{21}}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^4}, \end{aligned}$$

where in the last inequality we used $\prod_{k=1}^{m_2} x_k \leq \sum_{k=1}^{m_2} x_k^{m_2}$ for any $x_1 \geq 0, \dots, x_{m_2} \geq 0$.

Moreover, if $|\ell_1 - \ell_2| > 2m_0 + m_3 + m_4$, then there is another factor

$$\prod_{q=j_1-m_0-m_3-m_4}^{j_1-m_0-m_3} \phi_\varepsilon(a_{i_1-q}\lambda_1 + a_{i_2-q}\lambda_2 - a_{j_1-q}\lambda_1) = \prod_{p=m_0+m_3}^{m_0+m_3+m_4} \phi_\varepsilon((a_{i_1-j_1+p} - a_p)\lambda_1 + a_{i_2-j_1+p}\lambda_2)$$

in the expectation (4.13).

By Assumption (A1), $\lim_{n \rightarrow \infty} n^\beta a_n = c_0$, $|\ell_1 - \ell_2| > 2m_0 + m_3 + m_4$ and $0 \leq j_\theta - \ell_\theta \leq m_0$ for $\theta = 1, 2$, we can choose $m_3, m_4 \in \mathbb{N}$ independent of i_1, i_2, j_1 such that

$$\prod_{q=j_1-m_0-m_3-m_4}^{j_1-m_0-m_3} |\phi_\varepsilon(a_{i_1-q}\lambda_1 + a_{i_2-q}\lambda_2 - a_{j_1-q}\lambda_1)| \leq \sum_{q=m_0+m_3}^{m_0+m_3+m_4} \frac{c_{22}}{1 + |(a_{i_1-i_2+i_2-j_1+q} - a_q)\lambda_1 + a_{i_2-j_1+q}\lambda_2|^4}.$$

Note that

$$i_2 - j_1 = i_1 - j_1 - (i_1 - i_2) > m - m_0.$$

So we can choose $m \in \mathbb{N}$ large enough such that

$$|a_{i_1-i_2+i_2-j_1+q} - a_q| \geq \frac{1}{2}|a_q| \quad (4.14)$$

and

$$\left| \det \left(\begin{pmatrix} a_{i_1-i_2+p} & a_p \\ a_{i_1-i_2+i_2-j_1+q} - a_q & a_{i_2-j_1+q} \end{pmatrix} \right) \right| \geq \frac{1}{4}|a_p a_q| > 0 \quad (4.15)$$

for all $0 \leq i_1 - i_2 \leq m_0$, $m_0 + 1 \leq p \leq m_0 + m_2$, $m_0 + m_3 \leq q \leq m_0 + m_3 + m_4$.

So for m large enough, in the first case $i_1 \geq i_2 > j_1 \geq j_2$, the right hand side of (4.12) is less than a constant multiple of

$$\text{III}_1 + \text{III}_2,$$

where

$$\begin{aligned} \text{III}_1 &= \frac{1}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \int_{\mathbb{R}^2} |\widehat{K}(\lambda_1 h_n)| |\widehat{K}(\lambda_2 h_n)| \\ &\quad \times \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| \leq 2m_0+m_3+m_4\}} \sum_{p=m_0+1}^{m_0+m_2} \frac{1}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^4} d\lambda_1 d\lambda_2 \end{aligned}$$

and

$$\begin{aligned}
\text{III}_2 &= \frac{1}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \int_{\mathbb{R}^2} \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| > 2m_0+m_3+m_4\}} \sum_{p=m_0+1}^{m_0+m_2} \frac{1}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^4} \\
&\quad \times \sum_{q=m_0+m_3}^{m_0+m_3+m_4} \frac{1}{1 + |(a_{i_1-i_2+i_2-j_1+q} - a_q)\lambda_1 + a_{i_2-j_1+q}\lambda_2|^4} \\
&\quad \times \left(\mathbf{1}_{\{\ell_1 < \ell_2\}} |\lambda_1 a_{i_1-\ell_2} + \lambda_2 a_{i_2-\ell_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1-\ell_1} + \lambda_2 a_{i_2-\ell_1} - \lambda_2 a_{j_2-\ell_1}|^{\frac{\alpha-\eta}{2}} \right. \\
&\quad \left. + \mathbf{1}_{\{\ell_1 > \ell_2\}} |\lambda_1 a_{i_1-\ell_1} + \lambda_2 a_{i_2-\ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1-\ell_2} + \lambda_2 a_{i_2-\ell_2} - \lambda_2 a_{j_1-\ell_2}|^{\frac{\alpha-\eta}{2}} \right) d\lambda_1 d\lambda_2.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\text{III}_1 &\leq \frac{1}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \int_{\mathbb{R}^2} \left(|\widehat{K}(\lambda_1 h_n)|^2 + |\widehat{K}(\lambda_2 h_n)|^2 \right) \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| \leq 2m_0+m_3+m_4\}} \\
&\quad \times \sum_{p=m_0+1}^{m_0+m_2} \frac{1}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^4} d\lambda_1 d\lambda_2 \\
&\leq \frac{c_{23}}{n^4} \left(\int_{\mathbb{R}} |\widehat{K}(\lambda_1 h_n)|^2 d\lambda_1 + \int_{\mathbb{R}} |\widehat{K}(\lambda_2 h_n)|^2 d\lambda_2 \right) \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| \leq 2m_0+m_3+m_4\}} \\
&\leq \frac{c_{24}}{n^2 h_n},
\end{aligned}$$

where we used Plancherel theorem for the kernel function K in the last inequality.

Moreover,

$$\begin{aligned}
\text{III}_2 &\leq \frac{2}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| > 2m_0+m_3+m_4\}} \int_{\mathbb{R}^2} (|\lambda_1|^{\alpha-\eta} + |\lambda_2|^{\alpha-\eta}) \\
&\quad \times \sum_{p=m_0+1}^{m_0+m_2} \frac{1}{1 + |a_{i_1-i_2+p}\lambda_1 + a_p\lambda_2|^4} \sum_{q=m_0+m_3}^{m_0+m_3+m_4} \frac{1}{1 + |(a_{i_1-i_2+i_2-j_1+q} - a_q)\lambda_1 + a_{i_2-j_1+q}\lambda_2|^4} \\
&\quad \times \left(\mathbf{1}_{\{\ell_1 < \ell_2\}} (|a_{i_1-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_2}|^{\frac{\alpha-\eta}{2}}) (|a_{i_1-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{j_2-\ell_1}|^{\frac{\alpha-\eta}{2}}) \right. \\
&\quad \left. + \mathbf{1}_{\{\ell_1 > \ell_2\}} (|a_{i_1-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_1}|^{\frac{\alpha-\eta}{2}}) (|a_{i_1-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{j_1-\ell_2}|^{\frac{\alpha-\eta}{2}}) \right) d\lambda_1 d\lambda_2 \\
&\leq \frac{c_{25}}{n^4} \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta \neq \ell_\theta \\ 0 \leq i_\theta - k_\theta \leq m_0, 0 \leq j_\theta - \ell_\theta \leq m_0, \theta=1,2}} \mathbf{1}_{\{k_1=k_2, |\ell_1-\ell_2| > 2m_0+m_3+m_4\}} \\
&\quad \times \left(\mathbf{1}_{\{\ell_1 < \ell_2\}} (|a_{i_1-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_2}|^{\frac{\alpha-\eta}{2}}) (|a_{i_1-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{j_2-\ell_1}|^{\frac{\alpha-\eta}{2}}) \right. \\
&\quad \left. + \mathbf{1}_{\{\ell_1 > \ell_2\}} (|a_{i_1-\ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_1}|^{\frac{\alpha-\eta}{2}}) (|a_{i_1-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2-\ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{j_1-\ell_2}|^{\frac{\alpha-\eta}{2}}) \right) \\
&\leq \frac{c_{26}}{n^{1+(\alpha-\eta)\beta}},
\end{aligned}$$

where in the second inequality we used (4.14) and (4.15) to make proper change of variables to get the finiteness of the integral with respect to λ_1 and λ_2 .

Therefore, for m large enough, in the first case $i_1 \geq i_2 > j_1 \geq j_2$, the right hand side of (4.12) is less than a constant multiple of $\frac{1}{n^2 h_n} + \frac{1}{n^{1+(\alpha-\eta)\beta}}$. Similarly, for m large enough, in the second case $i_1 \geq i_2 > j_2 \geq j_1$, we can also show that the right hand side of (4.12) is less than a constant multiple of $\frac{1}{n^2 h_n} + \frac{1}{n^{1+(\alpha-\eta)\beta}}$. Hence,

$$\mathbb{E}|B_{n,2}|^2 \leq c_{27} \left(\frac{1}{n^2 h_n} + \frac{1}{n^{1+(\alpha-\eta)\beta}} \right). \quad (4.16)$$

Now we estimate $\mathbb{E}|B_{n,3}|^2$. Note that

$$\begin{aligned} & \mathbb{E}|B_{n,3}|^2 \\ & \leq \frac{c_{28}}{n^4} \int_{\mathbb{R}^2} \left(\sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta > \ell_\theta \\ i_\theta - k_\theta > m_0 \text{ or } j_\theta - \ell_\theta > m_0, \theta=1,2}} + \sum_{\substack{1 \leq j_\theta < i_\theta \leq n, i_\theta - j_\theta > m, k_\theta < \ell_\theta \\ i_\theta - k_\theta > m_0 \text{ or } j_\theta - \ell_\theta > m_0, \theta=1,2}} \right) |\widehat{K}(\lambda_1 h_n)| |\widehat{K}(\lambda_2 h_n)| \\ & \quad \times \left| \mathbb{E}[\mathcal{P}_{k_1} H(X_{i_1})(\lambda_1) \mathcal{P}_{\ell_1} H(X_{j_1})(-\lambda_1) \mathcal{P}_{k_2} H(X_{i_2})(\lambda_2) \mathcal{P}_{\ell_2} H(X_{j_2})(-\lambda_2)] \right| d\lambda_1 d\lambda_2. \end{aligned} \quad (4.17)$$

Recall the choice of m_0 in the proof of Lemma 4.2. Then, by using Cauchy-Schwartz inequality and Lemma 4.1, we can show that the absolute value of the expectation in (4.17) is less than a constant multiple of

$$\begin{aligned} & \frac{1_{\{k_1 > \ell_1, k_2 > \ell_2, k_1 = k_2\}}}{(1 + |\lambda_1|^6)(1 + |\lambda_2|^6)} |\lambda_1 a_{i_1 - k_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{i_2 - k_2}|^{\frac{\alpha-\eta}{2}} \left(1_{\{\ell_1 = \ell_2\}} |\lambda_1 a_{j_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{j_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} \right. \\ & \quad + 1_{\{\ell_1 < \ell_2\}} |\lambda_2 a_{j_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - \ell_2} + \lambda_2 a_{i_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{j_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - \ell_1} + \lambda_2 a_{i_2 - \ell_1} - \lambda_2 a_{j_2 - \ell_1}|^{\frac{\alpha-\eta}{2}} \\ & \quad \left. + 1_{\{\ell_1 > \ell_2\}} |\lambda_1 a_{j_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - \ell_1} + \lambda_2 a_{i_2 - \ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{j_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - \ell_2} + \lambda_2 a_{i_2 - \ell_2} - \lambda_2 a_{j_1 - \ell_2}|^{\frac{\alpha-\eta}{2}} \right) \\ & + \frac{1_{\{\ell_1 > k_1, \ell_2 > k_2, \ell_1 = \ell_2\}}}{(1 + |\lambda_1|^6)(1 + |\lambda_2|^6)} |\lambda_1 a_{i_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{i_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} \left(1_{\{k_1 = k_2\}} |\lambda_1 a_{j_1 - k_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{j_2 - k_2}|^{\frac{\alpha-\eta}{2}} \right. \\ & \quad + 1_{\{k_1 < k_2\}} |\lambda_2 a_{j_2 - k_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - k_2} + \lambda_2 a_{i_2 - k_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{j_1 - k_1}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - k_1} + \lambda_2 a_{i_2 - k_1} - \lambda_2 a_{j_2 - k_1}|^{\frac{\alpha-\eta}{2}} \\ & \quad \left. + 1_{\{k_1 > k_2\}} |\lambda_1 a_{j_1 - k_1}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - k_1} + \lambda_2 a_{i_2 - k_1}|^{\frac{\alpha-\eta}{2}} |\lambda_2 a_{j_2 - k_2}|^{\frac{\alpha-\eta}{2}} |\lambda_1 a_{i_1 - k_2} + \lambda_2 a_{i_2 - k_2} - \lambda_2 a_{j_1 - k_2}|^{\frac{\alpha-\eta}{2}} \right) \\ & \leq \frac{1_{\{k_1 > \ell_1, k_2 > \ell_2, k_1 = k_2\}}}{(1 + |\lambda_1|^2)(1 + |\lambda_2|^2)} |a_{i_1 - k_1}|^{\frac{\alpha-\eta}{2}} |a_{i_2 - k_2}|^{\frac{\alpha-\eta}{2}} \left(1_{\{\ell_1 = \ell_2\}} |a_{j_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |a_{j_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} \right. \\ & \quad + 1_{\{\ell_1 \neq \ell_2\}} |a_{j_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} (|a_{i_1 - \ell_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2 - \ell_2}|^{\frac{\alpha-\eta}{2}}) |a_{j_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} (|a_{i_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2 - \ell_1}|^{\frac{\alpha-\eta}{2}} + |a_{j_2 - \ell_1}|^{\frac{\alpha-\eta}{2}}) \\ & \quad + \frac{1_{\{\ell_1 > k_1, \ell_2 > k_2, \ell_1 = \ell_2\}}}{(1 + |\lambda_1|^2)(1 + |\lambda_2|^2)} |a_{i_1 - \ell_1}|^{\frac{\alpha-\eta}{2}} |a_{i_2 - \ell_2}|^{\frac{\alpha-\eta}{2}} \left(1_{\{k_1 = k_2\}} |a_{j_1 - k_1}|^{\frac{\alpha-\eta}{2}} |a_{j_2 - k_2}|^{\frac{\alpha-\eta}{2}} \right. \\ & \quad \left. + 1_{\{k_1 \neq k_2\}} |a_{j_2 - k_2}|^{\frac{\alpha-\eta}{2}} (|a_{i_1 - k_2}|^{\frac{\alpha-\eta}{2}} + |a_{i_2 - k_2}|^{\frac{\alpha-\eta}{2}}) |a_{j_1 - k_1}|^{\frac{\alpha-\eta}{2}} (|a_{i_1 - k_1}|^{\frac{\alpha-\eta}{2}} + |a_{i_2 - k_1}|^{\frac{\alpha-\eta}{2}} + |a_{j_2 - k_1}|^{\frac{\alpha-\eta}{2}}) \right). \end{aligned}$$

Therefore, after simple calculations, we have

$$\mathbb{E}|B_{n,3}|^2 \leq c_{29} n^{2-2(\alpha-\eta)\beta}. \quad (4.18)$$

Combining (4.10), (4.11), (4.16) and (4.18) gives

$$\mathbb{E}|B_n - \mathbb{E}B_n| \leq c_{30} \left(n^{1-(\alpha-\eta)\beta} + \frac{1}{\sqrt{n^2 h_n}} \right). \quad (4.19)$$

Step 4. We estimate $\mathbb{E}[|N_n - \overline{N}_n|^2]$ where

$$\overline{N}_n = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{K}(0) (\phi_n(\lambda) - \phi(\lambda)) \phi(-\lambda) d\lambda.$$

Let

$$\tilde{N}_n = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{K}(\lambda h_n) (\phi_n(\lambda) - \phi(\lambda)) \phi(-\lambda) d\lambda.$$

Recall the definition of N_n in (4.2). $|N_n - \tilde{N}_n|$ is less than a constant multiple of $\frac{1}{n}$. Moreover, by Cauchy-Schwartz inequality and Lemma 4.2,

$$\begin{aligned} \mathbb{E} [|\tilde{N}_n - \bar{N}_n|^2] &\leq \mathbb{E} \left[\left| \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{K}(\lambda h_n) - \widehat{K}(0)) (\phi_n(\lambda) - \phi(\lambda)) \phi(-\lambda) d\lambda \right|^2 \right] \\ &\leq \left(\int_{\mathbb{R}} |\widehat{K}(\lambda h_n) - \widehat{K}(0)|^2 |\phi(\lambda)| d\lambda \right) \left(\int_{\mathbb{R}} \mathbb{E} |\phi_n(\lambda) - \phi(\lambda)|^2 |\phi(\lambda)| d\lambda \right) \\ &\leq c_{31} h_n^4 n^{1-(\alpha-\eta)\beta}. \end{aligned}$$

Hence

$$\mathbb{E} [|N_n - \bar{N}_n|^2] \leq c_{32} \left(\frac{1}{n^2} + h_n^4 n^{1-(\alpha-\eta)\beta} \right). \quad (4.20)$$

Step 5. It is easy to see that

$$\bar{N}_n = \frac{1}{n} \sum_{i=1}^n \left(f(X_i) - \int_{\mathbb{R}} f^2(x) dx \right).$$

Finally, combining (4.2), (4.9), (4.19) and (4.20) gives

$$\mathbb{E} \left| T_n(h_n) - \mathbb{E} T_n(h_n) - \frac{1}{n} \sum_{i=1}^n Y_i \right| \leq c_{33} \left(n^{1-(\alpha-\eta)\beta} + \frac{1}{\sqrt{n^3 h_n^2}} + \frac{1}{\sqrt{n^2 h_n}} + h_n^2 n^{\frac{1-(\alpha-\eta)\beta}{2}} \right).$$

This finishes the proof of Theorem 2.1. □

Finally, we give the proof of Theorem 2.2.

Proof of Theorem 2.2 According to Theorem 2.1, we only need to consider the asymptotic behavior of \bar{N}_n . In the region $1 < \alpha < 2$, (2.1) and (2.2) follow from Corollary 2.3 in [7] and Theorem 2.2 in [10], respectively. In the region $0 < \alpha < 1$, (2.3) follows from Theorem 2.1 in [4] and the paragraph after it. □

References

- [1] Ahmad, I.A. (1979). Strong consistency of density estimation by orthogonal series methods for dependent variables with applications. *Ann. Inst. Statist. Math.* **31**: 279–288.
- [2] Giraitis, L., Koul, H. L. and Surgailis, D. (1996). Asymptotic normality of regression estimators with long memory errors. *Statistics & Probability Letters*, **29**, 317–335.
- [3] Hsing, T. (1999). On the asymptotic distributions of partial sums of functionals of in nite-variance moving averages. *Annals of Probability*, **27**: 1579–1599.
- [4] Honda, T. (2009). A limit theorem for sums of bounded functionals of linear processes without finite mean. *Probability and Mathematical Statistics*, **29**: 337–351.

- [5] Ibragimov, I.A., Linnik, Y.V. (1971). Independent and stationary sequences of random variables. *Groningen: Wolters-Noordhoff*.
- [6] Källberg, D., Leonenko, N. and Seleznev, O. (2014). Statistical estimation of quadratic Rényi entropy for a stationary m -dependent sequence. *Journal of Nonparametric Statistics* **26**: 385–411.
- [7] Koul, H.L., Surgailis, D. (2001). Asymptotics of empirical processes of long memory moving averages with infinite variance. *Stochastic Processes and their Applications*, **91**: 309–336.
- [8] Samorodnitsky, G., Taqqu, M.S. (1994). Stable non-Gaussian processes: stochastic models with infinite variance. London: Chapman & Hall.
- [9] Sang, H., Sang, Y. and Xu, F. (2018). Kernel entropy estimation for linear processes. *Journal of Time Series Analysis*, **39**: 563–591.
- [10] Surgailis, D. (2002). Stable limits of empirical processes of moving averages with infinite variance. *Stochastic Processes and their Applications*, **100**: 255–274.

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