Birational invariance of the Chow-Witt group of zero-cycles

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Abstract

We prove that the Chow-Witt group of zero-cycles is a birational invariant of smooth proper schemes over a base field.

Contents

Introduction

The notion of *Milnor-Witt cycle modules* is introduced by the author in [Fel20, Fel21b] over a perfect field k which, after slight changes, can be generalized to more general base schemes (see [BHP22] for the case of a regular base scheme, and [DFJ22] for any base schemes).

The main example of a Milnor-Witt cycle module is given by the Milnor-Witt K-theory $\underline{\mathrm{K}}^{MW}$ (see [BCD$^+$20, Fel20, Fel21b, Fel21c, Fel21a] for more details).

To any MW-cycle module M and any k-scheme X equipped with a line bundle l_X, one can associated a Rost-Schmid complex $C_*(X, M, l_X)$ whose homology groups are called the called *the Chow-Witt groups with coefficient in* M. In particular, if $M = \underline{\mathrm{K}}^{MW}$, one recovers the Chow-Witt groups $\widetilde{\mathrm{CH}}_*(X, l)$ (see [Fas20]) which are, in some sense, a *quadratic refinement* of the classical Chow group $\mathrm{CH}_*(X)$.

A well-known consequence of intersection theory is that the Chow group $\mathrm{CH}_0(X)$ is a birational invariant. Indeed, this was proved in full generality in characteristic 0, and for surfaces in characteristic $p > 0$ in the fundamental work of Colliot-Thélène and Coray [CC79, Prop. 6.3]. The case of an algebraically closed base field can be found in [Ful98, Example 16.1.11], but the proof works verbatim for an arbitrary field.

A natural question is whether or not the birational invariance holds true for the Chow-Witt group and, more generally, of the Chow-Witt groups with coefficients in a Milnor-Witt cycle module). It is easy to see that the Chow-Witt group in **cohomological** degree zero $\widetilde{\mathrm{CH}}^0$ is a birational invariant for smooth proper k-scheme (see [Fel21b, Theorem 5.6]). In homological degree zero, the question is more complex.

Following ideas of Merkurjev [KM13], we prove that the Chow-Witt group of zero-cycles is a birational invariant for smooth proper schemes. More generally, we have:

Theorem 1 (see Theorem 2.2.12)**.** The group $A_0(X, M)$ is a birational invariant of the smooth proper scheme X.

In particular, the Chow-Witt group of zero-cycles $\widetilde{\mathrm{CH}}_0(X)$ is a birational invariant of the smooth proper scheme X.

Outline of the paper

In Section 1, we explain how to build a special type of Milnor-Witt cycle module from a fix MW-module. Moreover, we define a cup product for oriented schemes.

In Section 2, we prove that the two previous constructions are compatible with each other in some sense. This allows us to define a composition of *Milnor-Witt rational correspondences* and construct an associated pushforward map. Finally, we apply these results to prove that Chow-Witt group of zero-cycles is a birational invariant for smooth proper schemes.

In Appendix A, we recall the basic definitions of (cohomological) Milnor-Witt cycle modules along with the basic maps (pushforward, pullback, etc.). We then define the new class of *oriented* schemes.

Acknowledgments

The author deeply thanks Frédéric Déglise, Fangzhou Jin, Jean Fasel, Adrien Dubouloz, Marcello Bernardara, Marc Levine, Paul Arne Østvær, Bertrand Toën, Joana Cirici, Baptiste Calmès, Alexey Ananyevskiy.

Notations and conventions

In this paper, schemes are noetherian and finite dimensional. We fix a base field[1] k and put $S = \operatorname{Spec} k$, and we fix a base ring of coefficients R. If not stated otherwise, all schemes and morphisms of schemes are defined over S. A *point* (resp. *trait*, *singular trait*) of S will be a morphism of schemes $\operatorname{Spec}(k) \to S$ essentially of finite type and such

Conventions: a morphism $f : X \to S$ (sometime denoted by X/S) is:

- essentially of finite type if f is the projective limit of a cofiltered system $(f_i)_{i \in I}$ of morphisms of finite type with affine and étale transition maps
- lci if it is smoothable and a local complete intersection (*i.e.* admits a global factorization $f = p \circ i$, p smooth and i a regular closed immersion);
- essentially lci if it is a limit of lci morphisms with étale transition maps.

Let X/S be a scheme essentially of finite type. We put $X_{(p)}$ the set of p-dimensional points of X.

A point x of S is a map $x : \operatorname{Spec}(E) \to S$ essentially of finite type and such E is a field. We also say that E is a field over S.

Given a morphism of schemes $f : Y \to X$, we let L_f be its cotangent complex, an object of $\mathrm{D}^b_{\mathrm{coh}}(Y)$, and when the latter is perfect (e.g. if f is essentially lci), we let τ_f be its associated virtual vector bundle over Y, and by ω_f the determinant of τ_f.

If not stated otherwise, M is a (cohomological) Milnor-Witt cycle module, X is an S-scheme, l is a line bundle over X, and p, q are integers.

1 Main constructions

1.1 The relative perverse homology

We follow [Ros96, §7]. In this section, we show that new Milnor-Witt cycle modules can be obtained from the Chow groups of the fibers of a morphism.

[1] Many results of the present paper are in fact true over a more general base scheme.

1.1.1. Let $\rho : Q \to S$ be a morphism of finite type and let M be a cohomological MW-cycle module over Q. Fix l a line bundle over Q. For any field F over S, denote by $Q_F = Q \times_B \operatorname{Spec} F$. We define an object function $A_p[\rho, M, l]$ on $\mathbf{F}(S)$ by

$$A_p[\rho, M, l] = \bigoplus_{q \in \mathbf{Z}} A_p[\rho, M_q, l]$$

where

$$A_p[\rho, M_q, l](F) = A_p(Q_F, M_q, \omega^\vee_{Q_F/Q} \otimes l).$$

Our aim is to show that $A_p[\rho, M, l]$ is in a natural way a Milnor-Witt cycle module over S.

1.1.2. All the properties of Milnor-Witt cycle modules except axiom (C) hold already on complex level, i.e. for the groups $C_p(Q_F, M)$. Indeed, we denote by $\widehat{M}$ the object function on $\mathbf{F}(B)$ defined by

$$\widehat{M}(F) = C_p(Q_F, M, \omega^\vee_{Q_F/Q} \otimes l) = \bigoplus_{q \in \mathbf{Z}} C_p(Q_F, M_q, \omega^\vee_{Q_F/Q}).$$

We first describe its data as a Milnor-Witt cycle premodule. These will be denoted by $\widehat{\operatorname{res}}_{F/E}, \widehat{\operatorname{cores}}_{F/E}$, etc. in order to distinguish them from the data $\operatorname{res}_{F/E}, \operatorname{cores}_{F/E}$, etc. of M.

For a morphism of fields $\phi : E \to F$, let $\overline{\phi} : Q_F \to Q_E$ be the induced map.

1. DATA D1 Define

$$\widehat{\operatorname{res}}_{F/E} := \phi^! : C_p(Q_E, M_q, \omega^\vee_{Q_E/Q}) \to C_p(Q_F, M_q, \omega^\vee_{Q_F/Q}).$$

2. DATA D2 Assume ϕ finite. Define

$$\widehat{\operatorname{cores}}_{F/E} := \phi_* : C_p(Q_F, M_q, \mathcal{O}_{Q_F}) \to C_p(Q_E, M_q, \mathcal{O}_{Q_E}).$$

3. DATA D3 Simply take the $\underline{\mathrm{K}}^{MW}$-module structure on $C_p(Q_F, M)$ described in [DFJ22, §1.4 and §5.4].

4. DATA D4 Denote by $\widetilde{Q}_v = Q \times_S \operatorname{Spec} \mathcal{O}_v$, the generic fiber Q_F and the special fiber $Q_{\kappa(v)}$. Define

$$\widehat{\partial}_v : C_p(Q_F, M_q) \to C_{p-1}(Q_{\kappa(v)}, M_q)$$

by $(\widehat{\partial}_v)^x_y = \partial^x_y$ with ∂^x_y as in [DFJ22, §5.3.13] with respect to the scheme $\widetilde{Q_v}$.

Theorem 1.1.3. *Keeping the previous notations, the object functor $\widehat{M}$ along with these data form a Milnor-Witt cycle premodule over S.*

Proof. All the required properties follow from the rules and axioms for M and from the functorial properties studied in [DFJ22, §1.4 and §5.4]. □

1.1.4. Now, we want to relate the differentials for the MW-cycle premodule $\widehat{M}$ to the differentials for the MW-cycle module M.

Let $X \to S$ be a scheme over S and let $\widetilde{X} = Q \times_S X$. Then for x, y in X, there is a map

$$\widehat{\partial_y^x} : \widehat{M}(x) \to \widehat{M}(y)$$

as in [DFJ22, §5.3.13]. By definition, this is a map

$$\widehat{\partial}_y^x : C_p(Q_{\kappa(x)}, M) \to C_p(Q_{\kappa(y)}, M)$$

between cycle groups with coefficients in M.

Proposition 1.1.5. *Let $\widetilde{x}, \widetilde{y}$ in $\widetilde{X}$ be points lying over $x, y \in X$, respectively, and assume that $\widetilde{x} \in (Q_{\kappa(x)})_{(q)}$ and $\widetilde{y} \in (Q_{\kappa(y)})_{(q)}$. Denote by $(\widehat{\partial_y^x})_{\widetilde{y}}^{\widetilde{x}}$ the component of $\widehat{\partial_y^x}$ with respect to $\widetilde{x}$ and $\widetilde{y}$. Then*

$$(\widehat{\partial}_y^x)_{\widetilde{y}}^{\widetilde{x}} = \partial_{\widetilde{y}}^{\widetilde{x}} : M_q(\widetilde{x}, \omega_{\widetilde{x}/S}) \to M_{q-1}(\widetilde{y}, \omega_{\widetilde{y}/S}).$$

Proof. We may assume $\widetilde{y} \in \overline{\{\widetilde{x}\}}^{(1)}$, since otherwise both sides are trivial. The dimension inequality [Mat80, p. 85] shows then $y \in \overline{\{x\}}^{(1)}$. Let v run through the valuations of $\kappa(x)$ with center y in X. Moreover, let w run through the valuations on $\kappa(\widetilde{x})$ with center $\widetilde{y}$ in $\widetilde{X}$. The restriction of any w to $\kappa(x)$ is one of the valuations v. Let $\widetilde{w} \in Q_{\kappa(v)}$ be the center of w in $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_v$. Now the claim follows from

$$\begin{aligned}
(\widehat{\partial}_y^x)_{\widetilde{y}}^{\widetilde{x}} &= (\textstyle\sum_v \widehat{\mathrm{cores}}_{\kappa(v)/\kappa(y)} \circ \widehat{\partial}_v)_{\widetilde{y}}^{\widetilde{x}} \\
&= \textstyle\sum_v \sum_{w|v} (\widehat{\mathrm{cores}}_{\kappa(v)/\kappa(y)})_{\widetilde{y}}^{\widetilde{w}} \circ (\widehat{\partial}_v)_{\widetilde{w}}^{\widetilde{x}} \\
&= \textstyle\sum_v \sum_{w|v} \mathrm{cores}_{\kappa(\widetilde{w}/\kappa(\widetilde{y})} \circ \mathrm{cores}_{\kappa(w)|\kappa(\widetilde{w})} \circ \partial_w \\
&= \textstyle\sum_w \mathrm{cores}_{\kappa(w)/\kappa(\widetilde{y})} \circ \partial_w \\
&= \partial_{\widetilde{y}}^{\widetilde{x}}.
\end{aligned}$$

□

It follows from [DFJ22, Proposition 1.4.6] that the data of the MW-cycle premodule $\widehat{M}$ commute with the differentials of the complex $C_*(Q_F, M)$. Passing to homology, we obtain data D1-D4 for the object function $A_q[\rho, M]$.

Theorem 1.1.6. *Keeping the previous notations, the object function $A_p[\rho, M]$ together with these data is a Milnor-Witt cycle module over S.*

Proof. The rules for the data of the MW-cycle premodule $A_p[\rho, M]$ are immediate from the rules for $\widehat{M}$. Moreover, axiom (FD) for M and Proposition 1.1.5 show that (FD) holds for $\widehat{M}$ and thus for $A_p[\rho, M]$. It remains to verify axiom (C).

Consider the map

$$C_p(Q_{\kappa(\xi)}) \xrightarrow{\delta} C_{p-1}(Q_{\kappa(\xi)}) \oplus \bigoplus_{x\in X^{(1)}} C_p(Q_{\kappa(x)}) \oplus C_{p+1}(Q_{\kappa(x_0)}) \xrightarrow{\delta} C_p(Q_{\kappa(x_0)})$$

defined by $\delta^z_y = \partial^z_y$ with ∂^z_y as in [DFJ22, §5.3.13] with respect to the scheme $Q \times_B X$ (we have shortened the notation by omitting M).

By Proposition 1.1.5, we are reduced to show $\delta \circ \delta = 0$. It suffices to check that $(\delta \circ \delta)^z_y = 0$ for $z \in (Q_{\kappa(\xi)})_{(q)}$ and $y \in (Q_{\kappa(x_0)})_{(q)}$ with $y \in \overline{\{z\}}^{(2)}$ (here $\overline{\{z\}}$ is the closure of z in $\widetilde{X}$). The dimension inequality [Mat80, p. 85] shows

$$Z^{(1)} \subset (Q_{\kappa(\xi)})_{(q-1)} \cup \bigcup_x (Q_{\kappa(x)})_{(q)} \cup (Q_{\kappa(x_0)})_{(q+1)}$$

with $Z = \overline{\{z\}}_{(y)}$. We are done by axiom (C) for M. ☐

Definition 1.1.7. Keeping the previous notations, the Milnor-Witt cycle module $A_p[\rho, M]$ is called the *p-th relative perverse homology* of M with respect to ρ.

Remark 1.1.8. One should also obtain the results present in [Ros96, §8]. In particular, the MW-cycle module $A_q[\rho, M]$ could be used to give another proof of the homotopy invariance of the Rost-Schmid complex.

1.2 The cup product

1.2.1. We follow ideas of Merkurjev [Mer03]. We work over a base field k. We fix M a Milnor-Witt cycle module over k.

1.2.2. Let $M \times N \to P$ be a bilinear pairing of MW-cycle modules over k. Let X, Y and Z be smooth schemes over k with Y irreducible smooth and proper. Denote by $\Delta : Y \to Y \times Y$ the diagonal map. Let q be an integer and l_X (resp. l_Y, l'_Y, and l'_Z) a line bundle over X (resp. Y, Y, and Z). Assume that $\omega_{Y/k} \otimes l_Y \otimes l'_Y \simeq \mathcal{O}_Y$.

We have a $\cup$-product

$$\cup : A_r(X \times Y, M_s, l_X \otimes l_Y) \otimes A_p(Y \times Z, N_q, l'_Y \otimes l'_Z) \to \\ A_{r+p-d_Y}(X \times Z, P_{s+q+d_Y}, l_X \otimes l_Z)$$

defined as the composition

$$
\begin{array}{c}
A_r(X\times Y, M_s, l_X\otimes l_Y)\otimes A_p(Y\times Z, N_q, l'_Y\otimes l'_Z) \\
\Big\downarrow{\scriptstyle \times} \\
A_{r+p}(X\times Y\times Y\times Z, P_{s+q}, l_X\otimes l_Y\otimes l'_Y\otimes l'_Z) \\
\Big\downarrow{\scriptstyle (\mathrm{Id}_X\otimes\Delta\otimes\mathrm{Id}_Z)^*} \\
A_{r+p-d_Y}(X\times Y\times Z, P_{s+q+d_Y}, l_X\otimes\omega_{Y/k}\otimes l_Y\otimes l'_Y\otimes l'_Z) \\
\Big\downarrow{\scriptstyle \simeq} \\
A_{r+p-d_Y}(X\times Y\times Z, P_{s+q+d_Y}, l_X\otimes l'_Z) \\
\Big\downarrow{\scriptstyle \pi_{XZ*}} \\
A_{r+p-d_Y}(X\times Z, P_{s+q+d_Y}, l_X\otimes l'_Z)
\end{array}
$$

where $\times$ is the cross product (see [Fel21b, Section 10]), $\Delta : Y \to Y\times Y$ is the diagonal embedding and $\pi_{XZ} : X\times Y\times Z \to X\times Z$ is the projection. The pushforward p_{XZ*} is well-defined because Y is smooth and proper.

1.2.3. In particular, taking $N = M = P = \underline{\mathrm{K}}^{MW}$, $l_X = \omega^\vee_{X/k}$, $l_Y = \mathcal{O}_Y$, $l'_Y = \omega^\vee_{Y/k}$, $l'_Z = \mathcal{O}_Z$, $r = -s$ and $p = -q$, we have the product

$$\cup : \widetilde{\mathrm{CH}}_r(X\times Y, \omega^\vee_{X/S})\otimes\widetilde{\mathrm{CH}}_p(Y\times Z, \omega^\vee_{Y/S}) \to \widetilde{\mathrm{CH}}_{r+p-d_Y}(X\times Z, \omega^\vee_{X/S})$$

which could be taken as the composition law for the category of Milnor-Witt integral correspondences $\widetilde{\mathbf{Cor}}$ with objects the smooth proper schemes over k and morphisms

$$\mathrm{Hom}_{\widetilde{\mathbf{Cor}}}(X,Y) = \bigoplus_i \widetilde{\mathrm{CH}}_{d_i}(X_i\times Y, \omega^\vee_{X/S}),$$

where X_i are irreducible (connected) components of X with $d_i = \dim X_i$.

2 Milnor-Witt rational correspondences

Let X be a smooth and proper k-scheme and l_X (resp. l_Y) a line bundle over X (resp. Y). There is a canonical map of complexes

$$\Theta_M : C_p(X\times Y, M_q, l_X\otimes l_Y) \to C_p(X, A_0[Y, M_q, l_Y], l_X),$$

that takes an elements in $M(z, \omega_z\otimes l_{X|z}\otimes l_{Y|z})$ for $z\in(X\times Y)_{(p)}$ to zero if dimension of the projection x of z in X is strictly less than p, and identically to itself otherwise. In the latter case, we consider z as a point of dimension 0 in $Y_x := Y_{\kappa(x)}$ under the inclusion $Y_x\subset X\times Y$. Thus, $\Theta_{Y,M}$ "ignores" points in $X\times Y$ that lose dimension being projected to X.

We study various compatibility properties of Θ_M.

2.1 Cross products

Let $M \times N \to P$ be a bilinear pairing of MW-cycle modules over k. For a smooth scheme Y over k and l_Y a line bundle over Y, we can define a pairing

$$M \times A_0[Y, N, l_Y] \to A_0[Y, P, l_Y]$$

in an obvious way.

Lemma 2.1.1. *For X, Y, Z smooth k-schemes, and l_X (resp. l_Y, l_Y) a line bundle over X (resp. Y, Z), the following diagram is commutative:*

$$\begin{CD}
C_p(X, M_q, l_X) \otimes C_r(Y \times Z, N_s, l_Y \otimes l_Z) @>{\times}>> C_{p+r}(X \times Y \times Z, P_{q+s}, l_X \otimes l_Y \otimes l_Z) \\
@V{\mathrm{Id} \times \Theta_N}VV @VV{\Theta_P}V \\
C_p(X, M_q, l_X) \otimes C_r(Y A_0[Z, N_s, l_Z], l_Y) @>{\times}>> C_{p+r}(X \times Y, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y).
\end{CD}$$

Proof. Let $x \in X_{(p)}$ and $\mu \in C_p(X, M_q, l_X)$. Consider the following commutative diagram

$$\begin{CD}
C_r(Y \times Z, N_s, l_Y \otimes l_Z) @>{\Theta_N}>> C_r(Y, A_0[Z, N_s, l_Z], l_Y) \\
@V{\pi_x'^*}VV @VV{\pi_x^*}V \\
C_r((Y \times Z)_x, N_s, l_Y \otimes l_Z) @>{\Theta_N}>> C_r(Y_x, A_0[Z, N_s, l_Z], l_Y) \\
@V{m'_\mu}VV @VV{m_\mu}V \\
C_{p+r}((Y \times Z)_x, P_{q+s}, l_X \otimes l_Y \otimes l_Z) @>{\Theta_P}>> C_{p+r}(Y_x, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y) \\
@V{i'_{x,*}}VV @VV{i_{x,*}}V \\
C_{p+r}(X \times Y \times Z, P_{q+s}, l_X \otimes l_Y \otimes l_Z) @>{\Theta_P}>> C_{p+r}(X \times Y, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y)
\end{CD}$$

where $\pi_x : Y_x \to Y$ and $\pi'_x : (X \times Y)_x \to X \times Y$ are the natural projections, m_μ and m'_μ are the multiplications by μ, and $i_x : Y_x \to X \times Y$ and $i'_x : (Y \times Z)_z \to X \times Y \times Z$ are the inclusions. By the definition of the cross product, the compositions in the two rows of the diagram are the multiplications by μ.

$\square$

2.1.2. Pullback maps Let $f : Z \to X$ be a regular closed embedding of smooth schemes of dimension s and l a line bundle over X. We denote by $N_{X/Z}$ the normal bundle over Z. For an smooth scheme Y, the closed embedding

$$f' = f \times \mathrm{Id}_Y : Z \times Y \to X \times Y$$

is also regular and the normal bundle $N_{X\times Y/Z\times Y}$ is isomorphic to $N_{X/Z}\times Y$.

Lemma 2.1.3. *The following diagram is commutative:*

$$\begin{array}{ccc} A_p(X\times Y, M_q, l) & \xrightarrow{f'^*} & A_{p+s}(Z\times Y, M_{q-s}, l\otimes\omega_f^\vee) \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\ A_p(X, A_0[Y, M_q], l) & \xrightarrow{f^*} & A_{p+s}(Z, A_0[Y, M_{q-s}], l\otimes\omega_f^\vee). \end{array}$$

Proof. Let $\pi_X : \mathbf{G}_m\times X\to X$ and $\pi'_X : \mathbf{G}_m\times X\times Y\to X\times Y$ be the natural projections. The following diagram

$$\begin{array}{ccc} C_p(X\times Y, M_q, l) & \xrightarrow{(\pi'_X)^*} & C_{p+1}(\mathbf{G}_m\times X\times Y, M_{q-1}, l) \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\ C_p(X, A_0[Y, M_q], l) & \xrightarrow{\pi_X^*} & C_{q+1}(\mathbf{G}_m\times X, A_0[Y, M_{q-1}], l) \end{array}$$

is commutative.

Let t be the coordinate function on $\mathbf{G}_m$. The map Θ_M commutes with the multiplication by t, i.e. the following diagram

$$\begin{array}{ccc} C_p(\mathbf{G}_m\times X\times Y, M_q, l) & \xrightarrow{[t]} & C_p(\mathbf{G}_m\times X\times Y, M_{q+1}, l) \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\ C_p(\mathbf{G}_m\times X, A_0[Y, M_q], l) & \xrightarrow{[t]} & C_p(\mathbf{G}_m\times X, A_0[Y, M_{q+1}], l) \end{array}$$

is commutative.

Let $D = D(X,Z)$ be the deformation space of the embedding f (see e.g. [Ros96, §10]). There is a closed embedding $i : N_{X/Z}\to D$ with the open complement $j : \mathbf{G}_m\times X\to D$. Then $D' = D\times Y$ is the deformation space $D(X\times Y, Z\times Y)$ with the closed embedding

$$i' = i\times \mathrm{Id}_Y : N_{X\times Y/Z\times Y}\to D'$$

and the open complement $j' = j\times\mathrm{Id}_Y : \mathbf{G}_m\times X\times Y\to D'$.

The commutative diagram with exact rows

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
C_p(N_{X/Z} \times Y, M_q, l) & \xrightarrow{\Theta_M} & C_p(N_{X/Z}, A_0[Y, M_q], l) \\
\downarrow{\scriptstyle i'_*} & & \downarrow{\scriptstyle i_*} \\
C_p(D', M_q, l) & \xrightarrow{\Theta_M} & C_p(D, A_0[Y, M_q], l) \\
\downarrow{\scriptstyle j'^*} & & \downarrow{\scriptstyle j^*} \\
C_p(\mathbf{G}_m \times X \times Y, M_q, l) & \xrightarrow{\Theta_M} & C_p(\mathbf{G}_m \times X, A_0[Y, M_q], l) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

induces the commutative diagram

$$
\begin{array}{ccc}
C_p(\mathbf{G}_m \times X \times Y, M_q, l) & \xrightarrow{\partial} & C_{p-1}(N_{X/Z} \times Y, M_q, l) \\
\downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\
C_p(\mathbf{G}_m \times X, A_0[Y, M_q], l) & \xrightarrow{\partial} & C_{p-1}(N_{X/Z}, A_0[Y, M_q], l).
\end{array}
$$

Finally, we also have the commutative diagram

$$
\begin{array}{ccc}
C_p(Z \times Y, M_q, l \otimes \omega_f!!^\vee) & \xrightarrow{\pi^*} & C_{p+s}(N_{X/Z} \times Y, M_{q-s}, l) \\
\downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\
C_p(Z, A_0[Y, M_q], l \otimes \omega_f^\vee) & \xrightarrow{\pi'^*} & C_{p+s}(N_{X/Z}, A_0[Y, M_{q-s}], l)
\end{array}
$$

where $\pi : N_{X/Z} \to Z$ is the canonical projection and s its relative dimension (π is a quasi-isomorphism by homotopy invariance). By the definition of the pullback map (see [Fel20, Section 7]), the result follows from the composition of the previous commutative square. □

Remark 2.1.4. The previous lemma could be stated at the level of complexes with the use of Rost's coordinations or by using the homotopy complex defined in [DFJ22, §2.2], but we do not need this generality.

2.1.5. PUSHFORWARD MAPS Let $f : X \to Z$ be a map of smooth schemes (over k). and l a line bundle over Z. For an oriented smooth scheme Y, set

$$f' = f \times \mathrm{Id}_Y : X \times Y \to Z \times Y.$$

Lemma 2.1.6. *The following diagram*

$$\begin{array}{ccc} C_p(X\times Y, M_q, l) & \xrightarrow{f'_*} & C_p(Z\times Y, M_q, l) \\ \downarrow \Theta_M & & \downarrow \Theta_M \\ C_p(X, A_0[Y, M_q], l) & \xrightarrow{f_*} & C_p(Z, A_0[Y, M_q], l). \end{array}$$

is commutative.

Proof. Let $u \in (X \times Y)_{(p)}$, $a \in M(\kappa(u), \omega_u \otimes l)$. Set $v = f'(u) \in Z \times Y$. If $\dim(v) < p$ then $(f'_*)_u(a) = 0$. In this case, the dimension of the projection y of u in Y is less than p and hence $(\Theta_M)_u(a) = 0$.

Assume that $\dim(v) = p$. Then $\kappa(u)/\kappa(v)$ is a finite field extension and

$$b = (f'_*)_u(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) \in M(\kappa(v), \omega_v \otimes l).$$

If $\dim(y) < p$, then $(\Theta_M)_u(a) = 0$, and $\Theta_v(b) = 0$.

Assume that $\dim(y) = p$, then

$$(\Theta_M \circ f'_*)_u(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) = b$$

considered as an element of $A_0[Y, M_q](\kappa(z), \omega_z \otimes l) = A_0(Y_z, M_q, l)$, where z is the image of v in Z. On the other hand,

$$(f_* \circ \Theta_M)_u(a) = \phi_*(a),$$

where $\phi : Y_x \to Y_z$ is the natural map (where x is the image of u in X) and is considered as an element of $A_0[Y, M_q](\kappa(z), \omega_z \otimes l)$. It remains to notice that

$$\phi_*(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) = b.$$

□

2.2 Rational correspondences

Let Y and Z be smooth schemes over k. Assume Y irreducible and denote by d_Y the dimension of Y.By Lemma 2.1.1, for the pairing $M \times \underline{\mathrm{K}}^{MW} \to M$ and "$X = Y$" we have the commutative diagram

$$\begin{array}{ccc} A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega_{Y/k}^\vee) & \xrightarrow{\times} & A_{d_Y}(Y \times Y \times Z, M_{-d_Y+q}, \omega_{Y/k}^\vee) \\ \downarrow \mathrm{Id} \otimes \Theta_{\underline{\mathrm{K}}^{MW}} & & \downarrow \Theta_M \\ A_0(Y, M_q) \otimes A_{d_Y}(Y, A_0[Z, \underline{\mathrm{K}}^{MW}_{-d_Y}], \omega_{Y/k}^\vee) & \xrightarrow{\times} & A_{d_Y}(Y \times Y, A_0[Z, M_{-d_Y+q}], \omega_{Y/k}^\vee). \end{array}$$

Let $\Delta : Y \to Y \times Y$ be the diagonal embedding and $\Delta' = \Delta \otimes \mathrm{Id}_Z$. By Lemma 2.1.3, the following diagram

$$\begin{array}{ccc} A_{d_Y}(Y\times Y\times Z, M_{-d_Y+q}, \omega^\vee_{Y/k}) & \xrightarrow{\Delta'^*} & A_0(Y\times Z, M_q) \\ \downarrow{\Theta_M} & & \downarrow{\Theta_M} \\ A_{d_Y}(Y\times Y, A_0[Z, M_{-d_Y+q}], \omega^\vee_{Y/k}) & \xrightarrow{\Delta^*} & A_0(Y, A_0[Z, M_q]) \end{array}$$

is commutative.

Finally, assume that the structure map $f : Y \to \operatorname{Spec} k$ is proper and denote by $f' = \mathrm{Id}_X \times f$. Lemma 2.1.6 implies that the following diagram

$$\begin{array}{ccc} A_0(Y\times Z, M_q) & \xrightarrow{f'_*} & A_0(Z, M_q) \\ \downarrow{\Theta_M} & & \| \\ A_0(Y, A_0[Z, M_q]) & \xrightarrow{f_*} & A_0(\operatorname{Spec} k, A_0[Z, M_q]). \end{array}$$

is commutative.

Proposition 2.2.1. *Let Y and Z be smooth schemes over k, Y an irreducible smooth and proper, and M an MW-cycle module over k. Then the pairing*

$$\cup : A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y\times Z, \omega^\vee_{Y/k}) \to A_0(Z, M_q)$$

is trivial on all cycles in $\widetilde{\mathrm{CH}}_{d_Y}(Y\times Z, \omega^\vee_{Y/k})$ that are not dominant over Y. In other words, the $\cup$-product factors through a natural pairing

$$\cup : A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) \to A_0(Z, M_q)$$

Proof. This follows from composing all three diagrams and taking into account that

$$A_{d_Y}(Y, A_0[Z, \underline{\mathrm{K}}^{MW}_{-d_Y}], \omega^\vee_{Y/k}) = \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}).$$

□

2.2.2. Keeping the previous notations, for Z irreducible smooth scheme over k, the diagram

$$\begin{array}{ccc} A_{d_X}(X\times Y, M_q, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y\times Z, \omega^\vee_{Y/k}) & \xrightarrow{\cup} & A_{d_X}(X\times Z, M_q, \omega^\vee_{X/k}) \\ \downarrow & & \downarrow \\ A_0(Y_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y\times Z, \omega^\vee_{Y/k}) & \xrightarrow{\cup} & A_0(Z_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \\ \downarrow & & \downarrow \\ A_0(Y_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) & \xrightarrow{\cup} & A_0(Z_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \end{array}$$

is commutative.

2.2.3. In particular, we have a well defined pairing

$$\cup : \widetilde{\mathrm{CH}}_0(Y_{\kappa(X)}, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) \to \widetilde{\mathrm{CH}}_0(Z_{\kappa(X)}, \omega^\vee_{X/k})$$

that can be taken for the composition law in the category of Milnor-Witt rational correspondences $\widetilde{\mathbf{RatCor}}(k)$ whose objects are the smooth proper schemes over k and morphisms are given by

$$\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X, Y) = \bigoplus_i \widetilde{\mathrm{CH}}_0(Y_{\kappa(X_i)}, \omega^\vee_{X/k}),$$

where X_i are all irreducible (connected) components of X.

There is an obvious functor

$$\Xi : \widetilde{\mathbf{Cor}}(k) \to \widetilde{\mathbf{RatCor}}(k).$$

Theorem 2.2.4. *For an MW-cycle module M, there exists a well-defined covariant functor*

$$\widetilde{\mathbf{RatCor}}(k) \to \mathscr{A}b, X \mapsto A_0(X, M), a \mapsto - \cup a.$$

More precisely, the functor $\widetilde{\mathbf{Cor}}(k) \to \widetilde{\mathbf{RatCor}}(k)$ factors through Ξ.

Proof. This follows from Proposition 2.2.1. □

Remark 2.2.5. Assuming one works with *oriented* (see A.2.2) smooth proper k-schemes, then there is also a contravariant functor given by $a \mapsto {}^t a \cup -$. We won't need this result.

2.2.6. If $(\alpha : X \rightsquigarrow Y) \in \mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X, Y)$ is a MW-rational correspondence between two smooth proper k-schemes, we have a natural pushforward morphism

$$\alpha_* : A_0(X, M) \to A_0(Y, M).$$

Remark 2.2.7. If α et β are two composable Milnor-Witt rational correspondences, then

$$(\alpha \circ \beta)_* = \alpha_* \circ \beta_*.$$

2.2.8. Let $f : X \dashrightarrow Y$ be a rational morphism of irreducible smooth k-schemes. It defines a rational point of $Y_{\kappa(X)}$ over $\kappa(X)$ and hence a morphism in $\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X, Y)$ that we denote by $[f] : X \rightsquigarrow Y$. In fact, the rational correspondence $[f]$ is the image of the class of the (transposed of the) graph of f (as in [BCD$^+$20, Chapter 2, §4.3]) under the natural map

$$\widetilde{\mathrm{CH}}_{d_X}(X \times Y, \omega_{X/k}) \to \widetilde{\mathrm{CH}}_0(Y_{\kappa(X)}, \omega_{X/k}).$$

Lemma 2.2.9. *Let κ/k be a finite type extension of fields. Let $f : X \dashrightarrow Y$ be a rational morphism of smooth proper κ-schemes and let $x \in X$ be a rational point such that $f(x)$ is defined. Denote by $[x] \in \widetilde{\mathrm{CH}}_0(X, \omega_{\kappa/k})$ the 0-cycle associated to x. Then*

$$[f]_*([x]) = [f(x)]$$

in $\widetilde{\mathrm{CH}}_0(Y, \omega_{\kappa/k})$.

Proof. Let $\Gamma \subset X \times Y$ be the graph of f. The preimage of $\{x\} \times \Gamma$ under the morphism $\Delta_X \otimes \mathrm{Id}_Y : X \times Y \to X \times X \times Y$ is the reduced scheme $\{x\} \times \{f(x)\}$. Hence

$$[f]_*([x]) = [x] \cup [f] = \pi_*(\Delta_X \otimes \mathrm{Id}_Y)^*([x] \times [\Gamma]) = \pi_*([x] \times [f(x)]) = [f(x)]$$

where $\pi : X \times Y \to Y$ is the projection. □

Corollary 2.2.10. *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be composable rational morphisms of smooth proper schemes and let $h : X \dashrightarrow Z$ be the composition of f and g. Then $[g] \circ [f] = [h]$ in $\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X, Z)$.*

Proof. Let y be the rational point of $Y_{\kappa(X)}$ corresponding to f. By assumption, the rational morphism $g_{\kappa(X)} : Y_{\kappa(X)} \dashrightarrow Z_{\kappa(X)}$ is defined at y. By Lemma 2.2.9 (with "$\kappa = \kappa(X)$","$X = Y_{\kappa(X)}$", "$Y = Z_{\kappa(X)}$ and "$f = g_{\kappa(X)}$") we see that the composition of correspondences f and g takes $[y]$ to $[g_{\kappa(X)}(y)] \in \widetilde{\mathrm{CH}}_0(Z_{\kappa(X)}, \omega_{X/k}^\vee)$. Note that the latter class corresponds to h. □

Corollary 2.2.11. *For any two composable rational morphisms $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ of smooth proper schemes, we have*

$$[g \circ f]_* = [g]_* \circ [f]_*.$$

Proof. This is a consequence of Corollary 2.2.10. □

Theorem 2.2.12. *The group $A_0(X, M)$ is a birational invariant of the smooth proper scheme X.*

In particular, the Chow-Witt group of zero-cycles $\widetilde{\mathrm{CH}}_0(X)$ is a birational invariant of the smooth proper scheme X.

Proof. This is an immediate consequence of Corollary 2.2.11. □

Example 2.2.13. According to [Fas20, §5], we know that $\widetilde{\mathrm{CH}}_0(\mathbf{P}^n_k) = \mathrm{GW}(k)$ for any natural number n.

In particular, we recover the computations of $\widetilde{\mathrm{CH}}_0(Q_n)$ where Q_n is an n-dimensional split quadric (see [HXZ20, Corollary 9.5]).

Example 2.2.14. If M is K^M (the Milnor-Witt K-theory), then we recover the fact that the Chow group of zero-cycles $\mathrm{CH}_0(X)$ is a birational invariant of the smooth proper scheme X.

A Appendix

A.1 Cohomological Milnor-Witt cycle modules

Definition A.1.1. 1. If S is a scheme, call an S**-field** the spectrum of a field essentially of finite type over S, and a **morphism of** S**-fields** an S-morphism between the underlying schemes. The collection of S-fields together with morphisms of S-fields defines a category which we denote by $\mathcal{F}_S$. We say that a morphism of S-fields is **finite** (resp. **separable**) if the underlying field extension is finite (resp. separable).

   In what follows, we will denote for example $f : \operatorname{Spec} F \to \operatorname{Spec} E$ a morphism of S-fields, and $\phi : E \to F$ the underlying field extension.

   An S**-valuation** on an S-field $\operatorname{Spec} F$ is a discrete valuation v on F such that $\operatorname{Im}(\mathcal{O}(S) \to F) \subset \mathcal{O}_v$. We denote by $\kappa(v)$ the residue field, $\mathfrak{m}_v$ the valuation ideal and $N_v = \mathfrak{m}/\mathfrak{m}^2$.

2. Let S be a scheme and let R be a commutative ring with unit. An R**-linear cohomological Milnor-Witt cycle premodule** over S is a functor from $\mathcal{F}_S$ to the category of $\mathbf{Z}$-graded R-modules

$$\begin{aligned} M : (\mathcal{F}_S)^{op} &\to \mathrm{Mod}_R^{\mathbf{Z}} \\ \operatorname{Spec} E &\mapsto M(E) \end{aligned} \tag{A.1.1.a}$$

   for which we denote by $M_n(E)$ the n-the graded piece, together with the following functorialities and relations:

   Functorialities:

   (D1) For a morphism of S-fields $f : \operatorname{Spec} F \to \operatorname{Spec} E$ or (equivalently) $\phi : E \to F$, a map of degree 0

$$f^* = \phi_* = \mathrm{res}_{F/E} : M(E) \to M(F); \tag{A.1.1.b}$$

(D3) For an S-field $\operatorname{Spec} E$ and an element $x \in \mathrm{K}_m^{MW}(E)$, a map of degree m

$$\gamma_x : M(E) \to M(E) \tag{A.1.1.c}$$

making $M(E)$ a left module over the lax monoidal functor $\mathrm{K}_?^{MW}(E)$ (i.e. we have $\gamma_x \circ \gamma_y = \gamma_{x \cdot y}$ and $\gamma_1 = \mathrm{Id}$).

The axiom (D3) allows us to define, for every S-field $\operatorname{Spec} E$ and every 1-dimensional E-vector space $\mathcal{L}$, a graded R-module

$$M(E, \mathcal{L}) := M(E) \otimes_{R[E^\times]} R[\mathcal{L}^\times] \tag{A.1.1.d}$$

where $R[\mathcal{L}^\times]$ is the free R-module generated by the non-zero elements of $\mathcal{L}$, and the group algebra $R[E^\times]$ acts on $M(E)$ via $u \mapsto \langle u \rangle$ thanks to (D3).

(D2) For a finite morphism of S-fields $f : \operatorname{Spec} F \to \operatorname{Spec} E$ or $\phi : E \to F$, a map of degree 0

$$f_! = \phi^! = \operatorname{cores}_{F/E} : M(F, \omega_{F/E}) \to M(E); \tag{A.1.1.e}$$

(D4) For an S-field $\operatorname{Spec} E$ and an S-valuation v on E, a map of degree -1

$$\partial_v : M(E) \to M(\kappa(v), N_v^\vee). \tag{A.1.1.f}$$

Relations: We refer to [Fel20, Definition 3.1] for the list of relations.

A.1.2. Fix M a Milnor-Witt cycle premodule. If X is any scheme, let x, y be any points in X. We can define a map

$$\partial_y^x : M_q(\kappa(x), \omega_{\kappa(x)/k}) \to M_{q-1}(\kappa(y), \omega_{\kappa(y)/k})$$

thanks to (D2) and (D4).

Definition A.1.3. (see [Fel20, Definition 4.2])

A Milnor-Witt cycle module M over k is a Milnor-Witt cycle premodule M which satisfies the following conditions (FD) and (C).

(FD) FINITE SUPPORT OF DIVISORS. Let X be a normal scheme and ρ be an element of $M(\xi_X, X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(C) CLOSEDNESS. Let X be integral and local of dimension 2. Then

$$0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M(\kappa(\xi_X), \omega_{\kappa(\xi_X)/k}) \to M(\kappa(x_0), \omega_{\kappa(x_0)/k})$$

where ξ is the generic point and x_0 the closed point of X.

A.1.4. Let M be a Milnor-Witt cycle module over k. We can form a (cohomological) Rost-Schmid cycle complex $C_*(X, M, l)$ such that for any integer $p, q \in \mathbf{Z}$, and any line bundle l over X:

$$C_p(X, M_q, l) := \oplus_{X_{(p)}} M_{p+q}(\kappa(x), \omega_{\kappa(x)/k} \otimes l_{|x}). \tag{A.1.4.a}$$

We denote by $A_i(X, M_q, l)$ is the homology of $C_*(X, M_q, l)$ in degree i.

Remark A.1.5. Taking $M = \underline{\mathrm{K}}^{MW}$, we obtain

$$A_i(X, M_{-i}, l) = \widetilde{\mathrm{CH}}_i(X, l)$$

where the right-hand-side is known as the Chow-Witt group of X.

A.1.6. Fix M a Milnor-Witt cycle module and fix X a k-scheme with a dimensional pinning. We recall the basic maps that one can define on the cohomological Rost-Schmid complex.

A.1.7. PUSHFORWARD Let $f : Y \to X$ be a k-morphism of schemes. We have

$$f_* : C_p(Y, M_q, l) \to C_p(X, M_q, l)$$

as follows. If $x = f(y)$ and if $\kappa(y)$ is finite over $\kappa(x)$, then $(f_*)^y_x = \mathrm{cores}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f_*)^y_x = 0$.

A.1.8. PULLBACK Let $f : Y \to X$ be an *essentially smooth* morphism of schemes of relative dimension s. Suppose Y connected. Define

$$f^! : C_p(X, M_q, l) \to C_{p+s}(Y, M_{q-s}, l \otimes \omega_f^\vee)$$

as follows. If $f(y) = x$, then $(f^!)^x_y = \mathrm{res}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f^!)^x_y = 0$. If Y is not connected, take the sum over each connected component.

A.1.9. MULTIPLICATION WITH UNITS Let $a_1, \dots, a_n$ be global units in $\mathcal{O}_X^*$. Define

$$[a_1, \dots, a_n] : C_p(X, M_q, l) \to C_p(X, M_{q+n}, l)$$

as follows. Let x be in $X_{(p)}$ and $\rho \in \mathcal{M}(\kappa(x), *)$. We consider $[a_1(x), \dots, a_n(x)]$ as an element of $\underline{\mathrm{K}}^{MW}(\kappa(x))$. If $x = y$, then put $[a_1, \dots, a_n]^x_y(\rho) = [a_1(x), \dots, a_n(x)] \cdot \rho)$. Otherwise, put $[a_1, \dots, a_n]^x_y(\rho) = 0$.

A.1.10. MULTIPLICATION WITH η Define

$$\eta : C_p(X, M_q, l) \to C_p(X, M_{q-1}, l)$$

as follows. If $x = y$, then $\eta^x_y(\rho) = \gamma_\eta(\rho)$. Otherwise, $\eta^x_y(\rho) = 0$.

A.1.11. BOUNDARY MAPS Let X be a scheme of finite type over k, let $i : Z \to X$ be a closed immersion and let $j : U = X \setminus Z \to X$ be the inclusion of the open complement. We have a map

$$\partial = \partial^U_Z : C_p(U, M_q, *) \to C_{p-1}(Z, M_q, *).$$

which is called the boundary map for the closed immersion $i : Z \to X$.

A.1.12. A pairing $N \times M \to P$ between MW-cycle modules is given by maps

$$M_p(E, l) \otimes N_q(E, l') \to P_{p+q}(E, l \otimes l')$$

which are compatible with the data (D1),..., (D4) (see [Fel20, Definition 3.21] for more details).

A.1.13. PRODUCT If $M \times N \to P$ is a pairing of Milnor-Witt cycle modules, then there is a product map

$$C_p(X, M_q, l) \times C_r(Y, N_s, l') \to C_{p+r}(X \times Y, P_{q+s}, l \otimes l')$$

where X, Y are smooth schemes over k (see also [Fel20, §11]).

Remark A.1.14. The previous basic maps commute with the differentials of the Rost-Schmid complex and thus induce morphisms on the homology.

A.2 Oriented schemes

A.2.1. The notion of oriented real vector bundles was extended to the algebraic setting by Barges-Morel in [**?**]. We introduce a new category of oriented schemes. We refer to [DDØ22, Appendix §6.1] for similar results.

Definition A.2.2. Let X/S be a scheme. An orientation of X is an isomorphism $\sigma : \omega_{X/S} \to l_X^{\otimes 2}$, where l_X is an invertible sheaf over X.

An oriented S-scheme $(X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ is the data of a scheme X/S and an orientation $\sigma_X : \omega_{X/S} \to l_X^{\otimes 2}$.

A morphism of oriented schemes $(Y, \sigma_Y : \omega_{Y/S} \to l_Y^{\otimes 2}) \to (X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ is the data of an S-morphism $f : Y \to X$ along with an isomorphism of invertible sheaves $l_Y^{\otimes 2} \simeq f^{-1} l_X^{\otimes 2} \otimes \omega_f$ which makes the following diagram

$$\begin{array}{ccc} \omega_{Y/S} & \xrightarrow{\simeq} & f^{-1}\omega_{X/S} \otimes \omega_f \\ \downarrow{\sigma_Y} & & \downarrow{\sigma_X \otimes \mathrm{Id}_{\omega_f}} \\ l_Y^{\otimes 2} & \xrightarrow{\simeq} & f^{-1} l_X^{\otimes 2} \otimes \omega_f \end{array}$$

commutative.

Denote by **orSchm** the category of oriented schemes (along with morphisms of oriented schemes).

Remark A.2.3. Let $(X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ be an oriented scheme. By abuse of notation, we omit the orientation and simply write X.

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