A TRIPLE-POINT WHITNEY TRICK

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Abstract. We use a triple-point version of the Whitney trick to show that ornaments of three orientable $(2k-1)$-manifolds in $\mathbb{R}^{3k-1}$, $k>2$, are classified by the μ-invariant.

A very similar (but not identical) construction was found independently by I. Mabillard and U. Wagner, who also made it work in a much more general situation and obtained impressive applications. The present note is, by contrast, focused on a minimal working case of the construction.

1. Introduction

An *ornament* is a continuous map $f=\bigsqcup_{i=1}^n f_i$ from $X=\bigsqcup_{i=1}^n X_i$ to Y that has no $i=j=k$ *points*, i.e. $f(X_i)\cap f(X_j)\cap f(X_k)=\varnothing$, whenever i, j and k are pairwise distinct. Note that f is allowed to have *triple points* $f(x)=f(y)=f(z)$, where x,y,z belong one or two of the X_i's. We are interested in ornaments up to *ornament homotopy*, i.e. homotopy through ornaments.

Ornaments of circles in the plane were introduced by Vassiliev [14] as a generalization of doodles, previously studied by Fenn and Taylor [2]. Fenn and Taylor additionally required each circle to be embedded; however, Khovanov [4] redefined doodles as triple point free maps of circles in the plane, and Merkov proved that doodles in Khovanov's sense are classified by their finite-type invariants [12]. Further references and examples can be found in [11], which is a more thorough companion paper to this brief note.

The problem of classification of ornaments of spheres in $\mathbb{R}^m$ is motivated, in particular, by geometric and algebraic constructions that go from link maps and their "quadratic" invariants to ornaments and their "linear" invariants; and conversely [11]. Link maps are, in turn, related to links by the Jin suspension and its variations, which likewise reduce some "quadratic" invariants of links to "linear" invariants of link maps [9], [13; §3].

2. μ-invariant

We will consider only ornaments of the form $X_1\sqcup X_2\sqcup X_3\to\mathbb{R}^m$. If $f=f_1\sqcup f_2\sqcup f_3$ is such an ornament, let F be the composition

$$X_1\times X_2\times X_3\xrightarrow{f_1\times f_2\times f_3}\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta_{\mathbb{R}^m}\xrightarrow{\simeq}S^{2m-1},$$

where $\Delta_{\mathbb{R}^m}=\{(x,x,x)\mid x\in\mathbb{R}^m\}$ and the homotopy equivalence is given, for instance, by $(x,y,z)\mapsto\frac{(2x-y-z,\,2y-x-z)}{||(2x-y-z,\,2y-x-z)||}$. Let $\mu(f)\in H^{2m-1}(X_1\times X_2\times X_3)$ be the image under F^* of a fixed generator $\xi\in H^{2m-1}(S^{2m-1})$; to be precise, let us choose ξ to correspond to the orientation of S^{2m-1} given by its inwards co-orientation in the standardly oriented $\mathbb{R}^{2m}$. Clearly, $\mu(f)$ is invariant under ornament homotopy.

Let us now assume that each X_i is a connected closed oriented $(2k-1)$-manifold and $m = 3k-1$. Then F is a map between connected closed oriented $(6k-3)$-manifolds, and so $\mu(f)$ is an integer. In this simplest case, assuming additionally that each manifold X_i is either PL or smooth, one can compute $\mu(f)$ as follows.

First let us note that since each X_i is compact, for each ornament $f\colon X \to \mathbb{R}^m$ there exists an $\varepsilon > 0$ such that every map $f'\colon X \to \mathbb{R}^m$, ε-close to f (in the sup-metric), is also an ornament, and moreover the rectilinear homotopy between f and f' is an ornament homotopy. Thus we are free to replace ornaments by their generic (PL or smooth) approximations. Similarly, ornament homotopies can be replaced by their generic approximations.

Now let us consider a homotopy between f and the *trivial* ornament, which sends X_1, X_2 and X_3 to three distinct fixed points in $\mathbb{R}^m$. Its generic (PL or smooth) approximation h_t, if viewed as a map $X \times I \to \mathbb{R}^m \times I$, $(x,t) \mapsto \big(h_t(x),\, t\big)$, has only finitely many transverse $1 = 2 = 3$ points, which are naturally endowed with signs.[1] (See [1; II.4] concerning PL transversality.) The algebraic number of these $1 = 2 = 3$ points is easily seen to equal $\mu(f)$.[2]

Example 1. The inclusions of the unit disks in the coordinate $2k$-planes $\mathbb{R}^k \times \mathbb{R}^k \times 0$, $\mathbb{R}^k \times 0 \times \mathbb{R}^k$ and $0 \times \mathbb{R}^k \times \mathbb{R}^k$ in $\mathbb{R}^{3k}$ yield a smooth map $B^{2k} \sqcup B^{2k} \sqcup B^{2k} \to B^{3k}$ with one transverse $1 = 2 = 3$ point. Restricting to the boundaries, we get the *Borromean* ornament $b\colon S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to S^{3k-1}$. By stereographically projecting S^{3k-1} e.g. from $z = \frac{1}{\sqrt{3k}}(1,\dots,1)$ we also get an ornament $b_z\colon S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k-1}$.

On the other hand, the sphere of radius $\varepsilon\sqrt{k}$ centered at $(\varepsilon,\dots,\varepsilon)$ for a sufficiently small $\varepsilon > 0$ is tangent to each of the three unit $2k$-disks. By appropriately identifying the exterior of this sphere in the unit $3k$-disk B^{3k} with $S^{3k-1} \times I$, we get a smooth homotopy of b, and hence also of b_z, to the trivial ornament. It has one transverse $1 = 2 = 3$ point, which can be seen to be positive, and it follows that $\mu(b_z) = 1$.

In the case of doodles, the μ-invariant was introduced in [2]. See [11] concerning relations between the μ-invariant of ornaments and the triple μ-invariant of link maps.

3. Classification

Theorem 1. *Let $m = 3k-1$, $k > 2$ and let X_1, X_2, X_3 be connected closed oriented PL $(2k-1)$-manifolds. Then μ is a complete invariant of ornaments $X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m$.*

The proof is in the PL category. If the X_i are smooth manifolds, the same construction with minimal (straightforward) amendments can be carried out in the smooth category.

[1] Every triple point of a generic map $F\colon N \to M$ from a $2k$-manifold to a $3k$-manifold corresponds to a transversal intersection point between the $3k$-manifold Δ_M and the map $F^3\colon N^3 \to M^3$ from a $6k$-manifold to a $9k$-manifold.

[2] Each $1 = 2 = 3$ point of h_t corresponds to a transversal intersection point between $\Delta_{\mathbb{R}^m} \times I$ and the map $X_1 \times X_2 \times X_3 \times I \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times I$, $(x,y,z,t) \mapsto \big(h_t(x,t),\, h_t(y,t),\, h_t(z,t),\, t\big)$. It is easily seen to be of the same sign.

Proof. Let f and g be generic PL ornaments of $X := X_1 \sqcup X_2 \sqcup X_3$ in $\mathbb{R}^m$ with $\mu(f) = \mu(g)$. Let $h\colon X \times I \to \mathbb{R}^m \times I$ be a generic PL homotopy between them. Since $\mu(f) = \mu(g)$, the $1 = 2 = 3$ points of h can be paired up with opposite signs. Every such pair (p^+, p^-) will now be canceled by a triple-point Whitney trick.

Let $p_i^\pm$ be the preimage of $p^\pm$ in $M_i := X_i \times I$. We first arrange that (p_1^+, p_2^+) and (p_1^-, p_2^-) be in the same component of the double point set $\Delta_{12} := \{(x, y) \in M_1 \times M_2 \mid h(x) = h(y)\}$ (in case that initially they are not). To this end we pick points $(q_1^\pm, q_2^\pm)$ in the same components of Δ_{12} with $(p_1^\pm, p_2^\pm)$ and such that the double points $f(q_1^+) = f(q_2^+)$ and $f(q_1^-) = f(q_2^-)$ are not triple points.

Let us connect q_1^+ and q_1^- by an arc J_1 in M_1, disjoint from the preimages of any double points (using that $k > 1$). Now we attach a thin 1-handle to $h(M_2)$ along the image of J_1. That is, we modify $h(M_2)$ into $h'(M_2')$, where M_2' is obtained from M_2 by removing an oriented copy of $B^{2k} \times \partial I$ and pasting in $\partial B^{2k} \times I$. The embedded 1-handle $h'(\partial B^{2k} \times I)$ is constructed in a straightforward way. Namely, since h is generic, Δ_{12} is an oriented k-manifold, immersed into the $2k$-manifold M_1 by the projection $\pi\colon M_1 \times M_2 \to M_1$. Let us take an oriented connected sum of its components along a ribbon $r(D^k \times I)$ in M_1 (going near J_1).[3] Then $hr(D^k \times I)$ is naturally thickened to a solid rod $R(B^{2k} \times I)$ in $\mathbb{R}^m \times I$ whose lateral surface $R(\partial B^{2k} \times I)$ is the desired embedded 1-handle.[4]

To restore the topology of M_2, we cancel the 1-handle geometrically by attaching a 2-handle along an embedded 2-disk D, which is disjoint from $h(M_1 \sqcup M_3)$ and meets $h'(M_2')$ only in ∂D (such a disk exists since $k > 2$). That is, we modify $h'(M_2')$ into $h''(M_2'')$, where M_2'' is obtained from M_2' by removing an appropriately embedded copy of $B^{2k-1} \times \partial D^2$ and pasting in $\partial B^{2k-1} \times D^2$. As is well-known, this can be done so that M_2'' is homeomorphic to M_2.[5] Since we do not care about self-intersections of individual components, we may define h'' on $\partial B^{2k-1} \times D^2$ to be an arbitrary generic map into a small neighborhood of $D \cup h'(B^{2k-1} \times \partial D^2)$.

Thus we may assume that (p_1^+, p_2^+) and (p_1^-, p_2^-) are in the same component of Δ_{12}. To cancel the original $1 = 2 = 3$ points p^+ and p^-, let us connect (p_1^+, p_2^+) and (p_1^-, p_2^-) by an arc J_{12} in Δ_{12} and attach a thin 1-handle to $h(M_3)$ along the image of J_{12}. (This 1-handle is the spherical block normal bundle of $h(M_1) \cap h(M_2)$ over the image of J_{12}. It

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[3] Namely, $q_1^\pm$ has a regular neighborhood $N_\pm$ in M_1 that is homeomorphic to $[-1, 1]^{2k}$ by an orientation preserving homeomorphism $\varphi_\pm$ such that $\varphi_\pm^{-1}(\pi(\Delta_{12})) = [-1, 1]^k \times \{0\}^k$ and $\varphi_\pm^{-1}(J_1) = \{0\}^{2k-1} \times [0, 1]$. Let $Q = [-1, 1]^k \times \{0\}^{k-1}$ and let N be a regular neighborhood of $\overline{J_1 \setminus (N_+ \cup N_-)} \cup \varphi_+(Q \times 1) \cup \varphi_-(Q \times 1)$ in $\overline{M_1 \setminus (N_+ \cup N_-)}$. Since a k-ball unknots in the interior of a $(2k-1)$-ball, there is a homeomorphism $\psi\colon [-2, 2]^{2k} \to N$ such that $\psi^{-1}(\partial N_\pm) = [-2, 2]^{2k-1} \times \{\pm 2\}$ and $\psi(x, \pm 2) = \varphi_\pm(x, 1)$ for all $x \in Q$. Then $\varphi_+(Q \times I) \cup \varphi_-(Q \times I) \cup \psi(Q \times [-2, 2])$ is the desired ribbon $r(D^k \times I)$.

[4] If N_1 is a disk neighborhood of J_1 that is embedded by h, we may assume that $h(M_2)$ is transverse to a normal block bundle ν to $h(N_1)$, that is, $h(M_2)$ meets the total space $E(\nu)$ in $E(\nu|_{h(N_1) \cap h(M_2)})$. Since ν is trivial, there is a homeomorphism $R\colon B^{2k} \times I \to E(\nu|_{hr(D^k \times I)})$ sending $B^{2k} \times \partial I$ onto $E(\nu|_{hr(D^k \times \partial I)})$.

[5] In more detail, let us connect q_2^+ and q_2^- by an arc J_2 in M_2, disjoint from the preimages of any double points. Let H_1 be a small regular neighborhood of $J_1' := J_2 \times 1 \cup \partial J_2 \times [0, 1]$ in $M_2 \times [0, 2]$. Let H_2 be a small regular neighborhood of $D' := \overline{J_2 \times [0, 1] \setminus H_1}$ in $\overline{M_2 \times [0, 2] \setminus H_1}$. Then M_2' can be identified with the frontier of $M_2 \times [-1, 0] \cup H_1$ in $M_2 \times [-1, 2]$ so that $h'(\partial D')$ gets identified with ∂D; and M_2'' with the frontier of $M_2 \times [-1, 0] \cup H_1 \cup H_2$ in $M_2 \times [-1, 2]$, which is homeomorphic to M_2.

is attached orientably since the two $1 = 2 = 3$ points have opposite signs.) The topology of M_3 can be restored using another 2-disk like before. In particular, this 2-disk is disjoint from $h(M_1 \sqcup M_2)$, so no new $1 = 2 = 3$ points arise.

Finally, we need to apply the "ornament concordance implies ornament homotopy in codimension three" theorem [7], [8]. (Alternatively, it should be possible to rework the above construction so as to keep the levels preserved at every step — but it would be a rather laborious exercise; compare [9; proofs of Lemmas 5.1, 5.4, 5.5].) □

4. Discussion

Theorem 1 and its proof (in slightly less detailed form) were originally contained in the preprint [10], which I presented at conferences and seminars in 2006–07 and privately circulated at that time and in later years. For instance, the referee of the present paper (whose identity I know from his idiosyncratic remarks) does not deny that he received my preprint containing the proof of Theorem 1, exactly as it appears in [10], by email on May 23, 2006 and then again on July 7, 2006. I hesitated to publish [10] at that time as I hoped to get more progress on the conjectures stated in the introduction there; but other projects are still distracting me from this task.

In the meantime I. Mabillard and U. Wagner independently found and vastly generalized a version of the triple-point Whitney trick and also obtained nice applications leading to a disproof of the Topological Tverberg Conjecture [6]. (My only step in that direction was a feeble attempt to advertise the possibility of disproving the Topological Tverberg Conjecture by generalizing the construction of the present note — addressed, for instance, to P. Blagojević at the 2009 Oberwolfach Workshop on Topological Combinatorics.) Mabillard and Wagner call their construction the "triple Whitney trick", but I prefer to reserve this title for a certain other device, extending Koschorke's version of the Whitney–Haefliger construction [5; Proof of Theorem 1.15] and involving the triple-point Whitney trick as only one of several steps. It can be used to obtain a geometric proof of the Habegger–Kaiser classification of link maps in the 3/4 range [3], which will hopefully appear elsewhere (a sketch of this proof was presented in my talk at the Postnikov Memorial Conference in Będlewo, 2007).

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