Weak amenability of weighted measure algebras and their second duals

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Abstract. In this paper, we study the weak amenability of weighted measure algebras and prove that $M(G,\omega)$ is weakly amenable if and only if G is discrete and every bounded quasi-additive function is inner. We also study the weak amenability of $L^1(G,\omega)^{**}$ and $M(G,\omega)^{**}$ and show that the weak amenability of theses Banach algebras are equivalent to finiteness of G. This gives an answer to the question concerning weak amenability of $L^1(G,\omega)^{**}$ and $M(G,\omega)^{**}$.

1 Introduction

Let G be a locally compact group with an identity element e. Let us recall that a continuous function $\omega : G \to [1,\infty)$ is called a *weight function* if for every $x, y \in G$

$$\omega(xy) \leq \omega(x)\,\omega(y) \quad \text{and} \quad \omega(e) = 1.$$

Let $C_0(G, 1/\omega)$ be the space of all functions f on G such that $f/\omega \in C_0(G)$, the space of all bounded continuous functions on G that vanish at infinity. Let also $M(G,\omega)$ be the Banach space of all complex regular Borel measures μ on G for which $\omega\mu \in M(G)$, the measure algebra of G. It is well-known that $M(G,\omega)$ is the dual space of $C_0(G, 1/\omega)$ [4, 21, 26], see [22, 24] for study of weighted semigroup measure algebras; see also [13, 14, 17]. Note that $M(G,\omega)$ is a Banach algebra with the norm $\|\mu\|_\omega := \|\omega\mu\|$ and the convolution product "$*$"defined by

$$\mu * \nu(f) = \int_G \int_G f(xy)\, d\mu(x) d\nu(y) \qquad (\mu, \nu \in M(G,\omega), f \in C_0(G, 1/\omega)).$$

Let $L^1(G,\omega)$ be the Banach space of all Borel measurable functions f on G such that $\omega f \in L^1(G)$, the group algebra of G. Then $L^1(G,\omega)$ with the convolution product "$*$" and the norm $\|f\|_{1,\omega} = \|\omega f\|_1$ is a Banach algebras.

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A Borel measurable function p from $G \times G$ into $\mathbb{C}$ is called *quasi-additive* if for almost every where $x, y, z \in G$

$$p(xy, z) = p(x, yz) + p(y, zx).$$

If there exists $h \in L^\infty(G, 1/\omega)$ such that

$$p(x, y) = h(xy) - h(yx)$$

for almost every where $x, y \in G$, then p is called *inner*. Let $D(G, \omega)$ be the set of all quasi-additive functions p on G such that

$$\sup_{x,y \in G} \frac{|p(x, y)|}{\omega^\otimes(x, y)} < \infty.$$

We denote by $I(G, \omega)$ the set of inner quasi-additive functions. For $\mu \in M(G, \omega)$, let $L^\infty(|\mu|, \omega)$ be the Banach space of all $\omega-$bounded Borel measurable functions p on G such that $\|p\|_{\omega,\mu} = \|p/\omega\|_\mu < \infty$. An element

$$P = (p_\mu)_\mu \in \Pi\{L^\infty(|\mu|, \omega) : \mu \in M(G, \omega)\}$$

is called a $\omega-$*generlized function* on G if

$$\sup\{\|p_\mu\|_{\omega,\mu} : \mu \in M(G, \omega)\} < \infty$$

and for every $\mu, \nu \in M(G, \omega)$ with $|\mu| \ll |\nu|$ we have $p_\mu = p_\nu$, $|\mu| - a.e.$. The space of all $\omega-$generlized function on G is denoted by $GL(G, 1/\omega)$. It is well-known from [25] that $GL(G, 1/\omega)$ is the dual of $M(G, \omega)$ for the pairing

$$\langle (p_\mu)_\mu, \nu \rangle = \int_G p_\nu \, d\nu.$$

A function $F = (F_{\mu\otimes\nu})_{\mu,\nu \in M(G,\omega)} \in GL(G \times G, \omega^\otimes)$ is called a *generalized quasi-additive function* if

$$F_{(\mu*\nu)\otimes\eta}(xy, z) = F_{\mu\otimes(\nu*\eta)}(x, yz) + F_{\nu\otimes(\eta*\mu)}(y, zx)$$

for all $\mu, \nu, \eta \in M(G, \omega)$ and $x, y, z \in G$. The set of all generalized quasi-additive functions is denoted by $GD(G, \omega)$. If there exists $p = (p_\mu) \in GL(G, 1/\omega)$ such that for every $\mu, \nu \in M(G, \omega)$ and for almost every where $x, y \in G$

$$F_{\mu\otimes\nu}(x, y) = p_{\mu*\nu}(xy) - p_{\nu*\mu}(yx),$$

then F is said to be a *generalized inner quasi-additive function*. The set of all generalized inner quasi-additive functions is denoted by $GI(G, \omega)$.

Let A be a Banach algebra and $D : A \to A^*$ be a bounded linear operator. Then D is called *cyclic* if $\langle D(a), a\rangle = 0$ for all $a \in A$. Let us recall that a bounded linear operator $D : A \to A^*$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in A$. The space of all bounded continuous derivations from A into A^* is denoted by $\mathcal{Z}(\mathcal{A}, \mathcal{A}^*)$. If every element of $\mathcal{Z}(\mathcal{A}, \mathcal{A}^*)$ is cyclic, then A is called *cyclically weakly amenable*, however, A is called *weakly amenable* if every derivation $D \in \mathcal{Z}(\mathcal{A}, \mathcal{A}^*)$ is inner; that is, there exists $z \in A^*$ such that for every $a \in A$

$$D(a) = \mathrm{ad}_z(a) := z \cdot a - a \cdot z.$$

The weak amenability of group algebras have been study by several authors. For example, Brown and Moran [2] studied the weak amenability of measure algebra of locally compact Abelian groups and showed that if zero is the only continuous point derivation of $M(G)$, then G is discrete. Note that if G is discrete, then $M(G)$ is weakly amenable, because in this case $M(G) = \ell^1(G)$ is always weakly amenable [8]. One can prove that if d is a non-zero continuous point derivation of $M(G)$ at

$$\varphi \in \Delta(M(G)) \cup \{0\},$$

then the map $\mu \mapsto d(\mu)\varphi$ is a continuous non-inner derivation from $M(G)$ into $M(G)^*$. In other words, $M(G)$ is not weakly amenable. These facts give rise to the conjecture that for a locally compact group G, the Banach algebra $M(G)$ is weakly amenable if and only if G is discrete; or equivalently, zero is the only continuous point derivation of $M(G)$ at a character. Dales, Ghahramani and Helemskii [3] proved this conjecture. Some authors investigated the weak amenability of the second dual of Banach algebras. For instance, Ghahramani, Loy and Willis [7] proved that if G is a locally compact Abelian group and $L^1(G)^{**}$ is weakly amenable, then G is discrete. Forrest [6] investigated the weak amenability of the dual of a topological introverted subspace X of $VN(G)$. Under certain conditions, he showed that if $A(G)^{**}$ is weakly amenable, then every Abelian subgroup of G is finite. As a consequence of this result, he improved the result of Ghahramani, Loy and Willis. In fact, for a locally compact Abelian group G, he proved that weak amenability of $L^1(G)^{**}$ is equivalent to the finiteness of G. Lau and Loy [9] considered a left introverted subspace of $L^\infty(G)$ containing $AP(G)$, say X, and studied weak amenability of X^*. One can obtain the result of weak amenability of $L^1(G)^{**}$ from Lau-Loy's theorem. Finally, Dales, Lau and Strauss [5] proved that $L^1(G)^{**}$ is weakly amenable if and only if there is no non-zero continuous point derivation of $L^1(G)^{**}$ at the discrete augmentation character; or equivalently, G is finite.

This paper is organized as follow. In Section 2 we study the weak amenability of $M(G, \omega)$ and show that $M(G, \omega)$ is weakly amenable if and only if G is discrete and every bounded quasi-additive function is inner. We also prove that cyclic weak amenability and point amenability of $M(G,\omega)$ are equivalent to weak amenability of it. Section 3 is devoted to studies of the weak amenability of second dual of $L^1(G,\omega)$ and $M(G,\omega)$. We proved that $L^1(G,\omega)^{**}$ is weakly amenable if and only if $M(G,\omega)^{**}$ is weakly amenable; or equivalently, G is finite. We verify that cyclic weak amenability and point amenability of $L^1(G,\omega)^{**}$ and $M(G,\omega)^{**}$ are equivalent to finiteness of G.

2 Weighted measure algebras

Let ω_i be a weight function on locally compact group G_i for $i = 1, 2$. Define the weight function $\omega_1 \otimes \omega_2$ on $G_1 \times G_2$ by

$$\omega_1 \otimes \omega_2(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$$

for all $x_1 \in G_1$ and $x_2 \in G_2$. In the case where, $G_1 = G_2 = G$ and $\omega_1 = \omega_2 = \omega$, we set $\omega^{\otimes} = \omega_1 \otimes \omega_2$. The following result is needed to prove our results.

Proposition 2.1 *Let ω_i be a weight function on locally compact group G_i for $i = 1, 2$. Then*

$$M(G_1, \omega_1)\hat{\otimes} M(G_2, \omega_2) = M(G_1 \times G_2, \omega_1 \otimes \omega_2).$$

Proof. Let $\eta_i \in M(G_i, \omega_i)$, for $i = 1, 2$. Then for every $f \in C_0(G_1 \times G_2)$, we have

$$\langle \eta_1 \otimes \eta_2, f\rangle = \int_{G_1}\int_{G_2} f(x, y)\, d\eta_1(x)d\eta_2(y).$$

It is easy to prove that

$$\eta_1 \otimes \eta_2 \in C_0(G_1 \times G_2, 1/\omega_1 \otimes \omega_2)^* = M(G_1 \times G_2, \omega_1 \otimes \omega_2).$$

Conversely, let $\eta \in M(G_1 \times G_2, \omega_1 \otimes \omega_2)$. In view of Theorem Lusin's theorem, there exists sequences (f_n) and (g_n) in the unit ball $C_c(G_1, 1/\omega_1)$ and $C_c(G_2, 1/\omega_2)$ with compact support, respectively, such that for almost every where $x \in G_1$ and $y \in G_2$

$$f_n(x) \to 1 \quad \text{and} \quad g_n(y) \to 1$$

as $n \to \infty$. We define the functionals η_1 and η_2 by

$$\eta_1(f) = \lim_n \eta(f \otimes g_n) \quad \text{and} \quad \eta_2(g) = \lim_n \eta(f_n \otimes g)$$

for all $f \in C_0(G_1, 1/\omega_1)$ and $g \in C_0(G_2, 1/\omega_2)$. Then $\eta_1 \in M(G_1, \omega_1)$, $\eta_2 \in M(G, \omega_2)$. In fact,

$$|\eta_1(f)| \leq \|\eta\|\|f\|_{\infty,1/\omega} \quad \text{and} \quad |\eta_2(g)| \leq \|\eta\|\|g\|_{\infty,1/\omega}.$$

On the other hand,

$$\begin{aligned}\eta_1 \otimes \eta_2(f \otimes g) &= \lim_n \eta(f \otimes g_n)\eta(f_n \otimes g)\\ &= \lim_n \int_{G_1 \times G_2} f(x) f_n(x) g(y) g_n(y)\, d\eta(x,y).\end{aligned}$$

Since η is bounded, it follows from Lebesgue dominated convergence theorem that $\|f_n \otimes g_n\|_{\infty,1/\omega} \leq 1$ and $1 \in L^1(\eta)$. Furthermore, $f_n \otimes g_n(x,y) \to 1$ for every $x \in G_1, y \in G_2$. For every $f \in C_0(G_1, 1/\omega_1)$ and $g \in C_0(G_2, 1/\omega_2)$

$$\eta_1 \otimes \eta_2(f \otimes g) = \int_{G_1 \times G_2} f \otimes g(x,y)\, d\eta(x,y) = \eta(f \otimes g).$$

It follows that for every $h \in C_0(G_1, 1/\omega_1) \otimes C_0(G_2, 1/\omega_2)$

$$\eta_1 \otimes \eta_2(h) = \eta(h)$$

and so for every $h \in C_0(G_1 \times G_2, 1/\omega_1 \otimes \omega_2)$

$$\eta_1 \otimes \eta_2(h) = \eta(h).$$

Therefore, $\eta_1 \otimes \eta_2 = \eta$. □

For every $f \in L^1(G, \omega)$, we define the seminorm $T_f : M(G, \omega) \to [0, \infty)$ by

$$T_f(\mu) = \|f * \mu\|_{1,\omega} + \|\mu * f\|_{1,\omega}.$$

The locally convex topology defined by the family of seminorms $(T_f)_{f \in L^1(G,\omega)}$ is called the *strict topology* on $M(G, \omega)$ with respect to $L^1(G, \omega)$ (or briefly strict topology).

Proposition 2.2 *Let G be a locally compact group and ω be a weight function on G. If $p \in D(G, \omega)$, then there exists a unique bounded derivation $D \in \mathcal{Z}(M(G,\omega), M(G,\omega)^*)$ such that $p(x,y) = \langle D(\delta_x), \delta_y \rangle$ for all $x, y \in G$, where $\delta_\cdot$ is the Diract measure at $\cdot$.*

Proof. Let $p \in D(G, \omega)$. Then $\Gamma(D_1) = p$ for some $D_1 \in \mathcal{Z}(L^1(G,\omega), L^\infty(G, 1/\omega))$. By Proposition 2.1.6 [23], there exists $D_2 \in \mathcal{Z}(M(G,\omega), L^\infty(G, 1/\omega))$ such that D_2 is strict-weak* continuous and $D_2|_{L^1(G,\omega)} = D_1$. Hence for every $f \in L^1(G, \omega)$,

$$\begin{aligned}\langle D_2(\delta_x), f \rangle &= \lim \langle D_1(e_\alpha * \delta_x), f \rangle\\ &= \lim \int_G \int_G p(x,y)(e_\alpha * \delta_x)(z) f(z)\, dz dy\\ &= \int_G \int_G p(x,y) e_\alpha(z x^{-1}) f(y)\, dz dy.\end{aligned}$$

On the other hand, there exists a linear functional $T_1 : L^1(G \times G, \omega^\otimes) \to \mathbb{C}$ such that

$$\langle T_1, f \otimes g \rangle = \langle D_1(f), g \rangle$$

for all $f, g \in L^1(G,\omega)$. Since $L^1(G \times G, \omega^{\otimes})$ is a closed ideal in $M(G \times G, \omega^{\otimes})$, it follows that T_1 has a strict continuous extension, say $T_2 : M(G,\omega)\hat{\otimes}M(G,\omega) \to \mathbb{C}$. Define $D : M(G,\omega) \to M(G,\omega)^*$ by

$$\langle D(\mu), \nu\rangle = \langle T_2, \mu \otimes \nu\rangle$$

for all $\mu, \nu \in M(G,\omega)$. If (e_α) is a bounded approximate identity of $L^1(G,\omega)$, then for every $x \in G$, $e_\alpha * \delta_x \to \delta_x$ in the strict topology. So

$$T_2(e_\alpha * \delta_x \otimes e_\alpha * \delta_y) \to T_2(\delta_x \otimes \delta_y).$$

Therefore,

$$\begin{aligned} \langle D(\delta_x), \delta_y\rangle &= \lim \langle T_2(e_\alpha * \delta_x \otimes e_\alpha * \delta_y) \\ &= \lim \langle D_2(e_\alpha * \delta_x), e_\alpha * \delta_y\rangle = p(x, y), \end{aligned}$$

as claimed. □

In the following, let $\mathcal{I}_{nn}(M(G,\omega), M(G,\omega)^*)$ be the set of all inner derivations from $M(G,\omega)$ into $M(G,\omega)^*$, and let $\mathcal{B}(M(G,\omega), M(G,\omega)^*)$ be the space of bounded linear operators from $M(G,\omega)$ into $M(G,\omega)^*$. Define the isometric isomorphism Γ from Banach space $\mathcal{B}(M(G,\omega), M(G,\omega)^*)$ onto $(M(G,\omega)\hat{\otimes}M(G,\omega))^*$ by

$$\langle \Gamma(T), \mu \otimes \nu\rangle = \langle T(\mu), \nu\rangle,$$

$M(G,\omega)\hat{\otimes}M(G,\omega)$ is the projective tensor product of $M(G,\omega)$; see Proposition 13 VI in [1].

Proposition 2.3 *Let G be a locally compact group and ω be a weight function on G. Then the following statements hold.*

(i) *The function $\Gamma : \mathcal{Z}(M(G,\omega), M(G,\omega)^*) \to GD(G,\omega)$ is an isometric isomorphism. Furthermore, $\Gamma(\mathcal{I}_{nn}(M(G,\omega), M(G,\omega)^*)) = GI(G,\omega)$.*

(ii) *If $D \in \mathcal{Z}(M(G,\omega), M(G,\omega)^*)$, then for every $\mu \in M(G,\omega)$ there exists $F = (F_{\mu\otimes\nu})_\nu \in GD(G,\omega)$ such that $D(\mu) = (p_{\mu,\nu})_\nu$ and $p_{\mu,\nu}(y) = \int_G F_{\mu\otimes\nu}(x, y)\ d\mu(x)$ for almost every where $y \in G$.*

Proof. Let $D \in \mathcal{Z}(M(G,\omega), M(G,\omega)^*)$. Then $D \in \mathcal{B}(M(G,\omega), M(G,\omega)^*)$. Putting $A = B = M(G,\omega)$ in the definition of Γ, we have

$$F := \Gamma(D) \in (M(G,\omega)\hat{\otimes}M(G,\omega))^* = GL(G \times G, 1/\omega^{\otimes})$$

and

$$\begin{aligned} \langle D(\mu), \nu\rangle &= \langle F, \mu \otimes \nu\rangle \\ &= \int_G F_{\mu\otimes\nu}(x, y)\ d(\mu \otimes \nu)(x, y) \qquad (1) \\ &= \int_G \int_G F_{\mu\otimes\nu}(x, y)\ d\mu(x) d\nu(y). \end{aligned}$$

On the other hand, if $P = (p_\mu)_{\mu\in M(G,\omega)}$, then

$$\begin{aligned}\langle \mathrm{ad}_P(\mu), \nu\rangle &= \langle P\cdot\mu - \mu\cdot P, \nu\rangle \\ &= \langle P, \mu*\nu\rangle - \langle P, \nu*\mu\rangle \\ &= \int_G p_{\mu*\nu}\, d(\mu*\nu) - \int_G p_{\nu*\mu}\, d(\nu*\mu) \\ &= \int_G\int_G (p_{\mu*\nu}(xy) - p_{\nu*\mu}(yx))\, d\mu(x)d\nu(y).\end{aligned}$$

Now, by the argument used in the proof of Theorem 2.3 in [18], it can be shown that the statement (i) holds. For (ii), assume that $D \in \mathcal{Z}(M(G,\omega), M(G,\omega)^*)$ and $\mu \in M(G,\omega)$. Then

$$D(\mu) \in M(G,\omega)^* = GL(G, 1/\omega).$$

Thus $D(\mu) = (p_{\mu,\nu})_\nu$ for some $(p_{\mu,\nu})_\nu \in GL(G, 1/\omega)$. Hence for every $\nu \in M(G,\omega)$, we have

$$\langle D(\mu), \nu\rangle = \int_G p_{\mu,\nu}\, d\nu.$$

This together with (1) s shows that

$$p_{\mu,\nu}(y) = \int_G F_{\mu\otimes\nu}(x,y)\, d\mu(x)$$

for almost every where $y \in G$. □

We are now in a position to prove the main result of this section.

Theorem 2.4 *Let G be a locally compact group and ω be a weight function on G. Then the following assertions are equivalent.*

(a) *$M(G,\omega)$ is weakly amenable.*

(b) *For every $D \in \mathcal{Z}(M(G,\omega), M(G,\omega)^*)$ there exists $P = (p_\mu)_{\mu\in M(G,\omega)}$ such that $\langle D(\mu), \nu\rangle = \int_G\int_G (p_{\mu*\nu}(xy) - p_{\nu*\mu}(yx))\, d\mu(x)d\nu(y)$ for all $\mu, \nu \in M(G,\omega)$.*

(c) *Every generalized quasi-additive function is inner.*

(d) *$M(G)$ is weakly amenable and $D(G,\omega) = I(G,\omega)$.*

(e) *G is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.*

Proof. The implications (a)⇒(b)⇒(c)⇒(a) follow from Proposition 2.3. By Theorem 1.2 in [3] the implication (d)⇔(e) holds. Also, the implication (e)⇒(a) follows from Corollary 2.5 in [18]. For (a)⇒(e), let $M(G,\omega)$ be weakly amenable and φ be a character of $M(G)$. If d is a continuous point derivation at φ on $M(G)$, then $d|_{M(G,\omega)}$ is a continuous point derivation of $M(G,\omega)$ at $\varphi|_{M(G,\omega)}$. Hence d is zero on $M(G,\omega)$. Since $M(G,\omega)$ is dense in $M(G)$, we have $d = 0$ on $M(G)$ which is implies G is discrete.

Apply Theorem 2.4 in [18] to conclude that $D(G,\omega)=I(G,\omega)$. $\square$

From Theorem 4.8 in [19] and Theorem 2.4 and its proof, we may prove the next result.

Corollary 2.5 *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *$M(G,\omega)$ is weakly amenable.*

(b) *$M(G,\omega)$ is cyclically weakly amenable.*

(c) *$M(G,\omega)$ is point amenable.*

(d) *G is discrete and every non-inner quasi-additive function in $L^\infty(G,1/\omega)$ is unbounded.*

An elementary computation shows that the functions ω' and ω^* defined by

$$\omega'(x)=\omega(x^{-1}) \quad \text{and} \quad \omega^*(x)=\omega\otimes\omega'(x,x)$$

are weight functions on G. Combining Theorem 2.4 and the result of [18] we have the following result.

Corollary 2.6 *Let ω and ω_0 be weight functions on a locally compact group G. Then the following statements hold.*

(i) *If $\omega\leq m\omega_0$ for some $m>0$, $M(G,\omega_0)$ is weakly amenable and $I(G,\omega_0)=D(G,\omega_0)$, then $M(G,\omega)$ is weakly amenable.*

(ii) *If ω and ω_0 are equivalent, then weak amenability of $M(G,\omega)$ is equivalent to weak amenability of $M(G,\omega_0)$.*

(iii) *$M(G,\omega')$ is weakly amenable if and only if $M(G,\omega)$ is weakly amenable.*

(iv) *If $M(G,\omega^*)$ is weakly amenable and $I(G,\omega^*)=D(G,\omega^*)$, then $M(G,\omega)$ is weakly amenable.*

(v) *If G is Abelian, then $M(G,\omega^*)$ is weakly amenable if and only if $M(\mathfrak{D},\omega^{\otimes})$ is weakly amenable, where $\mathfrak{D}:=\{(x,x^{-1}):x\in G\}$.*

Let $\phi:G\to G$ be a group epimorphism and ω be a weight function on G. Then the function $\overleftarrow{\omega}:G\to[1,\infty)$ defined by $\overleftarrow{\omega}(x)=\omega(\phi(x))$ is a weight function on G. For every quasi-additive function p, let $\mathfrak{S}(p)$ be the quasi-additive function defined by

$$\mathfrak{S}(p)(x,y)=p(\phi(x),\phi(y)) \qquad (x,y\in G).$$

Theorem 2.4 together with Proposition 4.1 and Theorem 4.6 in [18] proves the next result.

Corollary 2.7 *Let ω be weight function on locally compact group G. Then the following statements hold.*

(i) *If $\phi : G \to G$ is a continuous group epimorphism, $M(G, \overleftarrow{\varpi})$ is weakly amenable and $\mathfrak{S}(I(G,\omega)) = I(G, \overleftarrow{\varpi})$, then $M(G,\omega)$ is weakly amenable.*

(ii) *If G is Abelian and $M(G,\tilde{\omega})$ is weakly amenable, then $M(H,\omega|_H)$ is weakly amenable, where H is a subgroup of G.*

Corollary 2.8 *Let ω_i be a weight function on a locally compact group G_i, for $i = 1, 2$. Then the following assertions are equivalent.*

(a) *$M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$ is weakly amenable.*

(b) *$M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$ is cyclically weakly amenable.*

(c) *$M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$ is point amenable.*

(d) *$M(G_i,\omega_i)$ is weakly amenable and G_i is discrete, for $i = 1, 2$.*

Proof. Let $M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$ be point amenable. Since $M(G_i,\omega_i)$ is unital, for $i = 1, 2$, from Proposition 2.1 we infer that then $M(G_1\times G_2,\omega_1\otimes\omega_2)$ is point amenable. By Theorem 2.4, $G_1 \times G_2$ is discrete. It follows that G_i is discrete, for $i = 1, 2$. Hence $M(G_i,\omega_i) = \ell^1(G_i,\omega_i)$ and so

$$\ell^1(G_1,\omega_1)\hat{\otimes}\ell^1(G_1,\omega_1)) = M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$$

is weakly amenable. In view of Corollary 4.8 in [18], $\ell^1(G_i,\omega_i)$ is weakly amenable. So (c) implies (d).

Let $M(G_i,\omega_i)$ is weakly amenable and G_i is discrete, for $i = 1, 2$. By Corollary 2.5, $M(G_i,\omega_i)$ is point amenable, for $i = 1, 2$. It follows from Theorem 4.1 in [20] and Proposition 2.1 that

$$M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2) = M(G_1 \times G_2,\omega_1 \otimes \omega_2)$$

is point amenable. Again, apply Corollary 2.5 to conclude that $M(G_1,\omega_1)\hat{\otimes}M(G_2,\omega_2)$ is weakly amenable. That is, (d) implies (a). □

As a consequence of Corollary 2.8, we give the next result.

Corollary 2.9 *Let ω_i be a weight function on a locally compact discrete group G_i, for $i = 1, 2$. Then the following assertions are equivalent.*

(a) *$\ell^1(G_1,\omega_1)\hat{\otimes}\ell^1(G_2,\omega_2)$ is weakly amenable.*

(b) *$\ell^1(G_1,\omega_1)\hat{\otimes}\ell^1(G_2,\omega_2)$ is cyclically weakly amenable.*

(c) *$\ell^1(G_1,\omega_1)\hat{\otimes}\ell^1(G_2,\omega_2)$ is point amenable.*

(d) *$\ell^1(G_i,\omega_i)$ is weakly amenable and G_i is discrete, for $i = 1, 2$.*

We say that $T \in M(G,\omega)^*$ *vanishes at infinity* if for every $\varepsilon > 0$, there exists a compact subset K of G, for which $|\langle T, \mu\rangle| < \varepsilon$, where $\mu \in M(G,\omega)$ with $|\mu|(K) = 0$ and $\|\mu\|_\omega = 1$. We denote by $M_*(G,\omega)$ the subspace of $M(G,\omega)^*$ consisting of all functionals that vanish at infinity. In the case where, $\omega(x) = 1$ for all $x \in G$, we write

$$M_*(G,\omega) := M_*(G).$$

The space $M_*(G,\omega)$ is a norm closed subspace of $M(G,\omega)^*$. It is proved that $M_*(G,\omega)^*$ with the first Arens product is a Banach algebra [16]. For each $f \in L^1(G,\omega)$, we may consider f as a linear functional in $M_*(G)^*$. One can prove that $L^1(G,\omega)$ is a closed ideal in $M_*(G,\omega)^*$ and $M_*(G,\omega)^* = L^1(G,\omega)$ if and only if G is discrete [16]; see [15] for the case $\omega = 1$.

Corollary 2.10 *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) $M_*(G,\omega)^*$ *is weakly amenable.*

(b) $M_*(G,\omega)^*$ *is cyclically weakly amenable.*

(c) $M_*(G,\omega)^*$ *is point amenable.*

(d) *G is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.*

Proof. Let $M_*(G,\omega)^*$ be point amenable. Since $M(G,\omega)$ is a direct summand of $M_*(G,\omega)^*$, by Theorem 3.7 in [20], $M(G,\omega)$ is point amenable. Hence G is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded. Thus (c) implies (d). It is easy to see that if G discrete, then

$$M_*(G,\omega)* = \ell^1(G,\omega) = M(G,\omega).$$

It follows that (d) implies (a). □

Let $L^\infty(G, 1/\omega)$ be the space of all Borel measurable functions f on G with $f/\omega \in L^\infty(G)$, the Lebesgue space of bounded Borel measurable functions on G. Let also $L_0^\infty(G, 1/\omega)$ denote the subspace of $L^\infty(G, 1/\omega)$ consisting of all functions $f \in L^\infty(G, 1/\omega)$ that vanish at infinity. It is proved that $L_0^\infty(G, 1/\omega)$ is left introverted in $L^\infty(G, 1/\omega)$. So $L_0^\infty(G, 1/\omega)^*$ is a Banach algebra with the first Arens product [10]; see also [4, 11, 12, 21].

Corollary 2.11 *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) $L_0^\infty(G, 1/\omega)^*$ *is weakly amenable.*

(b)$L_0^\infty(G, 1/\omega)^*$ *is cyclically weakly amenable.*

(c) $L_0^\infty(G, 1/\omega)^*$ *is point amenable.*

(d) *G is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.*

Corollary 2.12 *Let ω_i be a weight function on a locally compact group G_i, for $i = 1, 2$. Then the following assertions are equivalent.*

(a) *$M_*(G_1, \omega_1)^*$ and $M_*(G_2, \omega_2)^*$ are weakly amenable.*

(b) *$L_0^\infty(G, 1/\omega)^*$ and $L_0^\infty(G, 1/\omega)^*$ are weakly amenable.*

(c) *$M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^*$ is weakly amenable and G_i is discrete, for $i = 1, 2$.*

(d) *$L_0^\infty(G, 1/\omega)^* \hat{\otimes} L_0^\infty(G, 1/\omega)^*$ is weakly amenable and G_i is discrete, for $i = 1, 2$.*

Proof. Assume that $M_*(G_1, \omega_1)^*$ and $M_*(G_2, \omega_2)^*$ are weakly amenable. By Corollary 2.10, G_i is discrete and $M(G_i, \omega_i) = M_*(G_i, \omega_i)^*$ is weakly amenable, for $i = 1, 2$. It follows from Corollary 2.8 that

$$M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^* = M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$$

is weakly amenable. So (a) implies (c).

Let $M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^*$ be weakly amenable and G_i is discrete, for $i = 1, 2$. This implies that $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is weakly amenable. Thus $M_*(G_i, \omega_i)^* = M(G_i, \omega_i)$ is weakly amenable. Hence (c) implies (a). Similarly, (b) and (d) are equivalent. □

Let LUC$(G, 1/\omega)$ be the space of all continuous function f on G such that f/ω is a left uniformly continuous functions on G; for study of this space see [27]. Let $WAP(\mathfrak{A})$ be the space of all weakly almost periodic functionals on Banach algebra $\mathfrak{A}$, that is, $f \in \mathfrak{A}^*$ such that the map $a \mapsto af$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ is weakly compact, where $\langle af, b\rangle = \langle f, ba\rangle$ for all $b \in \mathfrak{A}$.

Corollary 2.13 *Let $WAP(L^1(G, \omega))^*$ or $LUC(G, \omega)^*$ be 0-point amenable. Then G is discrete.*

Let $\mathfrak{A}$ be one of the Banach algebras $M(G, \omega)$, $M_*(G, \omega)^*$, $L_0^\infty(G, 1/\omega)^*$, WAP$(L^1(G, \omega))^*$ or LUC$(G, \omega)^*$.

Proposition 2.14 *Let G be a locally compact group. If $\mathfrak{A}$ is cyclically amenable, then every element of $CD(G, \omega)$ is inner.*

Proof. Let $M(G, \omega)$ be cyclically amenable. Since $L^1(G, \omega)$ is a direct summand of $M(G, \omega)$, by Theorem 3.7 in [20], the Banach algebra $L^1(G, \omega)$ is cyclically amenable. It follows from Theorem 5.6 in[18] that every element of $CD(G, \omega)$ is inner. For the other cases, we only need to recall that

$$\mathfrak{A} = M(G, \omega) \oplus \mathfrak{B}$$

for some closed subspace $\mathfrak{B}$ of $\mathfrak{A}$. □

3 The second dual of Banach algebras

The main result of this section is the following which solves an open problem posed in [9].

Theorem 3.1 *Let G be a locally compact group. Then the following assertion are equivalent.*
(a) $L^1(G,\omega)^{**}$ *is weakly amenable.*
(b) $L^1(G,\omega)^{**}$ *is cyclically weakly amenable.*
(c) $L^1(G,\omega)^{**}$ *is point amenable.*
(d) G *is finite.*

Proof. Let $\iota : L^1(G,\omega) \to L^1(G)$ be the inclusion map. Since $L^1(G,\omega)$ is dense in $L^1(G)$, ι is a continuous homomorphism with dense range. So $\iota^{**} : L^1(G,\omega)^{**} \to L^1(G)^{**}$ is epimorphism. Hence if $L^1(G,\omega)^{**}$ is point amenable, then by Theorem 2.1 in [20] the Banach algebra $L^1(G)^{**}$ is point amenable. It follows that every continuous point derivation of $L^1(G)^{**}$ at the discrete augmentation character is zero. From Theorem 11.17 in [5] infer that G is finite. So (c)⇒(d). The implications (a)⇒(b)⇒(c) follows from Theorem 4.1 in [19]. □

Corollary 3.2 *Let G be a locally compact group. Then the following assertion are equivalent.*
(a) $M(G,\omega)^{**}$ *is weakly amenable.*
(b) $M(G,\omega)^{**}$ *is cyclic amenable.*
(c) $M(G,\omega)^{**}$ *is point amenable.*
(d) G *is finite.*

Proof. Let $M(G,\omega)^{**}$ is point amenable. By Proposition 5.2 in [20], the Banach algebra $M(G,\omega)$ is point amenable. In view of Corollary 2.5, G is discrete. Hence $L^1(G,\omega)^{**}$ is weakly amenable. Now, apply Theorem 3.1. □

Let us recall that if there exists a compact invariant neighborhood of e in G, then G is called an $[IN]-$*group.* The following result is an improvement of Theorem 3.4 in [9].

Theorem 3.3 *Let G be a connected locally compact group. If either G_d is amenable or G is an $[IN]-$group, then the following assertions are equivalent.*
(a) $L^1(G,\omega)^{**}$ *is weakly amenable.*
(b) $M(G,\omega)$ *is weakly amenable.*
(c) $G = \{e\}$.

Proof. Let $L^1(G,\omega)^{**}$ be weakly amenable. Since

$$L^1(G,\omega)^{**} = M(G,\omega) \oplus C_0(G,\omega)^{\perp}$$

and $C_0(G,\omega)^{\perp}$ is an ideal in $L^1(G,\omega)^{**}$, we have $M(G,\omega)$ is weakly amenable. So (a)⇒(b). Let's show that (b)⇒(c). To this end, let $M(G,\omega)$ be weakly amenable. It follows from Theorem 2.4 that G discrete and $M(G)$ is weakly amenable. If G_d is amenable, then from Theorem 3.3 in [9] we infer that $G = \{e\}$. If G is an $[IN]-$group, then by Theorem 3.4 in [9], G is compact. Since G is also discrete, it follows that G is finite. Hence G_d is amenable. Thus $G = \{e\}$. So (b)⇒(c). The implication (c)⇒(a) is clear. □

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