GRAPHS WITH EQUAL GIRTH AND CIRCUMFERENCE

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Abstract. We characterise the form of all simple, finite graphs for which the girth of the graph is equal to the circumference of the graph. We apply this to prove a bound on the number of edges in such a graph.

1. Introduction

Cycles are a natural object which arise from the definition of a graph, and the invariants related to cycles have been extensively studied. In particular, the girth and circumference of a graph (being the length of the shortest and longest cycle in a graph respectively) are invariants which arise naturally from the definition of a cycle. An interesting problem is to try to determine the structure of graphs for which the girth and circumference have specific values. In this paper, we characterise the graphs for which the girth and circumference are equal. In other words, graphs for which all of the cycles contained within it have the same length.

Similar questions to this have been studied before. For example, [AH] considered graphs G for which G and its complement have the same girth. Graphs G such that G and its complement have circumference 3 or 4 were also characterised. The opposite extreme to having a graph G with equal cycle lengths has also been studied. Erdös posed the problem of determining the maximum number $f(n)$ of edges in a simple graph with n vertices in which any two cycles are of different lengths [BM][p. 247, Problem 11]. There are currently lower bounds [S] and upper bounds [BCFY] on $f(n)$, however this is still an open problem.

In this paper, we fully characterise graphs G for which the girth and circumference are equal. First it is shown that G must be planar. Now let G be a finite, simple and planar graph containing at least one cycle. Two faces F and F' of G are adjacent if the cycles bounding them share an edge, and F and F' are connected if there is a sequence of faces $F_1, \cdots, F_m$ where $F_1 = F$, $F_m = F'$ and F_k is adjacent to F_{k+1} for all $1 \leq k \leq m-1$. A *strict face component* G_F containing a face F of G consists of the vertices and edges of the cycles bounding the faces connected to F. The characterisation of graphs with equal girth and circumference is in terms of its strict face components, as in Corollary 3.13. This leads to an alternative characterisation in Theorem 3.14 in terms of blocks, which can be used to give an algorithm for determining if a given graph has equal

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girth and circumference. Finally in Section 4, we give an upper bound on the number of edges in a finite, simple, connected and planar graph G on n vertices with equal girth and circumference r. In particular, if the value of r is known, it is proved in Theorem 4.9 that if r is even, then $|E(G)| \leq n-1+\left\lfloor \frac{n-\frac{r}{2}-1}{\frac{r}{2}-1} \right\rfloor$, or if r is odd, then $|E(G)| \leq n-1+\left\lfloor \frac{n-1}{r-1} \right\rfloor$. If the value of r is unknown, it is proved in Corollary 4.10 that $|E(G)| \leq 2n-4$. This has applications in showing whether a given graph must have two cycles of different lengths, solely in terms of the number of vertices and edges.

2. Basic Definitions

In this section, we define the basic notions in graph theory which will be required in this paper. We will use the definitions from Gross, Yellen and Zhang [GYZ]. We restrict our attention to simple, finite and connected graphs. There are two invariants of graphs related to cycles known as the girth and the circumference. These invariants are well defined when the graph G contains a cycle. Connected graphs which do not contain cycles are known as *trees*. The *girth* of G, denoted $\text{girth}(G)$, is the length of the shortest cycle in G. The *circumference* of G, denoted $\text{circum}(G)$, is the length of the longest cycle in G.

Let H be a subgraph of a graph G. The values of the girth and circumference of H and G can be related through an inequality. A graph invariant μ is *monotone* if $\mu(H) \leq \mu(G)$ for all simple, connected and finite subgraphs H of a simple, connected and finite graph G. A graph invariant μ is *anti-monotone* if $\mu(G) \leq \mu(H)$ for all simple, connected and finite subgraphs H of a simple, connected and finite graph G. It follows from the definitions of girth and circumference that the girth of a graph is anti-monotone and the circumference of a graph is monotone.

It will be shown that a necessary condition for a graph with $\text{girth}(G) = \text{circum}(G)$ is for a graph to be planar. A graph G is *planar* if it can be embedded in $\mathbb{R}^2$. An embedding of a planar graph G into the plane is known as a *plane representation* of G. A *face* of a given plane representation of a graph G is a component of the complement of the plane representation in the plane. For a given plane embedding of a graph G, we will need to differentiate between two types of faces. An *internal face* of a graph G is a bounded component of the complement of the plane representation in the plane. The *external face* of a graph G is the unbounded component of the complement of the plane representation in the plane. The structure of the internal faces of G will turn out to be closely related to subgraphs known as blocks. A *cut vertex* v of G is a vertex v such that removing it disconnects G. A *block* in a graph G is a maximally connected subgraph that does not contain a cut vertex. If G is connected and contains no cut vertices, then G is called a block graph.

We will also require the notion of homeomorphic graphs. This involves an operation on graphs known as subdivision of an edge. For any edge $e = x \cdot y \in E(G)$ in a graph G, we can form a new graph G' by introducing a new vertex z in the interior of e. This forms two new edges in G', namely

$e_1 = x \cdot z$ and $e_2 = z \cdot y$. A graph G' is a *subdivision* of a graph G if we can obtain G' by subdividing the edges of G.

Before defining a homeomorphism, we define the notion of a homomorphism between graphs. Let G and H be graphs. A *homomorphism* between G and H is a function $\phi : V(G) \to V(H)$ such that if v is adjacent to w in G, $\phi(v)$ is adjacent to $\phi(w)$ in H. A bijective homomorphism is called an *isomorphism*. If there exists an isomorphism between two graphs G and H, we say that G and H are *isomorphic* and we denote this by $G \cong H$. Two graphs are homeomorphic if there exist subdivisions G' and H' of G and H respectively such that $G' \cong H'$.

Finally, we define special types of graphs which will be used throughout the paper. The *complete graph* on n vertices, denoted K_n, is the graph such that every pair of distinct vertices are adjacent. There is a variant of the complete graph for bipartite graphs. For $m \geq 1$ and $n \geq 1$, the *complete bipartite graph*, denoted $K_{m,n}$, is the graph whose vertex set is the union of two sets $V(K_{m,n}) = A \cup B$, where A and B are disjoint sets with $|A| = m$ and $|B| = n$. The edge set $E(K_{n,m})$ consists of all edges of the form $e = a \cdot b$ where $a \in A$ and $b \in B$. The *path graph* P_n is a tree on $n+1$ vertices with 2 vertices of degree 1 and $n-1$ vertices of degree 2. Finally, the *cycle graph* on n vertices, denoted C_n is the graph consisting of a single cycle of length n.

3. Characterising graphs with equal girth and circumference

3.1. **Non-Planar Graphs.** In this section, it is proven that non-planar graphs G have girth$(G) \neq$ circum(G). From this point onwards, any references to faces will be taken to be internal faces unless otherwise specified. To do this, a well known characterisation of planar graphs will be required known as Kuratowski's theorem [K].

Theorem 3.1. *A simple, connected and finite graph G is non-planar if and only if G contains a subgraph that is homeomorphic to either K_5 or to $K_{3,3}$.* □

We will also require the following lemma.

Lemma 3.2. *Let G be a simple, connected and finite graph with subgraph H which contains a cycle. If girth(G) = circum(G), then girth(G) = girth(H) = circum(H).*

Proof. Let $\mathcal{C}$ be the set of cycles in G and $\mathcal{C}' \subseteq \mathcal{C}$ be the set of cycles in H. Note that by assumption, $\mathcal{C}'$ is non-empty. Each cycle $C \in \mathcal{C}$ must all be of the same length l = girth(G) since girth(G) = circum(G). Therefore, all the cycles in H must have length l = girth(G) and so girth(G) = girth(H) = circum(H). □

Using Theorem 3.1 and Lemma 3.2, we can prove the desired result.

Theorem 3.3. *Let G be a simple, connected and finite non-planar graph. Then girth$(G) \neq$ circum(G).*

Proof. It suffices to find a subgraph H with $\text{girth}(H) \neq \text{circum}(H)$. This is because if H is a subgraph with $\text{girth}(H) \neq \text{circum}(H)$, then H contains two cycles which have two distinct lengths. These cycles will also be contained in G and so $\text{girth}(G) \neq \text{circum}(G)$. By Theorem 3.1, there exists a subgraph H in G which is homeomorphic to either K_5 or to $K_{3,3}$. Therefore, it suffices to prove that any subgraph H homeomorphic to K_5 or to $K_{3,3}$ has $\text{girth}(H) \neq \text{circum}(H)$.

First, consider the case where H is homeomorphic to K_5 and suppose H satisfies $\text{girth}(H) = \text{circum}(H)$. This means H can be obtained by subdividing the edges of K_5. Let $e_1, \cdots, e_{10}$ be the edges of K_5 and denote by d_i the number of times the edge e_i is subdivided to obtain H. Let $C_1, \cdots, C_{10}$ be the 3-cycles in K_5.

To ensure that $\text{girth}(H) = \text{circum}(H)$, the length of each 3-cycle after subdividing must equal a fixed number $N \geq 3$. In particular, this means that for each 3-cycle C_k where $k \in \{1, \cdots, 10\}$, $\sum_{e_i \in C_k} d_i = N$. Each edge e_i of K_5 is contained in at least one cycle of K_5 and so this gives a system of 10 equations with 11 unknowns. As the reader can easily check, the dimension of the null space of the matrix associated with this system of equations is 1. Since each cycle C_k contains three edges, letting $d_i = d$ for any $d \in \mathbb{N}_0$ is the only solution to this system of equations. Hence for $\text{girth}(H)$ to be equal to $\text{circum}(H)$, each edge must be subdivided d times. However, K_5 contains a 5-cycle. Subdividing the edge of a 5-cycle d times gives a cycle of length $5 + 5d$, and so $\text{circum}(H) = 5 + 5d$. Subdividing the edge of a 3-cycle d times gives a cycle of length $3 + 3d$, and so $\text{girth}(H) = 3 + 3d$. These are not equal for $d \geq 0$. Hence, $\text{girth}(H) \neq \text{circum}(H)$.

Now, consider the case where H is homeomorphic to $K_{3,3}$ and suppose H satisfies $\text{girth}(H) = \text{circum}(H)$. This means H can be obtained by subdividing the edges of $K_{3,3}$. Let $e_1, \cdots, e_9$ be the edges of $K_{3,3}$ and denote by d_i the number of times the edge e_i must be subdivided to obtain H. Let $C_1, \cdots, C_9$ be the 4-cycles in K_5.

To ensure that $\text{girth}(H) = \text{circum}(H)$, the length of each 4-cycle after subdividing must equal a fixed number $N \geq 4$. In particular, this means that for each 4-cycle C_k where $k \in \{1, \cdots, 9\}$, $\sum_{e_i \in C_k} d_i = N$. Each edge e_i of $K_{3,3}$ is contained in at least one cycle of $K_{3,3}$ and so this gives a system of 9 equations with 10 unknowns. As the reader can easily check, the dimension of the null space of the matrix associated with this system of equations is 1. Since each cycle C_k contains four edges, letting $d_i = d$ for any $d \in \mathbb{N}_0$ is the only solution to this system of equations. Therefore for $\text{girth}(G)$ to be equal to $\text{circum}(H)$, each edge must be subdivided d times. However, $K_{3,3}$ contains a 6-cycle. Subdividing the edge of a 6-cycle d times gives a cycle of length $6 + 6d$, and so $\text{circum}(H) = 6 + 6d$. Subdividing the edge of a 4-cycle d times gives a cycle of length $4 + 4d$, and so $\text{girth}(H) = 4 + 4d$. These are not equal for $d \geq 0$. Hence $\text{girth}(H) \neq \text{circum}(H)$.

□

3.2. **Planar Graphs.** Theorem 3.3 implies that any graph with $\text{girth}(G) = \text{circum}(G)$ must be planar. To characterise planar graphs with $\text{girth}(G) = \text{circum}(G)$, we first define subgraphs of G

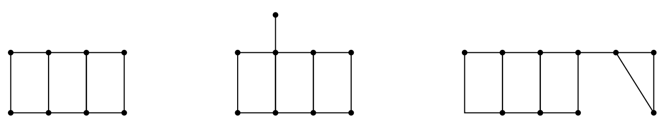
which we will consider as building blocks for G. This involves introducing the notion of a strict face-connected graph.

Definition 3.4. Suppose G is a finite, simple and planar graph. Let F and F' be two faces of G and let C and C' be the cycles which bound them. The faces F and F' are *adjacent* if C and C' share at least one common edge.

Definition 3.5. Let G be a finite, simple, planar graph containing at least one cycle. Two faces F and F' in G are connected if there is a sequence of faces $F_1, \cdots, F_m$ where $F_1 = F$, $F_m = F'$ and F_k is adjacent to F_{k+1} for all $1 \leq k \leq m-1$. The graph G is *face-connected* if any two faces F_i and F_j of G are connected.

Definition 3.6. A finite, simple and planar graph G containing a cycle is *strict face-connected* if it is face-connected and the vertices and edges of G are the union of the vertices and edges of the cycles bounding the internal faces.

The following figure gives an example of a graph that is strict face-connected (G_1), a graph that is face-connected but not strict face-connected (G_2) and a graph that is not face-connected (G_3).



(1) G_1 G_2 G_3.

Definition 3.7. Let G be a finite, simple and planar graph containing at least one cycle. Let F be a face of G. A *strict face component* G_F containing F consists of the vertices and edges of the cycles bounding the faces connected to F.

In (1), since G_1 and G_2 are face-connected, they have one strict face component although note that the strict face component of G_2 is not the whole graph. The graph G_3 has two strict face components, namely the sequence of 4-cycles on the left and the one 3-cycle on the right.

The notion of a strict face component is closely related to blocks. To see this, we first prove how strict face components are related.

Lemma 3.8. *Let G be a simple, finite and planar graph and let $\mathcal{F}_1, \cdots, \mathcal{F}_k$ be the strict face components of G. Then for any two distinct strict face components $\mathcal{F}_i$ and $\mathcal{F}_j$, $E(\mathcal{F}_i) \cap E(\mathcal{F}_j) = \emptyset$. Moreover, $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \{v\}$, where v is a cut vertex of G or $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$.*

Proof. Let $\mathcal{F}_i$ and $\mathcal{F}_j$ be two distinct strict face components and suppose $E(\mathcal{F}_i) \cap E(\mathcal{F}_j) \neq \emptyset$. This implies there exist faces F_i in $\mathcal{F}_i$ and F_j in $\mathcal{F}_j$ which are adjacent, and so $\mathcal{F}_i = \mathcal{F}_j$.

Now suppose $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \{v_1, \cdots, v_l\}$ for $l \geq 2$. Observe that $\mathcal{F}_i$ is connected as if not, for a face F in a connected component C, any vertex not contained in C can not be in the strict face component containing F. Therefore, there exists a path P_i contained in $\mathcal{F}_i$ between v_1 and v_2. Similarly, there exists a path P_j in $\mathcal{F}_j$ between v_1 and v_2. Since the edge sets of $\mathcal{F}_i$ and $\mathcal{F}_j$ are disjoint, the edge sets of these paths must be disjoint. This implies that v_1 and v_2 are part of a cycle C which has edges contained in both $\mathcal{F}_i$ and $\mathcal{F}_j$. Therefore, C has an edge adjacent to a face in $\mathcal{F}_i$ and an edge adjacent to a face in $\mathcal{F}_j$, and so it follows that $\mathcal{F}_i = \mathcal{F}_j$. Hence, either $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \{v\}$ or $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$.

Finally, suppose $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \{v\}$ and v is not a cut vertex. Since $\mathcal{F}_i$ and $\mathcal{F}_j$ are connected, there exist vertices $v_i \in \mathcal{F}_i$ and $v_j \in \mathcal{F}_j$ which are adjacent to v. Since v is not a cut vertex, removing v from G results in a graph G' which is still connected. Therefore there exists a path P between v_i and v_j contained in G'. Adding v back in gives a cycle C containing v in G which contains the edges $v \cdot v_i \in E(\mathcal{F}_i)$ and $v \cdot v_j \in E(\mathcal{F}_j)$. Hence, C is a cycle containing an edge in both $\mathcal{F}_i$ and $\mathcal{F}_j$, which is a contradiction. □

In the following, we will require the notion of the wedge sum of graphs.

Definition 3.9. Let G and H be simple graphs and let $v_G \in V(G)$ and $v_H \in V(H)$ be chosen vertices, known as *base vertices*. The wedge sum $G \vee H$ of G and H is the graph formed from the disjoint union of G and H by identifying the base vertices. Define the identified vertex in $G \vee H$ as the base vertex of the wedge sum.

Observe that the base vertex of $G \vee H$ is a cut vertex.

Proposition 3.10. *A planar graph G containing a cycle is strict face-connected iff G is a planar block graph.*

Proof. Suppose G is a planar block graph containing a cycle but not strict face-connected. By definition of block, this implies that G contains no cut vertices and so also has no bridges. Therefore, the only possibility for G is to contain at least 2 strict-face components. Since G is connected and contains no bridges, by Lemma 3.8, there must exist two strict face-components $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \{v\}$ where v is a cut vertex, which is a contradiction.

Now suppose G is strict face-connected but not a block graph. This implies that G must contain a cut vertex v. Removing v gives a graph G' with two connected components C_1 and C_2. Therefore, every path in G between vertices $v_1 \in C_1$ and $v_2 \in C_2$ must pass through v. It follows that $G = H_1 \vee H_2$ for some graphs H_1 and H_2. However, no face F_1 in H_1 can be adjacent to a face F_2 in H_2 since $E(H_1) \cap E(H_2) = \emptyset$. Therefore, G is not strict face-connected which is a contradiction. □

Proposition 3.10 shows that each strict face component of a planar graph G is a block. However, the converse is not true. For example for G_2 in (1), the induced subgraph on the vertex of degree one along with its adjacent vertex forms a block, however this is not part of a strict face component.

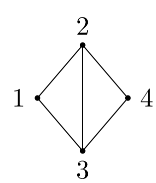
Let G be a finite, simple and planar graph with strict face components $\mathcal{F}_1, \cdots, \mathcal{F}_m$. Observe that each cycle of G is contained within a single strict face component. This observation gives the following result.

Proposition 3.11. *Let G be a finite, simple and planar graph with strict face components denoted $\mathcal{F}_1, \cdots, \mathcal{F}_m$. Then*

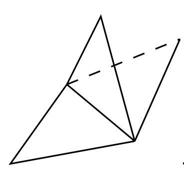
$$\textit{girth}(G) = \min\{\textit{girth}(\mathcal{F}_1), \cdots, \textit{girth}(\mathcal{F}_m)\} \text{ and } \textit{circum}(G) = \min\{\textit{circum}(\mathcal{F}_1), \cdots, \textit{circum}(\mathcal{F}_m)\}.$$

□

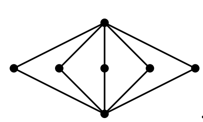
Proposition 3.11 implies that if G is a finite, simple and planar graph with girth(G) = circum(G), then each face component must also have equal girth and circumference. Therefore, it suffices to consider strict face-connected graphs. First, we will introduce a collection of graphs known as generalised book graphs. A *generalised book graph* is denoted $B(n, L, p)$ where $1 \leq n \leq L-2$, $L \geq 3$ and $p \geq 2$. These graphs will be the building blocks for graphs with equal girth and circumference. The graph $B(n, L, p)$ is p cycles C_L of length L glued together over a common path P_n of length n. For example, $B(1, 3, 2)$ is the graph



and the generalised book graph $B(1, 3, 3)$ is



For the theorem that follows, we draw the generalised book graph in its planar form. For example $B(2, 4, 4)$ is a graph of the form

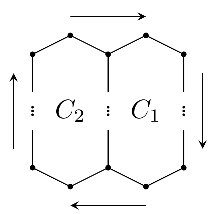


Observe that the generalised book graph $B(n, L, p)$ has p faces and is strict face-connected. We can now prove the characterisation theorem for planar graphs with girth equal to circumference.

Theorem 3.12. *Let G be a finite, simple and strict face-connected graph with girth(G) = circum(G). Then if G has $p = 1$ face, then $G = C_l$ for some $l \geq 3$. If G has $p \geq 2$ faces, then girth(G) = circum$(G) = 2k$ for some $k \in \mathbb{N}$, $k \geq 2$ and $G = B(k, 2k, p)$.*

Proof. First, if $G = C_m$ for some m then the result is clear. Therefore we consider the case where G has two or more faces.

We proceed by induction. Let G be a simple, strict face-connected and finite graph with two faces, F_1 and F_2 with cycles C_1 bounding F_1 and C_2 bounding F_2. Since girth(G) = circum(G), C_1 and C_2 have length L for some $L \geq 3$. The faces F_1 and F_2 are adjacent, which implies that the cycles C_1 and C_2 must intersect along a common path P_k for some $k \geq 1$. This creates a new cycle C_3 which starts at the endpoint of P_k, goes around C_1 (in the direction which does not travel along P_k) to the other endpoint of P_k and then around C_2 (in the direction which does not travel along P_k). This is illustrated by the following diagram.



This cycle has length $2(L-k)$. Since girth(G) = circum(G), $2(L-k) = L$ which implies that $L = 2k$. Therefore, the cycle must have length $2k$ and so $G = B(k, 2k, 2)$.

Now consider a simple, strict face-connected and finite graph G with three faces F_1, F_2 and F_3 bounded by cycles C_1, C_2 and C_3. By assumption, girth(G) = circum(G) and so these cycles have a common length L. Since G is face-connected, C_1 must be adjacent to another face. Suppose without loss of generality C_1 is adjacent to C_2. Considering the induced subgraph H on the vertices of C_1 and C_2, the $n = 2$ case implies $H = B(k, 2k, 2)$ for some $k \geq 2$. By Lemma 3.2, the lengths of the cycles in H must be equal to the lengths of the cycles in G and so $L = 2k$. By definition of $B(k, 2k, 2)$, C_1 and C_2 intersect over a common path of length k.

Now consider the cycle C_3 and assume without loss of generality that C_3 is adjacent to C_2. By a similar argument, considering the induced subgraph H' on the vertices of C_2 and C_3, C_2 and C_3 must intersect over a common path of length k. Therefore, C_1 and C_3 are both adjacent to C_2 over a path of length k. However, the edge set of these paths of length k must be disjoint as G is planar and so $G = B(k, 2k, 3)$.

Suppose the result is true for a strict face-connected graph with $k \leq m$ faces and consider a simple, strict face-connected and finite graph G with $m + 1$ faces, $m \geq 3$. Let $F_1, \cdots, F_{m+1}$ be the faces of G and let C_i be the cycle bounding the face F_i for $1 \leq i \leq m$. Since girth(G) = circum(G), C_i must have a common length L for all i. Suppose without loss of generality that F_{m+1} is adjacent to the external face of G. Consider the induced subgraph H on the vertices of the remaining cycles

$C_1, \cdots, C_m$. Observe that H is a graph with m faces and so by the inductive hypothesis has the form $B(k, 2k, m)$ for some $k \geq 2$. By Lemma 3.2, the lengths of the cycles in H must be equal to the lengths of the cycles in G and so $L = 2k$. Also by the structure of the generalised book graph, we can order the cycles $C_1, \cdots, C_m$ such that consecutive cycles intersect along a common path of length k and any two non-consecutive cycles C_i and C_j intersect at two vertices $\{v_0, v_1\}$ independent of i and j.

Now consider the remaining face F_{m+1}. Since G is face-connected, there exists $1 \leq l \leq m$ such that F_{n+1} is adjacent to F_l. Suppose $1 < l < m$. The induced subgraph H_l on the vertices of C_l and C_{m+1} has two faces and so $H_l = B(k_l, 2k_l, 2)$. By Lemma 3.2, the lengths of the cycles in H_l must be equal to the lengths of the cycles in G and so $k_l = k$. This implies that C_{m+1} intersects C_l over a path of length k. However by definition of H, this implies that F_{m+1} is also adjacent to F_{l-1} or F_{l+1}. Suppose without loss of generality, that F_{m+1} is adjacent to F_{l+1} and consider the induced subgraph H_{l+1} on the vertices of F_{m+1} and F_{l+1}. By the same argument, $H_{l+1} = B(k, 2k, 2)$ and C_{m+1} intersects C_{l+1} over a path of length k. The cycle C_{m+1} is adjacent to both C_l and C_{l+1} over paths of length k. However since G is planar, the edge set of these paths must be disjoint which cannot happen as F_l and F_{l+1} are adjacent. Therefore, C_{m+1} cannot be adjacent to C_l for $1 < l < m$.

Now suppose without loss of generality that F_{m+1} is adjacent to F_m and is not adjacent to F_l for $1 < l < m$. Consider the induced subgraph H on the vertices of C_m and C_{m+1}. By a similar argument to the previous case, the cycles C_m and C_{m+1} must intersect over a common path of length k. Now, suppose F_{m+1} is also adjacent to F_1. By considering the induced subgraph H' on the vertices of C_m and C_1, the cycles C_m and C_1 must intersect over a common path of length k. However since G is planar, the edge set of these paths must be disjoint which cannot happen as F_{m+1} is an internal face. Therefore, C_{m+1} cannot be adjacent to C_1. Hence $G = B(k, 2k, m+1)$ as desired. □

Combining Proposition 3.11 and Theorem 3.12 gives the following corollary.

Corollary 3.13. *Let G be a finite, simple and planar graph with r = girth(G) = circum(G). Let $\mathcal{F}_1, \cdots, \mathcal{F}_m$ be the strict face components of G. If r is even, then $\mathcal{F}_i = B\left(\frac{r}{2}, r, p\right)$ or $\mathcal{F}_i = C_r$. If r is odd, then $\mathcal{F}_i = C_r$.* □

This fully characterises the structure of a graph with girth equal to circumference. However given a simple, planar graph G, it may not be easy to determine if G has the required form. Proposition 3.10 can be used to rephrase Corollary 3.13 in terms of blocks in order to determine if G has the required form algorithmically. In particular, let G be a finite, simple and planar graph and define G_B to be the graph obtained from removing the bridges in G, and then removing the isolated vertices. Since an edge is a bridge iff it is not contained in a cycle, G_B is the union of the strict face components of G. By Proposition 3.10, these strict face components are blocks and we obtain the following result.

Theorem 3.14. *Let G be a finite, simple and planar graph and $H_1, \cdots, H_k$ be the blocks of G_B. Then G has $r = girth(G) = circum(G)$ iff for all $1 \leq i \leq k$, $H_i = B\left(\frac{r}{2}, r, p_i\right)$ or $H_i = C_r$ and r is even, or $H_i = C_r$ for all $1 \leq i \leq k$, if r is odd.* □

4. Upper bounds on the number of edges in a graph with equal girth and circumference

In this section, we prove upper bounds on the number of edges in a graph with girth equal to circumference. To do this, we will consider a general construction of planar graphs. Let $G_1, \cdots, G_k$ be graphs. Construct a graph G_2' from the disjoint union of G_1 and G_2 by identifying a vertex in G_1 and a vertex in G_2. Iteratively define G_i' for $3 \leq i \leq k$ from the disjoint union of G_{i-1}' and G_i by identifying a vertex in G_{i-1}' and a vertex in G_i. Since at each stage of the construction, we are identifying a single vertex of two graphs together, we obtain the following result.

Lemma 4.1. *The number of vertices and edges in the graphs $G_1', \cdots, G_k'$ are invariant under the choice of vertex identification.* □

Construction 4.2. Let G be a simple, finite and planar graph and let $\mathcal{F}_1, \cdots, \mathcal{F}_k$ be the strict face components of G. Lemma 3.8 implies that any edge contained in a cycle in G must be contained in a unique strict face component of G. Conversely, any edge not contained in any cycle is a bridge. Therefore, we can view G as strict face components joined together by a single cut vertex (by Lemma 3.8) or trees. This implies that we can construct any planar graph G iteratively in the following way. Define a graph G_2 as the disjoint union of a strict face component and another strict face component or a tree as appropriate and identifying a single vertex in each graph. Continue this process by defining G_i to be the disjoint union of G_{i-1} and a strict face component or a tree and identifying a single vertex in each graph. Since G is a finite graph, this process will terminate giving $G = G_n$ for some $n \geq 1$.

In what follows, we will only be concerned with the number of edges and vertices contained within G. Lemma 4.1 implies that the choice of vertex identification does not affect the number of vertices or edges in the graph. Therefore, it will be simplest to consider the wedge sum of the strict face components and trees.

The number of vertices and edges of a wedge sum can be determined easily in terms of the summands. At each stage, only one vertex is being identified. Therefore, the number of edges is unaffected, and two vertices in the disjoint union become one vertex in the wedge sum. Therefore, we obtain the following.

Lemma 4.3. *Let $G_1, \cdots, G_n$ be graphs, then*

$$\left|V\left(\bigvee_{i=1}^{n} G_i\right)\right| = \sum_{i=1}^{n} |V(G_i)| - (n-1), \quad \left|E\left(\bigvee_{i=1}^{n} G_i\right)\right| = \sum_{i=1}^{n} |E(G_i)|.$$

□

By Corollary 3.13, Construction 4.2 and Lemma 4.1, to consider the number of edges in a finite, simple and planar graph G with $r = \mathrm{girth}(G) = \mathrm{circum}(G)$, it suffices to consider $\bigvee_{i=1}^{k_1} \mathcal{F}_i \vee \bigvee_{j=1}^{k_2} T_j$ where T_j is a tree and $\mathcal{F}_i = B(\frac{r}{2}, r, p_i)$ for some $p_i \geq 2$ or $\mathcal{F}_i = C_r$ if r is even, or $\mathcal{F}_i = C_r$ if r is odd. Further, without loss of generality we can consider $G' = \bigvee_{i=1}^{m} \mathcal{F}_i \vee P_k$ by Lemma 4.1. In the case that r is even, we can view C_r as two paths of length $\frac{r}{2}$ glued together by their endpoints, therefore it makes sense to define $B(\frac{r}{2}, r, 1)$ as C_r in this case.

We now study the properties of the wedge sum of generalised book graphs and path graphs which will then be applied to determine a bound on the number of edges in G. Viewing $B(k, 2k, p)$ as $p+1$ path graphs P_k of length k glued together by their end points, we obtain the following result.

Lemma 4.4. *Let $B(k, 2k, p)$ be a generalised book graph with $k \geq 1$ and $p \geq 1$. Then*

$$|V(B(k, 2k, p))| = (k-1)(p+1) + 2, \quad |E(B(k, 2k, p))| = k(p+1).$$

□

Now, for a fixed number of vertices, we bound the number of edges of the wedge sum of two generalised book graphs.

Lemma 4.5. *Let $B(\frac{r}{2}, r, a)$ and $B(\frac{r}{2}, r, b)$ be generalised book graphs with $r \geq 4$, r even and $a, b \geq 1$. Let*

$$n = \left|V\left(B\left(\frac{r}{2}, r, a\right) \vee B\left(\frac{r}{2}, r, b\right)\right)\right|, \; p = \left\lfloor \frac{\left(\frac{r}{2}-1\right)(a+b+1)+1}{\frac{r}{2}-1} \right\rfloor, \; c = n - 2 - \left(\frac{r}{2}-1\right)(p+1).$$

Then

$$(i) \left|V\left(B\left(\frac{r}{2}, r, a\right) \vee B\left(\frac{r}{2}, r, b\right)\right)\right| = \left|V\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|$$
$$(ii) \left|E\left(B\left(\frac{r}{2}, r, a\right) \vee B\left(\frac{r}{2}, r, b\right)\right)\right| \leq \left|E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|.$$

Proof. First, by Lemma 4.3 and Lemma 4.4,

$$n = \left(\frac{r}{2}-1\right)(a+1) + 2 + \left(\frac{r}{2}-1\right)(b+1) + 2 - 1 = 2 + \left(\frac{\left(\frac{r}{2}-1\right)(a+b+1)+1}{\frac{r}{2}-1} + 1\right)\left(\frac{r}{2}-1\right)$$
$$\geq 2 + (p+1)\left(\frac{r}{2}-1\right) = \left|V(B(\frac{r}{2}, r, p))\right|.$$

Therefore $c \geq 0$, and so P_c is well defined. Moreover, (i) follows from Lemma 4.3 and Lemma 4.4.

For (ii), consider $\left|E\left(B\left(\frac{r}{2}, r, a\right) \vee B\left(\frac{r}{2}, r, b\right)\right)\right|$. By Lemma 4.3 and Lemma 4.4,

$$\left|E\left(B\left(\frac{r}{2}, r, a\right) \vee B\left(\frac{r}{2}, r, b\right)\right)\right| = \frac{r}{2}(a+1) + \frac{r}{2}(b+1)$$

$$= \left\lfloor \frac{\left(\frac{r}{2}-1\right)(a+b+1)}{\frac{r}{2}-1} \right\rfloor + \left(\frac{r}{2}-1\right)(a+b+2)+1 \leq \left\lfloor \frac{\left(\frac{r}{2}-1\right)(a+b+1)+1}{\frac{r}{2}-1} \right\rfloor + \left(\frac{r}{2}-1\right)(a+b+2)+1$$

$$< \left\lfloor \frac{\left(\frac{r}{2}-1\right)(a+b+1)+1}{\frac{r}{2}-1} \right\rfloor + \left(\frac{r}{2}-1\right)(a+b+2)+2 = \left\lfloor \frac{\left(\frac{r}{2}-1\right)(a+b+1)+1}{\frac{r}{2}-1} \right\rfloor + n - 1$$

$$= \left| E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|.$$

□

Next, for a fixed number of vertices, we bound the number of edges of the wedge sum of a generalised book graph and a path graph.

Lemma 4.6. *Let $B(\frac{r}{2}, r, p)$ and $B(\frac{r}{2}, r, p')$ be generalised book graphs with $r \geq 4$ where r is even, and $1 \leq p' \leq p$. Let P_c and $P_{c'}$ be path graphs with $0 \leq c \leq c'$. If $\left|V\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right| = \left|V\left(B\left(\frac{r}{2}, r, p'\right) \vee P_{c'}\right)\right|$, then*

$$\left|E\left(B\left(\frac{r}{2}, r, p'\right) \vee P_{c'}\right)\right| \leq \left|E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|.$$

Proof. By Lemma 4.3 and Lemma 4.4, the assumption $\left|V\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right| = \left|V\left(B\left(\frac{r}{2}, r, p'\right) \vee P_{c'}\right)\right|$ implies that $c' = \left(\frac{r}{2}-1\right)(p-p') + c$. Now by Lemma 4.3 and Lemma 4.4,

$$\left|E\left(B\left(\frac{r}{2}, r, p'\right) \vee P_{c'}\right)\right| = \frac{r}{2}(p'+1) + c' = p' - p + \frac{r}{2}(p+1) + c$$
$$\leq \frac{r}{2}(p+1) + c = \left|E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|$$

where the last inequality follows since $p' \leq p$. □

This gives us everything we need to bound the number of edges for a simple, connected, finite and planar graph G with $r = \text{girth}(G) = \text{circum}(G)$. We first consider the case that r is even.

Lemma 4.7. *Let G be a simple, connected, finite and planar graph with $n \geq 4$ vertices and $r =$ girth$(G) =$ circum(G) where r is even. Let*

$$p = \left\lfloor \frac{n - \frac{r}{2} - 1}{\frac{r}{2} - 1} \right\rfloor \text{ and } c = n - 2 - \left(\frac{r}{2}-1\right)(p+1)$$

Then

$$|E(G)| \leq \left|E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|.$$

Proof. First by Lemma 4.4,

$$\left|V\left(B\left(\frac{r}{2}, r, p\right)\right)\right| = \left(\frac{r}{2}-1\right)(p+1) + 2 \leq \left(\frac{r}{2}-1\right)\left(\frac{n - \frac{r}{2} - 1}{\frac{r}{2}-1} + 1\right) + 2 = n.$$

Therefore by Lemma 4.3 and Lemma 4.4, $c \geq 0$, and so P_c is well defined. Moreover, it follows that $\left|V\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right| = n$. Now, we show that the number of vertices in the graph $B\left(\frac{r}{2}, r, p+1\right)$ is greater than n. In particular by Lemma 4.6, this implies that any other graph H of the form $B \vee T$ where B is a generalised book graph and T is a tree with $n = |V(H)| = \left|V\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|$ has

$$|E(H)| \leq \left|E\left(B\left(\frac{r}{2}, r, p\right) \vee P_c\right)\right|. \tag{2}$$

By Lemma 4.4,

$$\left|V\left(B\left(\frac{r}{2},r,p+1\right)\right)\right| = \frac{r}{2}+1+\left(\left\lfloor\frac{n-\frac{r}{2}-1}{\frac{r}{2}-1}\right\rfloor+1\right)\left(\frac{r}{2}-1\right)$$

$$> \frac{r}{2}+1+\left(\frac{n-\frac{r}{2}-1}{\frac{r}{2}-1}-1+1\right)\left(\frac{r}{2}-1\right)=n.$$

Since we are only considering the number of edges in G, by Lemma 4.1 and Construction 4.2, it suffices to consider the graph $G' = \bigvee_{i=1}^{m} \mathcal{F}_i \vee P_j$ where $\mathcal{F}_i$ are the strict face components of G and P_j is a path graph. Corollary 3.13 implies that $\mathcal{F}_i = B(\frac{r}{2}, r, p_i)$ for $p_i \geq 1$, with the convention that $B(\frac{r}{2}, r, 1) = C_r$.

Now consider $|E(G')| = |E(\bigvee_{i=1}^{m} \mathcal{F}_i \vee P_k)|$. Observe that by definition, the wedge sum is associative and commutative, up to isomorphism. Applying Lemma 4.5 to $\mathcal{F}_{m-1} \vee \mathcal{F}_m$, we obtain

$$\left|E\left(\bigvee_{i=1}^{m} \mathcal{F}_i \vee P_k\right)\right| \leq \left|E\left(\bigvee_{i=1}^{m-2} \mathcal{F}_i \vee B' \vee P_{c_1} \vee P_k\right)\right|$$

for some generalised book graph B' and path graph P_{c_1}. Moreover, Lemma 4.1 implies that $\left|E\left(\bigvee_{i=1}^{m-2} \mathcal{F}_i \vee B' \vee P_{c_1} \vee P_k\right)\right| = \left|E\left(\bigvee_{i=1}^{m-2} \mathcal{F}_i \vee B' \vee P'\right)\right|$ for some path graph P'. Iterate this process by considering the last strict face component in the wedge summand and B' to obtain

$$\left|E\left(\bigvee_{i=1}^{m} \mathcal{F}_i \vee P_k\right)\right| \leq \left|E(\overline{B} \vee \overline{P})\right|$$

for some generalised book graph $\overline{B}$ and path graph $\overline{P}$. However, by (2)

$$E(\overline{B} \vee \overline{P}) \leq \left|E\left(B\left(\frac{r}{2},r,p\right) \vee P_c\right)\right|.$$

□

The case for r odd is similar.

Lemma 4.8. *Let G be a simple, connected, finite and planar graph with $n \geq 3$ vertices and $r =$ girth$(G) =$ circum(G) where r is odd. Let*

$$p' = \left\lfloor\frac{n-1}{r-1}\right\rfloor.$$

Then

$$|E(G)| \leq \left|E\left(\bigvee_{i=1}^{p'} C_r \vee P_{n-1+p'(1-r)}\right)\right|.$$

Sketch. It follows from Lemma 4.3,

$$n-1+p'(1-r) \geq 0 \text{ and } \left|V\left(\bigvee_{i=1}^{p'} C_r \vee P_{n-1+p'(1-r)}\right)\right| = n.$$

Similar to Lemma 4.6, it can be shown that for graphs $\bigvee_{i=1}^{k} C_r \vee P_l$ and $\bigvee_{i=1}^{k'} C_r \vee P_{l'}$ with $k \geq k'$, $l \leq l'$ and $\left|V\left(\bigvee_{i=1}^{k} C_r \vee P_l\right)\right| = \left|V\left(\bigvee_{i=1}^{k'} C_r \vee P_l\right)\right|$, that

$$\left|E\left(\bigvee_{i=1}^{k'} C_r \vee P_{l'}\right)\right| \leq \left|E\left(\bigvee_{i=1}^{k} C_r \vee P_l\right)\right|. \tag{3}$$

By Lemma 4.1 and Construction 4.2, it suffices to consider the graph $G' = \bigvee_{i=1}^{m} \mathcal{F}_i \vee P_j$ where $\mathcal{F}_i$ are the strict face components of G and P_j is a path graph. Corollary 3.13 implies that $\mathcal{F}_i = C_r$. Therefore by (3), to maximise $E(G')$ we wish to maximise m which is achieved by $m = p'$. □

Let G be a simple, connected, finite and planar graph with n vertices and $r = \text{girth}(G) = \text{circum}(G)$. Lemma 4.7 and Lemma 4.8 gives us a sharp upper bound for $|E(G)|$.

Theorem 4.9. *Let G be a simple, connected, finite and planar graph with n vertices and $r =$ girth$(G) =$ circum(G). Then if r is even and $n \geq 4$,*

$$|E(G)| \leq n - 1 + \left\lfloor \frac{n - \frac{r}{2} - 1}{\frac{r}{2} - 1} \right\rfloor,$$

and if r is odd and $n \geq 3$,

$$|E(G)| \leq n - 1 + \left\lfloor \frac{n-1}{r-1} \right\rfloor.$$

Proof. For r even, Lemma 4.7 shows that the graph $G' = B\left(\frac{r}{2}, r, p\right) \vee P_c$ where $p = \left\lfloor \frac{n - \frac{r}{2} - 1}{\frac{r}{2} - 1} \right\rfloor$ and $c = n - 2 - \left(\frac{r}{2} - 1\right)(p+1)$ has $|E(G)| \leq |E(G')|$ for all graphs G with $n \geq 4$ vertices and $r = \text{girth}(G) = \text{circum}(G)$ for r even. By Lemma 4.3 and Lemma 4.4,

$$|E(G')| = n - 1 + \left\lfloor \frac{n - \frac{r}{2} - 1}{\frac{r}{2} - 1} \right\rfloor.$$

For r odd, Lemma 4.8 shows that the graph $G' = \bigvee_{i=1}^{p'} C_r \vee P_{n-1+p'(1-r)}$ where $p' = \left\lfloor \frac{n-1}{r-1} \right\rfloor$ has $|E(G)| \leq |E(G')|$ for all graphs G with $n \geq 3$ vertices and $r = \text{girth}(G) = \text{circum}(G)$ for r odd. By Lemma 4.3 and Lemma 4.4,

$$|E(G')| = n - 1 + \left\lfloor \frac{n-1}{r-1} \right\rfloor.$$

□

Theorem 4.9 can be used to prove an upper bound that is independent of r.

Corollary 4.10. *Let G be a simple, connected, finite and planar graph with $n \geq 4$ vertices and $r =$ girth$(G) =$ circum(G). Then*

$$|E(G)| \leq 2n - 4.$$

Proof. By Theorem 4.9, if r is even, then $|E(G)| \leq n - 1 + \left\lfloor \frac{n - \frac{r}{2} - 1}{\frac{r}{2} - 1} \right\rfloor$. This is monotonically decreasing for $r \geq 4$ and so

$$|E(G)| \leq n - 1 + \left\lfloor \frac{n - \frac{4}{2} - 1}{\frac{4}{2} - 1} \right\rfloor = 2n - 4.$$

If r is odd, then by Theorem 4.9, $|E(G)| \leq n-1+\left\lfloor\frac{n-1}{r-1}\right\rfloor$. This is monotonically decreasing for $r \geq 3$ and so

$$|E(G)| \leq n-1+\left\lfloor\frac{n-1}{2}\right\rfloor.$$

Consider $n-1+\left\lfloor\frac{n-1}{2}\right\rfloor$. By definition of floor, $n-1+\left\lfloor\frac{n-1}{2}\right\rfloor \leq n-1+\frac{n-1}{2}=\frac{3}{2}n-\frac{3}{2}$. For all $n \geq 5$, $\frac{3}{2}n-\frac{3}{2} \leq 2n-4$ and so the bound holds for all $r \geq 3$ and $n \geq 5$. For $n=4$ and r odd, by Corollary 3.13, the only possibility for G is $G=C_3 \vee P_2$ which has $4 \leq 2\cdot 4-4=4$ edges. Therefore, the bound holds for all $n \geq 4$ and $r \geq 3$. □

Theorem 4.9 and Corollary 4.10 can be used to prove the existence of two cycles of different lengths in a planar graph.

Example 4.11. Let G be a simple, connected, finite and planar graph with 16 vertices and 29 edges. Suppose all the cycles in G are of the same length, in other words, $\text{girth}(G)=\text{circum}(G)$. Then by Corollary 4.10, $|E(G)| \leq 2\cdot 16-4=28$ which is a contradiction. Therefore, G must contain two cycles of different lengths.

Example 4.12. Let G be a simple, connected, finite and planar graph with 16 vertices, 22 edges and a cycle of length 6. Suppose all the cycles in G are of the same length, in other words, $6=\text{girth}(G)=\text{circum}(G)$. Then by Corollary 4.9, $|E(G)| \leq 16-1+\left\lfloor\frac{16-3-1}{3-1}\right\rfloor=21$ which is a contradiction. Therefore, G must contain two cycles of different lengths.

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