KOLLÁR'S PACKAGE FOR TWISTED SAITO'S S-SHEAVES

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Abstract. We generalize Kollár's conjecture (including torsion freeness, injectivity theorem, vanishing theorem and decomposition theorem) to Saito's S-sheaves twisted by a $\mathbb{Q}$-divisor. This gives a uniform treatment for various kinds of Kollár's package in different topics in complex geometry. As a consequence we prove Kollár's package of pluricanonical bundles twisted by a certain multiplier ideal sheaf. The method of the present paper is L^2-theoretic.

1. Introduction

Let $f : X \to Y$ be a proper morphism between complex spaces[1] such that Y is irreducible and each irreducible component of X is mapped onto Y. We say that a coherent sheaf $\mathscr{F}$ on X satisfies **Kollár's package** with respect to f if the following statements hold.

Torsion Freeness: $R^q f_*(\mathscr{F})$ is torsion free for every $q \geq 0$ and vanishes if $q > \dim X - \dim Y$.

Injectivity Theorem: If L is a semipositive holomorphic line bundle on X so that $L^{\otimes l}$ admits a nonzero holomorphic global section s for some $l > 0$, then the canonical morphism

$$R^q f_*(\times s) : R^q f_*(\mathscr{F} \otimes L^{\otimes k}) \to R^q f_*(\mathscr{F} \otimes L^{\otimes(k+l)})$$

is injective for every $q \geq 0$ and every $k \geq 1$.

Vanishing Theorem: If Y is a projective algebraic variety and L is an ample line bundle on Y, then

$$H^q(Y, R^p f_*(\mathscr{F}) \otimes L) = 0, \quad \forall q > 0, \forall p \geq 0.$$

Decomposition Theorem: Assume moreover that X is a compact Kähler space. Then $Rf_*(\mathscr{F})$ splits in the derived category $D(Y)$ of $\mathscr{O}_Y$-modules, i.e.

$$Rf_*(\mathscr{F}) \simeq \bigoplus_q R^q f_*(\mathscr{F})[-q] \in D(Y).$$

As a consequence, the spectral sequence

$$E_2^{pq} : H^p(Y, R^q f_*(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F})$$

degenerates at the E_2 page.

[1]All the complex spaces are assumed to be separated, reduced, paracompact, countable at infinity and of pure dimension throughout the present paper. We would like to point out that the complex spaces are allowed to be reducible.

These statements date back to J. Kollár [5, 6], who shows that the dualizing sheaf ω_X satisfies Kollár's package when X is smooth projective and Y is projective. Aiming at various geometric applications, Kollár's results have been further generalized in two directions.

The first direction is to show Kollár's package for the dualizing sheaf twisted by a $\mathbb{Q}$-divisor, or a multiplier ideal sheaf more generally. These kinds of (ad-hoc) Kollár's package have applications in E. Viehweg's works on the quasi-projective moduli of polarized manifolds [21, 22], O. Fujino's project of minimal model program for log-canonical varieties [3] and Kollár-Kovács' splitting criterion for du Bois singularities [7], etc. Besides, K. Takegoshi [20] proves Kollár's package for the dualizing sheaf twisted by a Nakano semi-positive vector bundle. The injectivity theorem for the dualizing sheaf twisted by a general multiplier ideal sheaf has been investigated by S. Matsumura [8] and Fujino-Matsumura [4]. However the complete proof of the Kollár's package (listed as above) for the dualizing sheaf twisted by a multiplier ideal sheaf is still missing.

The other direction is to generalize Kollár's package to certain Hodge theoretic objects such as variations of Hodge structure and Hodge modules. Assume that $f : X \to Y$ is a morphism between projective varieties. Let $\mathbb{V}$ be an $\mathbb{R}$-polarized variation of Hodge structure on some dense Zariski open subset $X^o \subset X_{\rm reg}$ of the regular loci $X_{\rm reg}$. M. Saito [15] constructs a coherent sheaf $S_X(\mathbb{V})$ and shows that $S_X(\mathbb{V})$ satisfies Kollár's package with respect to f. When $\mathbb{V}$ is the trivial variation of Hodge structure, $S_X(\mathbb{V}) \simeq \omega_X$. Saito's work gives an affirmative answer to Kollár's conjecture [6, §4]. Together with other deep results of Hodge modules, Kollár's package for $S_X(\mathbb{V})$ plays important roles in the series works of Popa-Schnell [12–14]. Recently the authors of the present paper [17] generalize Saito's result to the context of non-abelian Hodge theory.

The purpose of the present article is to show that Kollár's package holds for Saito's S-sheaves twisted by a multiplier ideal sheaf associated with a $\mathbb{Q}$-divisor. This gives a uniform and systematic treatment for various Kollár's package twisted by a $\mathbb{Q}$-divisor. This package contains new results even in the case of the dualizing sheaf twisted by a multiplier ideal sheaf. The main tool is the abstract Kollár's package established in [17].

1.1. **Main result.** Before stating the main results let us recall Saito's construction of $S_X(\mathbb{V})$, with two generalizations:

(1) We generalize Saito's construction to complex variations of Hodge structure. In particular we do not make assumptions on the local monodromy. This is interesting in the view of nonabelian Hodge theory because complex variations of Hodge structure are precisely the $\mathbb{C}^*$ fixed points on the moduli space of certain tame harmonic bundles ([19, Theorem 8], [9, Proposition 1.9]).
(2) We generalize Saito's construction with respect to the Deligne-Manin prolongations of the variation of Hodge structure with indices other than $(-1, 0]$. This is a combination of Saito's $S_X(\mathbb{V})$ with the multiplier ideal sheaf associated to a boundary $\mathbb{Q}$-divisor.

Let X be a complex space. Let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure (Definition 3.1) on some dense regular Zariski open subset X^o of X. Let A be an effective $\mathbb{Q}$-Cartier divisor on X. We define a coherent sheaf $S_X(\mathbb{V}, -A)$ as follows.

Log smooth case: Assume that X is smooth, $E := X\backslash X^o$ is a simple normal crossing divisor and $\mathrm{supp}(A) \subset E$. Denote by $E = \cup_{i=1}^{l} E_i$ the irreducible decomposition and denote $A = \sum_{i=1}^{l} r_i E_i$ with $r_1, \dots, r_l \in \mathbb{Q}_{\geq 0}$. Let $\boldsymbol{r} = (r_1, \dots, r_l)$. Let $\mathcal{V}_{>\boldsymbol{r}-1}$ be the Deligne-Manin prolongation with indices $> \boldsymbol{r} - 1$. It is a locally free $\mathscr{O}_X$-module extending $\mathcal{V}$ such that ∇ induces a connection with logarithmic singularities

$$\nabla : \mathcal{V}_{>\boldsymbol{r}-1} \to \mathcal{V}_{>\boldsymbol{r}-1} \otimes \Omega_X(\log E)$$

where the real part of the eigenvalues of the residue of ∇ along E_i belongs to $(r_i - 1, r_i]$ for each i. Let $j : X^o \to X$ be the open immersion. Denote $S(\mathbb{V}) := \mathcal{V}^{p_{\max}, k-p_{\max}}$ where $p_{\max} = \max\{p | \mathcal{V}^{p,k-p} \neq 0\}$. Define

$$S_X(\mathbb{V}, -A) := \omega_X \otimes (j_* S(\mathbb{V}) \cap \mathcal{V}_{>\boldsymbol{r}-1}).$$

General case: Let $\pi : \widetilde{X} \to X$ be a proper bimeromorphic morphism such that $\pi^o := \pi|_{\pi^{-1}(X^o \backslash \mathrm{supp}(A))} : \pi^{-1}(X^o\backslash \mathrm{supp}(A)) \to X^o \backslash \mathrm{supp}(A)$ is biholomorphic and the exceptional loci $E := \pi^{-1}((X\backslash X^o) \cup \mathrm{supp}(A)))$ is a simple normal crossing divisor. Then

$$S_X(\mathbb{V}, -A) \simeq \pi_* \left(S_{\widetilde{X}}(\pi^{o*}\mathbb{V}, -\pi^* A)\right). \tag{1.1}$$

When $A = \emptyset$, $S_X(\mathbb{V}, \emptyset)$ is canonically isomorphic to Saito's $S_X(\mathbb{V})$ (see [15], at least when $\mathbb{V}$ is $\mathbb{R}$-polarizable). The main result of the present article is

Theorem 1.1. (1) *$S_X(\mathbb{V}, -A)$ is a torsion free coherent sheaf on X which is independent of the choice of the desingularization $\pi : \widetilde{X} \to X$.*

(2) *Let $f : X \to Y$ be a locally Kähler proper morphism between complex spaces such that Y is irreducible and each irreducible component of X is mapped onto Y. Let L be a line bundle on X such that some multiple $mL = B + D$ where B is a semipositive line bundle and D is an effective Cartier divisor on X. Let F be an arbitrary Nakano-semipositive vector bundle on X. Then $S_X(\mathbb{V}, -\frac{1}{m}D) \otimes F \otimes L$ satisfies Kollár's package with respect to f.*

1.2. **Multiplier Grauert-Riemenschneider sheaf.** When $\mathbb{V} = \mathbb{C}_{X_{\mathrm{reg}}}$ is the trivial variation of Hodge structure, $S_X(\mathbb{C}_{X_{\mathrm{reg}}}, -A)$ is exactly the Grauert-Riemenschneider sheaf twisted by the multiplier ideal sheaf associated with A. This is called the multiplier ideals by Viehweg [21, 22] and it also appear in the Nadel vanishing theorem on complex spaces [2]. Let us recall its construction for the convenience of readers.

Log smooth case: Assume that X is smooth and $\mathrm{supp}(A)$ is a simple normal crossing divisor. Then

$$\mathscr{K}_X(-A) := \omega_X \otimes \mathscr{O}_X(-\lfloor A \rfloor).$$

General case: Let $\pi : \widetilde{X} \to X$ be a proper bimeromorphic morphism such that $\pi^o := \pi|_{\pi^{-1}(X^o \backslash \mathrm{supp}(A))} : \pi^{-1}(X^o\backslash \mathrm{supp}(A)) \to X^o \backslash \mathrm{supp}(A)$ is biholomorphic and the exceptional loci $E := \pi^{-1}((X\backslash X^o) \cup \mathrm{supp}(A)))$ is a simple normal crossing divisor. Then

$$\mathscr{K}_X(-A) := \pi_* \left(\mathscr{K}_{\widetilde{X}}(-\pi^* A)\right). \tag{1.2}$$

Certainly $\mathscr{K}_X(-A) \simeq S_X(\mathbb{C}_{X_{\mathrm{reg}}}, -A)$ and one has

$$\mathscr{K}_X(-A) \simeq \omega_X \otimes \mathscr{I}(-A)$$

when X is smooth ($\mathscr{I}(-A)$ is the multiplier ideal sheaf associated with A). In this case, by Theorem 1.1 one has the following.

Theorem 1.2. *Let $f: X \to Y$ be a locally Kähler proper morphism between complex spaces, such that Y is irreducible and each irreducible component of X is mapped onto Y. Let L be a line bundle such that some multiple $mL = B + D$ where B is a semipositive line bundle and D is an effective Cartier divisor on X. Let F be an arbitrary Nakano-semipositive vector bundle on X. Then $\mathscr{K}_X(-\frac{1}{m}D) \otimes F \otimes L$ satisfies Kollár's package with respect to f.*

Theorem 1.2 has an application to Kollár's package of pluricanonical bundles.

Corollary 1.3. *Let $f: X \to Y$ be a morphism from a compact Kähler manifold to an analytic space. Assume that $\omega_X^{\otimes km} \simeq A \otimes \mathscr{O}_X(D)$, $k, m > 0$ where A is a semipositive line bundle (e.g. a semiample line bundle) and D is an effective Cartier divisor. Let F be an arbitrary Nakano-semipositive vector bundle on X. Then $\mathscr{K}_X(-\frac{1}{m}D) \otimes \omega_X^{\otimes k} \otimes F$ satisfies Kollár's package with respect to f. In particular if ω_X is semipositive, then $\omega_X^{\otimes k} \otimes F$ satisfies Kollár's package with respect to f for every $k \geq 1$.*

2. Abstract Kollár's package

In this section we recall the abstract Kollár's package established in [17].

Let X be a complex space of dimension n and $X^o \subset X_{\mathrm{reg}}$ a dense Zariski open subset. Let (E, h) be a hermitian vector bundle on X^o. Define the $\mathscr{O}_X$-module $S_X(E, h)$ as follows. Let $U \subset X$ be an open subset. $S_X(E, h)(U)$ is the space of holomorphic E-valued $(n, 0)$-forms α on $U \cap X^o$ such that for every point $x \in U$, there is a neighborhood V_x of x so that

$$\int_{V_x \cap X^o} \alpha \wedge \overline{\alpha} < \infty.$$

Lemma 2.1. *(Functoriality,*[17, Proposition 2.5]*) Let $\pi: X' \to X$ be a proper holomorphic map between complex spaces which is biholomorphic over X^o. Then*

$$\pi_* S_{X'}(\pi^* E, \pi^* h) = S_X(E, h).$$

Lemma 2.2. *(*[17, Lemma 2.6]*) Let (F, h_F) be a hermitian vector bundle on X (in particular h_F is smooth on X). Then*

$$S_X(E, h) \otimes F \simeq S_X(E \otimes F, h \otimes h_F).$$

Definition 2.3. (E, h) is tame on X if, for every point $x \in X$, there is an open neighborhood U containing x, a proper bimeromorphic morphism $\pi: \widetilde{U} \to U$ which is biholomorphic over $U \cap X^o$, and a hermitian vector bundle (Q, h_Q) on $\widetilde{U}$ such that

(1) $\pi^* E|_{\pi^{-1}(X^o \cap U)} \subset Q|_{\pi^{-1}(X^o \cap U)}$ as a subsheaf.
(2) There is a hermitian metric h'_Q on $Q|_{\pi^{-1}(X^o \cap U)}$ so that $h'_Q|_{\pi^* E} \sim \pi^* h$ on $\pi^{-1}(X^o \cap U)$ and

$$(\sum_{i=1}^{r} \|\pi^* f_i\|^2)^c h_Q \lesssim h'_Q \tag{2.1}$$

for some $c \in \mathbb{R}$. Here $\{f_1, \ldots, f_r\}$ is an arbitrary set of local generators of the ideal sheaf defining $\widetilde{U} \backslash \pi^{-1}(X^o) \subset \widetilde{U}$.

The tameness condition (2.1) is independent of the choice of the set of local generators. In the present paper, a tame hermitian vector bundle (E, h) is constructed as a subsheaf of the underlying holomorphic bundle of a variation of Hodge structure. This is a special case of tame harmonic bundles in the sense of Simpson [19] and Mochizuki [10, 11]. In this case, Condition (2.1) comes from the theory of degeneration of variation of Hodge structure [1].

Theorem 2.4. *(*[17, Proposition 2.9 and §4]*) Let $f : X \to Y$ be a proper locally Kähler morphism from a complex space X to an irreducible complex space Y. Assume that every irreducible component of X is mapped onto Y, $X^o \subset X_{\mathrm{reg}}$ is a dense Zariski open subset and (E, h) is a hermitian vector bundle on X^o with Nakano semipositive curvature. Assume that (E, h) is tame on X. Then $S_X(E, h)$ is a coherent sheaf which satisfies Kollár's package with respect to $f : X \to Y$.*

3. Twisted Saito's S-sheaf and its Kollár package

3.1. Complex variation of Hodge structure.

Definition 3.1. [18, §8] Let X^o be a complex manifold. Denote by $\mathscr{A}^0_{X^o}$ the sheaf of C^∞ functions on X^o. A polarized complex variation of Hodge structure on X^o of weight k is a flat holomorphic connection $(\mathcal{V}, \nabla)$ on X^o together with a decomposition $\mathcal{V} \otimes_{\mathscr{O}_{X^o}} \mathscr{A}^0_{X^o} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}$ of C^∞ bundles and a flat hermitian form Q on $\mathcal{V}$ such that

1. The hermitian form h_Q which equals $(-1)^p Q$ on $\mathcal{V}^{p,q}$ is a hermitian metric on the C^∞ complex vector bundle $\mathcal{V} \otimes_{\mathscr{O}_{X^o}} \mathscr{A}^0_{X^o}$.
2. The decomposition $\mathcal{V} \otimes_{\mathscr{O}_{X^o}} \mathscr{A}^0_{X^o} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}$ is orthogonal with respect to h_Q.
3. The Griffiths transversality condition

$$\nabla(\mathcal{V}^{p,q}) \subset \mathscr{A}^{0,1}(\mathcal{V}^{p+1,q-1}) \oplus \mathscr{A}^{1,0}(\mathcal{V}^{p,q}) \oplus \mathscr{A}^{0,1}(\mathcal{V}^{p,q}) \oplus \mathscr{A}^{1,0}(\mathcal{V}^{p-1,q+1}) \tag{3.1}$$

holds for every p and q. Here $\mathscr{A}^{i,j}(\mathcal{V}^{p,q})$ denotes the sheaf of smooth (i, j)-forms with values in $\mathcal{V}^{p,q}$.

Denote $S(\mathbb{V}) := \mathcal{V}^{p_{\max}, k-p_{\max}}$ where $p_{\max} = \max\{p | \mathcal{V}^{p,k-p} \neq 0\}$.

Let X be a complex manifold and $\cup_{i=1}^l E_i = E := X \backslash X^o \subset X$ a simple normal crossing divisor where $E_1, \dots, E_l$ are irreducible components. Let $(\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure on $X^o := X \backslash E$. There is a system of prolongations of $\mathcal{V}$. Let $\boldsymbol{a} = (a_1, \dots, a_l) \in \mathbb{R}^l$. Let $\mathcal{V}_{>\boldsymbol{a}}$ be the Deligne-Manin prolongation with indices $> \boldsymbol{a}$. It is a locally free $\mathscr{O}_X$-module extending $\mathcal{V}$ such that ∇ induces a connection with logarithmic singularities

$$\nabla : \mathcal{V}_{>\boldsymbol{a}} \to \mathcal{V}_{>\boldsymbol{a}} \otimes \Omega_X(\log E)$$

whose real part of the eigenvalues of the residue of ∇ along E_i belongs to $(a_i, a_i + 1]$. Denote

$$R_X(\mathbb{V}) := \mathcal{V}_{>-\mathbf{1}} \cap j_*(S(\mathbb{V}))$$

where $j : X^o \to X$ is the open immersion and $-\mathbf{1} = (-1, \dots, -1)$. By the nilpotent orbit theorem [1] $R_X(\mathbb{V})$ is a subbundle of $\mathcal{V}_{>-\mathbf{1}}$, i.e. both $R_X(\mathbb{V})$ and $\mathcal{V}_{>-\mathbf{1}}/R_X(\mathbb{V})$ are locally free.

3.2. **L^2-adapted local frame on $R_X(\mathbb{V})$.** Let $\mathbb{V}=(\mathcal{V},\nabla,\{\mathcal{V}^{p,q}\},Q)$ be a polarized complex variation of Hodge structure over $(\Delta^*)^n\times\Delta^m$. Denote by h_Q the associated Hodge metric. Let $s_1,\dots,s_n$ be holomorphic coordinates of $(\Delta^*)^n$ and denote $D_i:=\{s_i=0\}\subset\Delta^{n+m}$. Let N_i be the unipotent part of $\mathrm{Res}_{D_i}\nabla$ and let

$$p:\mathbb{H}^n\times\Delta^m\to(\Delta^*)^n\times\Delta^m,$$
$$(z_1,\dots,z_n,w_1,\dots,w_m)\mapsto(e^{2\pi\sqrt{-1}z_1},\dots,e^{2\pi\sqrt{-1}z_n},w_1,\dots,w_m)$$

be the universal covering. Let $W^{(1)}=W(N_1),\dots,W^{(n)}=W(N_1+\cdots+N_n)$ be the monodromy weight filtrations (centered at 0) on $V:=\Gamma(\mathbb{H}^n\times\Delta^m,p^*\mathcal{V})^{p^*\nabla}$. The following norm estimate for flat sections is proved by Cattani-Kaplan-Schmid [1, Theorem 5.21] for the case when $\mathbb{V}$ has quasi-unipotent local monodromy and by Mochizuki [10, Part 3, Chapter 13] for the general case.

Theorem 3.2. *For any $0\neq v\in\mathrm{Gr}_{l_n}^{W^{(n)}}\cdots\mathrm{Gr}_{l_1}^{W^{(1)}}V$, one has*

$$|v|^2_{h_Q}\sim\left(\frac{\log|s_1|}{\log|s_2|}\right)^{l_1}\cdots(-\log|s_n|)^{l_n}$$

over any region of the form

$$\left\{(s_1,\dots s_n,w_1,\dots,w_m)\in(\Delta^*)^n\times\Delta^m\middle|\frac{\log|s_1|}{\log|s_2|}>\epsilon,\dots,-\log|s_n|>\epsilon,(w_1,\dots,w_m)\in K\right\}$$

for any $\epsilon>0$ and an arbitrary compact subset $K\subset\Delta^m$.

The rest of this part is devoted to the norm estimate of the singular hermitian metric h_Q on $R_X(\mathbb{V})$.

Lemma 3.3. *Assume that $n=1$. Then $W_{-1}(N_1)\cap R_X(\mathbb{V})_{\mathbf{0}}=0$.*

Proof. Assume that $W_{-1}(N_1)\cap R_X(\mathbb{V})_{\mathbf{0}}\neq0$. Let k be the weight of $\mathbb{V}$. Let $l=\max\{l|W_{-l}(N_1)\cap R_X(\mathbb{V})_{\mathbf{0}}\neq0\}$. Then $l\geq1$. By [16, 6.16], the decomposition $\mathcal{V}_{>-\mathbf{1}}\simeq\bigoplus_{p+q=k}j_*\mathcal{V}^{p,q}\cap\mathcal{V}_{>-\mathbf{1}}$ induces a pure Hodge structure of weight $m+k$ on $W_m(N_1)/W_{m-1}(N_1)$. Moreover

$$N^l:W_l(N_1)/W_{l-1}(N_1)\to W_{-l}(N_1)/W_{-l-1}(N_1)\tag{3.2}$$

is an isomorphism of type $(-l,-l)$. Denote $S(\mathbb{V})=\mathcal{V}^{p,k-p}$. By the definition of l, any nonzero element $\alpha\in W_{-l}(N_1)\cap R_X(\mathbb{V})_{\mathbf{0}}$ induces a nonzero $[\alpha]\in W_{-l}(N_1)/W_{-l-1}(N_1)$ of Hodge type $(p,k-l-p)$. Since (3.2) is an isomorphism, there is $\beta\in W_l(N_1)/W_{l-1}(N_1)$ of Hodge type $(p+l,k-p)$ such that $N^l(\beta)=[\alpha]$. However, $\beta=0$ since $\mathcal{F}^{p+l}=0$. This contradicts to the fact that $[\alpha]\neq0$. $W_{-1}(N_1)\cap R_X(\mathbb{V})_{\mathbf{0}}$ therefore has to be zero. □

Denote by T_i the local monodromy operator of $\mathbb{V}$ around D_i. Since $T_1,\dots,T_n$ are pairwise commutative, there is a finite decomposition

$$\mathcal{V}_{>-\mathbf{1}}|_{\mathbf{0}}=\bigoplus_{-1<\alpha_1,\dots,\alpha_n\leq0}\mathbb{V}_{\alpha_1,\dots,\alpha_n}$$

such that $(T_i-e^{2\pi\sqrt{-1}\alpha_i}\mathrm{Id})$ is unipotent on $\mathbb{V}_{\alpha_1,\dots,\alpha_n}$ for each $i=1,\dots,n$.

Let

$$v_1,\dots,v_N\in R_X(\mathbb{V})|_{\mathbf{0}}\cap\bigcup_{-1<\alpha_1,\dots,\alpha_n\leq0}\mathbb{V}_{\alpha_1,\dots,\alpha_n}$$

be an orthogonal basis of $R_X(\mathbb{V})|_{\mathbf{0}} \simeq \Gamma(\mathbb{H}^n, p^*S(\mathbb{V}))^{p^*\nabla}$. Then $\widetilde{v_1}, \dots, \widetilde{v_N}$ that are determined by

$$\widetilde{v_j} := \exp\left(\sum_{i=1}^{n} \log z_i(\alpha_i \mathrm{Id} + N_i)\right) v_j \text{ if } v_j \in \mathbb{V}_{\alpha_1,\dots,\alpha_n}, \quad \forall j = 1, \dots, N \tag{3.3}$$

form a frame of $\mathcal{V}_{>-\mathbf{1}} \cap j_*S(\mathbb{V})$. To be precise, we always use the notation $\alpha_{E_i}(\widetilde{v_j})$ instead of α_i in (3.3). By (3.3) we acquire that

$$\begin{aligned} |\widetilde{v_j}|^2_{h_Q} &\sim \left| \prod_{i=1}^{n} z_i^{\alpha_{E_i}(\widetilde{v_j})} \exp\left(\sum_{i=1}^{n} N_i \log z_i\right) v_j \right|^2_{h_Q} \\ &\sim |v_j|^2_{h_Q} \prod_{i=1}^{n} |z_i|^{2\alpha_{E_i}(\widetilde{v_j})}, \quad j = 1, \dots, N \end{aligned}$$

where $\alpha_{E_i}(\widetilde{v_j}) \in (-1, 0]$, $\forall i = 1, \dots n$.

By Theorem 3.2 and Lemma 3.3 one has

$$|v_j|^2_{h_Q} \sim \left(\frac{\log|s_1|}{\log|s_2|}\right)^{l_1} \cdots (-\log|s_n|)^{l_n}, \quad l_1 \le l_2 \le \cdots \le l_{n-1},$$

over any region of the form

$$\left\{(s_1, \dots s_n, w_1, \dots, w_m) \in (\Delta^*)^n \times \Delta^m \middle| \frac{\log|s_1|}{\log|s_2|} > \epsilon, \dots, -\log|s_n| > \epsilon, (w_1, \dots, w_m) \in K\right\}$$

for any $\epsilon > 0$ and an arbitrary compact subset $K \subset \Delta^m$. Therefore we obtain that

$$1 \lesssim |v_j| \lesssim |z_1 \cdots z_r|^{-\epsilon}, \quad \forall \epsilon > 0.$$

The local frame $(\widetilde{v_1}, \dots, \widetilde{v_N})$ is L^2-adapted in the sense of S. Zucker [23, page 433].

Definition 3.4. Let (E, h) be a vector bundle with a possibly singular hermitian metric h on a hermitian manifold (X, ds_0^2). A holomorphic local frame $(v_1, \dots, v_N)$ of E is called L^2-adapted if, for every set of measurable functions $\{f_1, \dots, f_N\}$, $\sum_{i=1}^{N} f_i v_i$ is locally square integrable if and only if $f_i v_i$ is locally square integrable for each $i = 1, \dots, N$.

To see that $(\widetilde{v_1}, \dots, \widetilde{v_N})$ is L^2-adapted, let us consider the measurable functions $f_1, \dots, f_N$. If

$$\sum_{j=1}^{N} f_j \widetilde{v_j} = \exp\left(\sum_{i=1}^{n} N_i \log z_i\right) \left(\sum_{j=1}^{N} f_j \prod_{i=1}^{n} |z_i|^{\alpha_{E_i}(\widetilde{v_j})} v_j\right)$$

is locally square integrable, then

$$\sum_{j=1}^{N} f_j \prod_{i=1}^{n} |z_i|^{\alpha_{E_i}(\widetilde{v_j})} v_j$$

is locally square integrable because the entries of the matrix $\exp\left(-\sum_{i=1}^{n} N_i \log z_i\right)$ are L^∞-bounded. Since $(v_1, \dots, v_N)$ is an orthogonal basis, $|f_j \widetilde{v_j}|_{h_Q} \sim \prod_{i=1}^{n} |z_i|^{\alpha_{E_i}(\widetilde{v_j})} |f_j v_j|_{h_Q}$ is locally square integrable for each $j = 1, \dots, N$.

In conclusion, we obtain the following proposition.

Proposition 3.5. *Let (X, ds_0^2) be a hermitian manifold and E a normal crossing divisor on X. Let $\mathbb{V}$ be a polarized complex variation of Hodge structure on $X^o := X\backslash E$. Then there is an L^2-adapted holomorphic local frame $(\widetilde{v_1}, \dots, \widetilde{v_N})$ of $R_X(\mathbb{V})$ at every point $x \in E$. There are moreover $\alpha_{E_i}(\widetilde{v_j}) \in (-1, 0]$, $i = 1, \dots, r$, $j = 1, \dots, N$ and positive real functions $\lambda_j \in C^\infty(X\backslash E)$, $j = 1, \dots, N$ such that*

$$|\widetilde{v_j}|^2 \sim \lambda_j \prod_{i=1}^{r} |z_i|^{2\alpha_{E_i}(\widetilde{v_j})}, \quad \forall j = 1, \dots, N \tag{3.4}$$

and

$$1 \lesssim \lambda_j \lesssim |z_1 \cdots z_r|^{-\epsilon}, \quad \forall \epsilon > 0$$

for each $j = 1, \dots, N$. Here $z_1, \cdots, z_n$ are holomorphic local coordinates on X so that $E_i = \{z_i = 0\}$, $i = 1, \cdots r$ and $E = \{z_1 \cdots z_r = 0\}$.

3.3. **Twisted Saito's S-sheaf.** Let X be a complex space and $X^o \subset X_{\rm reg}$ a dense Zariski open subset. Let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure on X^o. Let A be an effective $\mathbb{Q}$-Cartier divisor on X. We define a coherent sheaf $S_X(\mathbb{V}, -A)$ as follows.

Log smooth case: Assume that X is smooth, $E := X\backslash X^o$ is a simple normal crossing divisor and $\mathrm{supp}(A) \subset E$. Denote by $E = \cup_{i=1}^{l} E_i$ the irreducible decomposition and denote $A = \sum_{i=1}^{l} r_i E_i$ with $r_1, \dots, r_l \in \mathbb{Q}_{\geq 0}$. Let $\boldsymbol{r} = (r_1, \dots, r_l)$. Let $\mathcal{V}_{>\boldsymbol{r}-1}$ be the Deligne-Manin prolongation with indices $> \boldsymbol{r} - 1$. It is a locally free $\mathscr{O}_X$-module extending $\mathcal{V}$ such that ∇ induces a connection with logarithmic singularities

$$\nabla : \mathcal{V}_{>\boldsymbol{r}-1} \to \mathcal{V}_{>\boldsymbol{r}-1} \otimes \Omega_X(\log E)$$

where the real part of the eigenvalues of the residue of ∇ along E_i belongs to $(r_i - 1, r_i]$ for each i. Let $j : X^o \to X$ be the open immersion. Denote $S(\mathbb{V}) := \mathcal{V}^{p_{\max}, k-p_{\max}}$ where $p_{\max} = \max\{p | \mathcal{V}^{p,k-p} \neq 0\}$. Define

$$S_X(\mathbb{V}, -A) := \omega_X \otimes (j_* S(\mathbb{V}) \cap \mathcal{V}_{>\boldsymbol{r}-1}).$$

General case: Let $\pi : \widetilde{X} \to X$ be a proper bimeromorphic morphism such that $\pi^o := \pi|_{\pi^{-1}(X^o \backslash \mathrm{supp}(A))} : \pi^{-1}(X^o \backslash \mathrm{supp}(A)) \to X^o \backslash \mathrm{supp}(A)$ is biholomorphic and the exceptional loci $E := \pi^{-1}((X\backslash X^o) \cup \mathrm{supp}(A)))$ is a simple normal crossing divisor. Then

$$S_X(\mathbb{V}, -A) \simeq \pi_* \left(S_{\widetilde{X}}(\pi^{o*}\mathbb{V}, -\pi^* A) \right). \tag{3.5}$$

Let L be a line bundle such that some multiple $mL = B + D$ where B is a semipositive line bundle and D is an effective Cartier divisor on X. Let h_B be a hermitian metric on B with semipositive curvature and h_D the unique singular hermitian metric on $\mathscr{O}_X(D)$ determined by the effective divisor D. h_D is a singular hermitian metric, smooth over $X\backslash D$, defined as follows. Let $s \in H^0(X, \mathscr{O}_X(D))$ be the defining section of D and let h_0 be an arbitrary smooth hermitian metric on $\mathscr{O}_X(D)$. Then h_D is defined by $|\xi|_{h_D} = |\xi|_{h_0}/|s|_{h_0}$ which is independent of the choice of h_0. Denote $h_L := (h_B h_D)^{\frac{1}{m}}$. The main result of this section is

Theorem 3.6. *$S_X(\mathbb{V}, -\frac{1}{m}D) \otimes L \simeq S_X(S(\mathbb{V}) \otimes L, h_Q h_L)$. In particular $S_X(\mathbb{V}, -\frac{1}{m}D)$ is independent of the choice of the desingularization $\pi : \widetilde{X} \to X$.*

Proof. By Lemma 2.1, the proof can be reduced to the log smooth case, that is, X is smooth, $E := X\backslash X^o$ is a simple normal crossing divisor and $\mathrm{supp}(D) \subset E$. Denote $j : X^o := X\backslash E \to X$ to be the inclusion. We are going to show that

$$S_X(\mathbb{V}, -\frac{1}{m}D) \otimes L = S_X(S(\mathbb{V}) \otimes L, h_Q h_L)$$

as subsheaves of $\omega_X \otimes j_*(S(\mathbb{V})) \otimes L$. Since the problem is local, we assume that $X = \Delta^n$ is the polydisc. Denote $E := \{z_1 \cdots z_l = 0\}$ where $E_i := \{z_i = 0\}$ for each $i = 1, \ldots, l$. Let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, h_Q)$ be a polarized complex variation of Hodge structure on X^o. Let $\mathbf{0} = (0, \ldots, 0) \in X$ and let $(\widetilde{v_1}, \ldots, \widetilde{v_N})$ be an L^2-adapted local frame of $R_X(\mathbb{V})$ at $\mathbf{0}$ as in Proposition 3.5. Let $f_1, \ldots, f_N \in (j_*\mathscr{O}_{X^o})_{\mathbf{0}}$ and let e be the local frame of L at $\mathbf{0}$. We are going to prove that

$$\sum_{i=1}^{N} f_i[\widetilde{v_i} dz_1 \wedge \cdots \wedge dz_n \otimes e]_{\mathbf{0}} \in S_X(S(\mathbb{V}) \otimes L, h_Q h_L)_{\mathbf{0}}$$

if and only if

$$f_i \in \mathscr{O}_X\big(-\sum_{j=1}^{l} \lfloor \frac{r_i}{m} - \alpha_{E_j}(\widetilde{v_i}) \rfloor E_j\big)_{\mathbf{0}}$$

for every $i = 1, \ldots, N$.

Denote $ds_0^2 = \sum_{i=1}^n dz_i d\bar{z}_i$. Since $(\widetilde{v_1}, \ldots, \widetilde{v_N})$ is an L^2-adapted frame as in Proposition 3.5, the integral

$$\int |\sum_{i=1}^{N} f_i \widetilde{v_i} dz_1 \wedge \cdots \wedge dz_n|^2 |e|^2_{h_L} \mathrm{vol}_{ds_0^2} = \int |\sum_{i=1}^{N} f_i \widetilde{v_i}|^2 |e|^2_{h_L} \mathrm{vol}_{ds_0^2}$$

is finite near $\mathbf{0}$ if and only if

$$\int |f_i \widetilde{v_i}|^2 |e|^2_{h_L} \mathrm{vol}_{ds_0^2} \sim \int |f_i|^2 \prod_{j=1}^{r} |z_j|^{2\alpha_{E_j}(\widetilde{v_i}) - \frac{2r_i}{m}} \lambda_i \mathrm{vol}_{ds_0^2} \tag{3.6}$$

is finite near $\mathbf{0}$ for every $i = 1, \ldots, N$. Here λ_i is a positive real function so that

$$1 \lesssim \lambda_i \lesssim |z_1 \cdots z_r|^{-\epsilon}, \quad \forall \epsilon > 0. \tag{3.7}$$

Denote

$$v_j(f) := \min\{l | f_l \neq 0 \text{ in the Laurant expansion } f = \sum_{i \in \mathbb{Z}} f_i z_j^i\}.$$

By Lemma 3.7, the local integrability of (3.6) is equivalent to that

$$v_j(f_i) + \alpha_{E_i}(\widetilde{v_j}) - \frac{r_i}{m} > -1, \quad \forall j = 1, \ldots, l. \tag{3.8}$$

This is equivalent to

$$v_j(f_i) \geq \lfloor -\alpha_{E_i}(\widetilde{v_j}) + \frac{r_i}{m} \rfloor, \quad \forall j = 1, \ldots, l. \tag{3.9}$$

As a consequence, $S_X(S(\mathbb{V}) \otimes L, h_Q h_L)_{\mathbf{0}}$ is generated by

$$dz_1 \wedge \cdots \wedge dz_n \otimes e \otimes \exp\left(\sum_{i=1}^{n} \log z_i (\lfloor -\alpha_{E_i}(\widetilde{v_j}) + \frac{r_i}{m} \rfloor \mathrm{Id} + N_i) \right) \widetilde{v_j}, \quad \forall j = 1, \ldots, N.$$

These are exactly the generators of $\omega_X \otimes L \otimes (j_*(S(\mathbb{V})) \cap \mathcal{V}_{>\frac{\mathbf{r}}{m}-1})$ at $\mathbf{0}$. The proof is finished. □

The proof of the following lemma is omitted.

Lemma 3.7. *Let f be a holomorphic function on $\Delta^* := \{z \in \mathbb{C} | 0 < |z| < 1\}$ and $a \in \mathbb{R}$. Then*

$$\int_{|z|<\frac{1}{2}} |f|^2 |z|^{2a} dz d\bar{z} < \infty$$

if and only if $v(f) + a > -1$. Here

$$v(f) := \min\{l | f_l \neq 0 \text{ in the Laurant expansion } f = \sum_{i \in \mathbb{Z}} f_i z^i\}.$$

3.4. **Kollár package.** In this section we prove the main theorem (Theorem 1.1) of the present paper. Let X be a complex space and $X^o \subset X_{\mathrm{reg}}$ a dense Zariski open subset. Let $\mathbb{V} := (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure of weight k on X^o. Let

$$\nabla = \overline{\theta} + \partial + \bar{\partial} + \theta$$

be the decomposition according to (3.1). For the reason of degrees, $S(\mathbb{V})$ is a holomorphic subbundle of $\mathcal{V}$ and $\overline{\theta}(S(\mathbb{V})) = 0$.

Lemma 3.8. *$(S(\mathbb{V}), h_Q)$ is a Nakano semipositive vector bundle which is tame on X.*

Proof. To see that $(S(\mathbb{V}), h_Q)$ is Nakano semipositive, we take the decomposition

$$\nabla = \overline{\theta} + \partial + \bar{\partial} + \theta$$

according to (3.1). Since $\overline{\theta}(S(\mathbb{V})) = 0$, it follows from Griffiths' curvature formula

$$\Theta_h(S(\mathbb{V})) + \theta \wedge \overline{\theta} + \overline{\theta} \wedge \theta = 0$$

that

$$\sqrt{-1}\Theta_h(S(\mathbb{V})) = -\sqrt{-1}\overline{\theta} \wedge \theta \geq_{\mathrm{Nak}} 0.$$

To prove the tameness we use Deligne's extension. Since the problem is local, we assume that there is a desingularization $\pi : \widetilde{X} \to X$ such that π is biholomorphic over X^o and $D := \pi^{-1}(X \backslash X^o)$ is a simple normal crossing divisor. By abuse of notations we identify X^o and $\pi^{-1}(X^o)$. There is an inclusion $S(\mathbb{V}) \subset \mathcal{V}_{>-\mathbf{1}}|_{X^o}$. Let $z_1, \dots, z_n$ be holomorphic local coordinates such that $D_i = \{z_i = 0\}$, $i = 1, \dots, k$ and $D = \{z_1 \cdots z_k = 0\}$. By Theorem 3.2, one has the norm estimate

$$|z_1 \cdots z_k| |s|_{h_0} \lesssim |s|_h \tag{3.10}$$

for any holomorphic local section s of $\mathcal{V}_{>-\mathbf{1}}$. Here h_0 is an arbitrary (smooth) hermitian metric on $\mathcal{V}_{>-\mathbf{1}}$. This shows that $(S(\mathbb{V}), h_Q)$ is tame. □

Theorem 3.9. *Let X be a complex space and $X^o \subset X_{\mathrm{reg}}$ a dense Zariski open subset. Let $\mathbb{V} := (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure of weight k on X^o. Let L be a line bundle such that some multiple $mL = A + D$ where A is a semipositive line bundle and D is an effective Cartier divisor on X. Let F be an arbitrary Nakano-semipositive vector bundle on X. Then $S_X(\mathbb{V}, -\frac{1}{m}D) \otimes F \otimes L$ satisfies Kollár's package with respect to any locally Kähler proper morphism $f : X \to Y$ such that Y is irreducible and each irreducible component of X is mapped onto Y.*

Proof. Let h_A be a hermitian metric on A with semipositive curvature and h_D the singular hermitian metric on $\mathscr{O}_X(D)$ determined by the effective divisor D. Denote $h_L := (h_A h_D)^{\frac{1}{m}}$. Then

$$\sqrt{-1}\Theta_{h_L}(L)|_{X\backslash D} = \frac{\sqrt{-1}}{m}\Theta_{h_A}(A)|_{X\backslash D} \geq 0.$$

Hence $(L|_U, h_L|_U)$ has semipositive curvature and is tame on X. Therefore by Lemma 3.8 $(S(\mathbb{V})\otimes L\otimes F|_U, h_Q h_L h_F|_U)$ has semipositive curvature on $U = X^o\backslash\mathrm{supp}(D)$ and is tame on X. By Lemma 2.2, Theorem 2.4 and Theorem 3.6 we obtain that $S_X(\mathbb{V}, -\frac{1}{m}D)\otimes F\otimes L \simeq S_X(S(\mathbb{V})\otimes L\otimes F|_U, h_Q h_L h_F|_U)$ satisfies Kollár's package with respect to $f : X \to Y$. □

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