COGRAPHS AND 1-SUMS

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Abstract. A graph that can be generated from K_1 using joins and 0-sums is called a cograph. We define a sesquicograph to be a graph that can be generated from K_1 using joins, 0-sums, and 1-sums. We show that, like cographs, sesquicographs are closed under induced minors. Cographs are precisely the graphs that do not have the 4-vertex path as an induced subgraph. We obtain an analogue of this result for sesquicographs, that is, we find those non-sesquicographs for which every proper induced subgraph is a sesquicograph.

1. Introduction

In this paper, we only consider finite and simple graphs. The notation and terminology follows [3] except where otherwise indicated. For graphs G and H having disjoint vertex sets, the **0-sum** $G \oplus H$ of G and H is their disjoint union. A **1-sum** $G \oplus_1 H$ of G and H is obtained by identifying a vertex of G with a vertex of H. The **join** $G \bigtriangledown H$ of two disjoint graphs G and H is obtained from the 0-sum of G and H by joining every vertex of G to every vertex of H. A **cograph** is a graph that can be generated from the single- vertex graph K_1 using the operations of join and 0-sum. We define a graph to be a **sesquicograph** if it can be generated from K_1 using the operations of join, 0-sum, and 1-sum. The class of cographs has been extensively studied over the last fifty years (see, for example, [2, 4, 9]). Due to the following characterization, cographs are also called P_4-free graphs [1].

Theorem 1.1. *A graph G is a cograph if and only if G does not contain the path P_4 on four vertices as an induced subgraph.*

Since we consider only simple graphs in this paper, when we write G/e for an edge e of a graph G, we mean the simple graph obtained from the multigraph that results from contracting the edge e by deleting all but one edge from each class of parallel edges. An **induced minor** of a graph G is a graph H that can be obtained from G by a sequence of operations each consisting of a vertex deletion or an edge contraction. In Section 2, we show that every induced minor of a sesquicograph is a sesquicograph. In addition, we provide an alternative definition of a sesquicograph in terms of the vertex connectivities of its induced subgraphs and their complements. The graph obtained from a 6-cycle by adding a chord to create two 4-cycles is called

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the **domino** graph. We let C_6^+ denote the domino; $\overline{P_5}$ is the complement of a 5-vertex path. The next theorem is the main result of the paper.

Theorem 1.2. *A graph G is a sesquicograph if and only if G does not contain any of the following graphs as an induced subgraph:*

(i) cycles of length exceeding four, and

(ii) $\overline{P_5}, C_6^+, H_1, H_2, H_3, H_4$, and H_5,

where the graphs in (ii) are shown in Figure 1.

Its proof occupies most of Section 3. As a consequence of Theorem 1.2, we have the following characterization of sesquicographs in terms of forbidden induced minors.

Corollary 1.3. *A graph G is a sesquicograph if and only if G has no induced minor isomorphic to a graph in $\{C_5, \overline{P_5}, H_1, H_2, H_3, H_4, H_5\}$, where C_5 is the cycle of length five.*

A graph G is a **2-cograph** if it can be generated from K_1 using the operations of complementation, 0-sum, and 1-sum. The class of 2-cographs has been studied in [7]. This paper has some similarities with [7] although the arguments for sesquicographs are not as complex as they are for 2-cographs. Since the class of sesquicographs is the smallest class of graphs that contains K_1 and is closed under the operations of join, 0-sum, and 1-sum, it is a proper subclass of 2-cographs and, thus, of the class of perfect graphs. Note the path P_5 on five vertices is a sesquicograph but its complement $\overline{P_5}$ is not. It follows that the class of sesquicographs is not closed under complementation unlike the classes of cographs and 2-cographs.

2. Preliminaries

Let G be a graph. A vertex u of G is a **neighbour** of a vertex v of G if uv is an edge of G. The **neighbourhood** $N_G(v)$ of v in G is the set of all neighbours of v in G. If G is connected, a **t-cut** of G is set X_t of vertices of G such that $|X_t| = t$ and $G - X_t$ is disconnected. A graph that has no t-cuts for all t less than k is **k-connected**. Viewing G as a subgraph of K_n where $n = |V(G)|$, we colour the edges of G green while assigning the colour red to the non-edges of G. Similar to the terminology in [7], we use the terms **green graph** and **red graph** for G and its complementary graph $\overline{G}$, respectively. An edge of G is called a **green edge** while a **red edge** refers to an edge of $\overline{G}$. The **green degree** of a vertex v of G is the number of **green neighbours** of v, while the **red degree** of v is its number of **red neighbours**.

We omit the straightforward proofs of the next three results.

Lemma 2.1. *All graphs having at most four vertices are sesquicographs.*

Lemma 2.2. *A graph G is a join of two graphs if and only if its complement $\overline{G}$ is disconnected.*

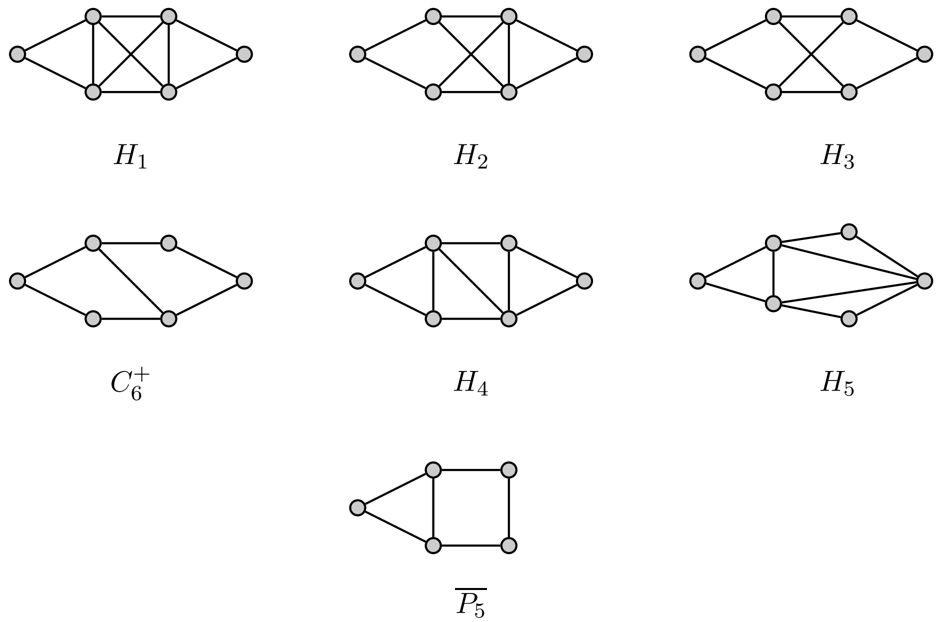

FIGURE 1. The induced-subgraph-minimal non-sesquicographs.

Lemma 2.3. *Let G be a graph and let uv be an edge e of G. Then $\overline{G/e}$ is the graph obtained by adding a vertex w with neighbourhood $N_{\overline{G}}(u) \cap N_{\overline{G}}(v)$ to the graph $\overline{G} - \{u, v\}$.*

Lemma 2.4. *Every induced subgraph of a sesquicograph is a sesquicograph.*

Proof. Let G be a sesquicograph. It is enough to show that, for every vertex v of G, the graph $G - v$ is a sesquicograph. Note that if $|V(G)| \le 5$, then our result follows by Lemma 2.1. Let $|V(G)| = n$. We proceed via induction on $|V(G)|$ and assume that the result is true for all sesquicographs with order less than n. Since G is a sesquicograph, G is a 0-sum, a 1-sum, or a join of proper induced subgraphs X and Y of G. Observe that if G is $X \oplus Y$ or $X \bigtriangledown Y$, then $G - v$ equals $(X - v) \oplus Y$ or $(X - v) \bigtriangledown Y$, and so the result follows by induction. Therefore we may assume that $G = X \oplus_1 Y$. Note that, in this case, $G - v$ is either $(X - v) \oplus (Y - v)$ or $(X - v) \oplus_1 Y$. Thus our result follows by induction. □

A graph is **trivial** if it contains only one vertex and no edge. Cographs can also be characterized as the graphs in which every non-trivial connected induced subgraph has a disconnected complement. Similarly, a graph G is a 2-cograph if G has no non-trivial induced subgraph H such that both H and $\overline{H}$ are 2-connected. Next we show that sesquicographs can be characterized in a similar way.

Proposition 2.5. *A graph G is a sesquicograph if and only if, for every non-trivial induced subgraph H of G, the graph H is not 2-connected or $\overline{H}$ is disconnected.*

Proof. Let G be a sesquicograph and let H be a non-trivial induced subgraph of G. By Lemma 2.4, H is a sesquicograph. Since H can be decomposed as a 0-sum, a 1-sum, or a join, it follows by Lemma 2.2, that H is not 2-connected or $\overline{H}$ is disconnected.

Conversely, let G be a graph such that, for every non-trivial induced subgraph H of G, the graph H is not 2-connected or $\overline{H}$ is disconnected. By Lemma 2.2, it follows that every non-trivial subgraph of G can be written as a 0-sum, a 1-sum, or a join of smaller induced subgraphs of G. Therefore G can be generated from K_1 using the operations of 0-sum, 1-sum, and join. Thus G is a sesquicograph. □

A slight variation of the proof of the closure of 2-cographs under contractions [7, Proposition 2.8] shows that sesquicographs are also closed under contractions.

Proposition 2.6. *Let G be a sesquicograph and e be an edge of G. Then G/e is a sesquicograph.*

Proof. Assume to the contrary that G/e is not a sesquicograph. Then there is a non-trivial induced subgraph H of G/e such that H is 2-connected and $\overline{H}$ is connected. Let $e = uv$ and let w denote the vertex in G/e obtained by identifying u and v. We may assume that w is a vertex of H, otherwise H is an induced subgraph of G, a contradiction. We assert that the subgraph H' of G induced on the vertex set $(V(H) \cup \{u, v\}) - \{w\}$ is 2-connected and its complement $\overline{H'}$ is connected. To see this, note that, since H is 2-connected, H' is 2-connected unless one of u and v, say u, is a leaf of H'. In the exceptional case, we have $H' - u \cong H$, so G has a 2-connected induced subgraph for which its complement is connected, a contradiction. We deduce that H' is 2-connected.

Note that, by Lemma 2.3, $\overline{H}$ is obtained from $\overline{H'}$ by adding a vertex w with neighbourhood $N_{\overline{H'}}(u) \cap N_{\overline{H'}}(v)$ to the graph $\overline{H'} - \{u, v\}$. Since $\overline{H}$ is connected, it follows that $\overline{H'}$ is connected, a contradiction. □

It now follows that the class of sesquicographs is closed under taking induced minors. Since we can compute the components and blocks of a graph in polynomial time [10, 4.1.23], the algorithm in Figure 2 recognizes sesquicographs in polynomial time.

3. Induced-subgraph-minimal non-sesquicographs

We noted in Section 2 that sesquicographs are closed under induced subgraphs. In this section, we consider those non-sesquicographs for which every proper induced subgraph is a sesquicograph. We call these graphs

Require: Input a simple graph G
  Set $H \leftarrow G$, BlocksList $\leftarrow [G]$
  if $|V(H)| \leq 4$ **then**
    remove H from BlocksList
    if BlocksList is empty **then**
      return G is a sesquicograph and exit the algorithm
    else
      update H to be an element of BlocksList
  if H is not 2-connected **then**
    remove H from BlocksList
    Decompose H into 2-connected blocks and add all the blocks of H to BlocksList
    update H to be an element of BlocksList
  else if $\overline{H}$ is not connected **then**
    remove H from BlocksList
    Decompose $\overline{H}$ into connected components and add the complements of all the components to BlocksList
    update H to be an element of BlocksList
  else
    return G is not a sesquicograph and exit the algorithm

FIGURE 2. Algorithm for recognizing a sesquicograph.

induced-subgraph-minimal non-sesquicographs. The goal of this section is to characterize such graphs. We begin by showing that all cycles of length exceeding four are examples of such graphs.

Lemma 3.1. *Let G be a cycle of length exceeding four. Then G is an induced-subgraph-minimal non-sesquicograph.*

Proof. Note that both G and $\overline{G}$ are 2-connected and so, by Proposition 2.5, G is not a sesquicograph. It is now enough to show that, for any vertex v of G, the graph $G - v$ is a sesquicograph. Observe that $G - v$ is a path and so is a sesquicograph. □

The next result can be easily checked.

Lemma 3.2. *The graphs $\overline{P_5}, C_6^+, H_1, H_2, H_3, H_4,$ and H_5 are induced-subgraph-minimal non-sesquicographs.*

Lemma 3.3. *Let G be an induced-subgraph-minimal non-sesquicograph. Then G is 2-connected and $\overline{G}$ is connected.*

Proof. Assume the contrary. Then for some proper induced subgraphs X and Y of G, we can decompose G as $X \oplus Y$, as $X \oplus_1 Y$, or, by Lemma 2.2, as $X \bigtriangledown Y$. Since G is an induced-subgraph-minimal non-sesquicograph, both X and Y are sesquicographs. It now follows that G is a sesquicograph, a contradiction. □

A 2-connected graph H is **critically** 2**-connected** if $H - v$ is not 2-connected for all vertices v of H.

Lemma 3.4. *Let G be an induced-subgraph-minimal non-sesquicograph. Then G is critically 2-connected, or G has vertex connectivity two and $\overline{G}$ has vertex connectivity one.*

Proof. Note that, by Lemma 3.3, G is 2-connected and $\overline{G}$ is connected, and, by Proposition 2.5, for each vertex v of G, the graph $G - v$ is not 2-connected or $\overline{G} - v$ is disconnected. Observe that G has a vertex v such that $\overline{G} - v$ is connected and so $G - v$ is not 2-connected. Therefore G has vertex connectivity two. Suppose that G is not critically 2-connected. Then there is a vertex w of G such that $G - w$ is 2-connected and so $\overline{G} - w$ is disconnected. Therefore the vertex connectivity of $\overline{G}$ is one. □

Next we find those induced-subgraph-minimal non-sesquicographs G such that G is critically 2-connected. We will use the following result of Nebesky [6].

Lemma 3.5. *Let G be a critically 2-connected graph such that $|V(G)| \geq 6$. Then G has at least two distinct paths of length exceeding two such that the internal vertices of these paths have degree two in G.*

Lemma 3.6. *Let G be an induced-subgraph-minimal non-sesquicograph such that G is not isomorphic to a cycle and let $wxyz$ be a path P of G such that both x and y have degree two in G. Then w and z are adjacent.*

Proof. Assume that w and z are not adjacent. By Lemma 3.3, G is 2-connected, so there is a path P' joining w and z such that P and P' are internally disjoint. We may assume that P' is a shortest such path. It now follows that G has a cycle C of length exceeding four as an induced subgraph. Since a cycle of length exceeding four is not a sesquicograph, $G = C$, a contradiction. □

Proposition 3.7. *Let G be an induced-subgraph-minimal non-sesquicograph such that G is critically 2-connected. Then G is isomorphic to a cycle of length exceeding four or to the domino.*

Proof. We may assume that G is not isomorphic to a cycle exceeding four otherwise we have our result. Note that, by Lemma 2.1, $|V(G)| \geq 5$. Since the cycle of length five is the only critically 2-connected graph on five vertices, we may assume that $|V(G)| \geq 6$. By Lemma 3.5, G has two distinct paths $P_1 = abcd$ and $P_2 = wxyz$ of length three such that their internal vertices have degree two. By Lemma 3.6, a and d are adjacent, and w and z are adjacent. Consider the graph $G' = G - \{b, c\}$. Note that G' is 2-connected and so, by Lemma 2.5, $\overline{G}$ is disconnected. It is now easy to check that $|V(G')| = 4$ and so G is isomorphic to the domino. □

Set FinalList $\leftarrow \emptyset$, $i \leftarrow 0$
Generate all two connected graphs of order 6 using nauty geng [5] and store in an iterator L
for g in L such that vertex connectivity of g is 2 and $\overline{g}$ is 1 **do**
  for v in $V(g)$ **do**
    $h = g \backslash v$
    if vertex connectivity of $h < 2$ or vertex connectivity of $\overline{h} < 1$ **then**
      $i \leftarrow i + 1$
  if i equals $|V(g)|$ **then**
    Add g to FinalList

FIGURE 3. Finding induced-subgraph-minimal non-sesquicographs of order six.

Proof of Theorem 1.2. We may assume that G is not critically 2-connected otherwise we are done by Proposition 3.7. By Lemma 3.3, G has vertex connectivity two and $\overline{G}$ has vertex connectivity one. We first show the following.

3.7.1. *$\overline{G}$ has at most three cut vertices.*

Let $\{u, v\}$ be a 2-cut of G and let the components of $G - \{u, v\}$ be partitioned into subgraphs A and B such that $|V(A)| \geq |V(B)|$ and $|V(A)| - |V(B)|$ is a minimum. Observe that $\overline{G} - x$ is connected for a vertex x in $V(G)$ unless x is the only red neighbour of u or the only red neighbour of v, or $|V(B)| = 1$ and x is in $V(B)$. Thus 3.7.1 holds.

We show next that the number of vertices of G can be bounded.

3.7.2. $|V(G)| \leq 6$.

Assume that $|V(G)| > 6$. By 3.7.1, $\overline{G}$ has at most three cut vertices. First suppose that $\overline{G}$ has one cut vertex x. Let the components of $\overline{G} - x$ be partitioned into subgraphs R_1 and R_2 such that $|V(R_1)| \geq |V(R_2)|$ and $|V(R_1)| - |V(R_2)|$ is a minimum. Since $|V(G)| \geq 7$, we have $|V(R_1)| \geq 3$. Observe that, if $|V(R_2)| \geq 2$, then there exists a vertex r in R_1 such that x has two green neighbours in $G - r$. Note that every edge joining a vertex in R_1 to a vertex in R_2 is a green edge and so $G - r$ is connected. Since every vertex in $V(G) - x$ is in a green 2-cut, this is a contradiction. Therefore $|V(R_2)| = 1$ and so $|V(R_1)| \geq 5$. Let $R_2 = \{\alpha\}$. Note that $G - x$ is 2-connected since G is not critically 2-connected. It is now clear that $G - \{x, \alpha\}$ is connected. If $G - \{x, \alpha\}$ has a vertex r such that $G - \{x, \alpha, r\}$ is connected and contains two green neighbours of x, then $G - \alpha$ is 2-connected, a contradiction. It now follows that $G - \{x, \alpha\}$ is a path and its leaves are the only green neighbours of x. Note that $G - \alpha$ is a cycle of length exceeding four, a contradiction.

Next suppose that $\overline{G}$ has two cut vertices x_1 and x_2. For $\{i,j\} = \{1,2\}$, let R_i be the disjoint union of the components of $\overline{G} - x_i$ that do not contain x_j. Let R_3 be the subgraph induced on $V(G) - (V(R_1) \cup V(R_2) \cup \{x_1, x_2\})$. We first consider the case when $V(R_3)$ is empty. We may assume that $|V(R_1)| \geq |V(R_2)|$ and so $|V(R_1)| \geq 3$. Note that if $|V(R_2)| \geq 2$, then there is a vertex r in R_1 such that $G - r$ is 2-connected, a contradiction. Therefore $|V(R_2)| = 1$ and so $|V(R_1)| \geq 4$. Let β be a green neighbour of x_1 in R_1. Note that $G - r$ is 2-connected for every vertex r in $V(R_1) - \beta$, a contradiction. Therefore $V(R_3)$ is non-empty. Observe that, if both R_1 and R_2 have at least two vertices, then $G - r$ is 2-connected for any vertex r in R_3, a contradiction. Therefore we may assume that $|V(R_1)| = 1$. We show that neither R_2 nor R_3 has more than two vertices. Assume that R_i has more than two vertices for some i in $\{2,3\}$. Then there exists a vertex r in $V(R_i)$ such that both x_1 and x_2 have at least two green neighbours in $G-r$. Note that $G-r$ is 2-connected, a contradiction. Therefore $|V(R_2)| = |V(R_3)| = 2$. Observe that there is a vertex r in R_3 such that both x_1 and x_2 have green degree at least two in $G-r$. It follows that $G-r$ is 2-connected, a contradiction. Thus $\overline{G}$ has three cut vertices.

Let $X = \{x_1, x_2, x_3\}$ be the set of cut vertices of $\overline{G}$. We may assume that for the cut vertex x_1 of $\overline{G}$, the components of $\overline{G} - x_1$ can be partitioned into subgraphs P and Q such that x_2 is in P and x_3 is in Q, and $|V(P)| \geq |V(Q)| \geq 2$. Note that all vertices in P are green neighbours of x_3 and all vertices in Q are green neighbours of x_2. If $|V(P)| \geq 4$, then there is a vertex r in P such that all vertices in X have at least two green neighbours in $G-r$ and so $G-r$ is 2-connected, a contradiction. Therefore $|V(P)| = |V(Q)| = 3$. Note that there is a vertex r in $P \cup Q$ such that all vertices in X have at least two green neighbours in $G-r$ and so $G-r$ is 2-connected, a contradiction. Thus 3.7.2 holds.

By Lemma 2.1, it is clear that $|V(G)| \geq 5$ and so $|V(G)|$ is either 5 or 6. Suppose $|V(G)| = 5$. Since $\overline{P_5}$ is the only graph on five vertices that is not critically 2-connected, has vertex connectivity two, and whose complement has vertex connectivity one, by Lemma 3.2, we have $G \cong \overline{P_5}$. Next suppose that $|V(G)| = 6$. Implementing the algorithm in Figure 3 in Sagemeth [8], it can be easily checked that G is isomorphic to one of the graphs in $\{H_1, H_2, H_3, H_4, H_5\}$. This completes the proof. □

Proof of Corollary 1.3. Note that every cycle of length exceeding five has the cycle of length five as an induced minor. Also, the domino graph C_6^+ contains $\overline{P_5}$ as an induced minor. The result now follows by Theorem 1.2. □

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