arXiv:2210.04160v1 [math.CO] 9 Oct 2022

Regular graphs with a complete bipartite graph as a star complement*

Xiaona Fang[a], Lihua You[a]†, Rangwei Wu[a], Yufei Huang[b]

[a]School of Mathematical Sciences, South China Normal University,
Guangzhou, 510631, P. R. China

[b]Department of Mathematics Teaching, Guangzhou Civil Aviation College,
Guangzhou, 510403, P. R. China

Abstract: Let G be a graph of order n and μ be an adjacency eigenvalue of G with multiplicity $k\geq 1$. A star complement H for μ in G is an induced subgraph of G of order $n-k$ with no eigenvalue μ, and the vertex subset $X=V(G-H)$ is called a star set for μ in G. The study of star complements and star sets provides a strong link between graph structure and linear algebra. In this paper, we study the regular graphs with $K_{t,s}$ ($s\geq t\geq 2$) as a star complement for an eigenvalue μ, especially, characterize the case of $t=3$ completely, obtain some properties when $t=s$, and propose some problems for further study.

Keywords: Adjacency eigenvalue; Star set; Star complement; Regular graph.

AMS classification: 05C50

1 Introduction

Let G be a simple graph with vertex set $V(G)=\{1,2,\ldots,n\}=[n]$ and edge set $E(G)$. The adjacency matrix of G is an $n\times n$ matrix $A(G)=(a_{ij})$, where $a_{ij}=1$ if vertex i is adjacency to vertex j, and 0 otherwise. We use the notation $i\sim j$ ($i\nsim j$) to indicate that i,j are adjacent (not-adjacent) and the notation $d_G(i)$ to indicate the degree of vertex i in G. The adjacency eigenvalues of G are just the eigenvalues of $A(G)$. For more details on graph spectra, see [6].

Let μ be an eigenvalue of G with multiplicity k. A *star set* for μ in G is a subset X of $V(G)$ such that $|X|=k$ and μ is not an eigenvalue of $G-X$, where $G-X$ is the subgraph of G induced by $\overline{X}=V(G)\setminus X$. In this situation $H=G-X$ is called a *star complement* corresponding to μ. Star sets and star complements exist for any eigenvalue of a graph, and they need not to be unique. The basic properties of star sets are established in Chapter 7 of [7].

*This work is supported by the National Natural Science Foundation of China (Grant No. 11971180), the Guangdong Provincial Natural Science Foundation (Grant No. 2019A1515012052).

†Corresponding author: ylhua@scnu.edu.cn

There is another equivalent geometric definition for star sets and star complements. Let G be a graph with vertex set $V(G) = \{1, \ldots, n\}$ and adjacency matrix A. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$, μ be an eigenvalue of G, and P be the matrix which represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu) = \{x \in \mathbb{R}^n : A(G)x = \mu x\ \}$ of A with respect to $\{e_1, \ldots, e_n\}$. Since $\mathcal{E}(\mu)$ is spanned by the vectors $Pe_j (j = 1, \ldots, n)$, there exists $X \subseteq V(G)$ such that the vectors $Pe_j (j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a subset X of $V(G)$ is called a *star set* for μ in G. In this situation $H = G - X$ is called a *star complement* for μ.

For any graph G of order n with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, there exists a partition $V(G) = X_1 \bigcup \cdots \bigcup X_m$ such that X_i is a star set for eigenvalue λ_i $(i = 1, \ldots, m)$. Such a partition is called a *star partition* of G. For any graph G, there exists at least one star partition ([10]). Each star partition determines a basis for $\mathbb{R}^n$ consisting of eigenvectors of an adjacency matrix. It provides a strong link between graph structure and linear algebra.

There are a lot of literatures about using star complements to construct and characterize certain graphs([1, 2, 3, 8, 12, 14, 16, 18, 19, 20, 21, 22, 23, 24, 25]), especially, regular graphs with a prescribed graph such as $K_{1,s}$, $K_1 \nabla hK_q$, $K_{2,5}$, $K_{2,s}$, $K_{1,1,t}$, $K_{1,1,1,t}$, $\overline{sK_1 \cup K_t}$, P_t $(\mu = 1)$, $K_{r,r,r}$ $(\mu = 1)$ or $K_{r,s} + tK_1$ $(\mu = 1)$ as a star complement were well studied in the literature. Motivated by the above research, in this paper, we introduce the fundamental properties of the theory of star complements in Section 2, study the regular graphs with the bipartite graph $K_{t,s}$ $(s \geq t \geq 1)$ as a star complement for an eigenvalue μ in Section 3, completely characterize the regular graphs with $K_{3,s}$ $(s \geq 3)$ as a star complement for an eigenvalue μ in Section 4, study some properties of $K_{s,s}$ in Section 5, and propose some problems for further research.

2 Preliminaries

In this section, we introduce some results of star sets and star complements that will be required in the sequel. The following fundamental result combines the Reconstruction Theorem ([7, Theorem 7.4.1]) with its converse ([7, Theorem 7.4.4]).

Theorem 2.1. ([7]) *Let μ be an eigenvalue of G with multiplicity k, X be a set of vertices in the graph G. Suppose that G has adjacency matrix*

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{2.1}$$

In this situation, $\mathcal{E}(\mu)$ consists of the vectors

$$\begin{pmatrix} \boldsymbol{x} \\ (\mu I - C)^{-1}B\boldsymbol{x} \end{pmatrix}, \tag{2.2}$$

where $\boldsymbol{x} \in \mathbb{R}^k$.

Note that if X is a star set for μ, then the corresponding star complement $H(= G - X)$ has adjacency matrix C, and (2.1) tells us that G can be determined by μ, H and the H-neighbourhood of vertices in X, where the H-neighbourhood of the vertex $u \in X$, denoted by $N_H(u)$, is defined as $N_H(u) = \{v \mid v \sim u, v \in V(H)\}$.

It is usually convenient to apply (2.1) in the form

$$m(\mu)(\mu I - A_X) = B^T m(\mu)(\mu I - C)^{-1}B,$$

where $m(x)$ is the minimal polynomial of C. This is because $m(\mu)(\mu I - C)^{-1}$ is given explicitly as follows.

Proposition 2.2. ([8], Proposition 0.2) *Let C be a square matrix with minimal polynomial*

$$m(x) = x^{d+1} + c_d x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0.$$

If μ is not an eigenvalue of C, then

$$m(\mu)(\mu I - C)^{-1} = a_d C^d + a_{d-1}C^{d-1} + \cdots + a_1 C + a_0 I,$$

where $a_d = 1$ and for $0 < i \leq d$, $a_{d-i} = \mu^i + c_d\mu^{i-1} + c_{d-1}\mu^{i-2} + \cdots + c_{d-i+1}$.

In order to find all the graphs with a prescribed star complement H for μ, we need to find all solution A_X, B for given μ and C. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^q$, where $q = |V(H)|$, let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T(\mu I - C)^{-1}\mathbf{y}. \tag{2.3}$$

Let $\mathbf{b}_u$ be the column of B for any $u \in X$. By Theorem 2.1, we have

Corollary 2.3. ([10], Corollary 5.1.8) *Suppose that μ is not an eigenvalue of the graph H, where $|V(H)| = q$. There exists a graph G with a star set $X = \{u_1, u_2, \dots, u_k\}$ for μ such that $G - X = H$ if and only if there exist $(0,1)$-vectors $\boldsymbol{b}_{u_1}, \boldsymbol{b}_{u_2}, \dots, \boldsymbol{b}_{u_k}$ in $\mathbb{R}^q$ such that*

(1) $\langle \boldsymbol{b}_u, \boldsymbol{b}_u \rangle = \mu$ *for all $u \in X$, and*

(2) $\langle \boldsymbol{b}_u, \boldsymbol{b}_v \rangle = \begin{cases} -1, & u \sim v \\ 0, & u \nsim v \end{cases}$ *for all pairs u, v in X.*

In view of the two equations in the above corollary, we have

Lemma 2.4. ([7]) *Let X be a star set for μ in G, and $H = G - X$.*

(1) *If $\mu \neq 0$, then $V(H)$ is a dominating set for G, that is, the H-neighbourhood of any vertex in X are non-empty;*

(2) *If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set for G, that is, the H-neighbourhood of distinct vertices in X are distinct and non-empty.*

It follows from (2) of Lemma 2.4 that there are only finitely regular graphs with a prescribed star complement for $\mu \notin \{-1, 0\}$. If $\mu = 0$ and X has distinct vertices u and v with the same neighbourhood in G, then u and v are called *duplicate vertices*. If $\mu = -1$ and X has distinct vertices u and v with the same closed neighbourhood in G, then u and v are called *co-duplicate vertices* (see [11]).

Recall that if the eigenspace $\mathcal{E}(\mu)$ is orthogonal to the all-1 vector $\mathbf{j}$ then μ is called a *non-main* eigenvalue. From (2.2), we have the following result.

Lemma 2.5. ([8], Proposition 0.3) *The eigenvalue μ is a non-main eigenvalue if and only if*

$$\langle \boldsymbol{b}_u, \boldsymbol{j} \rangle = -1 \quad \text{for all } u \in X, \tag{2.4}$$

where $\boldsymbol{j}$ is the all-1 vector.

Lemma 2.6. ([10], Corollary 3.9.12) *In an r-regular graph, all eigenvalues other than r are non-main.*

In the rest of this paper, we let $H \cong K_{t,s}$ $(s \geq t \geq 1)$, (V, W) be a bipartition of the graph $K_{t,s}$ with $V = \{v_1, v_2, \dots, v_t\}$, $W = \{w_1, w_2, \dots, w_s\}$. We say that a vertex $u \in X$ is of type (a, b) if it has a neighbours in V and b neighbours in W. Clearly $(a, b) \neq (0, 0)$ and $0 \leq a \leq t$, $0 \leq b \leq s$.

Let C be the adjacency matrix of H, then C has minimal polynomial $m(x) = x(x^2 - ts)$. Since μ is not an eigenvalue of C, we have $\mu \neq 0$ and $\mu^2 \neq ts$. From Proposition 2.2, we have

$$m(\mu)(\mu I - C)^{-1} = C^2 + \mu C + (\mu^2 - ts)I. \tag{2.5}$$

If μ is a non-main eigenvalue of G, then by (2.4) and (2.5) we have

$$\mu^2(a+b)+\mu(as+tb)=-\mu(\mu^2-ts). \tag{2.6}$$

Using (2.5) to compute $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$, we obtain the following equation

$$(\mu^2-ts)(a+b)+a^2s+tb^2+2ab\mu=\mu^2(\mu^2-ts). \tag{2.7}$$

Let u, v be distinct vertices in X of type (a,b), (c,d), respectively. Let $\rho_{uv}=|N_H(u)\cap N_H(v)|$, $a_{uv}=1$ or 0 according as $u\sim v$ or $u\nsim v$. Using (2.5) to compute $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -a_{uv}$, we have

$$(\mu^2-ts)\rho_{uv}+acs+bdt+\mu(ad+bc)=-\mu(\mu^2-ts)a_{uv}. \tag{2.8}$$

3 Regular graphs with $K_{t,s}$ as a star complement

An r-regular graph G with n vertices is said to be strongly regular with parameters (n,r,e,f) if every two adjacent vertices in G have e common neighbours and every two non-adjacent vertices have f common neighbours. For example, Petersen graph is strongly regular with parameters $(10,3,0,1)$.

For the regular graphs with the complete bipartite graph $K_{t,s}$ as a star complement, the case of $t=1$ was solved by Rowlinson and Tayfeh-Rezaie in 2010 ([19]), the case of $t=2,\ s=5$ was solved by Rowlinson and Jackson in 1999 ([18]), the case of $t=2,\ s\neq 5$ was solved by Yuan, Zhao, Liu and Chen in 2018 ([25]), and the conclusions are listed below.

Theorem 3.1. ([19]) If the r-regular graph G has $K_{1,s}$ $(s>1)$ as a star complement for an eigenvalue μ, then one of the following holds:
(1) $\mu=\pm 2, r=s=2$ and $G\cong K_{2,2}$;
(2) $\mu=\frac{1}{2}(-1\pm\sqrt{5}), r=s=2$ and G is a 5-cycle;
(3) $\mu\in\mathbb{N}_+, r=s$ and G is strongly regular with parameters $\Big((\mu^2+3\mu)^2, \mu\left(\mu^2+3\mu+1\right), 0, \mu(\mu+1)\Big)$.

Theorem 3.2. ([18, 25]) Let $s\geq 2$. If the r-regular graph G has $K_{2,s}$ as a star complement for an eigenvalue μ, then one of the following holds:
(1) $\mu=\pm 3, r=s=3$ and $G\cong K_{3,3}$;
(2) $1\neq\mu\in\mathbb{N}_+, r=s$ and G is an r-regular graph of order $\left(\mu^4+10\mu^3+27\mu^2+10\mu\right)/4$, where $r=\mu(\mu+1)(\mu+4)/2$.

(3) $\mu = 1$, $s = 5$ and either $G \cong Sch_{10}$ or G is isomorphic to one of the eleven induced regular subgraphs of Sch_{10}.
(4) $\mu = -1$, $r \equiv -1 \pmod{2s-1}$ and $G \cong G'(r)$ (see [25] for specific definitions).

In this section, we consider the general case. We prove that there is no regular graph G with $K_{t,s}$ ($s \geq t \geq 1$) as a star complement for $\mu = -t$, characterize the graph G when $\mu = r$, $\mu = -1$, and the case with all vertices in X of type $(0, b)$ for $\mu \notin \{-t, r, -1\}$. Furthermore, we propose a question for further research.

Proposition 3.3. *There is not an r-regular graph G with $K_{t,s}$ ($s \geq t \geq 1$) as a star complement for $\mu = -t$.*

Proof. Let $\mu = -t$. Since $\mu^2 \neq ts$, we have $s \neq t$ and then $s > t$. Let $u \in X$ be a vertex of type (a, b), thus $(a, b) \neq (0, 0)$ and $0 \leq a \leq t$, $0 \leq b \leq s$.

If $t = 1$, from Theorem 2.2 of [19], there is no r-regular graph G with $K_{1,s}$ as a star complement for $\mu = -1$.

If $t \geq 2$, by Lemma 2.6, we know that $\mu = -t$ is a non-main eigenvalue of G, and by (2.6), we have

$$t(t - s)(a - t) = 0. \tag{3.1}$$

Since $s > t$ and $t \geq 2$, (3.1) implies that $a = t$, and thus $s(b - t^2) = b^2 - tb - t^3 + t^2$ by (2.7).

If $b = t^2$, then $t^4 - 2t^3 + t^2 = 0$, thus $t = 0$ or 1, a contradiction.

If $b \neq t^2$, then $s = \frac{b^2 - tb - t^3 + t^2}{b - t^2} = b - t^2 + \frac{t^2(t-1)^2}{b - t^2} + 2t^2 - t$. If $b < t^2$, then $s \leq -2\sqrt{t^2(t-1)^2} + 2t^2 - t = t$, which contradicts with $s > t$. Thus $b > t^2$ and $s - (b + t - 1) = \frac{b(t-1)^2}{b - t^2} > 0$. Considering degrees, we have

$$d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t) = s + |X|,$$

and

$$d_G(u) \leq a + b + |X| - 1 = b + t - 1 + |X|, \ u \in X.$$

Hence, $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t) > d_G(u)$ which contradicts to the regularity of G.

Combining the above arguments, there is not an r-regular graph G with $K_{t,s}$ ($s \geq t \geq 1$) as a star complement for $\mu = -t$. $\square$

Theorem 3.4. *If the r-regular graph G has $K_{t,s}$ ($s \geq t \geq 1$) as a star complement for an eigenvalue $\mu = r$, then $s = t = 1$, $G \cong C_3$ or $r = s = t + 1$, $G \cong K_{t+1,t+1}$.*

Proof. Since μ is not an eigenvalue of $H \cong K_{t,s}$ $(s \geq t \geq 1)$, we have $\mu \neq 0$ and $\mu^2 \neq ts$. By Lemma 2.4, $V(K_{t,s})$ is a location-dominating set, and so G is connected.

By G is r-regular and connected, $\mu = r$, we know $k = 1$ and then $|X| = 1$. Since G is regular, we have $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t)$. Let $X = \{u\}$. Then either $u \sim v_1$, $u \sim v_2$, $\ldots$,$u \sim v_t$, or $u \nsim v_1$, $u \nsim v_2$,$\ldots$, $u \nsim v_t$.

If $u \sim v_1$, $u \sim v_2$, $\ldots$,$u \sim v_t$, then $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t) = s+1$, which implies that $d_G(u) = s+1$. It follows that the vertex u is adjacent to $s-t+1(\geq 1)$ vertices of W, and thus $t = 1$ by $d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$. Since $d_G(w_1) = t+1 = d_G(v_1)$, we have $s = t = 1$ and $G \cong C_3$.

If $u \nsim v_1$, $u \nsim v_2$, $\ldots$, $u \nsim v_t$, in view of the regularity, we have $d_G(u) = d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t) = s$, and then $d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s) = t+1$. Hence we have $s = r = t+1$ and $G \cong K_{t+1,t+1}$.□

Let $H \cong K_{t,s}$, (V, W) be a bipartition of the graph $K_{t,s}$ with $V = \{v_1, v_2, \ldots, v_t\}$, $W = \{w_1, w_2, \ldots, w_s\}$. We obtain an r-regular graph $G(r)$ with $V(G(r)) = X \cup V(H)$, $X = V_1 \cup \cdots \cup V_t \cup W_1 \cup \cdots \cup W_s$ where V_i is the set of vertices of type $(1, s)$ adjacent to $v_i \in V$ with $|V_i| = (r+1)(s-1)/(ts-1) - 1$, V_i induces a clique for $1 \leq i \leq t$, W_i is the set of vertices of type $(t, 1)$ adjacent to $w_i \in W$ with $|W_i| = (r+1)(t-1)/(ts-1) - 1$, W_i induces a clique for $1 \leq i \leq s$. For any i, j, each vertex in V_i is adjacent to all vertices in W_j.

The greatest common divisor of a and b is denoted by $\gcd(a, b)$. For $\mu = -1$, we have the following theorem.

Theorem 3.5. *If G is an r-regular graph with $H = K_{t,s}$ $(s \geq t \geq 2)$ as a star complement for an eigenvalue -1, then $r \equiv -1 \ (\mathrm{mod} \frac{ts-1}{\gcd(s-1,t-1)})$ and $G \cong G(r)$.*

Proof. Since $K_{t,s}$ is connected and $V(K_{t,s})$ is a dominating set (see Lemma 2.4), we know G is connected. Let $H \cong K_{t,s}$, (V, W) be a bipartition of the graph $K_{t,s}$ as above. Let $u \in X$ be a vertex of type (a, b), thus $(a, b) \neq (0, 0)$ and $0 \leq a \leq t$, $0 \leq b \leq s$. Let $\mu = -1$ in (2.7), so that

$$(1 - ts)(a + b - 1) + a^2 s + tb^2 - 2ab = 0. \tag{3.2}$$

By Lemma 2.6, we know that $\mu = -1$ is a non-main eigenvalue of G, thus from (2.6), we have

$$1 - ts + a(s-1) + b(t-1) = 0. \tag{3.3}$$

Combining (3.2) and (3.3), if $a = 1$, then $b = s$; if $a = t$, then $b = 1$; if $1 < a < t$, then $b = a - t + 1,\ s = 2 - t$ or $b = \frac{a}{t},\ s = \frac{1}{t}$. It is obvious that $s = 2 - t$ or $s = \frac{1}{t}$ contradicts with $s \in \mathbb{Z}_+$. Therefore, the possible types of vertices in X are $(1, s)$, $(t, 1)$, and the feasible solution of (2.8) are shown in Table 1.

(a, b)	(c, d)	a_{uv}	ρ_{uv}
$(1, s)$	$(1, s)$	0	s
$(1, s)$	$(1, s)$	1	$s + 1$
$(t, 1)$	$(t, 1)$	0	t
$(t, 1)$	$(t, 1)$	1	$t + 1$
$(1, s)$	$(t, 1)$	1	2

Table 1: The feasible solution of (2.8)

We observe that when u, v are of different types, they must be adjacent; when u, v are of the same type, $u \sim v$ if and only if they have the same H-neighbourhoods, and thus u, v are co-duplicate vertices. We can add arbitrarily many co-duplicate vertices when constructing graphs with a prescribed star complement for -1.

Now we partition the vertices in X. Let V_i be the set of vertices of type $(1, s)$ in X adjacent to $v_i \in V$, W_i be the set of vertices of type $(t, 1)$ in X adjacent to $w_i \in W$. Then any two vertices in V_i (W_i) are co-duplicate vertices. We do not exclude the possibility that some of the sets V_i, W_i are empty. Then for any $v_i \in V$, we have $d_G(v_i) = s + |V_i| + \sum_{i=1}^{s} |W_i|$; and for any $w_i \in W$, we have $d_G(w_i) = t + \sum_{i=1}^{t} |V_i| + |W_i|$.

Since G is r-regular, we have $|V_1| = |V_2| = \cdots = |V_t|$ by $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_t)$ and $|W_1| = |W_2| = \cdots = |W_s|$ by $d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$. Then we have

$$r = d_G(v_1) = s + |V_1| + s \cdot |W_1| \text{ and } r = d_G(w_1) = t + t \cdot |V_1| + |W_1|.$$

It turns out that

$$|V_1| = \frac{(s-1)(r+1)}{ts-1} - 1,\ \ |W_1| = \frac{(t-1)(r+1)}{ts-1} - 1.$$

Since $|V_1| \in \mathbb{N}$, $|W_1| \in \mathbb{N}$ and

$$\gcd(t-1, ts-1) = \gcd(s-1, ts-1) = \gcd(t-1, s-1),$$

we have $r \equiv -1 \pmod{\frac{ts-1}{\gcd(s-1,t-1)}}$. Consequently we obtain an r-regular graph $G(r)$. $\square$

Remark 3.6. *Note that if* $V_i^* = V_i \cup \{v_i\}$, $v_i \in V$ *and* $W_i^* = W_i \cup \{w_i\}$, $w_i \in W$, *then each of sets* V_i^*, W_i^* *induces a clique in* $G(r)$.

Next, we consider the case $\mu \notin \{-1,-t,r\}$. The following lemma lists all possible types of vertices in X.

Lemma 3.7. *Let G be a graph with $H = K_{t,s}$ $(s \geq t \geq 1)$ as a star complement for μ. If μ is a non-main eigenvalue of G and $\mu \notin \{-1,-t\}$, then the possible types of vertices in the star set X are $(a, \frac{\mu^3+t\mu^2-ta+a^2}{\mu+a})$, where $0 \leq a \leq t-1$ and $a \neq -\mu$.*

Proof. Let $u \in X$ be a vertex of type (a,b), thus $(a,b) \neq (0,0)$ and $0 \leq a \leq t$, $0 \leq b \leq s$. By (2.6), (2.7) and $\mu \notin \{0,-1,-t\}$, we have: (1) $a=t$, $b=-\mu$, $s=\frac{\mu^2}{t}$; (2) $a=-\mu$, $b=\frac{\mu^2}{t}$, $s=\frac{\mu^2}{t}$; (3) $0 \leq a \leq t-1$, $a \neq -\mu$, $\begin{cases} b = \frac{-a\mu}{t}, \\ s = \frac{\mu^2}{t}, \end{cases}$ or $\begin{cases} b = \frac{\mu^3+t\mu^2-ta+a^2}{\mu+a}, \\ s = \frac{\mu^4+(2t+1)\mu^3+(2a+t^2)\mu^2+(2a^2-at)\mu+a^2t-at^2}{(t-a)\mu+at-a^2}. \end{cases}$ Since μ is not an eigenvalue of H, we have $\mu^2 \neq ts$. Thus the possible types of vertices in the star set X are $(a, \frac{\mu^3+t\mu^2-ta+a^2}{\mu+a})$. □

For $\mu \notin \{-1,-t,r\}$, we consider the case $a=0$ in the following by Lemma 3.7.

Theorem 3.8. *If the r-regular graph G has $K_{t,s}$ $(s \geq t \geq 1)$ as a star complement for $\mu \notin \{-1,-t,r\}$ and all vertices in X are of type $(0,b)$, then one of the following holds:*
(1) $\mu=-r, r=s=t+1$ and $G \cong K_{t+1,t+1}$;
(2) $\mu=\frac{1}{2}(-1\pm\sqrt{5}), t=1, r=s=2$ and G is a 5-cycle;
(3) $\mu \in \mathbb{N}_+, r=s$ and G is an r-regular graph of order $\mu(\mu+2t+1)(\mu^2+2t\mu+\mu+t^2-t)/t^2$, where $r=\mu(\mu^2+2t\mu+\mu+t^2)/t$.

Proof. Since μ is not an eigenvalue of $K_{t,s}$, we have $\mu \neq 0$ and $\mu^2 \neq ts$. By Lemma 2.4, $V(K_{t,s})$ is a location-dominating set, and so G is connected. Then by $a=0$ and Lemma 3.7, we have

$$\begin{cases} b = \mu^2+t\mu, \\ s = \mu(\mu^2+2t\mu+\mu+t^2)/t. \end{cases}$$

Now we consider $H = K_{t,\mu(\mu^2+2t\mu+\mu+t^2)/t}$ and all vertices in X are of type $(0,\mu^2+t\mu)$. Then $r=s$. Counting the edges between X and $V(H)$, we have $|X|(\mu^2+t\mu) = s(r-t)$. Thus

$$|X| = \frac{s(r-t)}{(\mu^2+t\mu)} = \frac{1}{t^2}(\mu^2+2t\mu+\mu+t^2)(\mu^2+t\mu+\mu-t). \tag{3.4}$$

Case 1: $|X|=1$.

Then $(\mu+t+1)(\mu^3+(2t+1)\mu^2+(t^2-t)\mu-t^2)=0$ from (3.4). When $\mu^3+(2t+1)\mu^2+(t^2-t)\mu-t^2=0$, then $\mu \notin \mathbb{Z}$ and thus $r=\frac{\mu}{t}(\mu^2+2t\mu+\mu+t^2)-\frac{1}{t}(\mu^3+(2t+1)\mu^2+(t^2-t)\mu-t^2) = \mu+t \notin \mathbb{Z}$, a contradiction. When $\mu=-(t+1)$, then $r=s=t+1$ and $G \cong K_{t+1,t+1}$.

Case 2: $|X| \geq 2$.

We apply the compatibility condition (2.8) to vertices u, v in X, we find that

$$\rho_{uv} = \begin{cases} t\mu, & u \nsim v, \\ (t-1)\mu, & u \sim v. \end{cases} \tag{3.5}$$

If $t = 1$ and X induces a clique then $|X| - 1 = r - \mu^2 - \mu$, whence $(\mu+1)(\mu+2)(\mu^2+\mu-1) = 0$. Thus either $\mu = -2$, $r = s = t+1 = 2$ and $G \cong K_{2,2}$ which belongs to case (1), or $\mu = \frac{1}{2}(-1 \pm \sqrt{5})$ and we have case (2) ([19]).

Otherwise, it follows from (3.5) that $\mu \in \mathbb{N}_+$ and G is $\mu\left(\mu^2 + 2t\mu + \mu + t^2\right)/t$-regular of order $\mu(\mu + 2t + 1)(\mu^2 + 2t\mu + \mu + t^2 - t)/t^2$ with $K_{t,\mu(\mu^2+2t\mu+\mu+t^2)/t}$ as a star complement for μ. $\square$

In [19], Rowlinson gave a lemma to determine whether a connected r-regular graph with $K_{1,s}$ as a star complement is a strongly regular graph. Now we extend it to $K_{t,s}$.

Lemma 3.9. *Let G be a connected r-regular graph with $\mu (\neq r)$ as an eigenvalue of multiplicity k. Suppose that $|V(G)| = k + t + s$. If $k + t + s - 1 > r$, then*

$$(k + t + s)r - r^2 - k\mu^2 - \frac{(k\mu + r)^2}{s + t - 1} \geq 0,$$

with equality if and only if G is strongly regular.

Proof. Since $k + t + s - 1 > r$, neither G nor $\overline{G}$ is complete. Let $\theta_1, \ldots, \theta_{s+t-1}$ be the eigenvalue of G other than μ and r. We have

$$\sum_{i=1}^{s+t-1} \theta_i + k\mu + r = 0 \text{ and } \sum_{i=1}^{s+t-1} \theta_i^2 + k\mu^2 + r^2 = (k + t + s)r.$$

It follows that if $\overline{\theta} = \frac{1}{s+t-1} \sum\limits_{i=1}^{s+t-1} \theta_i$, then

$$\sum_{i=1}^{s+t-1} (\theta_i - \overline{\theta})^2 = \sum_{i=1}^{s+t-1} \theta_i^2 - (s + t - 1)\overline{\theta}^2 = (k + t + s)r - r^2 - k\mu^2 - \frac{(k\mu + r)^2}{s + t - 1} \geq 0.$$

Equality holds if and only if $\theta_i = \overline{\theta}$ ($i \in [s+t-1]$), equivalently G has just three distinct eigenvalue. By [9, Theorem 1.2.20], a non-complete connected regular graph is strongly regular if and only if it has exactly three distinct eigenvalues. The proof is completed. $\square$

Remark 3.10. *From Lemma 3.9, we know that the r-regular graph G in* (3) *of Theorem 3.8 is strongly regular when $t = 1$, and the graph G isn't strongly regular when $t \geq 2$.*

For the cases $1 \le a \le t-1$, we cannot give a characterization. Thus we propose a question for further research.

Question 3.11. *Let $s \ge t \ge 1$, $\mu \notin \{-1,-t,r\}$ and $1 \le a \le t-1$. Can we give a characterization of the r-regular graphs with $K_{t,s}$ as a star complement?*

4 Regular graphs with $K_{3,s}$ as a star complement

In this section, we completely solve Question 3.11 when $t = 3$. Since the cases when $\mu \in \{-1,-3,r\}$ have been solved in Section 3, we consider the cases $\mu \notin \{-1,-3,r\}$ in the following.

Let G be an r-regular graph with $H = K_{3,s}$ $(s \ge 3)$ as a star complement for $\mu \notin \{-1,-3,r\}$. Then $\mu \ne 0$ and $\mu^2 \ne 3s$ for μ is not an eigenvalue of $K_{3,s}$. By Lemma 3.7, the possible types of vertices in X are shown in Table 2:

Type	(a,b)	s
I	$(0, \mu^2+3\mu)$	$\mu(\mu^2+7\mu+9)/3$
II	$(1, \mu^2+2\mu-2)$	$(\mu+2)(\mu^2+4\mu-3)/2$
III	$(2, \frac{\mu^3+3\mu^2-2}{\mu+2})$	$\frac{\mu^4+7\mu^3+13\mu^2+2\mu-6}{\mu+2}$

Table 2: The possible types of vertices in X

Lemma 4.1. *Let G be a graph with $H = K_{3,s}$ $(s \ge 3)$ as a star complement for μ. If μ is a non-main eigenvalue of G and $\mu \notin \{-1,-3\}$, then all the vertices in the star set X are of the same type, say, Type I, Type II or Type III.*

Proof. Now we show it is impossible that there are two or three types of vertices in X.

Case 1. There exist vertices of Type I and Type III in X.

From Table 2, we have

$$s = \frac{1}{3}\mu(\mu^2+7\mu+9) = \frac{\mu^4+7\mu^3+13\mu^2+2\mu-6}{\mu+2}$$

with solution $\mu = -3$, -1 or 1. Since $\mu \notin \{-1,-3\}$, we have $\mu = 1$, and thus $s = 17/3$, a contraction.

Case 2. There exist vertices of Type II and Type III in X.

From Table 2, we have

$$s = \frac{1}{2}(\mu+2)(\mu^2+4\mu-3) = \frac{\mu^4+7\mu^3+13\mu^2+2\mu-6}{\mu+2}$$

with solution $\mu = -3$ or 0, a contraction.

Case 3. There exist vertices of Type I and Type II in X.

By Case 1 and Case 2, we know there does not exist vertices of Type III in X.

By Table 2, we have

$$s = \frac{1}{3}\mu(\mu^2 + 7\mu + 9) = \frac{1}{2}(\mu + 2)(\mu^2 + 4\mu - 3)$$

with solution $\mu = -3$ or 2. Since $\mu \neq -3$, we have $\mu = 2$, and thus $s = 18$. The vertices in X are of type $(0, 10)$, $(1, 6)$. From (2.8), Table 3 is obtained.

(a, b)	(c, d)	a_{uv}	ρ_{uv}
$(0, 10)$	$(1, 6)$	0	4
$(0, 10)$	$(1, 6)$	1	2
$(0, 10)$	$(0, 10)$	0	2
$(0, 10)$	$(0, 10)$	1	0
$(1, 6)$	$(1, 6)$	0	$-11/5$
$(1, 6)$	$(1, 6)$	1	$-21/5$

Table 3: The possible adjacency of vertices in X

For any two vertices of type $(1, 6)$, $(1, 6)$ in X, $\rho_{uv} \notin \mathbb{Z}$, so there is at most one vertex of type $(1, 6)$ in X. For any two vertices of type $(0, 10)$, $(0, 10)$ in X, $\rho_{uv} = 0$ or 2. Since $s = 18$, there are at most two vertices of type $(0, 10)$ in X, and $\rho_{uv} = 2$ if there are two vertices of type $(0, 10)$. Therefore, X has at most three vertices.

Let $V(K_{3,18}) = V \cup W$ with $|V| = 3$, $|W| = 18$. For any $v_i \in V$ and $w_i \in W$, we have $d_G(v_i) \geq 18$ and $d_G(w_i) \leq 3 + 3 = 6$, which contradicts with the regularity of G.

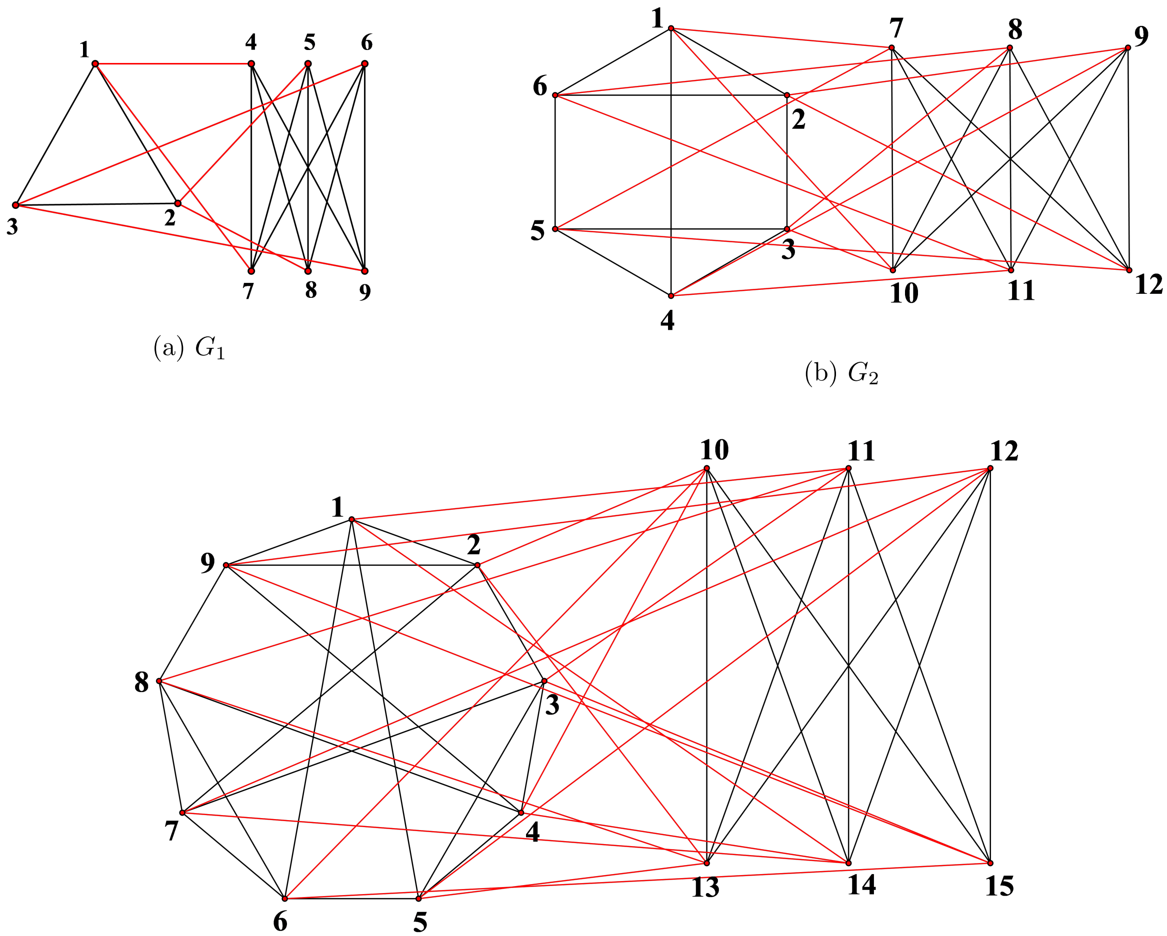
The proof is completed. $\square$

Now we define three special graphs G_1, G_2 and G_3 (see Figure 1) as follows. Clearly, the graph G_1 is a 4-regular graph of order 9, its spectrum is $[-3, -2^2, 0^2, 1^3, 4]$; G_2 is a 5-regular graph of order 12, its spectrum is $[-3^3, -1^2, 1^6, 5]$; G_3 is a 6-regular graph of order 15, its spectrum is $[-3^5, 1^9, 6]$. By Lemma 3.9, we know that G_1 and G_2 isn't strongly regular while G_3 is strongly regular with parameters $(15, 6, 1, 3)$.

Theorem 4.2. *Let $s \geq 3$, G be an r-regular graph with $K_{3,s}$ as a star complement for an eigenvalue μ. Then one of the following holds:*

(1) $\mu = -1$, $r \equiv -1 \pmod{\frac{3s-1}{(s-1,2)}}$ *and* $G \cong G(r)$, *where* $G(r)$ *is defined in Theorem 3.5;*

(2) $\mu = \pm 4$, $r = s = 4$ *and* $G \cong K_{4,4}$*;*



(a) G_1

(b) G_2

(c) G_3

Figure 1: Regular graphs G_1, G_2, G_3 of Theorem 4.2

(3) $\mu \in \mathbb{N}_+$, $r = s$ *and* G *is an* r*-regular graph of order* $\mu(\mu+7)(\mu+6)(\mu+1)/9$*, where* $r = \mu(\mu^2+7\mu+9)/3$*;*

(4) $\mu = 1$, $s = 3$, *and* $G \cong G_1$ $(r = 4)$, $G \cong G_2$ $(r = 5)$ *or* $G \cong G_3$ $(r = 6)$ (see Figure 1).

Proof. By Theorem 3.5 and $\mu = -1$, we have (1) holds.

By Theorem 3.4 and $\mu = r$, we have $s = r = 4$ and $G \cong K_{4,4}$.

If $\mu \notin \{-1, r\}$, then μ is non-main. From Proposition 3.3, we know $\mu \neq -3$. Since μ is not an eigenvalue of $H \cong K_{3,s}$, we have $\mu \neq 0$ and $\mu^2 \neq 3s$. By Lemma 2.4, $V(K_{3,s})$ is a location-dominating set, and so G is connected. Let $V(K_{3,s}) = V \cup W$ with $|V| = 3$, $|W| = s$. Denote the vertices in V and W by v_1, v_2, v_3 and $w_1, w_2, \ldots, w_s$, respectively. In the following, we suppose that $\mu \neq \{-1, r, -3, 0\}$ and $\mu^2 \neq 3s$. From Table 2, there are only three possible types of vertex in X. By Lemma 4.1, we know all the vertices in X are of the same type, and we consider the following three cases:

Case 1. The vertices in X are of Type I.

Then (1) or (3) of Theorem 3.8 holds by $s \geq 3$, say, either $\mu = -4$, $r = s = 4$ and $G \cong K_{4,4}$, or $\mu \in \mathbb{N}_+$ and G is $\mu(\mu^2+7\mu+9)/3$-regular of order $\mu(\mu+7)(\mu+6)(\mu+1)/9$ with $K_{3,(\mu^3+7\mu^2+9\mu)/3}$ as a star complement for μ.

Combining the above case of $\mu = r$, result (2) or (3) holds.

Case 2. The vertices in X are of Type II.

Then $H = K_{3,(\mu+2)(\mu^2+4\mu-3)/2}$ and all vertices in X are of type $(1, \mu^2+2\mu-2)$. Applying the compatibility condition (2.8) to vertices u, v of X, since $\mu \neq -3$, we find that

$$\rho_{uv} = \begin{cases} \mu - 1, & u \sim v, \\ 2\mu - 1, & u \nsim v. \end{cases} \tag{4.1}$$

By regularity of G, we have $d_G(v_1) = d_G(v_2) = d_G(v_3)$. This implies the vertices in X are equally divided into three parts, and each vertex in V is adjacent to every vertex of one part, so

$$r = s + |X|/3. \tag{4.2}$$

Now we compute the edges between X and $V(H)$ by two ways, and we have

$$|X|(\mu^2+2\mu-1) = 3(r-s) + s(r-3). \tag{4.3}$$

By (4.2), (4.3) and $s = (\mu+2)(\mu^2+4\mu-3)/2$, we have $(\mu+3)(\mu-1)(\mu-2)|X| = -\frac{3}{2}(\mu+2)(\mu^2+4\mu-3)(\mu+3)(\mu-1)(\mu+4)$. Since $\mu \neq -3$, we consider the following three subcases.

Subcase 2.1. $\mu \neq 1, 2$.

Then

$$|X| = -\frac{3}{2}(\mu+2)(\mu^2+4\mu-3)\frac{\mu+4}{\mu-2} = -3s(1+\frac{6}{\mu-2}).$$

Since $|X| \in \mathbb{Z}$ and $s \in \mathbb{Z}$, we have $\mu \in \mathbb{Q}$. Notice that μ is an algebraic integer, then $\mu \in \mathbb{Z}$. From (4.1), we know that $\mu \in \mathbb{N}_+ \setminus \{1, 2\}$. Thus $|X| < 0$, a contradiction.

Subcase 2.2. $\mu = 2$.

Then $s = \frac{(\mu+2)(\mu^2+4\mu-3)}{2} = 18$, $H = K_{3,18}$ and all vertices in X are of type $(1, 6)$. By (4.2) and (4.3), we obtain $0 = 270$, it is a contradiction.

Subcase 2.3. $\mu = 1$.

Then $s = 3$, $H = K_{3,3}$ with $V = \{v_1, v_2, v_3\}$, $W = \{w_1, w_2, w_3\}$ and all vertices in X are of type $(1, 1)$. Thus $|X| \leq \binom{3}{1}\binom{3}{1} = 9$. From (4.1), we have

$$\rho_{uv} = \begin{cases} 0, & u \sim v, \\ 1, & u \nsim v. \end{cases} \tag{4.4}$$

Since G and H are regular, X induces a r'-regular graph, denoted by $G[X]$.

Claim 1. The graph $G[X]$ cannot contain S_1, S_2 or S_3 as an induced subgraph (see Figure 2).

Proof. If $G[X]$ contains S_1 as an induced subgraph, without loss of generality, we suppose that $u_1 \sim v_1, u_1 \sim w_1$. Then by (4.4), $\rho_{u_1u_2} = \rho_{u_1u_3} = \rho_{u_1u_4} = 0$, and thus vertices u_2, u_3 and u_4 are adjacent to one vertex in $V \setminus \{v_1\}$ and one vertex in $W \setminus \{w_1\}$ such that $\rho_{u_2u_3} = \rho_{u_2u_4} = \rho_{u_3u_4} = 1$, it is impossible.

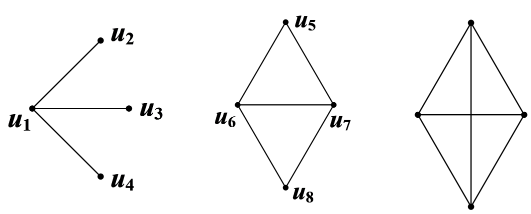
If $G[X]$ contains S_2 as an induced subgraph, since the vertices u_5, u_6 and u_7 are adjacent in pairs, by (4.4), without loss of generality, we can suppose that $u_5 \sim v_1, u_5 \sim w_1, u_6 \sim v_2, u_6 \sim w_2, u_7 \sim v_3, u_7 \sim w_3$. Since $u_8 \sim u_6, u_8 \sim u_7$, by (4.4), vertex u_8 is adjacent to vertices in $V(K_{3,3}) \setminus \{v_2, w_2, v_3, w_3\}$. Thus the H-neighbourhood of u_5 and u_8 is the same, a contradiction.

Similar to the proof of S_2, it's obvious that $G[X]$ cannot contain S_3 as an induced subgraph. □

By $G[X]$ is r'-regular and (4.2), for any $u \in X$, $d_G(u) = 2 + r' = r = 3 + |X|/3$. Then $2 \le r' = 1 + |X|/3 \le 4$ by $|X| \le 9$.

If $r' = 2$, then $|X| = 3$, $G[X] \cong C_3$, $G \cong G_1$ by (4.4);

If $r' = 3$, then $|X| = 6$ and $G[X]$ is connected since the minimum degree of $G[X]$ is equal to $\frac{|X|}{2}$. By [13], there are two non-isomorphic connected 3-regular graphs with 6 vertices (see Figure 3). Since A_2 contains S_1 as an induced subgraph, by Claim 1, $G[X] \ncong A_2$. Then $G[X] \cong A_1$, and the graph G_2 in Figure 1 is the only non-isomorphic graph satisfying (4.4).



(a) S_1 (b) S_2 (c) S_3

Figure 2: induced subgraph

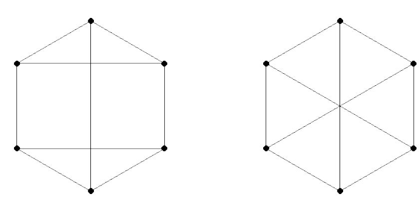
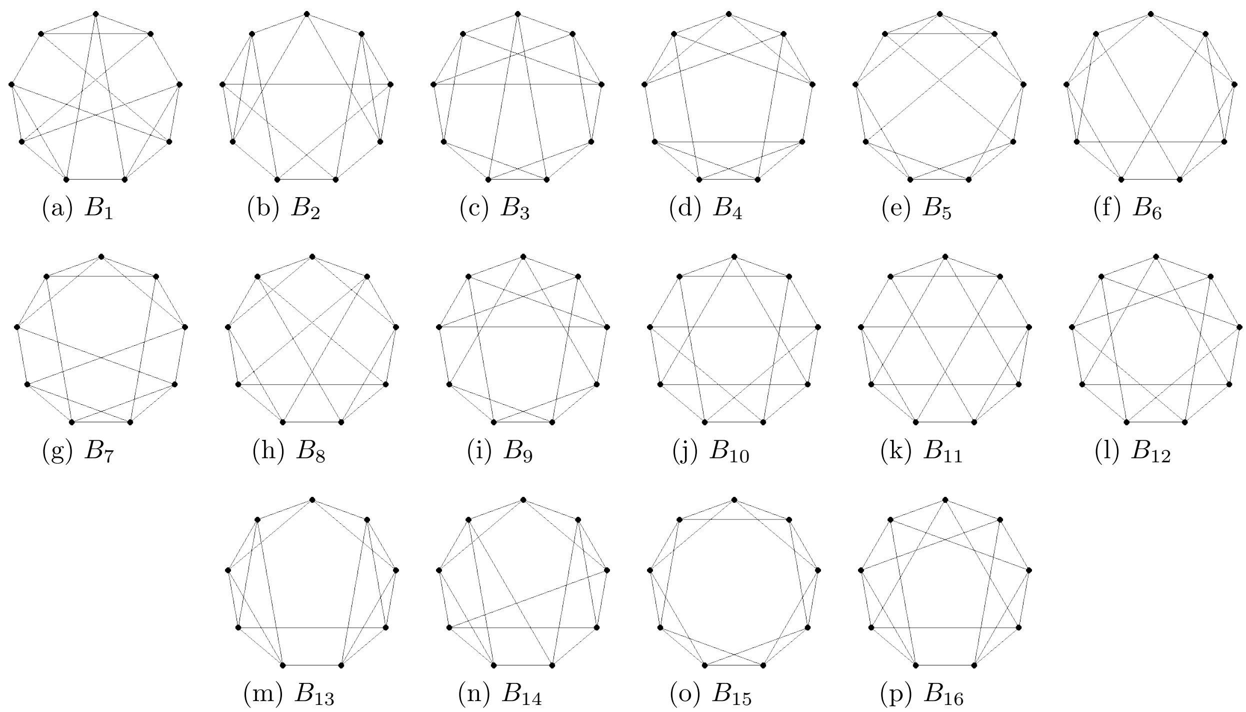

(a) A_1 (b) A_2

Figure 3: 3-regular graphs on 6 vertices.

If $r' = 4$, then $|X| = 9$ and $G[X]$ is connected since the minimum degree of $G[X]$ is equal to $\left\lfloor \frac{|X|}{2} \right\rfloor$. By [13], there are 16 non-isomorphic connected 4-regular graphs with 9 vertices (see Figure 4). Since $B_i (i \in \{2, \ldots, 12\})$ contain S_1 as an induced subgraph, graph B_{13} contain S_3 as an induced subgraph, graph $B_j (j \in \{14, 15, 16\})$ contain S_2 as an induced subgraph, by Claim 1, we have $G[X] \ncong B_i (i \in \{2, \ldots, 16\})$. So $G[X] \cong B_1$, and the graph G_3 in Figure 1 is the only non-isomorphic graph satisfying (4.4).

Combining the proof of Subcase 2.3, (4) holds.



(a) B_1 (b) B_2 (c) B_3 (d) B_4 (e) B_5 (f) B_6 (g) B_7 (h) B_8 (i) B_9 (j) B_{10} (k) B_{11} (l) B_{12} (m) B_{13} (n) B_{14} (o) B_{15} (p) B_{16}

Figure 4: 4-regular graphs on 9 vertices

Case 3. The vertices in X are of Type III.

Then $H = K_{3,(\mu^4+7\mu^3+13\mu^2+2\mu-6)/(\mu+2)}$ and all vertices in X are of type $(2, (\mu^3+3\mu^2-2)/(\mu+2))$. Since $\mu \neq -3$, by (2.8), we have

$$\rho_{uv} = \begin{cases} \frac{2}{\mu+2}, & u \sim v, \\ \frac{\mu^2+2\mu+2}{\mu+2}, & u \nsim v. \end{cases}$$

Then $(\mu+2) \mid 2$ by $\rho_{uv} \in \mathbb{Z}$. Noting that $\rho_{uv} \geq 0$ and $\mu \notin \{-1, -3, 0\}$, it is impossible. Therefore, there is not an r-regular graph with $K_{3,s}$ as a star complement in this case.

Combining the above argument, we complete the proof. $\square$

5 Regular graphs with $K_{s,s}$ as a star complement

By (4) of Theorem 4.2, we know that there are three regular graphs with $K_{3,3}$ as a star complement for $\mu = 1$. In the following, we will study some properties of regular graphs with $K_{s,s}$ as a star complement for an eigenvalue μ, and then give a sharp upper bound for the multiplicity k of μ.

Since μ is not an eigenvalue of $K_{s,s}$, we have $\mu \neq 0$ and $\mu \neq \pm s$. When $\mu = -1$, we have $r \equiv -1(\text{mod } (s+1))$ and $G \cong G(r)$ by Theorem 3.5. So we discuss the case when $\mu \notin \{-1, 0\}$ in the following.

Proposition 5.1. *Let $s \geq 2$ and G be an r-regular graph with $K_{s,s}$ as a star complement for an eigenvalue μ, where $\mu \notin \{-1, 0\}$. Then*

(1) $\mu \in \mathbb{Z}$ and $|\mu| < s$.

(2) If $\mu = 1$, then $s = 3$ and $G \cong G_1$, G_2 or G_3.

(3) If all vertices in the star set X are of the same type, then G is an r-regular graph of order $(2\mu+1)(\frac{r}{\mu}-1)$.

Proof. Clearly, there is no regular graph G with $K_{s,s}$ as a star complement for $\mu = r$ by Theorem 3.4. Thus μ is a non-main eigenvalue. Let $t = s$ in (2.6), we have $0 < a + b = s - \mu \leq 2s$, thus $\mu \in \mathbb{Z}$ and $-s \leq \mu < s$. Since $\mu \neq \pm s$, we have $|\mu| < s$, (1) holds.

Since $t = s$, by (2.6) and (2.7), we have $a = x_1$, $b = x_2$ or $a = x_2$, $b = x_1$ where

$$x_1 = \frac{s - \mu + \sqrt{-(s+\mu)(2\mu^2+\mu-s)}}{2}, \ x_2 = \frac{s - \mu - \sqrt{-(s+\mu)(2\mu^2+\mu-s)}}{2}. \tag{5.1}$$

Since $a, b \in \mathbb{Z}$, $-(s+\mu)(2\mu^2+\mu-s) = (s-\mu^2)^2 - (\mu^2+\mu)^2$ must be a perfect square. Thus, when $\mu = 1$, $(s-1)^2 - 4$ must be a perfect square, so $s = 3$, and thus the graphs G_1, G_2 and G_3 (see Figure 1) are the regular graphs with $K_{3,3}$ as a star complement for $\mu = 1$ by (4) of Theorem 4.2, (2) holds.

Let $u \in X$ be a vertex of type (a, b). Since G is r-regular with $K_{s,s}$ as a star complement and all vertices in X are of the same type, we have $a = b$. By (2.6) and (2.7), we have $a = b = s = -\mu$ or $a = b = \mu^2, s = 2\mu^2 + \mu$.

If $a = b = s = -\mu$, then $|X| = 1$, $r = 2s = 1 + s$. Thus $s = 1$ and $\mu = -1$, a contradiction.

If $a = b = \mu^2, s = 2\mu^2 + \mu$, counting the edges between X and $V(H)$ in two ways, we have $(a+b) \cdot |X| = 2s(r-s)$. Thus $|V(G)| = |X| + 2s = (2\mu+1)(\frac{r}{\mu}-1)$, (3) holds. □

Remark 5.2. *In fact, there are a lot of regular graphs with $H = K_{s,s}$ as a star complement. For example, when $\mu = -2$, it follows from (5.1) that $(s-4)^2 - 4$ must be a perfect square, thus $s = 2$ or 6.*

When $s = 2$, we have $a = b = 2$ by (5.1). Thus $|X| = 1$ by Lemma 2.4. In this situation, G is not regular.

But when $s = 6$, we have $a = b = 4$ by (5.1). From (2.8), we have $\rho_{uv} = \begin{cases} 4, & u \nsim v, \\ 6, & u \sim v. \end{cases}$ Since $|X| \cdot 8 = 12(r-6)$, we have $|X| = \frac{3(r-6)}{2} \in \mathbb{Z}$, and then r is even.

Let (V, W) be the bipartition of $H = K_{6,6}$ with $V = \{v_1, v_2, \cdots, v_6\}$, $W = \{w_1, w_2, \cdots, w_6\}$.

If $r = 8$, then $|X| = 3$. The graph G_4 defined as follows is a regular graph with $K_{6,6}$ as a star complement for $\mu = -2$: $X = \{u_1, u_2, u_3\}$, $V(G_4) = V(H) \cup X$, $G_4[X] = 3K_1$, and $N_H(u_1) = \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$, $N_H(u_2) = \{v_3, v_4, v_5, v_6, w_3, w_4, w_5, w_6\}$, $N_H(u_3) = \{v_1, v_2, v_5, v_6, w_1, w_2, w_5, w_6\}$. The spectrum of G_4 is $[-6, -2^3, 0^8, 2^2, 8]$.

If $r = 10$, then $|X| = 6$. The graph G_5 defined as follows is a regular graph with $K_{6,6}$ as a star complement for $\mu = -2$: $X = \{u_1, u_2, \ldots, u_6\}$, $V(G_5) = V(H) \cup X$, $G_5[X] = C_6$ with the edge set $\{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1\}$, and $N_H(u_1) = \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$, $N_H(u_2) = \{v_2, v_3, v_4, v_5, w_2, w_3, w_4, w_5\}$, $N_H(u_3) = \{v_3, v_4, v_5, v_6, w_3, w_4, w_5, w_6\}$, $N_H(u_4) = \{v_1, v_4, v_5, v_6, w_1, w_4, w_5, w_6\}$, $N_H(u_5) = \{v_1, v_2, v_5, v_6, w_1, w_2, w_5, w_6\}$, $N_H(u_6) = \{v_1, v_2, v_3, v_6, w_1, w_2, w_3, w_6\}$. The spectrum of G_5 is $[-6, -2^6, 0^6, 1^2, 3^2, 10]$.

Since $|X| \leq \frac{1}{2}(q+1)(q-2) = 65$, where $q = |V(H)|$ ([4]), there are various choices of r. It can be predicted that there are a lot of graphs that satisfy the conditions. Then the commonalities of the regular graphs with $K_{s,s}$ as a star complement seems to be an interesting question worth studying.

It is shown in [17] that if G is a connected r-regular graph of order n with $\mu \notin \{-1, 0\}$ as an eigenvalue of multiplicity k and $r > 2$, $q = n-k$, then $k \leq \frac{1}{2}(r-1)q$. In the following, we will show that when G has $K_{s,s}$ ($q = 2s \geq 2$) as a star complement for μ, then $k \leq s(r-s) = \frac{1}{2}q(r - \frac{q}{2}) \leq \frac{1}{2}(r-1)q$.

For subsets U', V' of $V(G)$, we write $E(V', U')$ for the set of edges between U' and V'. The authors of [5] have determined all the graphs with a star set X for which $E(X, \overline{X})$ is a perfect matching. The result is as follows.

Theorem 5.3. ([5]) *Let G be a graph with X as a star set for an eigenvalue μ. If $E(X, \overline{X})$ is a perfect matching, then one of the following holds:*
(1) *$G = K_2$ and $\mu = \pm 1$;* (2) *$G = C_4$ and $\mu = 0$;* (3) *G is the Petersen graph and $\mu = 1$.*

Theorem 5.4. *Let G be an r-regular graph of order n with $K_{s,s}$ as a star complement for the eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity k. Then $k \leq s(r-s)$, equivalently $n \leq s(r-s+2)$, with equality if and only if $\mu = 1$, $G \cong G_1, G_2$ or G_3 (see Figure 1).*

Proof. Let $H = K_{s,s}$ and $V(H) = V \cup W$ with $|V| = |W| = s$. By Lemma 2.4, $V(K_{s,s})$ is a location-dominating set, and so G is connected. By Lemma 2.4, we have $|N_H(u)| \geq 1$. Thus $k = |X| \leq \sum_{u \in X} |N_H(u)| = |E(X, \overline{X})| = 2s(r-s)$.

When the equality holds, we have $|N_H(u)| = 1$ for all $u \in X$. Since the neighbourhoods $N_H(u)$ $(u \in X)$ are distinct, any vertex in H has at most one adjacent vertex in X, thus $2s = |\overline{X}| \geq |X| = 2s(r-s)$ which means $r \leq s+1$. On the other hand, we have $r \geq s+1$ by G is r-regular and connected. Thus $r = s+1$ and then $|X| = 2s$. Therefore $E(X, \overline{X})$ is a perfect matching, but there is no such graph G by Theorem 5.3.

Let $t = s$ in (2.6), we find that $|N_H(u)| = a + b = s - \mu$ is a constant, which means $|N_H(u_1)| = |N_H(u_2)|$ for any $u_1, u_2 \in X$. Therefore, we have $|N_H(u)| \geq 2$ for any $u \in X$ and $2k \leq \sum\limits_{u \in X} |N_H(u)| = |E(X, \overline{X})| = 2s(r-s)$, equivalently $n \leq s(r-s+2)$, with equality if and only if $|N_H(u)| = 2$ for any $u \in X$.

If $n = s(r-s+2)$, we have $\mu = s-2$ and the possible types for the vertices in X are $(1,1), (0,2), (2,0)$.

If there are two vertices of type $(0,2)$ (or $(2,0)$) in X, then it follows from (2.8) that $\rho_{uv} = \begin{cases} \frac{s}{s-1}, & u \nsim v, \\ \frac{s^2-4s+2}{1-s}, & u \sim v. \end{cases}$ By (2) of Lemma 2.4, we have $\rho_{uv} \neq 2$, then $\rho_{uv} = 0$ or 1, it is a contradiction with $s \in \mathbb{Z}_+$. Therefore, there is at most one vertex of type $(0,2)$ (or $(2,0)$).

If there is one vertex of type $(2,0)$ and one vertex of type $(0,2)$ in X, then it follows from (2.8) that $\rho_{uv} = \begin{cases} \frac{s-2}{s-1}, & u \nsim v, \\ \frac{s^2-4s+4}{1-s}, & u \sim v. \end{cases}$ Clearly, $\rho_{uv} = 0$, $s = 2$, and $\mu = 0$, it is a contradiction. Therefore, the vertex of type $(2,0)$ and the vertex of type $(0,2)$ cannot exist at the same time in X.

Suppose that X contains a vertex of type $(0,2)$ (or $(2,0)$), then $|E(X,V)| \neq |E(X,W)|$, a contradiction. Thus all vertices in X are of type $(1,1)$. From (2.8), we have $\rho_{uv} = \begin{cases} 1, & u \nsim v, \\ 3-s, & u \sim v. \end{cases}$

If for any $u, v \in X$, $u \nsim v$, then $r = 2$ and $H \cong K_{1,1}$, $G \cong C_3$ and thus $s = 1$ and $\mu = s-2 = -1$, a contradiction.

If there exists $u, v \in X$ such that $u \sim v$, then $3 - s = 0$ or 1 by (2) of Lemma 2.4. If $3 - s = 1$, then $s = 2$ and $\mu = 0$, a contradiction. If $3 - s = 0$, then $s = 3$, $\mu = 1$ and thus $G = G_1, G_2$ or G_3 (see Figure 1) by (4) of Theorem 4.2.

The proof is completed. □

Competing interests

The authors declare that they have no competing interests.

References

[1] L. Asgharsharghi, D. Kiani, On regular graphs with complete tripartite star complements, Ars Combin. 122 (2015) 431–437.

[2] F.K. Bell, Characterizing line graphs by star complements, Linear Algebra Appl. 296 (1999) 15-25.

[3] F.K. Bell, Line graphs of bipartite graphs with hamiltonian paths, J. Graph Theory. 43 (2003) 137-149.

[4] F.K. Bell, P. Rowlinson, On the multiplicities of graph eigenvalues, Bull. Lond. Math. Soc. 35 (2003) 401–408.

[5] N.E. Clarke, W.D. Garraway, C.A. Hickman, R.J. Nowakowski, Graphs where star sets are matched to their complements, J. Combin. Math. Combin. Comput. 37 (2001) 177–185.

[6] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Application, 3rd edition, Jonah Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.

[7] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of Graphs, Cambridge University Press, Cambridge, 1997.

[8] D. Cvetković, P. Rowlinson, S. Simić, Some characterization of graphs by star complements, Linear Algebra Appl. 301 (1999) 81–97.

[9] D. Cvetković, P. Rowlinson, S. Simić, Spectral Generalizations of Line Graphs, Cambridge University Press, Cambridge, 2004.

[10] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2010.

[11] M. Ellingham, Basic subgraphs and graph spectra, Australas. J. Combin. 8 (1993) 247–265.

[12] X. Fang, L. You, Y. Huang, Maximal graphs with a prescribed complete bipartite graph as a star complement, AIMS Math. 6 (2021) 7153–7169.

[13] M. Meringer, Fast generation of regular graphs and construction of cages, J Graph Theory. 30 (1999) 137-146.

[14] F. Ramezani, B. Tayfeh-Rezaie, Graphs with prescribed star complement for the eigenvalue 1, Ars Combin. 116 (2014) 129–145.

[15] P. Rowlinson, On bipartite graphs with complete bipartite star complements, Linear Algebra Appl. 458 (2014) 149–160.

[16] P. Rowlinson, An extension of the star complement technique for regular graphs, Linear Algebra Appl. 557 (2018) 496-507.

[17] P. Rowlinson, Eigenvalue multiplicity in regular graphs, Discrete Appl. Math. 269 (2019) 11–17.

[18] P. Rowlinson, P.S. Jackson, On graphs with complete bipartite star complements, Linear Algebra Appl. 298 (1999) 9–20.

[19] P. Rowlinson, B. Tayfeh-Rezaie, Star complements in regular graphs: old and new results, Linear Algebra Appl. 432 (2010) 2230–2242.

[20] Z. Stanić, On graphs whose second largest eigenvalue equals 1 – the star complement technique, Linear Algebra Appl. 420 (2) (2007) 700–710.

[21] Z. Stanić, S.K. Simić, On graphs with unicyclic star complement for 1 as the second largest eigenvalue, Faculty of Mathematics, University of Belgrade, 351 (2005) 475-484.

[22] J. Wang, X. Yuan, L. Liu, Regular graphs with a prescribed complete multipartite graph as a star complement, Linear Algebra Appl. 579 (2019) 302–319.

[23] Y. Yang, Q. Huang, J. Wang, On a conjecture for regular graphs with complete multipartite star complement, arXiv:1912.07594v2, 2021.

[24] X. Yuan, H. Chen, L. Liu, On the characterization of graphs by star complements, Linear Algebra Appl. 533 (2017) 491-506.

[25] X. Yuan, Q. Zhao, L. Liu, H. Chen. On graphs with prescribed star complements, Linear Algebra Appl. 559 (2018) 80-94.