Laplace hyperfunctions via Čech-Dolbeault cohomology

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Abstract

The paper studies several properties of Laplace hyperfunctions introduced by H. Komatsu in the one dimensional case and by the authors in the higher dimensional cases from the viewpoint of Čech-Dolbeault cohomology theory, which enables us, for example, to construct the Laplace transformation and its inverse in a simple way. We also give some applications to a system of PDEs with constant coefficients.

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1 Introduction

A Laplace hyperfunction on the one dimensional space was first introduced by H. Komatsu ([5]-[10]) to justify the operational calculus for arbitrary functions without any growth condition at infinity. After his great success of the one dimensional Laplace hyperfunctions, the authors of this paper established an Oka type vanishing theorem (Theorem 3.7 [2]) and an edge of the wedge type theorem (Theorem 3.12 [3]) for the sheaf of holomorphic functions of several variables with exponential growth at infinity. Thanks to these fundamental theorems, we were succeeded in defining the sheaf $\mathscr{B}^{\exp}$ of Laplace hyperfunction of several variables as a local cohomology groups along the radial compactification $\mathbb{D}_{\mathbb{R}^n} = \mathbb{R}^n \sqcup S^{n-1}$ of $\mathbb{R}^n$ with coefficients in the sheaf $\mathscr{O}^{\exp}$ of holomorphic functions with exponential growth, and also showing that $\mathscr{B}^{\exp}$ is a soft sheaf (Corollary 5.9 [3]).

Since a Laplace hyperfunction is defined as an element of the local cohomology group, to understand its concrete expression we need some interpretation of the local cohomology group, which is done by usually considering its Čech representation through the relative Čech cohomology group or more generally its “intuitive representation” introduced in [11] Section 4 (see Subsection 4.3 also).

Recently T. Suwa [12] proposed another method to compute a local cohomology group by using a soft resolution of a coefficient sheaf, which is called the Čech-Dolbeault cohomology when we distinguish it from the usual sheaf cohomology. This implies, in particular, the sheaf of Sato's hyperfunction can be computed with the famous Dolbeault resolution of holomorphic functions by using the Čech-Dolbeault cohomology theory. In fact, N. Honda, T. Izawa and T. Suwa [1] studies Sato's hyperfunctions from the viewpoint of Čech-Dolbeault cohomology theory and finds that several operations to a hyperfunction such as the integration of a hyperfunction along fibers, etc. have very simple and easily understandable descriptions in this framework because a hyperfunction is represented by a pair (μ_1, μ_{01}) of C^∞-differential forms.

The purpose of this paper is to study Laplace hyperfunctions from the viewpoint of Čech-Dolbeault cohomology theory, which gives us several advantages to their treatments like the case of Sato's hyperfunctions. To make this point more clear, we briefly explain, as such an example, an inverse Laplace transformation $\mathcal{IL}$ in the framework of Čech-Dolbeault cohomology: It is given by a quite simple form (see Definition 7.0.3 for details)

$$\mathcal{IL}_\omega(f) = \left[\left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\gamma^*} \rho(\omega)(\mathrm{Im}\,\zeta/|\mathrm{Im}\,\zeta|,\, z)\, e^{\zeta z} f(\zeta) d\zeta\right],$$

where γ^* is an appropriate real n-dimensional chain asymptotic to $\sqrt{-1}\mathbb{R}^n$ and a pair $\rho(\omega)(\theta, z)$ of C^∞-differential forms represents, roughly speaking, the constant function 1 in Čech-Dolbeault cohomology on $S^{n-1} \times \mathbb{D}_{\mathbb{C}^n}$ which also satisfies the support condition

$$\mathrm{supp}(\rho(\omega)) \subset \widehat{\ }\left\{(\theta, z) \in S^{n-1} \times \mathbb{C}^n;\ \langle \theta, \mathrm{Im}\, z\rangle > 0\right\} \subset S^{n-1} \times \mathbb{D}_{\mathbb{C}^n}.$$

Here $\mathbb{D}_{\mathbb{C}^n} = \mathbb{C}^n \sqcup S^{2n-1}$ is the radial compactification of $\mathbb{C}^n$, and see Definition 2.2.2 for the symbol $^\wedge(\bullet)$. Note that the above support condition for $\rho(\omega)$ guarantees the convergence of the integral. The existence of such a kernel $\rho(\omega)e^{\zeta z}$ with the desired support condition is crucial in the definition of the inverse Laplace transformation, which comes from the fact that in Čech-Dolbeault cohomology group the support of a representative can be cut off in a desired way.

Furthermore, as was seen in the proof of Lemma 7.0.5, we can estimate the support of a Laplace hyperfunction $\mathcal{IL}_\omega(f)$ by using the fact that any derivative of $\rho(\omega)$ becomes zero as a cohomology class because it is cohomologically constant. Thus Čech-Dolbeault cohomology theory gives us several new ideas and methods in a study of Laplace hyperfunctions.

The paper is organized as follows: In Section 2, after a short review of Čech-Dolbeault cohomology theory, we introduce several geometrical notations which are used through the whole paper. Then we establish the fundamental de-Rham and Dolbeault theorems in Section 3 and give the definition of the sheaf of Laplace hyperfunctions in Section 4. We also give several expressions of Laplace hyperfunctions via Čech cohomology and Čech-Dolbeault cohomology in the same section. The one of important facts in hyperfunction theory is the notion of boundary values of holomorphic functions. We construct a boundary value morphism for Laplace hyperfunctions in Section 5. The Laplace transformation and its inverse in the framework of Čech-Dolbeault cohomology are defined in Sections 6 and 7, and the fact that they are inverse to each other is shown in Section 8. The last section gives some applications to a system of PDEs with constant coefficients.

2 Preparations

Through the paper, we use the language of the derived categories: Notations $\mathrm{Mod}(\mathbb{Z})$, $\mathrm{Mod}(\mathbb{Z}_X)$, $\mathrm{C}^+(\mathrm{Mod}(\mathbb{Z}_X))$, $\mathrm{K}^+(\mathrm{Mod}(\mathbb{Z}_X))$, $\mathrm{D}^+(\mathrm{Mod}(\mathbb{Z}_X))$, etc. are the same as those in the book [4], for example, $\mathrm{Mod}(\mathbb{Z})$ denotes the category of abelian groups, $\mathrm{Mod}(\mathbb{Z}_X)$ the category of sheaves on X of abelian groups, $\mathrm{C}^+(\mathrm{Mod}(\mathbb{Z}_X))$ the category of complexes bounded below of sheaves on X of abelian groups, and $\mathrm{D}^+(\mathrm{Mod}(\mathbb{Z}_X))$ is the subcategory consisting of complexes bounded below of the derived category of $\mathrm{Mod}(\mathbb{Z}_X)$.

2.1 Čech-Dolbeault complex

In this subsection, we briefly recall the definition of a Čech-Dolbeault complex. For details, refer the readers to [1]. Let X be a locally compact and σ-compact Hausdorff space and K its closed subset, and let $\mathcal{S} = \{U_i\}_{i\in\Lambda}$ be an open covering of X and Λ' a subset of Λ such that $\mathcal{S}' = \{U_i\}_{i\in\Lambda'}$ $(\Lambda' \subset \Lambda)$ becomes an open covering of $X \setminus K$. For $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_k) \in \Lambda^{k+1}$, we set

$$U_\alpha = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}.$$

Let $\mathscr{F}$ be a sheaf of $\mathbb{Z}$ modules on X. We denote by $C(\mathcal{S}, \mathcal{S}'; \mathscr{F})$ the Čech complex of $\mathscr{F}$ with respect to the pair $(\mathcal{S}, \mathcal{S}')$ of coverings, that is, $C(\mathcal{S}, \mathcal{S}'; \mathcal{F})$ is the complex

$$\cdots \xrightarrow{\delta^{k-1}} C^k(\mathcal{S}, \mathcal{S}'; \mathcal{F}) \xrightarrow{\delta^k} C^{k+1}(\mathcal{S}, \mathcal{S}'; \mathcal{F}) \xrightarrow{\delta^{k+1}} C^{k+2}(\mathcal{S}, \mathcal{S}'; \mathcal{F}) \xrightarrow{\delta^{k+2}} \cdots .$$

Here $C^k(\mathcal{S}, \mathcal{S}'; \mathcal{F})$ consists of alternating sections $\{s_\alpha\}_{\alpha\in\Lambda^{k+1}}$ with $s_\alpha \in \mathscr{F}(U_\alpha)$ and $s_\alpha = 0$ if $\alpha \in (\Lambda')^{k+1}$, and the differential δ^k is defined by

$$\delta^k(\{s_\alpha\}_{\alpha\in\Lambda^{k+1}})_\beta = \sum_{i=1}^{k+2}(-1)^{i+1} s_{\beta^{\vee_i}}|_{U_\beta} \qquad (\beta \in \Lambda^{k+2}),$$

where $\beta^{\vee_i}$ denotes the sequence such that the i-th element of β is removed.

Let $\mathscr{F}^\bullet$ be a complex with bounded below of sheaves of $\mathbb{Z}$ modules

$$\cdots \xrightarrow{d^{k-1}} \mathscr{F}^k \xrightarrow{d^k} \mathscr{F}^{k+1} \xrightarrow{d^{k+1}} \mathscr{F}^{k+2} \xrightarrow{d^{k+2}} \cdots .$$

Then we denote by $C(\mathcal{S}, \mathcal{S}')(\mathscr{F}^\bullet)$ the single complex associated with the double complex

$$\begin{array}{ccccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\xrightarrow{d^{q-1}} & C^{p+1}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^q) & \xrightarrow{d^q} & C^{p+1}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^{q+1}) & \xrightarrow{d^{q+1}} & C^{p+1}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^{q+2}) & \xrightarrow{d^{q+2}} \\
 & \uparrow \delta^p & & \uparrow \delta^p & & \uparrow \delta^p & \\
\xrightarrow{d^{q-1}} & C^{p}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^q) & \xrightarrow{d^q} & C^{p}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^{q+1}) & \xrightarrow{d^{q+1}} & C^{p}(\mathcal{S}, \mathcal{S}'; \mathscr{F}^{q+2}) & \xrightarrow{d^{q+2}} \\
 & \uparrow & & \uparrow & & \uparrow &
\end{array},$$

that is, the complex is given by

$$C^k(\mathcal{S}, \mathcal{S}')(\mathscr{F}^\bullet) = \bigoplus_{p+q=k} C^p(\mathcal{S}, \mathcal{S}'; \mathscr{F}^q)$$

and, for $\omega = \underset{p+q=k}{\oplus} \omega^{p,q} \in C^k(\mathcal{S}, \mathcal{S}')(\mathscr{F}^\bullet)$,

$$d^k_{C(\mathcal{S},\mathcal{S}')(\mathscr{F}^\bullet)}(\omega) = \underset{p+q=k+1}{\oplus} (\delta^{p-1}(\omega^{p-1,q}) + (-1)^p d^{q-1}(\omega^{p,q-1})).$$

Let $\mathscr{F}$ be a sheaf of $\mathbb{Z}$ modules on X and $\mathscr{F} \to \mathscr{F}^\bullet$ a resolution of $\mathscr{F}$ by soft sheaves. Then we sometimes call the complex $C(\mathcal{S}, \mathcal{S}')(\mathscr{F}^\bullet)$ the Čech-Dolbeault complex of $\mathscr{F}$ (with respect to the pair $(\mathcal{S}, \mathcal{S}')$ of coverings).

Theorem 2.1.1 ([1])**.** Under the above situation, there exists the canonical isomorphism in $\mathrm{D}^+(\mathrm{Mod}(\mathbb{Z}_X))$

$$\mathbf{R}\Gamma_K(X; \mathscr{F}) \simeq C(\mathcal{S}, \mathcal{S}')(\mathscr{F}^\bullet).$$

Example 2.1.2. If we take

$$\mathcal{V} = \{V_0 = X \setminus K, V_1 = X\}, \qquad \mathcal{V}' = \{V_0\}$$

as coverings of X and $X \setminus K$, then the complex $C(\mathcal{V}, \mathcal{V}')(\mathscr{F}^\bullet)$ becomes quite simple as follows:

$$C^k(\mathcal{V}, \mathcal{V}')(\mathscr{F}^\bullet) = \mathscr{F}^k(V_1) \oplus \mathscr{F}^{k-1}(V_{01}),$$

where $V_{01} = V_0 \cap V_1$, and $d^k_{C(\mathcal{V},\mathcal{V}')(\mathscr{F}^\bullet)}$ is given by

$$\mathscr{F}^k(V_1) \oplus \mathscr{F}^{k-1}(V_{01}) \ni (\omega_1, \omega_{01}) \mapsto (d^k\omega_1, \omega_1|_{V_{01}} - d^{k-1}\omega_{01}) \in \mathscr{F}^{k+1}(V_1) \oplus \mathscr{F}^k(V_{01}).$$

2.2 Radial compactification

Let M be an n-dimensional real vector space with the norm $|\bullet|$ and $E = M \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $\mathbb{D}_E$ (resp. $\mathbb{D}_M$) the radial compactification $E \sqcup S_\infty^{2n-1}$ (resp. $M \sqcup S_\infty^{n-1}$) of E (resp. M) as usual (see Definition 2.1 [3]). Note that $\mathbb{D}_M = \overline{M}$ holds, where $\overline{M}$ is the closure of M in $\mathbb{D}_E$. We also set $M_\infty = \mathbb{D}_M \setminus M$ and $E_\infty = \mathbb{D}_E \setminus E$.

We define an $\mathbb{R}_+$-action on $\mathbb{D}_E$ by, for $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{D}_E$,

$$\lambda x = \begin{cases} \lambda x & \text{if } x \in E, \\ x & \text{if } x \in E_\infty. \end{cases}$$

The $\mathbb{R}_+$-action on $\mathbb{D}_M$ is defined to be the restriction of the one in $\mathbb{D}_E$ to $\mathbb{D}_M$. And we also define an addition for $a \in M$ (resp. $a \in E$) and $x \in \mathbb{D}_M$ (resp. $x \in \mathbb{D}_E$) by

$$a + x = \begin{cases} a + x & \text{if } x \in M \text{ (resp. } x \in E), \\ x & \text{if } x \in M_\infty \text{ (resp. } x \in E_\infty). \end{cases}$$

Definition 2.2.1. A subset K in $\mathbb{D}_M$ is said to be a cone with vertex $a \in M$ in $\mathbb{D}_M$ if there exists an $\mathbb{R}_+$-conic set $L \subset \mathbb{D}_M$ such that

$$K = a + L.$$

The notion of a cone in $\mathbb{D}_E$ is similarly defined. We often need to extend an open subset in E to the one in $\mathbb{D}_E$.

Definition 2.2.2. Let V be an open subset in E, we define the open subset $\widehat{V}$ in $\mathbb{D}_E$ by

$$\widehat{V} = \mathbb{D}_E \setminus \overline{(E \setminus V)}.$$

Note that we sometimes write $^\frown V$ instead of $\widehat{V}$. For an open subset U in M, we can define an open subset $\widehat{U}$ in $\mathbb{D}_M$ in the same way as that in $\mathbb{D}_E$.

Lemma 2.2.3. $\widehat{V}$ is the largest open subset W in $\mathbb{D}_E$ with $V = W \cap E$.

In Definition 3.4 of [2], we introduced the notion that an open subset U in $\mathbb{D}_E$ is regular at ∞. In this paper, we call such an open subset "1-regular at ∞" to distinguish it from the similar notion for a closed subset.

Definition 2.2.4. A closed subset $F \subset \mathbb{D}_E$ is said to be regular if $\overline{F \cap E} = F$ holds.

It is clear that a closed cone $K \subset \mathbb{D}_E$ with vertex a becomes regular if and only if

$$\pi_{E_\infty}(x) \in K \cap E_\infty \iff a + x \in K$$

holds for any $x \in E \setminus \{0\}$. Here $\pi_{E_\infty} : E \setminus \{0\} \to E_\infty = (E \setminus \{0\})/\mathbb{R}_+$ is the canonical projection. Note also that, for example, the set consisting of the only one point in E_∞ is a closed cone in our definition, however, which is not regular.

Lemma 2.2.5 ([2], Lemma 3.5)**.** Let $K \subset \mathbb{D}_E$ be a closed cone. The conditions below are equivalent:

1. K is regular.

2. $\hat{}(E \setminus K) = \mathbb{D}_E \setminus K$ holds.

3. $\mathbb{D}_E \setminus K$ is a 1-regular at ∞.

The following definition are often used through the paper: For open subsets U and Γ in M, define an open subset $U \hat{\times} \sqrt{-1}\Gamma$ in $\mathbb{D}_E$ by

$$U \hat{\times} \sqrt{-1}\Gamma = \hat{}(U \times \sqrt{-1}\Gamma) \subset \mathbb{D}_E. \tag{2.1}$$

Let M^* and E^* be dual vector spaces of M and E, respectively. Then we can define the radial compactification $\mathbb{D}_{M^*}$ and M^*_∞ (resp. $\mathbb{D}_{E^*}$ and E^*_∞) for a vector space M^* (resp. E^*) in the same way as those of $\mathbb{D}_M$ and M_∞ (resp. $\mathbb{D}_E$ and E_∞).

We also define the open subset $\widehat{V}$ in $\mathbb{D}_{E^*}$ for an open subset V in E^* in the same way as that in $\mathbb{D}_E$, that is,

$$\widehat{V} = \mathbb{D}_{E^*} \setminus \overline{(E^* \setminus V)}. \tag{2.2}$$

Now we introduce the subset $\mathrm{N}^*_{pc}(Z)$ in E^*_∞ and the canonical projection ϖ_∞ as follows: The canonical projection $\varpi_\infty : E^*_\infty \setminus \sqrt{-1}M^*_\infty \to M^*_\infty$ is defined by

$$E^*_\infty \setminus \sqrt{-1}M^*_\infty = ((M^* \setminus \{0\}) \oplus \sqrt{-1}M^*)/\mathbb{R}_+ \xrightarrow{\varpi_\infty} (M^* \setminus \{0\})/\mathbb{R}_+ = M^*_\infty, \tag{2.3}$$

which is induced from the canonical projection $E^* = M^* \oplus \sqrt{-1}M^* \to M^*$, that is, ϖ_∞ is given by

$$E^*_\infty \setminus \sqrt{-1}M^*_\infty \ni \xi + \sqrt{-1}\eta \ ((\xi, \eta) \in S^{2n-1},\ \xi \neq 0) \ \mapsto \ \xi/|\xi| \in M^*_\infty.$$

Let Z be a subset in $\mathbb{D}_E$.

Definition 2.2.6. The subset $\mathrm{N}^*_{pc}(Z)$ in E^*_∞ is defined by

$$\{\zeta \in E^*_\infty;\ \mathrm{Re}\,\langle z, \zeta\rangle > 0\ (\forall z \in \overline{Z} \cap E_\infty)\}.$$

Note that $\mathrm{N}^*_{pc}(Z)$ is an open subset in E^*_∞ and that $\mathrm{N}^*_{pc}(Z) = E^*_\infty$ holds if $\overline{Z} \cap E_\infty = \emptyset$.

Definition 2.2.7. We say that Z is properly contained in a half space of $\mathbb{D}_E$ with direction $\zeta \in E^*_\infty$ if there exists $r \in \mathbb{R}$ such that

$$\overline{Z} \subset \hat{}\{z \in E;\ \mathrm{Re}\langle z, \zeta\rangle > r\}, \tag{2.4}$$

where ζ is regarded as a unit vector in E^*. If a subset Z is properly contained in a half space of $\mathbb{D}_E$ with some direction, then Z is often said to be proper in $\mathbb{D}_E$.

Then it is easy to see:

Lemma 2.2.8. Let $\zeta \in E^*_\infty$ and $Z \subset \mathbb{D}_E$. The Z is properly contained in a half space of $\mathbb{D}_E$ with direction ζ if and only if $\zeta \in \mathrm{N}^*_{pc}(Z)$.

Example 2.2.9. Let G be an $\mathbb{R}_+$-conic closed subset in E and $a \in E$. Set $K = \overline{a + G} \subset \mathbb{D}_E$. Then we have

$$\mathrm{N}^*_{pc}(K) = \mathrm{N}^*_{pc}(G) = \hat{}(G^\circ) \cap E^*_\infty,$$

where G° is a dual open cone of G in E^*, that is,

$$G^\circ = \{\zeta \in E^*;\ \mathrm{Re}\,\langle z, \zeta\rangle > 0\ (\forall z \in G)\}.$$

3 Several variants of de-Rham and Dolbeault theorems on $\mathbb{D}_E$

Let V be an open subset in $\mathbb{D}_E$ and f a measurable function on $V \cap E$. We fix a coordinate system $z = x + \sqrt{-1}y$ of E in what follows.

We say that f is of exponential type (at ∞) on V if, for any compact subset K in V, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\, f(z)|$ is essentially bounded on $K \cap E$, i.e.,

$$||\exp(-H_K|z|)\, f(z)||_{L^\infty(K\cap E)} < +\infty. \tag{3.1}$$

Set

$$\mathscr{Q}_{\mathbb{D}_E}(V) := \left\{ f \in C^\infty(V \cap E);\ \begin{array}{l}\text{Any higher derivative of } f \text{ with respect to variables } z \text{ and } \bar{z}\\ \text{is of exponential type on } V\end{array} \right\}.$$

Then it is easy to see that $\{\mathscr{Q}_{\mathbb{D}_E}(V)\}_V$ forms the sheaf $\mathscr{Q}_{\mathbb{D}_E}$ on $\mathbb{D}_E$. The following easy lemma is crucial in our theory:

Lemma 3.0.1. The sheaf $\mathscr{Q}_{\mathbb{D}_E}$ is fine.

Let $\mathscr{Q}^{p,q}_{\mathbb{D}_E}$ denote the sheaf on $\mathbb{D}_E$ of $(p,\, q)$-forms with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$, and set

$$\mathscr{Q}^k_{\mathbb{D}_E} = \bigoplus_{p+q=k} \mathscr{Q}^{p,q}_{\mathbb{D}_E}.$$

Now we define the de-Rham complex $\mathscr{Q}^\bullet_{\mathbb{D}_E}$ on $\mathbb{D}_E$ with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$ by

$$0 \longrightarrow \mathscr{Q}^0_{\mathbb{D}_E} \xrightarrow{d} \mathscr{Q}^1_{\mathbb{D}_E} \xrightarrow{d} \dots \xrightarrow{d} \mathscr{Q}^{2n}_{\mathbb{D}_E} \longrightarrow 0,$$

and the Dolbeault complex $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}$ on $\mathbb{D}_E$ by

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \mathscr{Q}^{p,1}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathscr{Q}^{p,n}_{\mathbb{D}_E} \longrightarrow 0.$$

Let $\mathscr{O}^{\exp}_{\mathbb{D}_E}$ (resp. $\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}$) denote the sheaf of holomorphic functions (resp. p-forms) of exponential type (at ∞) on $\mathbb{D}_E$. The following proposition can be shown by the similar arguments as those in the proof of the usual de-Rham and Dolbeault theorems with bounds.

Proposition 3.0.2. Both the canonical morphisms of complexes below are quasi-isomorphic:

$$\mathbb{C}_{\mathbb{D}_E} \longrightarrow \mathscr{Q}^\bullet_{\mathbb{D}_E}, \qquad \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E} \longrightarrow \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}.$$

We show, in [2], the Oka type vanishing theorem of holomorphic functions of exponential type on a Stein domain. Hence the above proposition immediately concludes:

Corollary 3.0.3 ([2], Theorem 3.7)**.** Assume that $V \cap E$ is Stein and that V is 1-regular at ∞. Then we have the quasi-isomorphism

$$\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}(V) \longrightarrow \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(V).$$

Furthermore, the edge of the wedge type theorem of exponential type has been also established in our previous papers:

Theorem 3.0.4 ([3], Theorem 3.12, Proposition 4.1)**.** The complexes $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E})$ and $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$ are concentrated in degree n. Furthermore, $\mathscr{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$ is isomorphic to $\mathbb{Z}_{\mathbb{D}_M}$.

In subsequent sections, we need to extend our de-Rham theorem to the one with a parameter. Let T be a real analytic manifold and set $Y := T \times \mathbb{D}_E$ and $Y_\infty = T \times (\mathbb{D}_E \setminus E)$. We denote by $p_T : Y \to T$ (resp. $p_{\mathbb{D}_E} : Y \to \mathbb{D}_E$) the canonical projection to T (resp. $\mathbb{D}_E$).

Let W be an open subset in Y and $f(t,z)$ a measurable function on $W \setminus Y_\infty$. We say that $f(t,z)$ is of exponential type on W if, for any compact subset K in W, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\, f(t,z)|$ is essentially bounded on $K \setminus Y_\infty$.

Now we introduce the set $\mathscr{L}\mathscr{Q}_Y(W)$ consisting of a locally integrable function $f(t,z)$ on $W \setminus Y_\infty$ satisfying the condition that any higher derivative (in the sense of distributions, for example) of $f(t,z)$ with respect to the variables z and $\bar{z}$ is a locally integrable function of exponential type on W. Then, in the same way as in $\mathscr{Q}_{\mathbb{D}_E}$, the family $\{\mathscr{L}\mathscr{Q}_Y(W)\}_W$ forms the sheaf $\mathscr{L}\mathscr{Q}_Y$ on Y which is fine. Let $\mathscr{L}\mathscr{Q}^k_Y$ denotes the sheaf on Y of k-forms with respect to the variables in E, and let us define the de-Rham complex $\mathscr{L}\mathscr{Q}^\bullet_Y$ by

$$0 \longrightarrow \mathscr{L}\mathscr{Q}^0_Y \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}^1_Y \xrightarrow{d_{\mathbb{D}_E}} \dots \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}^{2n}_Y \longrightarrow 0,$$

where $d_{\mathbb{D}_E}$ is the differential on $\mathbb{D}_E$.

Let $\mathscr{E}\mathscr{Q}_Y$ be the subsheaf of $\mathscr{L}\mathscr{Q}_Y$ consisting of a C^∞-function (with respect to all the variables t, z and $\bar{z}$) whose any higher derivative also belongs to $\mathscr{L}\mathscr{Q}_Y$. Then we have also the de-Rham complex $\mathscr{E}\mathscr{Q}^\bullet_Y$:

$$0 \longrightarrow \mathscr{E}\mathscr{Q}^0_Y \xrightarrow{d_{\mathbb{D}_E}} \mathscr{E}\mathscr{Q}^1_Y \xrightarrow{d_{\mathbb{D}_E}} \dots \xrightarrow{d_{\mathbb{D}_E}} \mathscr{E}\mathscr{Q}^{2n}_Y \longrightarrow 0.$$

We denote by $\mathscr{L}^\infty_{loc,T}$ (resp. $\mathscr{E}_T$) the sheaf of L^∞_{loc}-functions (resp. C^∞-functions) on T. Then the following proposition follows from the same arguments as those of a usual de-Rham complex.

Proposition 3.0.5. We have the quasi-isomorphisms

$$p_T^{-1}\mathscr{L}^\infty_{loc,T} \longrightarrow \mathscr{L}\mathscr{Q}^\bullet_Y \qquad \text{and} \qquad p_T^{-1}\mathscr{E}_T \longrightarrow \mathscr{E}\mathscr{Q}^\bullet_Y.$$

We also have

Proposition 3.0.6. Let $\mathscr{F}$ be a sheaf of $\mathbb{Z}$ modules on T. The complexes $\mathbf{R}\Gamma_{p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M)}(p_T^{-1}\mathscr{F})$ is concentrated in degree n, and we have the canonical isomorphism

$$\tilde{p}_T^{-1}\mathscr{F} \ \otimes_{\mathbb{Z}_{p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M)}} \ or_{p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M)/Y} \longrightarrow \mathrm{H}^n_{p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M)}(p_T^{-1}\mathscr{F}),$$

where $\tilde{p}_T : p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M) = T \times \mathbb{D}_M \to T$ is the canonical projection.

Proof. Since $\mathbb{D}_M$ has an open neighborhood U in $\mathbb{D}_E$ which is topologically isomorphic to $\mathbb{D}_M \times \mathbb{R}^n$, we may replace $\mathbb{D}_E$ with $U = \mathbb{D}_M \times \mathbb{R}^n$, and we have the commutative diagram of topological spaces

$$\begin{array}{ccccc} T & \xleftarrow{\ p_T\ } & Y = T \times \mathbb{D}_M \times \mathbb{R}^n & \xleftarrow{\ i\ } & p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M) = T \times \mathbb{D}_M \\ & \nwarrow{\scriptstyle \tilde{p}_T} & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathrm{id}} & \\ & & p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M) = T \times \mathbb{D}_M & & \end{array},$$

where $i(t,x) = (t,x,0)$ and $\pi(t,x,y) = (t,x)$. Then, for a sheaf $\mathscr{F}$ on T, we have a chain of isomorphisms

$$\begin{aligned}\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1}\mathscr{F}) &= i^! p_T^{-1}\mathscr{F} = i^! \pi^{-1}\tilde{p}_T^{-1}\mathscr{F} \\ &= i^!\pi^!\tilde{p}_T^{-1}\mathscr{F} \otimes i^{-1} or_{Y/p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}[-n] \\ &= \tilde{p}_T^{-1}\mathscr{F} \otimes i^{-1} or_{Y/p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}[-n].\end{aligned}$$

The last isomorphism comes from the fact $\pi \circ i = \mathrm{id}$, which also implies

$$or_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)/Y} \otimes i^{-1} or_{Y/p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)} \simeq \mathbb{Z}_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}.$$

This completes the proof. □

Corollary 3.0.7. Let W be an open subset in Y and $s \in \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(W; p_T^{-1}\mathscr{L}^\infty_{loc,T})$, and let Δ be a subset in $\tilde{W} := W \cap p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)$. Assume the conditions below:

1. $\tilde{p}_T(\tilde{W}) \setminus \tilde{p}_T(\Delta)$ is a set of measure zero in T.
2. For any $q \in \Delta$, the stalk $s_q \in \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1}\mathscr{L}^\infty_{loc,T})_q$ of s is zero.
3. The set $\tilde{p}_T^{-1}\tilde{p}_T(q) \cap \tilde{W}$ is connected for any $q \in \tilde{W}$.

Then s is zero.

Proof. We have the commutative diagram, for any point $q \in \tilde{W}$,

$$\begin{array}{ccccc}\mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(W; p_T^{-1}\mathscr{L}^\infty_{loc,T}) & \simeq & \Gamma(\tilde{W}; \tilde{p}^{-1}\mathscr{L}^\infty_{loc,T}) & \simeq & \Gamma(\tilde{p}_T(\tilde{W}); \mathscr{L}^\infty_{loc,T}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1}\mathscr{L}^\infty_{loc,T})_q & \simeq & (\tilde{p}_T^{-1}\mathscr{L}^\infty_{loc,T})_q & \simeq & (\mathscr{L}^\infty_{loc,T})_{\tilde{p}_T(q)}.\end{array}$$

Hence s can be regarded as an L^∞_{loc}-function on $\tilde{p}_T(\tilde{W})$. Then, by the assumption, s is zero on $\tilde{p}_T(\Delta)$. Hence s is almost everywhere zero, and thus, s is zero as an L^∞_{loc}-function. This completes the proof. □

We can also define the Dolbeault complex with a parameter in the same way as $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}$. Let $\mathscr{L}\mathscr{Q}^{p,q}_Y$ and $\mathscr{E}\mathscr{Q}^{p,q}_Y$ be the sheaves of (p,q)-forms of z and $\bar{z}$ with coefficients in $\mathscr{L}\mathscr{Q}_Y$ and $\mathscr{E}\mathscr{Q}_Y$, respectively. Then we define the Dolbeault complex $\mathscr{L}\mathscr{Q}^{p,\bullet}$ with a parameter on Y by

$$0 \longrightarrow \mathscr{L}\mathscr{Q}^{p,0}_Y \xrightarrow{\bar{\partial}} \mathscr{L}\mathscr{Q}^{p,1}_Y \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathscr{L}\mathscr{Q}^{p,n}_Y \longrightarrow 0,$$

and $\mathscr{E}\mathscr{Q}^{p,\bullet}$ on Y by

$$0 \longrightarrow \mathscr{E}\mathscr{Q}^{p,0}_Y \xrightarrow{\bar{\partial}} \mathscr{E}\mathscr{Q}^{p,1}_Y \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathscr{E}\mathscr{Q}^{p,n}_Y \longrightarrow 0.$$

Then by standard arguments we have

Proposition 3.0.8. Both the canonical morphisms of complexes below are quasi-isomorphic:

$$\mathscr{L}\mathscr{O}^{\exp}_Y \longrightarrow \mathscr{L}\mathscr{Q}^{0,\bullet}_Y, \qquad \mathscr{E}\mathscr{O}^{\exp}_Y \longrightarrow \mathscr{E}\mathscr{Q}^{0,\bullet}_Y.$$

Here $\mathscr{L}\mathscr{O}^{\exp}_Y$ and $\mathscr{E}\mathscr{O}^{\exp}_Y$ are the subsheaves of $\mathscr{L}\mathscr{Q}_Y$ and $\mathscr{E}\mathscr{Q}_Y$ consisting of sections which are holomorphic with respect to the variables z, respectively.

4 Various expressions of Laplace hyperfunctions

Let M be an n-dimensional real vector space with the norm $|\bullet|$ and $E = M \otimes_{\mathbb{R}} \mathbb{C}$. Recall that $\mathbb{D}_E$ (resp. $\mathbb{D}_M$) denotes the radial compactification $E \sqcup S^{2n-1}_\infty$ (resp. $M \sqcup S^{n-1}_\infty$) of E (resp. M). Let U be an open subset in $\mathbb{D}_M$, and V an open subset in $\mathbb{D}_E$ with $V \cap \mathbb{D}_M = U$.

Definition 4.0.1. The sheaf on $\mathbb{D}_M$ of p-forms of Laplace hyperfunctions is defined by

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M} := \mathscr{H}^n_{\mathbb{D}_M}(\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_M/\mathbb{D}_E},$$

where $or_{\mathbb{D}_M/\mathbb{D}_E}$ is the relative orientation sheaf over $\mathbb{D}_M$, that is, it is given by $\mathscr{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$.

It follows from Theorem 3.0.4 that we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U) = \mathrm{H}^n_U(V;\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \ \otimes_{\mathbb{Z}_{\mathbb{D}_M}(U)}\ or_{\mathbb{D}_M/\mathbb{D}_E}(U).$$

The above cohomology groups have several equivalent expressions. We briefly recall those definitions which will be used in this paper.

4.1 Čech-Dolbeault representation

We first give a representation of a Laplace hyperfunction by Čech-Dolbeault cohomology groups. Set $V_0 = V \setminus \mathbb{D}_M$, $V_1 = V$ and $V_{01} = V_0 \cap V_1$ as usual. Then define the coverings

$$\mathcal{V}_{\mathbb{D}_M} = \{V_0,\, V_1\}, \qquad \mathcal{V}'_{\mathbb{D}_M} = \{V_0\}.$$

We denote by $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}')$ the Čech-Dolbeault complex $C(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')(\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E})$ (see Subsection 2.1 for the definition of the functor $C(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')(\bullet)$)

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \xrightarrow{\overline{\vartheta}} \mathscr{Q}^{p,1}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \xrightarrow{\overline{\vartheta}} \dots \xrightarrow{\overline{\vartheta}} \mathscr{Q}^{p,n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \longrightarrow 0,$$

where $\overline{\vartheta}$ is used to denote the differential of this complex. In the same way, we denote by $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}')$ the complex $C(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')(\mathscr{Q}^{\bullet}_{\mathbb{D}_E})$:

$$0 \longrightarrow \mathscr{Q}^{0}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \xrightarrow{D} \mathscr{Q}^{1}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \xrightarrow{D} \dots \xrightarrow{D} \mathscr{Q}^{2n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}') \longrightarrow 0,$$

where D is used to denote the differential of this complex. The $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}')$ is called the Čech - de-Rham complex. Then we have

Theorem 4.1.1. There exist the canonical quasi-isomorphisms:

$$\mathbf{R}\Gamma_U(V;\, \mathbb{C}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}'), \qquad \mathbf{R}\Gamma_U(V;\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}').$$

It follows from the theorem that we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U) \simeq \mathrm{H}^n(\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, \mathcal{V}_{\mathbb{D}_M}{}')) \otimes_{\mathbb{Z}_{\mathbb{D}_M(U)}} or_{\mathbb{D}_M/\mathbb{D}_E}(U). \tag{4.1}$$

This implies that any Laplace hyperfunction $u \in \mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U)$ is represented by a pair (ω_1, ω_{01}) of C^∞-forms which satisfies the following conditions 1. and 2.

1. $\omega_1 \in \mathscr{Q}^{p,n}(V)$ and $\omega_{01} \in \mathscr{Q}^{p,n-1}(V \setminus U)$
2. $\overline{\partial}\omega_{01} = \omega_1$ on $V \setminus U$.

Remark 4.1.2. Let $\mathcal{S} = \{S_i\}_{i\in\Lambda}$ be an open covering of V, and let $\Lambda' \subset \Lambda$. Assume $\mathcal{S}' = \{S_i\}_{i\in\Lambda'}$ is an open covering of $V \setminus \mathbb{D}_M$. Then, as did in this subsection, $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')$ (resp. $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')$) denotes the Čech-Dolbeault complex $C(\mathcal{S}, \mathcal{S}')(\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E})$ (resp. $C(\mathcal{S}, \mathcal{S}')(\mathscr{Q}^{\bullet}_{\mathbb{D}_E})$). For these complexes, we also have the isomorphisms

$$\mathbf{R}\Gamma_U(V;\, \mathbb{C}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}'), \qquad \mathbf{R}\Gamma_U(V;\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}').$$

4.2 Representation by Čech cohomology groups

Next we give a representation of a Laplace hyperfunction by Čech cohomology groups. We assume that, in this subsection, V is 1-regular at ∞ and $V \cap E$ is a Stein open subset. Let $\eta_0, \ldots, \eta_{n-1}$ be linearly independent vectors in M^* so that $\{\eta_0, \ldots, \eta_{n-1}\}$ forms a positive frame of M^*. Set $\eta_n := -(\eta_0 + \cdots + \eta_{n-1}) \in M^*$ and

$$S_k := \widehat{\ }\{z = x + \sqrt{-1}y \in E;\, z \in V,\, \langle y,\, \eta_k\rangle > 0\} \qquad (k = 0, 1, \cdots, n).$$

For convenience, we set $S_{n+1} = V$. Let $\Lambda = \{0, 1, 2, \ldots, n+1\}$ and set, for any $\alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda^{k+1}$,

$$S_\alpha := S_{\alpha_0} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}.$$

We define a covering of the pairs $(V,\, V \setminus U)$ by

$$\mathcal{S} := \{S_0, S_1, \ldots, S_{n+1}\}, \qquad \mathcal{S}' := \{S_0, \ldots, S_n\}.$$

Since $S_\alpha \cap E$ is an Stein open subset and S_α is 1-regular at ∞ for any $\alpha \in \Lambda^k$, by the theory of Čech cohomology, we have the isomorphism

$$\mathrm{H}^n_U(V;\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \mathrm{H}^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}).$$

Let Λ^{k+1}_* be the subset in Λ^{k+1} consisting of $\alpha = (\alpha_0, \ldots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n + 1.$$

Then we obtain

$$\mathrm{H}^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \frac{\bigoplus_{\alpha\in\Lambda^{n+1}_*} \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}(S_\alpha)}{\bigoplus_{\beta\in\Lambda^{n}_*} \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}(S_\beta)}.$$

Hence, any hyperfunction u has a representative $\underset{\alpha\in\Lambda^{n+1}_*}{\oplus} f_\alpha$ which is a formal sum of $(n+1)$-holomorphic functions defined on each S_α $(\alpha \in \Lambda^{n+1}_*)$.

Note that the Čech representation and the Čech-Dolbeault representation of Laplace hyperfunctions are linked by the following diagram whose morphisms are all quasi-isomorphisms.

$$C^\bullet(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \xrightarrow{\beta_1} \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}') \xleftarrow{\beta_2} \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, {\mathcal{V}_{\mathbb{D}_M}}'), \tag{4.2}$$

where the middle complex is the Čech-Dolbeault one associated with the covering $(\mathcal{S}, \mathcal{S}')$, β_1 is induced from the canonical morphism $\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E} \to \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}$ of complexes and β_2 follows from the fact that $\mathcal{S}$ is a finer covering of $\mathcal{V}_{\mathbb{D}_M}$.

4.3 Generalization of Čech representations

Representation by Čech cohomology groups can be generalized to the much more convenient one, that is "intuitive representation" of Laplace hyperfunctions introduced in [11]. Let us briefly recall this representation. Through this subsection, let U be an open subset in M (not the one in $\mathbb{D}_M$ as the previous sections).

Let Γ be an $\mathbb{R}_+$-conic connected open subset in M. Then:

Definition 4.3.1 ([11] Definition 4.8)**.** An open subset $W \subset \mathbb{D}_E$ is said to be an infinitesimal wedge of type $U \hat{\times} \sqrt{-1}\Gamma$ if and only if for any $\mathbb{R}_+$-conic open subset Γ' properly contained in Γ there exists an open neighborhood $O \subset \mathbb{D}_E$ of $\widehat{U}$ such that

$$(U \hat{\times} \sqrt{-1}\Gamma') \cap O \subset W.$$

holds (see (2.1) for the symbol $\hat{\times}$).

Remark 4.3.2. The definition of an infinitesimal wedge itself does not assume the inclusion $W \subset U \hat{\times} \sqrt{-1}\Gamma$.

We denote by $\mathcal{W}(U \hat{\times} \sqrt{-1}\Gamma)$ the set of all the infinitesimal wedges of type $U \hat{\times} \sqrt{-1}\Gamma$ which are additionally contained in $U \hat{\times} \sqrt{-1}\Gamma$. Furthermore, we set

$$\mathcal{W}(\hat{U}) := \bigcup_{\Gamma} \mathcal{W}(U \hat{\times} \sqrt{-1}\Gamma),$$

where Γ runs through all the $\mathbb{R}_+$-conic connected open subsets in M (in particular, Γ is non-empty).

Define the quotient vector space

$$\hat{\mathrm{H}}^n(\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) := \left(\bigoplus_{W \in \mathcal{W}(\hat{U})} \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W) \right) / \mathcal{R}, \tag{4.3}$$

where $\mathcal{R}$ is a $\mathbb{C}$-vector space generated by elements

$$f \oplus (-f|_{W_2}) \quad \in \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W_1) \oplus \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W_2)$$

for any $W_2 \subset W_1$ in $\mathcal{W}(\hat{U})$ and any $f \in \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W_1)$.

Theorem 4.3.3 ([11] Theorem 4.9)**.** Assume U is an open cone in M. Then there exists a family $b_{\mathcal{W}} = \{b_W\}_{W \in \mathcal{W}(\hat{U})}$ of morphisms $b_W : \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W) \to \mathscr{B}^{\mathrm{exp}}_{\mathbb{D}_M}(\hat{U})$ $(W \in \mathcal{W}(\hat{U}))$ which satisfies

$$b_{W_1}(f) = b_{W_2}(f|_{W_2}) \qquad \text{in } \mathscr{B}^{\mathrm{exp}}_{\mathbb{D}_M}(\hat{U})$$

for any $W_2 \subset W_1$ in $\mathcal{W}(\hat{U})$ and any $f \in \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(W_1)$. Furthermore the induced morphism

$$b_{\mathcal{W}} : \hat{\mathrm{H}}^n(\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) \to \mathscr{B}^{\mathrm{exp}}_{\mathbb{D}_M}(\hat{U})$$

becomes an isomorphism.

Remark 4.3.4. If $W \in \mathcal{W}(\hat{U})$ is cohomologically trivial, that is, it satisfies the conditions A1. and A2. given in Subsection 5.1, then b_W coincide with the boundary value map constructed in the subsection.

5 Boundary values in $\mathbb{D}_E$

One of the important features in hyperfunction theory is a boundary value map, by which we can regard a holomorphic function of exponential type on an wedge as a Laplace hyperfunction. We construct, in this section, the boundary value map in the framework of Čech-Dolbeault cohomology.

Let U be an open subset in $\mathbb{D}_M$ and V an open subset in $\mathbb{D}_E$ such that $V \cap \mathbb{D}_M = U$. Let Ω be an open subset in $\mathbb{D}_E$.

5.1 Functorial construction

We first construct the boundary value map in a functorial way. For an open subset W in $\mathbb{D}_E$ and a complex F of sheaves on W, we define its dual on W by

$$\mathrm{D}_W(F) := \mathbf{R}\mathcal{H}om_{\mathbb{C}_W}(F, \mathbb{C}_W).$$

Note that, for a complex F of sheaves on $\mathbb{D}_E$, we have $\mathrm{D}_{\mathbb{D}_E}(F)|_W = \mathrm{D}_W(F|_W)$. We assume:

A1. $U \subset \overline{\Omega}$.

A2. Ω is cohomologically trivial in V, that is,

$$\mathrm{D}_{\mathbb{D}_E}(\mathbb{C}_\Omega)|_V \simeq \mathbb{C}_{\overline{\Omega}}|_V, \qquad \mathrm{D}_{\mathbb{D}_E}(\mathbb{C}_{\overline{\Omega}})|_V \simeq \mathbb{C}_\Omega|_V.$$

Through this subsection, we always assume conditions A1. and A2. Following Schapira's construction (see Section 11.5 in [4]) of a boundary map morphism, we can construct the corresponding one for a Laplace hyperfunction as follows: Let $j_V : V \to \mathbb{D}_E$ be the canonical inclusion. By the assumption, we have the canonical morphism on V

$$j_V^{-1}\mathbb{C}_{\overline{\Omega}} \to j_V^{-1}\mathbb{C}_{\mathbb{D}_M}.$$

It follows from the assumption that we have

$$\mathrm{D}_V(j_V^{-1}\mathbb{C}_{\overline{\Omega}}) \simeq j_V^{-1}\mathbb{C}_\Omega, \qquad \mathrm{D}_V(j_V^{-1}\mathbb{C}_{\mathbb{D}_M}) \simeq j_V^{-1}(\mathbb{C}_{\mathbb{D}_M} \otimes or_{\mathbb{D}_M/\mathbb{D}_E})[-n].$$

Hence, applying the functor $\mathrm{D}_V(\bullet)$ to the above morphism, we obtain the canonical morphism

$$j_V^{-1}(\mathbb{C}_{\mathbb{D}_M} \otimes or_{\mathbb{D}_M/\mathbb{D}_E})[-n] \to j_V^{-1}\mathbb{C}_\Omega.$$

Now applying the functor $\mathbf{R}\mathrm{H}om_{\mathbb{C}_V}(\bullet, j_V^{-1}\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E})$ to the above morphism and taking the 0-th cohomology groups, we have obtained the boundary value map

$$b_\Omega : \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}(\Omega \cap V) \to \mathscr{B}^{\mathrm{exp}}_{\mathbb{D}_M}(U).$$

5.2 Čech-Dolbeault construction of a boundary value map

The construction of a boundary value map for Laplace hyperfunctions in the framework of Čech-Dolbeault cohomology is the almost same as that for hyperfunctions done in the paper [1]. First recall the coverings

$$\mathcal{V}_{\mathbb{D}_M} = \{V_0, V_1\}, \qquad \mathcal{V}_{\mathbb{D}_M}{}' = \{V_0\}$$

of V and $V \setminus U$, where $V_0 = V \setminus U$, $V_1 = V$ and $V_{01} = V_0 \cap V_1$. We now construct the boundary value morphism

$$b_\Omega : \mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega) \longrightarrow \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')) \otimes_{\mathbb{Z}_{\mathbb{D}_M(U)}} or_{\mathbb{D}_M/\mathbb{D}_E}(U)$$

in the framework of Čech-Dolbeault cohomology.

Let us first recall the morphism of complexes $\rho : \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}') \to \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')$, which is defined by the projection to the space of anti-holomorphic forms, that is,

$$\mathscr{Q}^{k}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}') \ni \sum_{|I|=i,\,|J|=j,\,i+j=k} f_{I,J} dz^I \wedge d\bar{z}^J \qquad \mapsto \qquad \sum_{|J|=k} f_{\emptyset,J} d\bar{z}^J \in \mathscr{Q}^{0,k}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}').$$

Then we have

Lemma 5.2.1. The following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}\Gamma_U(V;\, \mathbb{C}_{\mathbb{D}_E}) & \longrightarrow & \mathbf{R}\Gamma_U(V;\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) \\ \downarrow & & \downarrow \\ \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}') & \xrightarrow{\rho} & \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}') \end{array},$$

where the top horizontal arrow is the morphism associated with the canonical sheaf morphism $\mathbb{C}_{\mathbb{D}_E} \to \mathscr{O}^{\exp}_{\mathbb{D}_E}$.

Let us take a section $\mathbb{1} \in \mathrm{H}^n_U(V;\, \mathbb{Z}_{\mathbb{D}_E})$ such that, for each $x \in U$, the stalk $\mathbb{1}_x$ of $\mathbb{1}$ at x generates $\mathrm{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})_x$ as a $\mathbb{Z}$ module. Note that we have, in each connected component of U, two choices of such a $\mathbb{1}$, i.e., either $\mathbb{1}$ or $-\mathbb{1}$. Then the canonical sheaf morphism $\mathbb{Z}_{\mathbb{D}_E} \to \mathbb{C}_{\mathbb{D}_E}$ induces the injective morphism

$$\mathrm{H}^n_U(V;\, \mathbb{Z}_{\mathbb{D}_E}) \to \mathrm{H}^n_U(V;\, \mathbb{C}_{\mathbb{D}_E}).$$

Note that we still denote by $\mathbb{1}$ the image in $\mathrm{H}^n_U(V;\, \mathbb{C}_{\mathbb{D}_E})$ of $\mathbb{1}$ by this morphism.

Now we assume the following conditions to Ω.

B1. The canonical inclusion $(V \setminus \Omega) \setminus \mathbb{D}_M \hookrightarrow (V \setminus \Omega)$ gives a homotopical equivalence.

The following lemma can be proved in the same way as that in Lemma 7.10 in [1].

Lemma 5.2.2. Assume the conditions A1 and B1. Then there exists $\tau = (\tau_1, \tau_{01}) \in \mathscr{Q}^{n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')$ which satisfies the following conditions:

1. $D\tau = 0$ and $[\tau] = \mathbb{1}$ in $\mathrm{H}^n(\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}'))$.
2. $\mathrm{supp}_{V_{01}}(\tau_{01}) \subset \Omega$ and $\mathrm{supp}_{V_1}(\tau_1) \subset \Omega$.

Now we assume the conditions A1 and B1, and let $\tau = (\tau_1, \tau_{01})$ be the one given in the above Lemma. Then we can define the morphism

$$b_\Omega : \mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega) \longrightarrow \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')) \otimes_{\mathbb{Z}_{\mathbb{D}_M(U)}} or_{\mathbb{D}_M/\mathbb{D}_E}(U). \tag{5.1}$$

by

$$b_\Omega(f) = [f\rho(\tau)] \otimes \mathbb{1} \qquad (f \in \mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega)). \tag{5.2}$$

Lemma 5.2.3. The above b_Ω is well-defined.

To avoid a higher jet as an Ω, we also introduce the following condition

B2. For any point $x \in \mathbb{D}_M$, there exist an open neighborhood $W \subset \mathbb{D}_E$ of x and a non-empty open cone $\Gamma \subset M$ such that

$$((W \cap M) \hat{\times} \sqrt{-1}\Gamma) \cap W \subset \Omega.$$

Note that the condition B2 implies A1. We also introduced the localized version of the condition B1.

B1'. For any point $x \in \mathbb{D}_M$, there exist a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of fundamental open neighborhoods of x in V, for which the canonical inclusion $(V_\lambda \setminus \Omega) \setminus \mathbb{D}_M \hookrightarrow (V_\lambda \setminus \Omega)$ gives a homotopical equivalence.

The following theorem can be shown in the same way as that in Appendix A. in [1].

Theorem 5.2.4. Assume the conditions A2, B1, B1' and B2. Then the boundary value morphism constructed in the functorial way and the one in this subsection coincide.

Now we give a concrete construction of τ in a specific case.

Example 5.2.5. Let $M = \mathbb{R}^n$ and $E = \mathbb{C}^n$. Assume $U = \mathbb{D}_M$, $V = \mathbb{D}_E$ and $\Omega = M \hat{\times} \sqrt{-1}\Gamma$ with $\Gamma \subset M$ being an $\mathbb{R}_+$-conic non-empty open subset. Let $\eta_1, \ldots, \eta_n$ be unit vectors in M^* which satisfy the following conditions:

1. $\eta_1, \ldots, \eta_n$ are linearly independent, and the sequence of vectors in this order give a standard positive orientation of M^*.

2. $H_1 \cap H_2 \cap \cdots \cap H_n \subset \Gamma$, where $H_k = \{y \in M;\ \langle y, \eta_k \rangle > 0\}$.

Set $\eta_{n+1} = -(\eta_1 + \cdots + \eta_n)$ and define H_{n+1} in the same way as H_k $(k = 1, \ldots, n)$. Note that we have

$$H_1 \cup \cdots \cup H_n \cup H_{n+1} = M \setminus \{0\}.$$

Then we choose $(n+1)$-sections $\varphi_1, \ldots, \varphi_{n+1}$ in $\mathscr{Q}(\mathbb{D}_E \setminus \mathbb{D}_M)$ which satisfies

1. $\operatorname{supp}(\varphi_k) \subset M \hat{\times} \sqrt{-1} H_k$ holds for $k = 1, \ldots, n+1$.

2. $\varphi_1 + \varphi_2 + \cdots + \varphi_{n+1} = 1$ on $\mathbb{D}_E \setminus \mathbb{D}_M$.

Now we define

$$\tau_{01} = (-1)^n (n-1)!\, \chi_{E \setminus H_{n+1}}\, d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1}, \tag{5.3}$$

where χ_Z is the characteristic function of the set Z. We can see the following facts by the same reasoning as that of Example 7.14 in [1].

1. $\tau := (0, \tau_{01})$ belongs to $\mathscr{Q}^n_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')$.

2. $D\tau = 0$ and $[\tau] = \mathbb{1}$ in $\mathrm{H}^n(\mathscr{Q}^\bullet_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}'))$. Here we choose $\mathbb{1}$ so that it gives the standard positive orientation of M.

3. $\operatorname{supp}_{\mathbb{D}_E \setminus \mathbb{D}_M}(\tau_{01}) \subset \Omega$.

Hence this τ satisfies all the desired properties described in Lemma 5.2.2. Note that we have

$$\rho(\tau) = \left(0,\ (-1)^n (n-1)!\, \chi_{E \setminus H_{n+1}}\, \bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}\right). \tag{5.4}$$

6 Laplace transformation $\mathcal{L}$ for hyperfunctions

6.1 Preparation

Let $(z_1 = x_1 + \sqrt{-1}y_1, \cdots, z_n = x_n + \sqrt{-1}y_n)$ be a coordinate system of E. Hereafter, we fix the orientation of M and E so that $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\}$ gives the positive orientation on M, and $\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ give the one on E.

Remark 6.1.1. The above orientation of E is different from the usual standard orientation of $\mathbb{C}^n$, where $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$ is taken to be a positive frame.

We say that the boundary ∂D of a subset D in $\mathbb{D}_E$ is (partially) smooth if $\partial D \cap E$ is (partially) smooth. Note that, when the boundary ∂D is smooth, the orientation of ∂D is determined so that the outward-pointing normal vector of ∂D followed by a positive frame of ∂D determines the positive orientation of E.

Let $h : E^*_\infty \to \{-\infty\} \cup \mathbb{R}$ be an upper semi-continuous function, and let W be an open subset in $\mathbb{D}_{E^*}$ and f a holomorphic function on $W \cap E^*$.

Definition 6.1.2. We say that f is of infra-h-exponential type (at ∞) on W if, for any compact set $K \subset W$ and any $\epsilon > 0$, there exists $C > 0$ such that

$$e^{|\zeta|\, h(\pi_{E^*_\infty}(\zeta))} |f(\zeta)| \le C e^{\epsilon|\zeta|} \qquad (\zeta \in K \cap (E^* \setminus \{0\})),$$

where $\pi_{E^*_\infty} : E^* \setminus \{0\} \to (E^* \setminus \{0\})/\mathbb{R}_+ = E^*_\infty$ is the canonical projection, i.e., $\pi_{E^*_\infty}(\zeta) = \zeta/|\zeta|$, and we set $e^{-\infty} = 0$ for convenience. In particular, we say that f is simply called of infra-exponential type if $h \equiv 0$.

Define a sheaf on E^*_∞ by, for an open subset Ω in E^*_∞,

$$\mathscr{O}^{\mathrm{inf}}_{E^*_\infty}(\Omega) := \varinjlim_{W} \{f \in \mathscr{O}(W \cap E^*);\ f \text{ is of infra-exponential type on } W\},$$

where W runs through open neighborhoods of Ω in $\mathbb{D}_{E^*}$. Then the family $\{\mathscr{O}^{\mathrm{inf}}_{E^*_\infty}(\Omega)\}_\Omega$ forms the sheaf $\mathscr{O}^{\mathrm{inf}}_{E^*_\infty}$ on E^*_∞. Similarly we define the sheaf $\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}$ on E^*_∞ by, for an open subset $\Omega \subset E^*_\infty$,

$$\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}(\Omega) := \varinjlim_{W} \{f \in \mathscr{O}(W \cap E^*);\ f \text{ is of infra-}h\text{-exponential type on } W\},$$

where W runs through open neighborhoods of Ω in $\mathbb{D}_{E^*}$.

We also introduces the sheaf $\mathscr{A}^{\mathrm{exp}}_{\mathbb{D}_M} := \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}\big|_{\mathbb{D}_M}$ of real analytic functions of exponential type and the one $\mathscr{V}^{\mathrm{exp}}_{\mathbb{D}_M}$ of real analytic volumes of exponential type. The latter sheaf is defined by

$$\mathscr{V}^{\mathrm{exp}}_{\mathbb{D}_M} = \mathscr{O}^{\mathrm{exp},(n)}_{\mathbb{D}_E}\Big|_{\mathbb{D}_M} \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_M},$$

where $or_{\mathbb{D}_M} := (j_M)_*\, or_M$ with $j_M : M \hookrightarrow \mathbb{D}_M$ being the canonical inclusion. Note that we can also define the orientation sheaf $or_{\mathbb{D}_E}$ on $\mathbb{D}_E$ by $(j_E)_*\, or_E$ with the canonical inclusion $j_E : E \hookrightarrow \mathbb{D}_E$, for which we have the canonical isomorphism

$$or_{\mathbb{D}_M/\mathbb{D}_E} \otimes or_{\mathbb{D}_M} \simeq or_{\mathbb{D}_E}|_{\mathbb{D}_M}. \tag{6.1}$$

Let K be a subset in $\mathbb{D}_E$. Then we define the support function $h_K(\zeta) : E^*_\infty \to \{\pm\infty\} \cup \mathbb{R}$ by

$$h_K(\zeta) = \begin{cases} +\infty & \text{if } K \cap E \text{ is empty,} \\ \\ \inf_{z \in K \cap E} \operatorname{Re}\langle z,\, \zeta\rangle & \text{otherwise,} \end{cases} \tag{6.2}$$

where we identify $\zeta \in E^*_\infty$ with a unit vector in E^*. Note that if K is properly contained in a half space of $\mathbb{D}_E$ with direction $\zeta_0 \in E^*_\infty$ (which is equivalently saying $\zeta_0 \in \mathrm{N}^*_{pc}(K)$) and if $K \cap E$ is non-empty, then the subset

$$\overline{K} \cap \overline{\{z \in E;\, \operatorname{Re}\langle z,\, \zeta_0\rangle = h_K(\zeta_0)\}}$$

is a compact set in E. The following lemma easily follows from the definition.

Lemma 6.1.3. Let $K \subset \mathbb{D}_E$ with $\mathrm{N}^*_{pc}(K) \neq \emptyset$. Then $\mathrm{N}^*_{pc}(K)$ is a connected open subset in E^*_∞. The function $h_K(\zeta)$ is upper semi-continuous on E^*_∞, in particular, it is continuous on $\mathrm{N}^*_{pc}(K)$ and $h_K(\zeta) > -\infty$ there.

Remark 6.1.4. In the above lemma, if $K \subset \mathbb{D}_M$, then we have

$$\mathrm{N}^*_{pc}(K) = \varpi_\infty^{-1}(\mathrm{N}^*_{pc}(K) \cap M^*_\infty)$$

and $h_K(\zeta)$ is continuous on $\mathrm{N}^*_{pc}(K) \cup \sqrt{-1}M^*_\infty$ (for the definition of ϖ_∞, see (2.3)).

6.2 Laplace transformation

Let K be a closed subset in $\mathbb{D}_M$ such that $\mathrm{N}^*_{pc}(K)$ is non-empty. Take $\xi_0 \in \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and an open neighborhood V of K in $\mathbb{D}_E$. Set $U := \mathbb{D}_M \cap V$ and coverings

$$\mathcal{V}_K := \{V_0 := V \setminus K,\, V_1 := V\}, \qquad \mathcal{V}'_K := \{V_0\}. \tag{6.3}$$

In what follow, we assume that U and V are connected for simplicity. Note that we have

$$\Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}) \simeq \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M/\mathbb{D}_E}(U) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M}(U).$$

Let

$$u = \tilde{u} \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M} \in \Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}),$$

where $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M} \in or_{\mathbb{D}_M/\mathbb{D}_E}(U) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M}(U)$ and let $\nu = (\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')$ be a representative of $\tilde{u}$, i.e., $\tilde{u} = [\nu]$.

Here we may assume that $a_{\mathbb{D}_M/\mathbb{D}_E}$ and $a_{\mathbb{D}_M}$ are generators in each orientation sheaf. Hence, through the canonical isomorphism (6.1), the section $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$ determines the orientation of E. We perform the subsequent integrations under this orientation.

Remark 6.2.1. If $or_{\mathbb{D}_M/\mathbb{D}_E}$ gives the orientation so that $\{dy_1, \dots, dy_n\}$ is a positive frame and if or_M gives the orientation so that $\{dx_1, \dots, dx_n\}$ is a positive one. Then $\{dy_1, \dots, dy_n, dx_1, \dots, dx_n\}$ becomes a positive frame under the orientation determined by $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$.

Definition 6.2.2. The Laplace transform of u with a Čech-Dolbeault representative $\nu = (\nu_1, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}')$ is defined by

$$\mathcal{L}_D(u)(\zeta) := \int_{D\cap E} e^{-z\zeta}\nu_1 - \int_{\partial D\cap E} e^{-z\zeta}\nu_{01}, \tag{6.4}$$

where D is a contractible open subset in $\mathbb{D}_E$ with (partially) smooth boundary such that $K \subset D \subset \overline{D} \subset V$ and it is properly contained in a half space of $\mathbb{D}_E$ with direction ξ_0.

Note that the orientation of D and ∂D is taken in the usual way, that is, the orientation of D is that of E, and the one of ∂D is determined so that the outward pointing normal vector of D and a positive frame of ∂D form that of E.

Set $z = x + \sqrt{-1}y$ and $\zeta = \xi + \sqrt{-1}\eta$. We may assume $\xi_0 = (1, 0, \cdots, 0)$, and we write $x = (x_1, x')$ and $\xi = (\xi_1, \xi')$. Then there exist $b \in \mathbb{R}$ and $\kappa > 0$ such that

$$D \subset \{z = x + \sqrt{-1}y;\ |x'| + |y| \le \kappa(x_1 - b)\}.$$

Furthermore, it follows from the definition of ν that there exist $H > 0$ and $C \ge 0$ such that $|\nu_{01}| \le Ce^{Hx_1}$ on a neighborhood of ∂D and $|\nu_1| \le Ce^{Hx_1}$ on a neighborhood of $\overline{D}$. Hence, if $z \in D$, we have

$$|e^{-z\zeta}\nu_1| \le Ce^{-x\xi + y\eta + Hx_1} \le Ce^{-x_1\xi_1 + \kappa(|\xi'| + |\eta|)(x_1 - b) + Hx_1},$$

from which the integral $\int_{D\cap E} e^{-z\zeta}\nu_1$ converges if ξ_1 is sufficiently large. We also have the same conclusion for $\int_{\partial D\cap E} e^{-z\zeta}\nu_{01}$.

Remark 6.2.3. In what follows, we write $\int_D e^{-z\zeta}\nu_1$ instead of $\int_{D\cap E} e^{-z\zeta}\nu_1$, etc., for simplicity.

Lemma 6.2.4. $\mathcal{L}_D(u)$ is holomorphic at points $\zeta = R\xi_0$ if $R > 0$ is sufficiently large. Furthermore, $\mathcal{L}_D(u)$ is independent of the choices of a representative ν of u and D of the integral. Here we identify ξ_0 with the corresponding unit vector in M^*.

Proof. The convergence of the integration is already shown above. Let $\tilde{D}$ be another open subset in $\mathbb{D}_E$ which satisfies the conditions given in Definition 6.2.2. By replacing $\tilde{D}$ with $\tilde{D} \cap D$, we may assume $\tilde{D} \subset D$ from the beginning. Then, since $\partial(D \setminus \tilde{D}) = \partial D - \partial\tilde{D}$ holds, by the Stokes formula, we get

$$\begin{aligned}\mathcal{L}_D(u) - \mathcal{L}_{\tilde{D}}(u) &= \int_{D\setminus\tilde{D}} e^{-z\zeta}\nu_1 - \int_{\partial D - \partial\tilde{D}} e^{-z\zeta}\nu_{01} = \int_{D\setminus\tilde{D}} e^{-z\zeta}\overline{\partial}\nu_{01} - \int_{\partial D - \partial\tilde{D}} e^{-z\zeta}\nu_{01} \\ &= \int_{D\setminus\tilde{D}} d(e^{-z\zeta}\nu_{01}) - \int_{\partial D - \partial\tilde{D}} e^{-z\zeta}\nu_{01} = 0.\end{aligned}$$

By the same reasoning, we also have $\mathcal{L}_D(\overline{\vartheta}\tau) = 0$ for $\tau \in \mathscr{Q}^{n,n-1}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}')$. $\square$

Due to the above lemma, in what follows, we write $\mathcal{L}(\bullet)$ instead of $\mathcal{L}_D(\bullet)$. By taking an appropriate representative of u, we can make (6.4) much simpler form as follows: Let $\varphi \in \mathscr{Q}(\mathbb{D}_E)$ which satisfies

1. $\mathrm{supp}(\varphi) \subset D$, where D is the chain of the integration (6.4).

2. $\varphi = 1$ on $W \cap E$ for an open neighborhood W of K in $\mathbb{D}_E$,

and define

$$\tilde{\nu} = (\tilde{\nu}_{01},\, \tilde{\nu}_1) = \left(\varphi\nu_1 + \bar{\partial}\varphi \wedge \nu_{01},\, \varphi\nu_{01}\right).$$

Since we have

$$\nu - \tilde{\nu} = \overline{\vartheta}\,\left((1-\varphi)\nu_{01},\, 0\right),$$

representatives ν and $\tilde{\nu}$ give the same cohomology class. Furthermore, as the support of $\tilde{\nu}$ is contained in D, we have obtained

$$\mathcal{L}(u) = \int_E e^{-z\zeta}\tilde{\nu}_{01} = \int_E e^{-z\zeta}\left(\varphi\nu_1 + \bar{\partial}\varphi \wedge \nu_{01}\right). \tag{6.5}$$

Then, by the integration by parts, we get:

Corollary 6.2.5. For $u \in \Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})$ and $v \in \Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M})$, we have the formulas

$$\frac{\partial}{\partial\zeta_k}\mathcal{L}(u) = \mathcal{L}(-x_k u), \qquad \zeta_k\mathcal{L}(v\,dx \otimes a_{\mathbb{D}_M}) = \mathcal{L}\left(\frac{\partial v}{\partial x_k}dx \otimes a_{\mathbb{D}_M}\right) \qquad (k = 1, 2, \cdots, n).$$

Note that, by the definition of ϖ_∞ given in (2.3), we have, for $\xi_0 \in M^*_\infty$,

$$\varpi_\infty^{-1}(\xi_0) = \{\xi_0 + \sqrt{-1}\eta \in E^*;\, \eta \in M^*\}/\mathbb{R}_+ \ \subset E^*_\infty.$$

Proposition 6.2.6. Assume $K \cap M$ is non-empty. For any $a \in K \cap \{x \in M;\, \langle x,\, \xi_0\rangle = h_K(\xi_0)\}$, any $\epsilon > 0$ and any compact subset L in $\varpi_\infty^{-1}(\xi_0)$, there exist $C > 0$ and an open neighborhood $W \subset \mathbb{D}_{E^*}$ of L such that

$$|e^{a\zeta}\mathcal{L}(u)(\zeta)| \le Ce^{\epsilon|\zeta|} \qquad (\zeta \in W \cap E^*).$$

Proof. Take a point $\zeta_0 = (\xi_0 + \sqrt{-1}\eta_0)/|\xi_0 + \sqrt{-1}\eta_0| \in E^*_\infty$. In what follows, we sometimes identify a point in E^*_∞ with a unit vector in E^*. Denote by $B_\delta(\zeta_0)$ an open ball with radius $\delta > 0$ and center at ζ_0.

Since K is properly contained in a half space of $\mathbb{D}_M$ with direction ξ_0, there exist $\delta_1 > 0$, $\sigma_1 > 0$, a relatively compact open neighborhood $O \subset M$ of $K \cap \overline{\{x \in M;\, \langle x - a,\, \xi_0\rangle = 0\}}$ and an $\mathbb{R}_+$-conic proper closed cone $G \subset \mathbb{D}_M$ such that

$$K \subset O \cup (a + \mathrm{int}(G)),$$

$$O \subset \{x \in M;\, |\langle x - a,\, \xi\rangle| < \epsilon/2\} \qquad (\xi \in B_{\delta_1}(\xi_0) \cap M^*_\infty),$$

and

$$\langle x, \xi\rangle \ge \sigma_1|x| \qquad (x \in G \cap M,\ \xi \in B_{\delta_1}(\xi_0) \cap M^*_\infty).$$

For $\delta_2 > 0$, define open subsets D_O in E and D_G in $\mathbb{D}_E$ by

$$D_O = \left\{z = x + \sqrt{-1}y \in E;\, x \in O,\ |y| < \frac{\epsilon}{2\max\{1,\, 2|\eta_0|\}}\right\},$$

$$D_G = \widehat{\ }\left\{z = x + \sqrt{-1}y \in E;\, x \in a + \mathrm{int}(G), |y| < \delta_2 \mathrm{dist}(x, M \setminus (a + G))\right\}.$$

By deforming D of the Laplace integral, we may assume $D \subset D_O \cup D_G$. If we take $\delta_2 > 0$ sufficiently small, there exists $\sigma_2 > 0$ such that

$$\mathrm{Re}\,\langle z - a, t\zeta\rangle \ge \sigma_2 t|z - a| \qquad (t \in \mathbb{R}_+,\ z \in D_G \cap E,\ \zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty)$$

holds. Note that we also have

$$|\operatorname{Re}\langle z-a,\, t\zeta\rangle| < \epsilon t \qquad (t \in \mathbb{R}_+,\ z \in D_O,\ \zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty).$$

Then, for $t \in \mathbb{R}_+$ and $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$, we get

$$\begin{aligned}\mathcal{L}(u)(t\zeta) &= \int_D e^{-tz\zeta}\nu_1 - \int_{\partial D} e^{-tz\zeta}\nu_{01} \\ &= e^{-ta\zeta}\left(\int_D e^{-t(z-a)\zeta}\nu_1 - \int_{\partial D} e^{-t(z-a)\zeta}\nu_{01}\right).\end{aligned}$$

The rightmost integration in the above equation is estimated as follows. Note that the integration of ν_1 is also estimated by the same arguments. We have

$$\int_{\partial D} e^{-t(z-a)\zeta}\nu_{01} = \int_{T_{<\epsilon}} + \int_{T_{\geq\epsilon}},$$

where we set

$$T_{<\epsilon} = \partial D \cap \overline{D_O}, \qquad T_{\geq\epsilon} = \partial D \cap \overline{D_G \setminus D_O}.$$

Then, for $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$ and $t \in \mathbb{R}_+$, there exists a positive constant $C_1 > 0$ such that

$$\left|\int_{T_{<\epsilon}} e^{-t(z-a)\zeta}\nu_{01}\right| \leq C_1 e^{\epsilon t}.$$

Furthermore, since there exist a constant $C_2, H > 0$ such that

$$|\nu_{01}| \leq C_2 e^{H|z-a|} \qquad (z \in T_{\geq\epsilon}),$$

we get, for $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$ and $t \in \mathbb{R}_+$,

$$\left|\int_{T_{\geq\epsilon}} e^{-t(z-a)\zeta}\nu_{01}\right| \leq C_2 \int_{T_{\geq\epsilon}} e^{(-\sigma_2 t + H)|z-a|}\, dS,$$

where dS denotes a surface volume element of ∂D. Hence the last integral converges if t is sufficiently large, which completes the proof. □

Recall that $\mathrm{N}^*_{pc}(K)$ is an open subset in E^*_∞. Since $h_K(\zeta)$ is upper semi-continuous, we have the following corollary as a consequence of the proposition:

Corollary 6.2.7. Assume $K \cap M$ is non-empty. Then we have $\mathcal{L}(u) \in \mathscr{O}^{\inf -h_K}_{E^*_\infty}(\mathrm{N}^*_{pc}(K))$.

Let G be an $\mathbb{R}_+$-conic proper closed subset in M and $a \in M$. We denote by $G^\circ \subset E^*$ the dual open cone of G in E^*, that is,

$$G^\circ := \{\zeta \in E^*;\, \operatorname{Re}\langle\zeta, x\rangle > 0 \ \text{ for any } x \in G\}.$$

Assume $K = \overline{\{a\} + G} \subset \mathbb{D}_M$. Since $\mathrm{N}^*_{pc}(K) = {}^\frown(G^\circ) \cap E^*_\infty$ and $h_K(\zeta) = \operatorname{Re} a\zeta$ on $\mathrm{N}^*_{pc}(K)$ hold (here we write $a\zeta = \langle a, \zeta\rangle$), the corollary immediately implies the following theorem.

Theorem 6.2.8. Under the above situation, $e^{a\zeta}\mathcal{L}(u)(\zeta)$ belongs to $\mathscr{O}^{\inf}_{E^*_\infty}({}^\frown(G^\circ) \cap E^*_\infty)$.

6.3 Several equivalent definitions of Laplace transform

We give, in this subsection, several equivalent definitions of Laplace transform previously defined for various expressions of a Laplace hyperfunction. The following proposition is quite important to obtain a good Čech representation of a Laplace hyperfunction with compact support. Recall the definition of a regular closed subset given in Definition 2.2.4 and the one of an infinitesimal wedge in Definition 4.3.1. Recall also that we use the word "1-regular at ∞" to indicate the notion "regular at ∞" introduced in Definition 3.4 [2].

Proposition 6.3.1. Let K be a regular closed cone in $\mathbb{D}_M$ and let $\eta \in M^*_\infty$. Then we can find an open subset $S \subset \mathbb{D}_E \setminus K$ such that

1. S is an infinitesimal wedge of type $M \hat{\times} \sqrt{-1}\Gamma$, where $\Gamma = \{y \in M;\ \langle y, \eta \rangle > 0\}$.
2. $S \cap E$ is a Stein open subset and S is 1-regular at ∞.
3. S is an open neighborhood of $\mathbb{D}_M \setminus K$ in $\mathbb{D}_E$.

Proof. The proof is the almost same as that of Theorem 4.10 [11]. For reader's convenience, we briefly explain how to construct the desired S. We may assume that the vertex of S is the origin and $\eta = (1, 0, \cdots, 0)$. Let σ be a sufficiently small positive number and set, for $\xi \in M$,

$$\varphi_\xi(z) = (z_1 - (\xi_1 + \sqrt{-1}\sigma|\xi|))^2 + (z_2 - \xi_2)^2 + \cdots + (z_n - \xi_n)^2 + \sigma^2|\xi|^2.$$

Note that

$$\operatorname{Re}\varphi_\xi(z) > 0 \iff (y_1 - \sigma|\xi|)^2 + y_2^2 + \cdots + y_n^2 < \sigma^2|\xi|^2 + |x - \xi|^2.$$

Then, by the same reasoning as in the proof of Theorem 4.10 [11], the set

$$O = \operatorname{Int}\left(\bigcap_{\xi \in K} \{z \in E;\ \operatorname{Re}\varphi_\xi(z) > 0\}\right)$$

is an $\mathbb{R}_+$-conic Stein open subset, and hence, $\widehat{O}$ is 1-regular at ∞. Define S by modifying O near the origin:

$$S = {}^\frown\operatorname{Int}\left(\left(\bigcap_{\xi \in K, |\xi| \geq 1} \{z \in E;\ \operatorname{Re}\varphi_\xi(z) > 0\}\right) \bigcap \left(\bigcap_{\xi \in K, |\xi| < 1} \{z \in E;\ \operatorname{Re}\psi_\xi(z) > 0\}\right)\right),$$

where

$$\psi_\xi(z) = (z_1 - (\xi_1 + \sqrt{-1}\sigma))^2 + (z_2 - \xi_2)^2 + \cdots + (z_n - \xi_n)^2 + \sigma^2.$$

Since $\widehat{O}$ and S coincide in an open neighborhood of E_∞, the S is still 1-regular at ∞ and $S \cap E$ is a Stein open subset. We can easily confirm that S satisfies the rest of required properties in the proposition. □

6.3.1 Laplace transform for Čech representation

We give here several examples to compute the Laplace transform of a Čech representative of a Laplace hyperfunction.

Example 6.3.2. Let $K \subset \mathbb{D}_M$ be a closed cone which is regular and proper in $\mathbb{D}_M$, and let $\eta_0, \dots, \eta_{n-1}$ be linearly independent vectors in M^* so that $\{\eta_0, \dots, \eta_{n-1}\}$ forms a positive frame of M^*. Set $\eta_n := -(\eta_0 + \cdots + \eta_{n-1}) \in M^*$.

Then, by applying Proposition 6.3.1 to the vector η_k, we obtain S_k satisfying the conditions in the proposition with $\eta = \eta_k$ $(k = 0, \dots, n)$. Since $S_0 \cup \cdots \cup S_n \cup \mathbb{D}_M$ is an open neighborhood of $\mathbb{D}_M$, it follows from Theorem 4.10 [11] that we can take an open neighborhood $S \subset \mathbb{D}_E$ of $\mathbb{D}_M$ such that

1. $S \cap E$ is a Stein open subset and it is 1-regular at ∞.
2. $\{S_0 \cap S,\, S_1 \cap S,\, \dots,\, S_n \cap S\}$ is a covering of the set $S \setminus K$.

For simplicity, we set $S_{n+1} := S$. Let $\Lambda = \{0, 1, 2, \dots, n+1\}$ and set, for any $\alpha = (\alpha_0, \dots, \alpha_k) \in \Lambda^{k+1}$,

$$S_\alpha := S_{\alpha_0} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}.$$

We already defined the covering $(\mathcal{V}_K, \mathcal{V}'_K)$ of $(S,\, S \setminus K)$ in (6.3) with $V = S$. We also define another covering of $(S,\, S \setminus K)$ by

$$\mathcal{S} := \{S_0, S_1, \dots, S_{n+1}\}, \qquad \mathcal{S}' := \{S_0, \dots, S_n\}.$$

Then, by the theories of Čech and Čech-Dolbeault cohomologies, we have

$$\mathrm{H}^n_K(S;\, \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) \simeq \mathrm{H}^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) \simeq \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')) \simeq \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')).$$

Let Λ^{k+1}_* be the subset in Λ^{k+1} consisting of $\alpha = (\alpha_0, \dots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n+1.$$

We take a proper open convex cone $U' \subset M$ with $K \subset \widehat{U'}$, and set, for a sufficiently small $\epsilon > 0$,

$$\rho(x) := \epsilon \operatorname{dist}(x,\, M \setminus U') \qquad (x \in M).$$

Then we define closed subsets in E by

$$\sigma_{n+1} := \bigcap_{0 \le k \le n} \overline{\{z = x + \sqrt{-1}y \in E;\, x \in U',\, \langle y,\, \eta_k \rangle < \rho(x)\}} \bigcap E$$

and, for $0 \le k \le n$,

$$\sigma_k := \overline{\{z = x + \sqrt{-1}y \in E;\, x \in U',\, \langle y,\, \eta_k \rangle > \rho(x)\}} \bigcap E.$$

We may assume that, by taking $\epsilon > 0$ sufficiently small,

$$\overline{\sigma_{n+1}} \cap \overline{\sigma_k} \subset S_k \qquad (k = 0, 1, \cdots, n+1) \tag{6.6}$$

holds in $\mathbb{D}_E$. For any $\alpha = (\alpha_0, \dots, \alpha_k) \in \Lambda^{k+1}_*$, we also define

$$\sigma_\alpha := \sigma_{\alpha_0} \cap \sigma_{\alpha_1} \cap \cdots \cap \sigma_{\alpha_k}.$$

Here we determine the orientation of σ_α in the following way:

1. σ_{n+1} has the same orientation as the one of E.

2. For $k > 0$ and $\alpha \in \Lambda_*^{k+1}$, the vectors $(-\eta_{\alpha_0})$, $(-\eta_{\alpha_1})$, $\cdots$, $(-\eta_{\alpha_{k-1}})$ followed by the the positive frame of σ_α form a positive frame of E. Note that $\alpha_k = n+1$ as $\alpha \in \Lambda_*^{k+1}$.

Remark 6.3.3. The above 2. is equivalently saying that, for a point x in the smooth part of σ_α and taking points $x_j \in \mathrm{int}(\sigma_{\alpha_j})$ $(j = 0, 1, \cdots, k)$ sufficiently close to x, the positive frame of σ_α at x is determined so that the vectors $\overrightarrow{x_0 x_k}$, $\overrightarrow{x_1 x_k}$, $\cdots$, $\overrightarrow{x_{k-1} x_k}$ and the positive frame of σ_α at x form that of E at x.

Then, for any $\alpha \in \Lambda^{k+1}$ which contains the index $n+1$, we can define σ_α with orientation by extending the above definition in the alternative way, that is, $\sigma_\alpha = 0$ if the same index appears twice in α, and otherwise

$$\sigma_\alpha = \mathrm{sgn}(\alpha, \tilde{\alpha})\, \sigma_{\tilde{\alpha}},$$

where $\tilde{\alpha} \in \Lambda_*^{k+1}$ is obtained by a permutation of σ and $\mathrm{sgn}(\alpha, \tilde{\alpha})$ denotes the signature of this permutation.

Now let us consider the Čech-Dolbeault complex $\mathscr{Q}_{\mathbb{D}_E}^{n,\bullet}(\mathcal{S}, \mathcal{S}')$ for the covering $(\mathcal{S}, \mathcal{S}')$. Then, for any

$$\omega = \{\omega_\alpha\}_{0 \le k \le n,\, \alpha \in \Lambda_*^{k+1}} \in \bigoplus_{0 \le k \le n} C^k(\mathcal{S}, \mathcal{S}';\, \mathscr{Q}_{\mathbb{D}_E}^{n,n-k}) = \mathscr{Q}_{\mathbb{D}_E}^{n,n}(\mathcal{S}, \mathcal{S}'),$$

we define the Laplace transform of ω by

$$I(\omega) := \sum_{0 \le k \le n} \sum_{\alpha \in \Lambda_*^{k+1}} \int_{\sigma_\alpha} e^{-z\zeta}\, \omega_\alpha.$$

By our convention of orientation of σ_α and the fact

$$\dim_{\mathbb{R}} \sigma_{n+1} \cap \{z = x + \sqrt{-1}y \in E;\, x \in \partial U'\} < n,$$

we have, for any $\alpha \in \Lambda_*^{k+1}$,

$$\partial \sigma_\alpha = \sum_{0 \le j \le n+1} \sigma_{[\alpha\, j]},$$

where $[\alpha\, j]$ denotes a sequence in Λ^{k+2} whose last element is j.

Hence it follows from Stokes's formula that we obtain

$$I(\overline{\vartheta}\omega) = 0 \qquad (\omega \in \mathscr{Q}_{\mathbb{D}_E}^{n,n-1}(\mathcal{S}, \mathcal{S}')).$$

As a matter of fact, for $\omega_\alpha \in \mathscr{Q}_{\mathbb{D}_E}^{n,n-k-1}(S_\alpha)$ with $\alpha \in \Lambda_*^{k+1}$, we have

$$e^{-z\zeta}\overline{\vartheta}\omega_\alpha = (-1)^k \overline{\partial}(e^{-z\zeta}\omega_\alpha) + \delta(e^{-z\zeta}\omega_\alpha) = (-1)^k d(e^{-z\zeta}\omega_\alpha) + \delta(e^{-z\zeta}\omega_\alpha),$$

and thus, by noticing $\sigma_{[j\,\alpha]} = (-1)^{k+1}\sigma_{[\alpha\, j]}$,

$$\begin{aligned} I(\overline{\vartheta}\omega_\alpha) &= (-1)^k \int_{\sigma_\alpha} d(e^{-z\zeta}\omega_\alpha) + \sum_{j=0}^{n+1} \int_{\sigma_{[j\,\alpha]}} e^{-z\zeta}\omega_\alpha \\ &= (-1)^k \sum_{j=0}^{n+1} \int_{\sigma_{[\alpha\, j]}} e^{-z\zeta}\omega_\alpha + \sum_{j=0}^{n+1} \int_{\sigma_{[j\,\alpha]}} e^{-z\zeta}\omega_\alpha = 0. \end{aligned}$$

Summing up, if ω and ω' in $\mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{S}, \mathcal{S}'))$ give the same cohomology class, we have

$$I(\omega) = I(\omega').$$

Now let us consider the canonical quasi-isomorphisms of complexes

$$C^\bullet(\mathcal{S}, \mathcal{S}'; \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) \xrightarrow{\beta_1} \mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{S}, \mathcal{S}') \xleftarrow{\beta_2} \mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}').$$

It is easy to see, for $\nu_2 \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}')$ with $\overline{\vartheta}\nu_2 = 0$,

$$\mathcal{L}([\nu_2]) = I(\beta_2(\nu_2)).$$

Let $\nu_1 = \{\nu_{1,\alpha}\}_{\alpha\in\Lambda_*^{n+1}} \in C^n(\mathcal{S}, \mathcal{S}'; \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E})$ with $\delta\nu_1 = 0$. If $\beta_1(\nu_1)$ and $\beta_2(\nu_2)$ give the same cohomology class in $\mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{S}, \mathcal{S}'))$, by the above reasoning, we get

$$I(\beta_1(\nu_1)) = I(\beta_2(\nu_2)) = \mathcal{L}([\nu_2]).$$

It follows from the definition of $I(\bullet)$ that we have

$$I(\beta_1(\nu_1)) = \sum_{\alpha\in\Lambda_*^{n+1}} \int_{\sigma_\alpha} e^{-z\zeta}\nu_{1,\alpha}.$$

Furthermore, each integration can be rewritten to

$$\int_{\sigma_\alpha} e^{-z\zeta}\nu_{1,\alpha} = (-1)^n \operatorname{sgn}(\det(\eta_{\alpha_0}, \dots, \eta_{\alpha_{n-1}})) \int_{L_\alpha} e^{-z\zeta}\nu_{1,\alpha}, \tag{6.7}$$

where L_α is a real n-chain in E

$$L_\alpha = \{z = x + \sqrt{-1}y \in E;\ x \in \overline{U'} \cap M,\ y = \rho_\alpha(x)\} \tag{6.8}$$

with a smooth function $\rho_\alpha : \overline{U'} \cap M \to M$ satisfying the conditions

1. $\rho_\alpha(x) = 0$ for $x \in \partial U' \cap M$,
2. $\overline{L_\alpha} \subset S_\alpha$ in $\mathbb{D}_E$,

and its orientation is the same as the one of U'.

Summing up, for a Čech representation $\{\nu_{1,\alpha}\}_{\alpha\in\Lambda_*^{n+1}}$ of a Laplace hyperfunction u, its Laplace transform is given by

$$\mathcal{L}(u) = (-1)^n \sum_{\alpha\in\Lambda_*^{n+1}} \operatorname{sgn}(\det(\eta_{\alpha_0}, \dots, \eta_{\alpha_{n-1}})) \int_{L_\alpha} e^{-z\zeta}\nu_{1,\alpha}. \tag{6.9}$$

Remark 6.3.4. In our settings, the last index of a covering is assigned to the one for an open neighborhood S of $\mathbb{D}_M$, i.e., $S_{n+1} = S$. In usual hyperfunction theory, however, the first index 0 is assigned to it, i.e., $S_0 = S$. This is the reason why the factor $(-1)^n$ appeared in the above expression.

Example 6.3.5. Now we consider another useful example. Set

$$\Gamma_{+^n} = \{y = (y_1, \cdots, y_n) \in M\,;\, y_k > 0\ (k = 1, 2, \cdots, n)\}$$

and $K = \overline{\Gamma_{+^n}}$ in $\mathbb{D}_M$. Let $S \subset \mathbb{D}_E$ be an open neighborhood of $\mathbb{D}_M$ such that $S \cap E$ is a Stein open subset and S is 1-regular at ∞. Define, for $k = 0, 1, \cdots, n-1$,

$$S_k := \widehat{\ }\{z = (z_1, z_2, \cdots, z_n) \in S;\ z_{k+1} \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}\}\ \subset \mathbb{D}_E.$$

Set $S_n = S$. Then

$$\mathcal{S} = \{S_0, S_1, \ldots, S_n\}, \qquad \mathcal{S}' = \{S_0, \ldots, S_{n-1}\}$$

are coverings of $(S,\, S \setminus K)$.

Define the $n \times n$ matrix $B := (1+\epsilon)I - \epsilon C$ for sufficiently small $\epsilon > 0$, where I is the identity matrix and C is the $n \times n$ matrix with entries being all 1. We define the $\mathbb{R}$-linear transformation T on $E = M \times \sqrt{-1}M$ by

$$x + \sqrt{-1}y \in E \longrightarrow B\,x + \sqrt{-1}\,y \in E.$$

Let $\gamma \subset \mathbb{C}$ be the open subset defined by

$$\gamma := \{z = x + \sqrt{-1}y \in \mathbb{C};\ |y| < \epsilon(x - \epsilon)\}.$$

Then we introduce real $2n$-dimensional chains in E by

$$\sigma_n := \overline{T(\gamma \times \cdots \times \gamma)} \bigcap E,$$

and, for $k = 0, \ldots, n-1$,

$$\sigma_k := \overline{T(\mathbb{C} \times \cdots \times \underset{(k+1)\text{-th}}{(\mathbb{C} \setminus \gamma)} \times \cdots \times \mathbb{C})} \bigcap E.$$

Note that $\overline{\sigma_n}$ is a neighborhood of K in $\mathbb{D}_E$. One should aware that, however, $\overline{\gamma \times \gamma \times \cdots \times \gamma}$ is not.

Set $\Lambda = \{0, 1, \ldots, n\}$, and Λ_*^{k+1} is the subset of Λ^{k+1} consisting of an element $(\alpha_0, \alpha_1, \cdots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n.$$

Then, for any $\alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda_*^{k+1}$, the orientation of $\sigma_\alpha := \sigma_{\alpha_0} \cap \sigma_{\alpha_1} \cap \cdots \cap \sigma_{\alpha_k}$ is determined in the following way:

1. σ_n has the same orientation as the one of E.

2. the outward-pointing normal vector of σ_{α_0}, that of σ_{α_1}, $\cdots$, that of $\sigma_{\alpha_{k-1}}$ followed by the the positive frame of σ_α form a positive frame of E.

Note that, for any $\alpha \in \Lambda^{k+1}$ which contains the index n, we can define σ_α with orientation by extending the above definition in the alternative way as did in the previous example.

For any $\alpha \in \Lambda_*^{k+1}$, we have

$$\partial \sigma_\alpha = \sum_{0 \leq j \leq n} \sigma_{[\alpha\, j]},$$

where $[\alpha\, j]$ is the sequence in Λ^{k+2} whose last element is j. Therefore the rest of argument goes in the same way as in Example 6.3.2, and we finally obtain, for $u \in \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})$ and its Čech representative $\nu_{(012\ldots n)} \in C^n(\mathcal{S}, \mathcal{S}'; \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) = \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}(S_0 \cap S_1 \cap \cdots \cap S_n)$,

$$\mathcal{L}(u) = \int_{L_{(012\ldots n)}} e^{-z\zeta}\, \nu_{(012\ldots n)} \tag{6.10}$$

with the real n-chain

$$L_{(012\ldots n)} := T(\partial\gamma \times \cdots \times \partial\gamma) \subset E \tag{6.11}$$

whose orientation is given so that each arc $\partial\gamma \subset \mathbb{C}$ has anti-clockwise direction.

Example 6.3.6. Let us consider another kind of Čech covering: Let $K \subset \mathbb{D}_M$ be a closed cone which is regular and proper in $\mathbb{D}_M$, and let η_k's $(k = 0, \ldots, n-1)$ be a family of linearly independent vectors in M^*, for which the sequence $\eta_0,\ \eta_1,\ \cdots,\ \eta_{n-1}$ of vectors forms a positive frame of M^*. Set

$$\eta_{k,\pm} = \pm\eta_k \qquad (k = 0, \ldots, n-1).$$

Then, we take open subsets S and $S_{k,\pm}$ $(k = 0, 1, \cdots, n-1)$ in the same way as those in Example 6.3.2 by using Proposition 6.3.1 with $\eta = \eta_{k,\pm}$. Set $S_n = S$ and coverings

$$\mathcal{S} := \{S_{0,\pm}, \ldots, S_{n-1,\pm}, S_n\}, \qquad \mathcal{S}' := \{S_{0,\pm}, \ldots, S_{n-1,\pm}\}.$$

Let Λ be the set consisting of "n" and pairs "(i, ϵ)" with $i \in \{0, 1, \ldots, n-1\}$ and $\epsilon \in \{+, -\}$. We define the linear order $<$ on Λ by:

a. $\alpha < n$ for any $\alpha \in \Lambda \setminus \{n\}$.

b. $(i, e_i) < (j, e_j)$ if $i < j$ or if $i = j$ and $e_i = +$ and $e_j = -$.

Let Λ^{k+1}_* be the subset in Λ^{k+1} consisting of $\alpha = (\alpha_0, \ldots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n.$$

Furthermore, let Λ^{k+1}_{**} be the subset in Λ^{k+1}_* consisting of

$$\alpha = ((i_0, \epsilon_0),\ \cdots,\ (i_{k-1}, \epsilon_{k-1}),\ n) \in \Lambda^{k+1}_*$$

with $i_0 < i_1 < \cdots < i_{k-1}$. For $\alpha \in \Lambda^{k+1}_*$, the subset S_α is defined as usual, that is,

$$S_\alpha = S_{\alpha_0} \cap \cdots \cap S_{\alpha_k}.$$

Note that, in this example, the open subset S_α is not necessarily empty for $\alpha \in \Lambda^{k+1}_* \setminus \Lambda^{k+1}_{**}$ with $k > n$.

We take a proper open convex cone $U' \subset M$ with $K \subset \widehat{U'}$. For $\rho(x) := \epsilon \operatorname{dist}(x, M \setminus U')$ $(x \in M)$ with a sufficiently small $\epsilon > 0$, we define closed subsets in E by

$$\sigma_n = \bigcap_{0 \le k \le n-1} \overline{\{z = x + \sqrt{-1}y \in E;\ x \in U',\ -\rho(x) < \langle y, \eta_k\rangle < \rho(x)\}} \bigcap E$$

and, for $0 \le k \le n-1$,

$$\sigma_{(k,\pm)} = \overline{\{z = x + \sqrt{-1}y \in E;\, x \in U',\, \pm\langle y,\, \eta_k\rangle > \rho(x)\}} \bigcap E.$$

Note that $\overline{\sigma_n} \cap \overline{\sigma_\alpha} \subset S_\alpha$ holds for $\alpha \in \Lambda$ if ϵ is sufficiently small. Then, in the same way as in the previous example, we can define σ_α for $\alpha \in \Lambda^{k+1}_{**}$ and determine its orientation. For any

$$\omega = \{\omega_\alpha\}_{0\le k\le n,\, \alpha\in\Lambda^{k+1}_*} \in \bigoplus_{0\le k\le n} C^k(\mathcal{S}, \mathcal{S}';\, \mathscr{Q}^{n,n-k}_{\mathbb{D}_E}) = \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}'),$$

we define the Laplace transform of ω by

$$I(\omega) := \sum_{0\le k\le n} \sum_{\alpha\in\Lambda^{k+1}_{**}} \int_{\sigma_\alpha} e^{-z\zeta}\, \omega_\alpha,$$

for which one should aware that the sum ranges through indices only in $\Lambda^{k+1}_{**} \subset \Lambda^{k+1}_*$.

We have, for any $\alpha = ((i_0, e_0), \cdots, (i_{k-1},\, e_{k-1}),\, n) \in \Lambda^{k+1}_{**}$,

$$\partial\sigma_\alpha = \sum_{j\notin\{i_0,\cdots,i_{k-1},n\},\, \epsilon=\pm} \sigma_{[\alpha\,(j,\,\epsilon)]},$$

where $[\alpha\,(j,\,\epsilon)]$ is a sequence in Λ^{k+2} whose last element is $(j,\,\epsilon)$. The important fact here is that $\partial\sigma_\alpha$ ($\alpha \in \Lambda^{k+1}_{**}$) does not contain any cell σ_β with $\beta \in \Lambda^{k+2}_* \setminus \Lambda^{k+2}_{**}$. Hence, by Stokes's formula, we still obtain

$$I(\overline{\vartheta}\omega) = 0 \qquad (\omega \in \mathscr{Q}^{n,n-1}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')).$$

As a matter of fact, if $\alpha \in \Lambda^{k+1}_* \setminus \Lambda^{k+1}_{**}$, then $I(\overline{\vartheta}\omega_\alpha) = 0$ for $\omega_\alpha \in \mathscr{Q}^{n,n-k-1}_{\mathbb{D}_E}(S_\alpha)$ because $\overline{\partial}\omega_\alpha$ (resp. $\delta\omega_\alpha$) does not contain a non-zero term with an index in Λ^{k+1}_{**} (resp. Λ^{k+2}_{**}). If $\alpha = ((i_0, e_0), \cdots, (i_{k-1},\, e_{k-1}),\, n) \in \Lambda^{k+1}_{**}$, then we have for $\omega_\alpha \in \mathscr{Q}^{n,n-k-1}_{\mathbb{D}_E}(S_\alpha)$

$$\begin{aligned} I(\overline{\vartheta}\omega_\alpha) &= (-1)^k \int_{\sigma_\alpha} d(e^{-z\zeta}\omega_\alpha) + \sum_{j\notin\{i_0,\cdots,i_{k-1},n\},\, \epsilon=\pm} \int_{\sigma_{[(j,\epsilon)]\,\alpha]}} e^{-z\zeta}\omega_\alpha \\ &= (-1)^k \sum_{j\notin\{i_0,\cdots,i_{k-1},n\},\, \epsilon=\pm} \int_{\sigma_{[\alpha\,(j,\epsilon)]}} e^{-z\zeta}\omega_\alpha + \sum_{j\notin\{i_0,\cdots,i_{k-1},n\},\, \epsilon=\pm} \int_{\sigma_{[(j,\epsilon)\,\alpha]}} e^{-z\zeta}\omega_\alpha = 0. \end{aligned}$$

The rest of argument is the same as the one in the previous example: For $u \in \Gamma_K(U';\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})$ and its representative

$$\nu = \bigoplus_{\alpha\in\Lambda^{n+1}_*} \nu_\alpha \in C^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) \quad \text{with } \delta\nu = 0,$$

we obtain

$$\mathcal{L}(u) = (-1)^n \sum_{\alpha\in\Lambda^{n+1}_{**}} \operatorname{sgn}(\alpha) \int_{L_\alpha} e^{-z\zeta}\nu_\alpha. \tag{6.12}$$

Here, for $\alpha = ((0, \epsilon_0), \ldots, (n-1, \epsilon_{n-1}),\, n) \in \Lambda^{n+1}_{**}$, we set $\operatorname{sgn}(\alpha) = \epsilon_0\epsilon_1\cdots\epsilon_{n-1}$ and L_α is the real n-chain in E

$$L_\alpha = \{z = x + \sqrt{-1}y \in E;\, x \in \overline{U'} \cap M,\ y = \rho_\alpha(x)\} \tag{6.13}$$

with a smooth function $\rho_\alpha : \overline{U'} \cap M \to M$ satisfying the conditions

1. $\rho_\alpha(x) = 0$ for $x \in \partial U' \cap M$,
2. $\overline{L_\alpha} \subset S_\alpha$ in $\mathbb{D}_E$,

and its orientation is the same as the one of U'.

6.3.2 Laplace transform whose chain is of product type

Let us consider the Laplace transformation of a Laplace hyperfunction u whose support is contained in $\overline{\Gamma_{+^n}} \subset \mathbb{D}_M$. Here $\Gamma_{+^n} = \{(x_1, \cdots, x_n) \in M;\, x_k > 0\ (k = 1, 2, \cdots, n)\}$. In this case, one can expect the the path of the integration to be the product $\gamma_1 \times \cdots \times \gamma_n$ of the one dimensional paths γ_k. However, we cannot take such a path unless the support of u is contained in a cone strictly smaller than $\overline{\Gamma_{+^n}}$. In this subsection, we show that a chain of product type can be taken as an integral path of the Laplace transformation if the condition $\overline{\mathrm{supp}(u)} \setminus \{0\} \subset \widehat{\Gamma_{+^n}}$ is satisfied.

Let $K \subset \mathbb{D}_M$ be a regular closed cone satisfying

$$K \setminus \{0\} \subset \widehat{\Gamma_{+^n}}. \tag{6.14}$$

Let $\epsilon > 0$ and Let $D_k \subset \mathbb{C}$ be an open subset with smooth boundary satisfying $\overline{\mathbb{R}_+} \subset \widehat{D_k}$ and

$$D_k \subset \{z = x + \sqrt{-1}y \in \mathbb{C}\,;\, |y| < \epsilon(x - \epsilon)\}.$$

Set

$$D = D_1 \times D_2 \times \cdots \times D_n \subset E.$$

One should aware that $\widehat{D}$ is not an open neighborhood of $\overline{\Gamma_{+^n}}$ in $\mathbb{D}_E$. However, since $\widehat{D}$ becomes an open neighborhood of K in $\mathbb{D}_E$ because of (6.14), we can compute its Laplace transform by

$$\mathcal{L}(u)(\zeta) := \int_D e^{-z\zeta}\nu_1 - \int_{\partial D} e^{-z\zeta}\nu_{01}$$

for a Laplace hyperfunction $u = [(\nu_1, \nu_{01})]$ $((\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}'))$ with support in K.

Let $\eta_{k,\pm} = (0, \cdots, \pm 1, \cdots, 0)$ $(k = 0, \ldots, n-1)$ be a unit vector whose $(k+1)$-th element is ± 1. Recall the definitions of Λ^{k+1}_* and Λ^{k+1}_{**} given in Example 6.3.6, and let us introduce open subsets S, $S_{k,\pm}$ and the pair $(\mathcal{S}, \mathcal{S}')$ of coverings of $(S, S \setminus K)$ in the same way as those in Example 6.3.6. Set $\sigma_n = \overline{D} \cap E$ and, for $k = 0, \cdots, n-1$,

$$\sigma_{k,\pm} = \overline{\{z = (z_1, \cdots, z_n) \in E\,;\, z_{k+1} \in \mathbb{C} \setminus D_{k+1},\, \pm \mathrm{Im}\, z_{k+1} \geq 0\}} \bigcap E.$$

Then, as we did in the example, we define the Laplace transform by

$$I(\omega) := \sum_{0 \leq k \leq n} \sum_{\alpha \in \Lambda^{k+1}_{**}} \int_{\sigma_\alpha} e^{-z\zeta}\, \omega_\alpha$$

for $\omega = \{\omega_\alpha\}_{0 \leq k \leq n,\, \alpha \in \Lambda^{k+1}_*} \in \bigoplus_{0 \leq k \leq n} C^k(\mathcal{S}, \mathcal{S}';\, \mathscr{Q}^{n,n-k}_{\mathbb{D}_E}) = \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')$.

Note that we have, for $\alpha = ((i_0, \epsilon_0), \cdots, (i_{k-1}, \epsilon_{k-1}), n) \in \Lambda^{k+1}_*$,

$$\partial \sigma_\alpha = \sum_{j \notin \{i_0, \ldots, i_{k-1}, n\},\, \epsilon = \pm} \sigma_{[\alpha\, (j,\epsilon)]} + \sum_{j \in \{i_0, \ldots, i_{k-1}, n\},\, \epsilon = \pm} \sigma_{[\alpha\, (j,\epsilon)]}.$$

Define $\pi_j : \mathbb{C}^n \to \mathbb{C}$ to be $\pi_j(z_1, \cdots, z_n) = z_{j+1}$. Since $\pi_j(\sigma_{j,+} \cap \sigma_{j,-} \cap \sigma_n)$ $(j = 0, 1, \cdots, n-1)$ consists of the one point, for $j \in \{i_0, \ldots, i_{k-1}\}$ and $\epsilon = \pm$, the restriction of the holomorphic n-form dz to $\sigma_{[\alpha\,(j,\epsilon)]}$ becomes 0 and we get

$$\int_{\sigma_{[\alpha\,(j,\epsilon)]}} e^{-z\zeta}\tau = 0$$

for an $(n, n-k-1)$-form τ. Therefore we still have the same Stokes formula as the one in Example 6.3.6

$$\int_{\sigma_\alpha} e^{-z\zeta}\overline{\partial}\tau = \sum_{j\notin\{i_0,\dots,i_{k-1},n\},\, \epsilon=\pm} \int_{\sigma_{[\alpha\,(j,\epsilon)]}} e^{-z\zeta}\tau,$$

and hence, we obtain

$$I(\overline{\vartheta}\omega) = 0 \qquad (\omega \in \mathscr{Q}^{n,n-1}_{\mathbb{D}_E}(\mathcal{S}, \mathcal{S}')).$$

Summing up, for $u \in \Gamma_K(\mathbb{D}_M;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})$ and its Čech representative

$$\nu = \bigoplus_{\alpha\in\Lambda^{n+1}_*} \nu_\alpha \in C^n(\mathcal{S}, \mathcal{S}';\, \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}) \quad \text{with} \;\; \delta\nu = 0,$$

we have

$$\mathcal{L}(u) = (-1)^n \sum_{\alpha\in\Lambda^{n+1}_{**}} \operatorname{sgn}(\alpha) \int_{\gamma_\alpha} e^{-z\zeta}\nu_\alpha, \tag{6.15}$$

where, for $\alpha = ((0,\epsilon_0), \dots, (n-1,\epsilon_{n-1}),\, n) \in \Lambda^{n+1}_{**}$, we set $\operatorname{sgn}(\alpha) = \epsilon_0\epsilon_1\dots\epsilon_{n-1}$,

$$\begin{aligned} &\gamma_\alpha = (\partial D_1 \times \partial D_2 \times \cdots \times \partial D_n) \bigcap \overline{\Gamma_\alpha}, \\ &\Gamma_\alpha = \{z = (z_1, \cdots, z_n) \in E\,;\, \epsilon_k \operatorname{Im} z_{k+1} > 0 \quad (k = 0, 1, \cdots, n-1)\} \end{aligned} \tag{6.16}$$

and the orientation of γ_α is chosen to be the same as the one in M.

6.4 Reconstruction of a representative

By the same arguments as in the previous examples, we have a formula to reconstruct the corresponding Čech representative from a Čech-Dolbeault representative of a Laplace hyperfunction.

Recall the definition of Λ^{n+1}_* and Λ^{n+1}_{**} given in Example 6.3.6. Set

$$\Gamma_\alpha := \{x \in M;\, \epsilon_k x_{k+1} > 0 \;\; (k = 0, \dots, n-1)\}$$

for any $\alpha = ((0,\epsilon_0), (1,\epsilon_1), \cdots, (n-1,\epsilon_{n-1}), n) \in \Lambda^{n+1}_{**}$. In particular, we denote by $+^n$ the sequence $((0,+), (1,+), \cdots, (n-1,+), n)$. Thus Γ_{+^n} denotes the first orthant in M.

Let $K \subset \mathbb{D}_M$ be a regular closed cone such that $K \cap M$ is convex, and $V \subset \mathbb{D}_E$ an open cone such that V is 1-regular at ∞ and $V \cap E$ is a Stein open subset. Note that, since V is an open cone, the fact that V is 1-regular at ∞ is equivalent to saying that $\widehat{\ }(V \cap E) = V$. We also assume

$$K \setminus \{0\} \subset \widehat{\Gamma_{+^n}} \subset \overline{\Gamma_{+^n}} \subset V. \tag{6.17}$$

Let $U = V \cap M \subset M$, and let $\hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U})))$ denote the intuitive representation of Laplace hyperfunctions on $\hat{U}$.

Remark 6.4.1. In this subsection, we assume that $\mathcal{W}(\hat{U})$ consists of an infinitesimal wedge which satisfies the condition B1. in Section 5. For such a family $\mathcal{W}(\hat{U})$ of restricted open subsets, still Theorem 4.3.3 holds.

Then, we define $b : \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) \to \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}'))$ by

$$\mathscr{O}^{\exp}_{\mathbb{D}_E}(W) \ni f \ \mapsto\ b_W(f) \in \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')) \qquad (W \in \mathcal{W}(\hat{U})),$$

where $\mathcal{V}_{\mathbb{D}_M} = \{V \setminus \hat{U},\, V\}$, $\mathcal{V}'_{\mathbb{D}_M} = \{V \setminus \hat{U}\}$ and b_W is the boundary value map (5.2).

Recall that the isomorphism $b_{\mathcal{W}} : \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) \to \Gamma(\hat{U};\, \mathscr{B}^{\exp}_{\mathbb{D}_M})$ was given in Theorem 4.3.3, for which we have the commutative diagram

$$\begin{array}{ccccc}
 & & \Gamma(\hat{U};\, \mathscr{B}^{\exp}_{\mathbb{D}_M}) & \xleftarrow{\ \iota\ } & \Gamma_K(\hat{U};\, \mathscr{B}^{\exp}_{\mathbb{D}_M}) \\
 & \nearrow^{b_{\mathcal{W}}} & \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} \\
\hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) & \xrightarrow{\ b\ } & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')) & \xleftarrow{\ \iota\ } & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}'))
\end{array},$$

where $\mathcal{V}_K = \{V \setminus K,\, V\}$ and $\mathcal{V}'_K = \{V \setminus K\}$, the morphisms ι are injective and all the other morphisms are isomorphic. Set

$$\hat{\mathrm{H}}^n_K(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) := \{u \in \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U})));\, \mathrm{Supp}(b_{\mathcal{W}}(u)) \subset K\}.$$

Then the morphism b induces the isomorphism

$$b_K : \hat{\mathrm{H}}^n_K(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))) \xrightarrow{\sim} \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}')).$$

Now we give the inverse of b_K concretely. Let $u \in \Gamma_K(\hat{U};\, \mathscr{B}^{\exp}_{\mathbb{D}_M})$ and $\tau = (\tau_1, \tau_{01}) \in \mathscr{Q}^{0,n}_{\mathbb{D}_E}(\mathcal{V}_K, \mathcal{V}_K{}')$ be its representation. Define

$$h_u(z) = \frac{1}{(2\pi\sqrt{-1})^n}\left(\int_D \frac{\tau_1(w)e^{(z-w)a}}{w-z}dw - \int_{\partial D} \frac{\tau_{01}(w)e^{(z-w)a}}{w-z}dw\right),$$

where $\dfrac{1}{w-z}$ denotes $\dfrac{1}{(w_1-z_1)\cdots(w_n-z_n)}$, the vector a and the domain D are as follows:

1. D is a contractible open subset in $\mathbb{D}_E$ with the (partially) smooth boundary ∂D which satisfies
$$K \subset D \subset \overline{D} \subset V$$
and
$$(D \cap E) \subset \bigcup_{k=1}^n \{w \in E;\, |w_k - z_k| > \delta\}$$
for some $\delta > 0$. Furthermore, D is properly contained in an half space of $\mathbb{D}_E$ with direction $\dfrac{1}{\sqrt{n}}(1, 1, \cdots, 1)$.
2. $a = R(1, 1, \ldots, 1)$, where $R > 0$ is sufficiently large so that the integrals converge.

Note that the orientation of D is the same as the one of E, and that of ∂D is determined so that the outward-pointing normal vector of ∂D followed by a positive frame of ∂D form a positive frame of E.

Then it is easy to check that $h_u(z)$ remains unchanged when we take another D and representative τ of u if the integral converges for the same a. Hence, by deforming D suitably (here keep D unchanged near $K \cap E_\infty$, and hence, we do not need to change a in this deformation), we find that $h_u(z)$ belongs to $\mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega)$, where

$$\Omega := \widehat{\ }\{z = x + \sqrt{-1}y \in E;\ y_1 y_2 \cdots y_n \neq 0\}.$$

For $\alpha \in \Lambda^{n+1}_{**}$, set $\Omega_\alpha := M \hat{\times} \sqrt{-1}\Gamma_\alpha \subset \mathbb{D}_E$. Note that we have

$$\Omega := \bigsqcup_{\alpha \in \Lambda^{n+1}_{**}} \Omega_\alpha.$$

Now we define the inverse $b^\dagger_K$ of b_K by

$$u = [\tau] \longrightarrow (-1)^n \sum_{\alpha \in \Lambda^{n+1}_{**}} \operatorname{sgn}(\alpha) h_u(z)\big|_{\Omega_\alpha} \in \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))), \tag{6.18}$$

where $\operatorname{sgn}(\alpha) = \epsilon_0 \epsilon_1 \cdots \epsilon_{n-1}$ for $\alpha = ((0, \epsilon_0), (1, \epsilon_1), \cdots, (n-1, \epsilon_{n-1}), n) \in \Lambda^{n+1}_{**}$.

Lemma 6.4.2. $b^\dagger_K$ is independent of the choices of $a = R(1, \ldots, 1)$ if $R > 0$ is sufficiently large.

Proof. Let $a' = (R', R, \ldots, R)$ with $R' > R$. It is enough to show that $b^\dagger_K(u)$ gives the same result for both the a and a' because a general case is obtained by the repetition of application of this result. Clearly we have

$$\begin{aligned}
&\left(\int_D \frac{\tau_1(w) e^{(z-w)a'}}{w-z} dw - \int_{\partial D} \frac{\tau_{01}(w) e^{(z-w)a'}}{w-z} dw\right) - \\
&\qquad \left(\int_D \frac{\tau_1(w) e^{(z-w)a}}{w-z} dw - \int_{\partial D} \frac{\tau_{01}(w) e^{(z-w)a}}{w-z} dw\right) \\
&= (R' - R)\left(\int_D \int_0^1 \frac{\tau_1(w) e^{(z-w)(ta' + (1-t)a)}}{w' - z'} dt dw - \int_{\partial D} \int_0^1 \frac{\tau_{01}(w) e^{(z-w)(ta' + (1-t)a)}}{w' - z'} dt dw\right),
\end{aligned}$$

where $z' = (z_2, \ldots, z_n)$ and $w' = (w_2, \ldots, w_n)$. Since the last integral denoted by $\tilde{h}(z)$ hereafter belongs to $\mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega')$ with

$$\Omega' := \widehat{\ }\{z = x + \sqrt{-1}y \in E;\ y_2 \cdots y_n \neq 0\},$$

we have $\sum_\alpha \operatorname{sgn}(\alpha) \tilde{h}(z)\big|_{\Omega_\alpha} = 0$ in $\hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U})))$. This shows the result. □

Theorem 6.4.3. b_K and $b^\dagger_K$ are inverse to each other.

Proof. We use the same notations as those in Subsection 6.3.2, where we take an open subset V as S. Hence, the pair $(\mathcal{S}, \mathcal{S}')$ are coverings of $(V, V \setminus K)$. Set

$$Q_{k,\epsilon} = \{y = (y_1, \cdots, y_n) \in M\,;\ \epsilon y_{k+1} > L^{-1}|y|\} \qquad (k = 0, 1, \cdots, n-1, \epsilon = \pm)$$

for sufficiently large $L > 0$ and set

$$T_{k,\epsilon} = U \hat{\times} \sqrt{-1} Q_{k,\epsilon}.$$

Let $T \subset \mathbb{D}_E$ be an open neighborhood of $\hat{U}$ such that T is 1-regular at ∞ and $T \cap E$ is a Stein open subset. Furthermore, by shirking T if necessary, we may assume $T \subset S$ and

$$T_{k,\epsilon} \cap T \subset S_{k,\epsilon} \cap S \qquad (k = 0, 1, \cdots, n-1,\ \epsilon = \pm).$$

Set also $T_n = T$ and define the pair $(\mathcal{T}, \mathcal{T}')$ of coverings of $(T, T \setminus \mathbb{D}_M)$ by

$$\mathcal{T} = \{T_{0,+},\, T_{0,-},\, \cdots,\, T_{n-1,+},\, T_{n-1,-},\, T_n\}, \qquad \mathcal{T}' = \{T_{0,+},\, T_{0,-},\, \cdots,\, T_{n-1,+},\, T_{n-1,-}\}.$$

Using these coverings, we have the commutative diagram of complexes, where the horizontal arrows are all quasi-isomorphisms:

$$\begin{array}{ccccc} C^\bullet(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) & \xrightarrow{\beta_1} & \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}') & \xleftarrow{\beta_2} & \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K,\, {\mathcal{V}_K}') \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_2} \\ C^\bullet(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) & \xrightarrow{\alpha_1} & \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{T},\, \mathcal{T}') & \xleftarrow{\alpha_2} & \mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, {\mathcal{V}_{\mathbb{D}_M}}') \end{array}.$$

Then by taking n-th cohomology groups we get

$$\begin{array}{ccccc} \mathrm{H}^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) & \xrightarrow{\beta_1^n} & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{S},\, \mathcal{S}')) & \xleftarrow{\beta_2^n} & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K,\, {\mathcal{V}_K}')) \\ \downarrow{\scriptstyle \iota_1^n} & & \downarrow{\scriptstyle \iota^n} & & \downarrow{\scriptstyle \iota_2^n} \\ \mathrm{H}^n(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) & \xrightarrow{\alpha_1^n} & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{T},\, \mathcal{T}')) & \xleftarrow{\alpha_2^n} & \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, {\mathcal{V}_{\mathbb{D}_M}}')) \end{array},$$

where all the horizontal arrows are isomorphic and all the vertical arrows are injective. We first note that the canonical isomorphism

$$\mathrm{H}^n(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}) \xrightarrow{\sim} \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U})))$$

is given by

$$\left[\bigoplus_{\alpha \in \Lambda^{n+1}_{**}} g_\alpha\right] \mapsto \sum_{\alpha \in \Lambda^{n+1}_{**}} \operatorname{sgn}(\alpha) g_\alpha|_{T_\alpha}.$$

Furthermore, it follows from the construction of boundary values morphisms that, for $g = \{g_\alpha\}_{\alpha \in \Lambda^{n+1}_{**}} \in C^n(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E})$, we have

$$\sum_{\alpha \in \Lambda^{n+1}_{**}} \operatorname{sgn}(\alpha) b_{T_\alpha}(g_\alpha) = ((\alpha_2^n)^{-1} \circ \alpha_1^n)([g]) \qquad \text{in } \mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, {\mathcal{V}_{\mathbb{D}_M}}'))).$$

Hence the morphism b coincides with $(\alpha_2^n)^{-1} \circ \alpha_1^n$ as a morphism from $\mathrm{H}^n(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E})$ to $\mathrm{H}^n(\mathscr{Q}^{0,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M},\, {\mathcal{V}_{\mathbb{D}_M}}'))$.

The morphism ι_1 is induced from the restriction of coverings, that is, for $\{f_\alpha\}_{\alpha \in \Lambda^{n+1}_{*}} \in C^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E})$, we have

$$\iota_1^n([\{f_\alpha\}_{\alpha \in \Lambda^{n+1}_{*}}]) = [\{f_\alpha|_{T_\alpha}\}_{\alpha \in \Lambda^{n+1}_{**}}] \qquad \text{in } \mathrm{H}^n(\mathcal{T},\, \mathcal{T}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E}).$$

Note that, for $\alpha \in \Lambda^{n+1}_{*} \setminus \Lambda^{n+1}_{**}$, we have always $T_\alpha = \emptyset$ but S_α is not necessarily empty.

Let $u \in \Gamma_K(\hat{U};\, \mathscr{B}^{\exp}_{\mathbb{D}_M})$. Then we can find $f = \{f_\alpha\}_{\alpha\in\Lambda^{n+1}_*} \in C^n(\mathcal{S},\, \mathcal{S}';\, \mathscr{O}^{\exp}_{\mathbb{D}_E})$ such that $u = [f]$. Let $\tau = (\tau_1, \tau_{01}) \in \mathscr{Q}^{0,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K'))$ with $u = [\tau] = ((\beta^n_2)^{-1} \circ \beta^n_1)([f])$. Then, to show the theorem (i.e., $b^\dagger_K \circ b_K = \mathrm{id}$), it suffices to prove

$$(-1)^n \sum_{\alpha\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha) h_u(z)\big|_{\Omega_\alpha} = \sum_{\alpha\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha) f_\alpha|_{T_\alpha} \qquad \text{in } \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))).$$

Applying the result in Subsection 6.3.2, we have

$$h_u(z) = \frac{(-1)^n}{(2\pi\sqrt{-1})^n} \sum_{\alpha\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha) \int_{\Phi_\alpha(L(z))} \frac{f_\alpha(z)e^{(z-w)a}}{w-z} dw \qquad (z \in \Omega \cap E),$$

where, for $\alpha = ((0, \epsilon_0), \cdots, (n-1, \epsilon_{n-1}), n) \in \Lambda^{n+1}_{**}$, the mapping $\Phi_\alpha : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$\Phi_\alpha(x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n) = (x_1 + \epsilon_0\sqrt{-1}y_1, x_2 + \epsilon_1\sqrt{-1}y_2, \cdots, x_n + \epsilon_{n-1}\sqrt{-1}y_n)$$

and

$$L(z) = \ell_+(z_1) \times \ell_+(z_2) \times \cdots \times \ell_+(z_n).$$

Here, for $z_0 = x_0 + \sqrt{-1}y_0 \in \mathbb{C}$ with $y_0 \neq 0$, the path $\ell_+(z_0) \subset \mathbb{C}$ is defined as follows: Let $\gamma \subset \mathbb{D}_{\mathbb{C}}$ be a domain with smooth boundary such that it contains the real half line $\{z = x + \sqrt{-1}y \in \mathbb{C}; x \geq \min\{0, 2x_0\},\ y = 0\} \subset \mathbb{D}_{\mathbb{C}}$ and two points $x_0 \pm \sqrt{-1}y_0 \in \mathbb{C}$ are outside $\overline{\gamma}$. Then we set

$$\ell_+(z_0) = \partial\gamma \cap \{z \in \mathbb{C};\, \mathrm{Im}\, z \geq 0\}.$$

Furthermore, the orientation of $\ell_+(z)$ is the same as that of the real axis.

In the same way, we define $\ell_-(z_0) \subset \mathbb{C}$ by taking the domain γ as in the case of $\ell_+(z_0)$. However, in this case, we take γ so that the two points $x_0 \pm \sqrt{-1}y_0$ are also contained in γ. For any $\beta = ((0, \epsilon_0), (1, \epsilon_1), \cdots, (n-1, \epsilon_{n-1}), n) \in \Lambda^{n+1}_{**}$, we set

$$L_\beta(z) := \ell_{\epsilon_0}(z_1) \times \ell_{\epsilon_1}(z_2) \times \cdots \times \ell_{\epsilon_{n-1}}(z_n)$$

and

$$g_{\alpha,\beta}(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Phi_\alpha(L_\beta(z))} \frac{f_\alpha(z)e^{(z-w)a}}{w-z} dw.$$

It follows from the Cauchy integral formula that

$$\sum_{\beta\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\beta) g_{\alpha,\beta}(z) = \mathrm{sgn}(\alpha) f_\alpha(z) \quad (z \in T_\alpha). \tag{6.19}$$

For $\alpha = ((0, \epsilon_0), (1, \epsilon_1), \cdots, (n-1, \epsilon_{n-1}), n)$ and $\beta = ((0, \eta_0), (1, \eta_1), \cdots, (n-1, \eta_{n-1}), n)$ in Λ^{n+1}_{**}, we define

$$\alpha \cdot \beta = ((0, \epsilon_0\eta_0), (1, \epsilon_1\eta_1), \cdots, (n-1, \epsilon_{n-1}\eta_{n-1}), n) \in \Lambda^{n+1}_{**}.$$

Remember that $+^n$ denotes $((0, +), (1, +), \cdots, (n-1, +), n)$. If $\beta \in \Lambda^{n+1}_{**}$ is different from $+^n$, then $g_{\alpha,\beta}|_{T_\alpha}$ and $g_{\alpha,+^n}|_{T_{\alpha\cdot\beta}}$ can analytically extend to some common infinitesimal wedge in $\mathbb{D}_E$ and they coincide there. Hence we have, for any $\alpha, \beta \in \Lambda^{n+1}_{**}$,

$$g_{\alpha,\beta}|_{T_\alpha} = g_{\alpha,+^n}|_{T_{\alpha\cdot\beta}} \quad \text{in } \hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U}))),$$

from which we have obtained in $\hat{\mathrm{H}}^n(\mathscr{O}^{\exp}_{\mathbb{D}_E}(\mathcal{W}(\hat{U})))$

$$(-1)^n \sum_{\alpha\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha) h_u(z)\big|_{\Omega_\alpha} = \sum_{\alpha\in\Lambda^{n+1}_{**}} \sum_{\beta\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) g_{\beta,+^n}|_{T_\alpha}$$
$$= \sum_{\alpha\in\Lambda^{n+1}_{**}} \sum_{\beta\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\alpha\cdot\beta) g_{\beta,\,\alpha\cdot\beta}|_{T_\beta}$$
$$= \sum_{\beta\in\Lambda^{n+1}_{**}} \mathrm{sgn}(\beta) f_\beta|_{T_\beta}.$$

This completes the proof. □

7 Laplace inverse transformation $\mathcal{IL}$

Let S be a connected open subset in M^*_∞ and $a \in M$. Note that a connected subset is, in particular, non-empty. Recall the definition of the map ϖ_∞ given in (2.3), for which we have

$$\varpi_\infty^{-1}(S) = \{\xi + \sqrt{-1}\eta \in E^*;\, \xi \in S,\, \eta \in M^*\}/\mathbb{R}_+ \ \subset (E^* \setminus \{0\})/\mathbb{R}_+ = E^*_\infty.$$

Here we identify a point in M^*_∞ with a unit vector in M^*.

Let $h : M^*_\infty \to \{-\infty\} \cup \mathbb{R}$ be an upper semi-continuous function such that $h(\xi)$ is continuous on S and $h(\xi) > -\infty$ there. Now we extend h to the one on E^*_∞ in the following canonical way: Define $\hat{h}(\zeta)$ on E^*_∞ by, for $\zeta = \xi + \sqrt{-1}\eta \in E^*_\infty$ ($(\xi, \eta) \in S^{2n-1}$),

$$\hat{h}(\zeta) = \begin{cases} 0 & (\zeta \in \sqrt{-1}M^*_\infty), \\ |\xi| h(\varpi_\infty(\zeta)) & (\zeta \in E^*_\infty \setminus \sqrt{-1}M^*_\infty). \end{cases}$$

Note that $\hat{h}$ is also upper semi-continuous on E^*_∞ and continuous on $\varpi_\infty^{-1}(S) \cup \sqrt{-1}M^*_\infty$.

Let $f \in \mathscr{O}^{\inf\text{-}\hat{h}}_{E^*_\infty}(\varpi_\infty^{-1}(S))$. It follows from the definition of $\mathscr{O}^{\inf\text{-}\hat{h}}_{E^*_\infty}$ that we can find continuous functions $\psi : S \times [0,\infty) \to \mathbb{R}_{\geq 0}$ and $\varphi : [0,\infty) \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:

1. For any compact subset $L \subset S$, the function $\sup\limits_{\xi\in L} \psi(\xi, \lambda)$ is an infra-linear function of the variable λ and f is holomorphic on an open subset $W_\psi \cap E^*$, where

$$W_\psi := {}^\frown \left\{\zeta = \lambda\xi + \sqrt{-1}\eta \in E^*;\, \eta \in M^*,\ \xi \in S,\ \lambda > \psi(\xi, |\eta|)\right\}. \tag{7.1}$$

   Note that we identify a point in M^*_∞ with a unit vector in M^* here.

2. $\varphi(t)$ is a continuous infra-linear function on $[0,\infty)$ such that

$$|f(\zeta)| \leq e^{-|\zeta|\hat{h}(\pi_{E^*_\infty}(\zeta))+\varphi(|\zeta|)} = e^{-|\xi| h(\pi_{M^*_\infty}(\xi))+\varphi(|\zeta|)} \qquad (\zeta = \xi + \sqrt{-1}\eta \in W_\psi \cap E^*), \tag{7.2}$$

   where $\pi_{E^*_\infty} : E^* \setminus \{0\} \to (E^* \setminus \{0\})/\mathbb{R}_+ = E^*_\infty$ (resp. $\pi_{M^*_\infty} : M^* \setminus \{0\} \to M^*_\infty$) is the canonical projection.

We also define an n-dimensional real chain in E^* by

$$\gamma^* := \left\{\zeta = \xi + \sqrt{-1}\eta \in E^*;\, \eta \in M^* \setminus \{0\},\, \xi = \psi_{\xi_0}(|\eta|)\,\xi_0\right\}, \tag{7.3}$$

where $\xi_0 \in S$ and $\psi_{\xi_0}(\lambda)$ is a continuous infra-linear function on $[0,\infty)$ with $\psi_{\xi_0}(\lambda) > \psi(\xi_0,\lambda)$ $(\lambda \in [0,\infty))$ and $\psi_{\xi_0}(\lambda)/(\psi(\xi_0,\lambda)+1) \to \infty$ $(\lambda \to \infty)$. Note that the orientation of γ^* is chosen to be the same as that of $\sqrt{-1}M^*$.

Example 7.0.1. The following situation is the most important one considered in the paper: Let K be a regular closed subset in $\mathbb{D}_M$ such that $\mathrm{N}^*_{pc}(K) \cap M^*_\infty$ is connected (in particular, non-empty). Then we set $S = \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and

$$h(\xi) = h_K(\xi) = \inf_{x \in K \cap M} \langle x, \xi \rangle.$$

In this case, we have

$$\varpi_\infty^{-1}(S) = \mathrm{N}^*_{pc}(K), \quad \hat{h}(\zeta) = \inf_{x \in K \cap M} \mathrm{Re}\,\langle x, \zeta \rangle.$$

Furthermore, $\hat{h}(\zeta)$ is upper semi-continuous on E^*_∞ and continuous on $\mathrm{N}^*_{pc}(K) \cup \sqrt{-1}M^*_\infty$.

Now we consider the de-Rham theorem with a parameter in Section 3, for which we take $T = S^{n-1} = \{\eta \in M^*;\, |\eta| = 1\}$ and $Y = S^{n-1} \times \mathbb{D}_E$. Define coverings

$$\mathcal{W} = \{W_0 = Y \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M),\, W_1 = Y\}, \qquad \mathcal{W}' = \{W_0\}$$

with $W_{01} = W_0 \cap W_1$. Recall the isomorphisms given in Proposition 3.0.6

$$\Gamma(T;\, \mathscr{L}^\infty_{loc,T}) = \Gamma(Y;\, \tilde{p}_T^{-1}\mathscr{L}^\infty_{loc,T}) \xrightarrow{\sim} \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{L}^\infty_{loc,T}) = \mathrm{H}^n(\mathscr{L}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}')),$$

and set

$$\Omega := {}^\frown\left\{(\theta, z) \in S^{n-1} \times E;\, \langle \theta, \mathrm{Im}\, z\rangle > 0\right\} \subset Y.$$

Let $j : \Omega \hookrightarrow Y$ be the canonical open inclusion. Then we can take a specific $\omega = (\omega_1, \omega_{01}) \in \mathscr{L}\mathscr{Q}^n_Y(\mathcal{W},\mathcal{W}')$ satisfying the following conditions:

- D1. $D_{\mathbb{D}_E}\omega = 0$ and $[\omega]$ is the image of a constant function $1 \in \Gamma(T;\, \mathscr{L}^\infty_{loc,T})$ through the above isomorphisms.
- D2. We have $\mathrm{supp}_{W_1}(\omega_1) \subset \Omega$ and $\mathrm{supp}_{W_{01}}(\omega_{01}) \subset \Omega$.

The existence of the above ω comes from the following lemma:

Lemma 7.0.2. The canonical morphisms

$$j_!j^{-1}\mathscr{L}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}') \longrightarrow \mathscr{L}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}') \text{ and } j_!j^{-1}\mathscr{E}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}') \longrightarrow \mathscr{E}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}')$$

are quasi-isomorphic.

Proof. Let $\mathscr{F}$ be a $\mathscr{L}^\infty_{loc,T}$ or $\mathscr{E}_T$, and let $i : Y \setminus \Omega \to Y$ denote the closed embedding. Then the above isomorphism is equivalent to the following isomorphism:

$$\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, j_!j^{-1}p_T^{-1}\mathscr{F}) \longrightarrow \mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{F}),$$

which comes from the fact

$$\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, i_* i^{-1} p_T^{-1}\mathscr{F}) \simeq 0. \tag{7.4}$$

The fact itself can be shown by the following argument: Let us consider the distinguished triangle

$$\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, i_* i^{-1} p_T^{-1}\mathscr{F}) \to \mathbf{R}\Gamma(Y \setminus \Omega;\, p_T^{-1}\mathscr{F}) \overset{\beta}{\to}$$
$$\mathbf{R}\Gamma((Y \setminus \Omega) \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M);\, p_T^{-1}\mathscr{F}) \overset{+1}{\to}$$

Under the commutative diagram below,

$$\begin{array}{ccc} (Y \setminus \Omega) \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M) & \xrightarrow{\iota} & Y \setminus \Omega \\ & \searrow^{p_T} & \downarrow{\scriptstyle p_T} \\ & & T \end{array},$$

the morphism ι gives a homotopical equivalence over T, and hence, it follows from Corollary 2.7.7 (i) [KS] that the morphism β is isomorphic. This implies (7.4). The proof has been completed. $\square$

Note that we will give a concrete construction of such an ω later. Recall the standard coverings

$$\mathcal{V}_{\mathbb{D}_M} = \{V_0 = \mathbb{D}_E \setminus \mathbb{D}_M,\, V_1 = \mathbb{D}_E\}, \qquad \mathcal{V}'_{\mathbb{D}_M} = \{V_0\},$$

and the morphism $\rho = \{\rho_k\} : \mathscr{L}\mathscr{Q}_Y^{\bullet} \to \mathscr{L}\mathscr{Q}_Y^{0,\bullet}$ of complexes which is the projection to the space of anti-holomorphic forms, that is, each $\rho_k : \mathscr{L}\mathscr{Q}_Y^{k} \to \mathscr{L}\mathscr{Q}_Y^{0,k}$ is defined by

$$\sum_{|I|=i,\,|J|=j,\,i+j=k} f_{I,J}(\theta, z) dz^I \wedge d\bar{z}^J \qquad \mapsto \qquad \sum_{|J|=k} f_{\emptyset,J}(\theta, z) d\bar{z}^J.$$

Note that the following diagram commutes

$$\begin{array}{ccc} \mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{L}^{\infty}_{loc,T}) & \longrightarrow & \mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, \mathscr{L}\mathscr{O}_Y^{\exp}) \\ \downarrow & & \downarrow \\ \mathscr{L}\mathscr{Q}_Y^{\bullet}(\mathcal{W}, \mathcal{W}') & \xrightarrow{\rho} & \mathscr{L}\mathscr{Q}_Y^{0,\bullet}(\mathcal{W}, \mathcal{W}') \end{array},$$

where vertical arrows are quasi-isomorphic.

Let us take an $\omega = (\omega_1, \omega_{01}) \in \mathscr{L}\mathscr{Q}_Y^{n}(\mathcal{W}, \mathcal{W}')$ which satisfies the conditions D1. and D2.

Definition 7.0.3. The Laplace inverse transform $\mathcal{IL}$ is given by

$$\mathcal{IL}(f) = \big([\mathcal{IL}_{\omega}(f d\zeta)] \otimes a_{\mathbb{D}_M/\mathbb{D}_E}\big) \otimes \nu_{\mathbb{D}_M}$$

with

$$\mathcal{IL}_\omega(fd\zeta) := \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\gamma^*} \rho(\omega)(\frac{\eta}{|\eta|}, z)\, e^{\zeta z} f(\zeta) d\zeta$$
$$= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(\int_{\gamma^*} \rho_n(\omega_1)(\frac{\eta}{|\eta|}, z)\, e^{\zeta z} f(\zeta) d\zeta,\ \int_{\gamma^*} \rho_{n-1}(\omega_{01})(\frac{\eta}{|\eta|}, z)\, e^{\zeta z} f(\zeta) d\zeta\right).$$

Here $\zeta = \xi + \sqrt{-1}\eta$ are the dual variables of $z = x + \sqrt{-1}y$, $a_{\mathbb{D}_M} \in or_{\mathbb{D}_M}(\mathbb{D}_M)$, $a_{\mathbb{D}_M/\mathbb{D}_E} \in or_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M)$ so that $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$ has the same orientation as that of E through the isomorphism $or_{\mathbb{D}_M/\mathbb{D}_E} \otimes or_{\mathbb{D}_M} \simeq or_{\mathbb{D}_E}|_{\mathbb{D}_M}$, and the volume $\nu_{\mathbb{D}_M}$ is defined by $dz \otimes a_{\mathbb{D}_M}$ with $dz = dz_1 \wedge \cdots \wedge dz_n$ and $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$.

Lemma 7.0.4. We have

1. The integration $\mathcal{IL}_\omega(fd\zeta)$ converges and it belongs to $\mathscr{Q}^{0,n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')$. Furthermore, $\overline{\vartheta}(\mathcal{IL}_\omega(fd\zeta)) = 0$ holds.

2. $\mathcal{IL}_\omega(fd\zeta)$ does not depend on the choices of ω.

Proof. Since the support of ω_{01} (resp. ω_1) is a closed subset in W_{01} (resp. W_1) and $Y_\infty = S^{n-1} \times E_\infty$ is compact, we have the followings:

1. There exist an open neighborhood $O \subset \mathbb{D}_E$ of $\mathbb{D}_M$ and $\delta > 0$ such that

   $$\mathrm{supp}_{W_1}(\omega_1) \ \subset\ (S^{n-1} \times (\mathbb{D}_E \setminus O)) \bigcap {}^\frown\{(\eta, z) \in S^{n-1} \times E;\ \langle \eta, \mathrm{Im}\, z\rangle > \delta|\,\mathrm{Im}\, z|\}.$$

2. For any open neighborhood $O \subset \mathbb{D}_E$ of $\mathbb{D}_M$, there exists $\delta > 0$ such that

   $$\mathrm{supp}_{W_{01}}(\omega_{01}) \cap (S^{n-1} \times (\mathbb{D}_E \setminus O)) \ \subset\ {}^\frown\{(\eta, z) \in S^{n-1} \times E;\ \langle \eta, \mathrm{Im}\, z\rangle > \delta|\,\mathrm{Im}\, z|\}.$$

The fact $\mathcal{IL}_\omega(fd\zeta) \in \mathscr{Q}^{0,n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')$ and the claim 1. easily follows from these facts. Now let us show the claim 2. Let ω' be another choice of ω. Then, by the Lemma 7.0.2, we can find $\omega^{n-1} \in j_! j^{-1} \mathscr{L}\mathscr{Q}^{n-1}_Y(\mathcal{W}, \mathcal{W}')$ such that

$$\rho(\omega) - \rho(\omega') = \rho(D_{\mathbb{D}_E}\omega^{n-1}) = \overline{\vartheta}_{\mathbb{D}_E}\rho(\omega^{n-1}).$$

Since ω^{n-1} satisfies the same support conditions as those for ω, the integration $\mathcal{IL}_{\omega^{n-1}}(fd\zeta)$ which is defined by replacing ω with ω^{n-1} in the definition of $\mathcal{IL}_\omega(fd\zeta)$ also converges. Hence we have

$$\mathcal{IL}_\omega(fd\zeta) - \mathcal{IL}_{\omega'}(fd\zeta) = \overline{\vartheta}\mathcal{IL}_{\omega^{n-1}}(fd\zeta).$$

This completes the proof. □

Lemma 7.0.5. The $\mathcal{IL}(f)$ is independent of the choice of ξ_0 and ψ_{ξ_0} which appear in the definition of γ^*. As a consequence, we have

$$\mathrm{supp}(\mathcal{IL}(f)) \subset \bigcap_{\xi_0 \in S} \overline{\{x \in M;\ \langle x, \xi_0\rangle \geq h(\xi_0)\}}. \tag{7.5}$$

Proof. We first assume $n > 1$. Let us consider the commutative diagram below:

$$\begin{array}{ccccccc}
\Gamma(T;\, \mathscr{L}^{\infty}_{loc,T}) & = \Gamma(Y;\, \tilde{p}_T^{-1}\mathscr{L}^{\infty}_{loc,T}) & \xrightarrow{\sim} & \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{L}^{\infty}_{loc,T}) & = \mathrm{H}^n(\mathscr{L}\mathscr{Q}^{\bullet}_Y(\mathcal{W},\mathcal{W}')) \\
\uparrow & & & \uparrow & \uparrow \\
\Gamma(T;\, \mathscr{E}_T) & = \Gamma(Y;\, \tilde{p}_T^{-1}\mathscr{E}_T) & \xrightarrow{\sim} & \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{E}_T) & = \mathrm{H}^n(\mathscr{E}\mathscr{Q}^{\bullet}_Y(\mathcal{W},\mathcal{W}')),
\end{array}$$

where all the horizontal arrows are isomorphisms and every vertical arrow is injective. Furthermore, the bottom horizontal arrows are morphisms of $\mathscr{D}_T$ modules. Hence we can take $\omega = (\omega_1, \omega_{01}) \in \mathscr{E}\mathscr{Q}^n_Y(\mathcal{W},\mathcal{W}')$ that is a representative of the image of $1 \in \Gamma(T;\, \mathscr{E}_T)$ by the bottom horizontal arrows. It follows from Lemma 7.0.2 that the ω is assumed to satisfy the following conditions:

1. $\mathrm{supp}_{W_1}(\omega_1) \subset \Omega$ and $\mathrm{supp}_{W_{01}}(\omega_{01}) \subset \Omega$.
2. For any vector fields ν on T, we have $\nu[\omega] = 0$ since $[\omega]$ is the image of 1 and the bottom horizontal morphisms in the commutative diagram are $\mathscr{D}_T$-linear.

Let $(\theta_1, \cdots, \theta_n)$ be a homogeneous coordinate system of S^{n-1}, and let $\pi : M^* \setminus \{0\} \to S^{n-1}$ a smooth map defined by

$$(\eta_1, \cdots, \eta_n) \mapsto (\frac{\eta_1}{|\eta|}, \frac{\eta_2}{|\eta|}, \cdots, \frac{\eta_n}{|\eta|}),$$

which induces the morphism of vector bundles

$$\pi' : T(M^* \setminus \{0\}) \to (M^* \setminus \{0\}) \underset{S^{n-1}}{\times} TS^{n-1}.$$

By restricting the base space of the above bundle map to $S^{n-1} \subset M^* \setminus \{0\}$, we get the morphism of vector bundles

$$\varphi : TM^*|_{S^{n-1}} \to TS^{n-1}, \tag{7.6}$$

by which we define the vector fields ν_k on $T = S^{n-1}$ as

$$\nu_k = \varphi(\frac{\partial}{\partial \eta_k}\Big|_{S^{n-1}}) \qquad (k = 1, 2, \cdots, n). \tag{7.7}$$

Then, since $\nu_k[\omega] = 0$ holds, it follows from Lemma 7.0.2 that there exists $\tilde{\omega}_k = (\tilde{\omega}_{k,1}, \tilde{\omega}_{k,01}) \in \mathscr{E}\mathscr{Q}^{n-1}_Y(\mathcal{W},\mathcal{W}')$ with

$$\mathrm{supp}_{W_1}(\tilde{\omega}_{k,1}) \subset \Omega \quad \text{and} \quad \mathrm{supp}_{W_{01}}(\tilde{\omega}_{k,01}) \subset \Omega,$$

such that

$$\nu_k \omega = D_{\mathbb{D}_E} \tilde{\omega}_k,$$

from which we have ($\zeta = \xi + \sqrt{-1}\eta$)

$$\begin{aligned}
\frac{\partial}{\partial \overline{\zeta}_k}\big(\rho(\omega)(\eta/|\eta|, z)\big) &= \frac{\sqrt{-1}}{|\eta|}\rho(\nu_k\omega)(\eta/|\eta|, z) = \frac{\sqrt{-1}}{|\eta|}\rho(D_{\mathbb{D}_E}\tilde{\omega}_k)(\eta/|\eta|, z) \\
&= \overline{\vartheta}\big(\frac{\sqrt{-1}}{|\eta|}\rho(\tilde{\omega}_k)(\eta/|\eta|, z)\big).
\end{aligned} \tag{7.8}$$

Let us consider (ξ_0, ψ_{ξ_0}) and (ξ_1, ψ_{ξ_1}), which generate the n-dimensional chains γ_0^* and γ_1^*, respectively. Then, by taking a continuous path $s(\lambda)$ ($\lambda \in [0,1]$) in S with $s(0) = \xi_0$ and $s(1) = \xi_1$, we define an $(n+1)$-dimensional chain $\tilde{\gamma}^*$ by

$$\tilde{\gamma}^* := \{\xi + \sqrt{-1}\eta \in E^*;\, \xi = ((1-\lambda)\psi_{\xi_0}(|\eta|) + \lambda\psi_{\xi_1}(|\eta|))s(\lambda),\ 0 \le \lambda \le 1,\ \eta \in M^* \setminus \{0\}\}.$$

Here we may assume $\tilde{\gamma}^* \subset W_\psi$. In fact, we first consider the pair of chains generated by (ξ_0, ψ_{ξ_0}) and (ξ_0, g) where g is taken to be a sufficiently large infra-linear function. Then consider the pair of chains generated by (ξ_0, g) and (ξ_1, g) and finally that by (ξ_1, ψ_{ξ_1}) and (ξ_1, g).

By noticing that the function $\dfrac{1}{|\eta|}$ on $M^* \setminus \{0\}$ is integrable near the origin if $n > 1$ and that each $\tilde{\omega}_k$ satisfies the same support condition as that for ω, it follows from the Stokes formula that we obtain

$$\begin{aligned}&\int_{\tilde{\gamma}^*} f(\zeta)\, \overline{\partial}_\zeta\big(\rho(\omega)(\eta/|\eta|, z)\big)\, e^{\zeta z}\, d\zeta \\ &\qquad = \int_{\gamma_1^*} f(\zeta)\, \rho(\omega)(\eta/|\eta|, z)\, e^{\zeta z}\, d\zeta - \int_{\gamma_0^*} f(\zeta)\, \rho(\omega)(\eta/|\eta|, z)\, e^{\zeta z}\, d\zeta.\end{aligned}$$

It follows from (7.8) that we have

$$\begin{aligned}\int_{\tilde{\gamma}^*} f(\zeta)\, \overline{\partial}_\zeta\big(\rho(\omega)(\eta/|\eta|, z)\big)\, e^{\zeta z}\, d\zeta &= \int_{\tilde{\gamma}^*} f(\zeta)\, e^{\zeta z} \sum_{k=1}^{n} \frac{\partial}{\partial \overline{\zeta}_k}\big(\rho(\omega)(\eta/|\eta|, z)\big)\, d\overline{\zeta}_k \wedge d\zeta \\ &= \overline{\vartheta} \int_{\tilde{\gamma}^*} f(\zeta)\, e^{\zeta z} \frac{\sqrt{-1}}{|\eta|} \sum_{k=1}^{n} \rho(\tilde{\omega}_k)(\eta/|\eta|, z) d\overline{\zeta}_k \wedge d\zeta.\end{aligned}$$

Hence the Laplace transform of f with the chain γ_0^* and the one with the chain γ_1^* give the same cohomology class.

Let us show (7.5) in the lemma. Fix $\xi_0 \in S$ and take a sufficiently large $\ell > 0$ so that $\psi_{\xi_0}(t) \le t + \ell$ holds for $t \in [0, \infty)$. Let us consider the n-dimensional chain γ_ϵ^* for $1 \ge \epsilon > 0$

$$\gamma_\epsilon^* := \big\{\xi + \sqrt{-1}\eta \in E^*;\, \xi = (\epsilon^{-1}|\eta| + \ell)\xi_0,\, \eta \in M^* \setminus \{0\}\big\}$$

and the $(n+1)$-dimensional chain

$$\tilde{\gamma}_\epsilon^* = \left\{\xi + \sqrt{-1}\eta \in E^*;\, \begin{array}{l}\xi = \big((1-\lambda)\psi_{\xi_0}(|\eta|) + \lambda(\epsilon^{-1}|\eta| + \ell)\big)\ \xi_0 \\ 0 \le \lambda \le 1,\ \eta \in M^* \setminus \{0\}\end{array}\right\}.$$

Note that $\tilde{\gamma}_\epsilon^* \subset W_\psi$ holds for $1 \ge \epsilon > 0$. Then, on $\{z \in E;\, \mathrm{Re}\,\langle \xi_0, z\rangle < h(\xi_0)\}$, we have

$$\begin{aligned}&\int_{\tilde{\gamma}_\epsilon^*} f(\zeta)\, \overline{\partial}_\zeta\big(\rho(\omega)(\eta/|\eta|, z)\big)\, e^{\zeta z}\, d\zeta \\ &\qquad = \int_{\gamma_0^*} f(\zeta)\, \rho(\omega)(\eta/|\eta|, z)\, e^{\zeta z}\, d\zeta - \int_{\gamma_\epsilon^*} f(\zeta)\, \rho(\omega)(\eta/|\eta|, z)\, e^{\zeta z}\, d\zeta,\end{aligned}$$

where all the integrals converge. Hence, by letting $\epsilon \to 0+0$, we get

$$\int_{\tilde{\gamma}_{0+0}^*} f(\zeta)\, \overline{\partial}_\zeta\big(\rho(\omega)(\eta/|\eta|, z)\big)\, e^{\zeta z}\, d\zeta = \int_{\gamma_0^*} f(\zeta)\, \rho(\omega)(\eta/|\eta|, z)\, e^{\zeta z}\, d\zeta.$$

Here the $(n+1)$-dimensional chain $\tilde{\gamma}_{0+0}$ is

$$\tilde{\gamma}^*_{0+0} := \{\xi + \sqrt{-1}\eta \in E^*;\ \xi = \lambda\xi_0,\ \lambda \geq \psi_{\xi_0}(|\eta|),\ \eta \in M^* \setminus \{0\}\}$$

and all the integrals still converge. This implies that, as the left hand side of the above equation gives the zero cohomology class in $\widehat{\ }\{z \in E; \mathrm{Re}\,\langle \xi_0, z\rangle < h(\xi_0)\}$, and thus, $\mathrm{supp}(\mathcal{IL}(f))$ is contained in $\overline{\{x \in M\,;\, \langle \xi_0, x\rangle \geq h(\xi_0)\}}$. Since we can take any vector in S as ξ_0, we have concluded the second claim of this lemma when $n > 1$.

Now we consider the case $n = 1$. In this case, S^{n-1} consists of only two points $\{+1, -1\}$. Hence it follows from the definition of ω that $\tau = \omega(1, z)$ (resp. $\tau = \omega(-1, z)$) satisfies the conditions in Lemma 5.2.2 with $\Omega = \Omega^1_+$ (resp. $\Omega = \Omega^1_-$), where

$$\Omega^1_\pm = \widehat{\ }\{z \in \mathbb{C}; \pm\mathrm{Im}\, z > 0\} \subset \mathbb{D}_{\mathbb{C}}.$$

Hence we have obtained

$$\mathcal{IL}_\omega(f d\zeta) = b_{\Omega^1_+}\left(\frac{\sqrt{-1}}{2\pi}\int_{\gamma^*\cap\Omega^1_+} e^{\zeta z} f(\zeta) d\zeta\right) - b_{\Omega^1_-}\left(\frac{\sqrt{-1}}{2\pi}\int_{\gamma^*\cap\Omega^1_-} e^{\zeta z} f(\zeta) d\zeta\right)$$

for which we can easily see the claims of the lemma. This completes the proof. □

In particular, we get

Corollary 7.0.6. Let $a \in M$ and $G \subset M$ be an $\mathbb{R}_+$-conic proper closed convex subset. Set $K = \overline{a + G} \subset \mathbb{D}_M$ and let $e^{a\zeta} g(\zeta) \in \mathscr{O}^{\inf}_{E^*_\infty}(\mathrm{N}^*_{pc}(K)) = \mathscr{O}^{\inf}_{E^*_\infty}(\widehat{\ }(G^\circ) \cap E^*_\infty)$, where G° is the open dual cone of G, that is, $G^\circ = \{\zeta \in E^*; \mathrm{Re}\,\langle \zeta, x\rangle > 0\ (\forall x \in G)\}$. Then we have

$$\mathrm{supp}(\mathcal{IL}(g)) \subset K. \tag{7.9}$$

In fact, the corollary follows from the lemma by taking $S = \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and $h(\xi) = a\xi$ and by noticing the facts $\varpi^{-1}_\infty(S) = \mathrm{N}^*_{pc}(K)$ and $K = \bigcap_{\xi_0 \in S} \overline{\{x \in M;\, x\xi_0 \geq a\xi_0\}}$.

7.1 Concrete construction of ω

Now we give a method to construct ω concretely. Let O be a subset in $S^{n-1} = \{\xi \in M^*;\, |\xi| = 1\}$, and let $\theta_k : O \to S^{n-1} \subset M^*$ $(k = 1, \ldots, n)$ be continuous maps on O. Set, for $\xi \in O$,

$$\kappa(\xi) := \bigcap_{k=1}^{n} \{x \in \mathbb{R}^n; \langle x, \theta_k(\xi)\rangle > 0\} \ \subset M.$$

We assume that there exists $\delta > 0$ satisfying

C1. $S^{n-1} \setminus O$ is measure zero.

C2. $\kappa(\xi) \subset \{x \in M;\, \langle x, \xi\rangle > \sigma|x|\}$ for any $\xi \in O$.

C3. Let $A(\xi)$ be an $n \times n$-matrix $(\theta_1(\xi), \ldots, \theta_n(\xi))$. Then $\det(A(\xi)) \geq \delta$ for any $\xi \in O$.

Note that the condition C2 is equivalent to the following C2':

C2'. Set $G(\xi) := \sum_{k=1}^{n} \mathbb{R}_+\theta_k(\xi)$. Then we have

$$\mathrm{dist}(\xi, \mathbb{R}^n \setminus G(\xi)) > \delta \qquad (\xi \in O).$$

In fact, C2' implies

$$\left\{\tau \in \mathbb{R}^n;\ \left|\frac{\tau}{|\tau|} - \xi\right| \le \frac{\delta}{2}\right\} \subset G(\xi).$$

Then, by taking the dual of the above sets and by noticing $G(\xi)^\circ = \kappa(\xi)$, we can obtain C2.

Let $\varphi_1(z), \ldots, \varphi_{n+1}(z)$ be in $\mathscr{Q}_{\mathbb{D}_E}(\mathbb{D}_E \setminus \mathbb{D}_M)$ which are given in Example 5.2.5 with

$$\eta_k = (0, \ldots, 0, \overset{k\text{-th}}{1}, 0, \ldots, 0) \qquad (k = 1, \ldots, n).$$

Using these φ_k's, we define ω_{01} by

$$\omega_{01}(\xi, z) := (-1)^n (n-1)!\, \chi_{E \setminus H_{n+1}}({}^tA(\xi)z)\, \bar{\partial}_z(\varphi_1({}^tA(\xi)z)) \wedge \cdots \wedge \bar{\partial}_z(\varphi_{n-1}({}^tA(\xi)z)),$$

where H_{n+1} is also given in Example 5.2.5. Then, by the same reasoning as that of Example 7.14 in [1] and Corollary 3.0.7, we have

Lemma 7.1.1. Thus constructed $\omega = (0, \omega_{01})$ satisfies the conditions D1. and D2. described before Lemma 7.0.2.

We give some examples of such a family θ_k's.

Example 7.1.2. Let χ be a triangulation of S^{n-1}, and let $\{\sigma_\lambda\}_{\lambda\in\Lambda}$ be the set of $(n-1)$-cells of χ. For each $\lambda \in \Lambda$, we take linearly independent n-vectors $\nu_{\lambda,1}, \cdots, \nu_{\lambda,n} \in M^*$ which satisfies

$$\overline{\sigma_\lambda} \subset \sum_{k=1}^{n} \mathbb{R}_+\nu_{\lambda,k},$$

and $\det A_\lambda > 0$ for the constant matrix $A_\lambda := (\nu_{\lambda,1}, \nu_{\lambda,2}, \ldots, \nu_{\lambda,n})$. Note that such a family of constant vectors always exists if each σ_λ is sufficiently small. Furthermore, we may assume the frame $\nu_{\lambda,1}, \nu_{\lambda,2}, \ldots, \nu_{\lambda,n}$ determine the positive orientation in M^* for each λ, Then, we set $O := \cup_{\lambda\in\Lambda}\sigma_\lambda$ and, for $k = 1, \ldots, n$, define $\theta_k(\xi)$ on O by

$$\theta_k(\xi) = \nu_{\lambda,k} \qquad (\xi \in \sigma_\lambda).$$

Clearly these O and θ_k's satisfy the conditions C1, C2 and C3.

Example 7.1.3. Assume M^* has an inner product. Let p be a point in S^{n-1} and set $O := S^{n-1} \setminus \{p\}$. Then O becomes contractible, and hence, there exists a continuous orthogonal frame $\tilde{\theta}_1(\xi), \ldots, \tilde{\theta}_n(\xi) \in M^*$ on O. Here we may assume $\tilde{\theta}_1(\xi) = \xi$. Set, for some $\delta > 0$,

$$\begin{aligned}
\theta_1(\xi) &:= \tilde{\theta}_2(\xi) + \delta\tilde{\theta}_1(\xi),\\
\theta_2(\xi) &:= \tilde{\theta}_3(\xi) + \delta\tilde{\theta}_1(\xi),\\
&\vdots\\
\theta_{n-1}(\xi) &:= \tilde{\theta}_n(\xi) + \delta\tilde{\theta}_1(\xi),\\
\theta_n(\xi) &:= -(\tilde{\theta}_2 + \cdots + \tilde{\theta}_n(\xi)) + \delta\tilde{\theta}_1(\xi).
\end{aligned}$$

Then these O and θ_k's satisfy the conditions C1, C2 and C3.

Let us compute $\mathcal{IL}$ when ω comes from Example 7.1.2. In this case, on each σ_λ, $\omega_{01}(\xi, z)$ does not depend on the variables ξ. Hence we obtain

$$\mathcal{IL}(f) := \left[\left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(0,\ \sum_{\lambda\in\Lambda} \tau_{01,\lambda} \int_{\gamma^*_\lambda} f(\zeta)e^{\zeta z}d\zeta\right)\right] \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \otimes \nu_{\mathbb{D}_M}. \tag{7.10}$$

Here

$$\gamma^*_\lambda := \left\{\zeta = \xi + \sqrt{-1}\eta \in E^*;\ \eta \in \mathbb{R}_+\sigma_\lambda,\ \xi = \psi_{\xi_0}(|\eta|)\,\xi_0\right\},$$

and

$$\tau_{01,\lambda}(z) := (-1)^n(n-1)!\,\chi_{E\setminus H_{n+1}}({}^tA_\lambda z)\,\bar{\partial}(\varphi_1({}^tA_\lambda z)) \wedge \cdots \wedge \bar{\partial}(\varphi_{n-1}({}^tA_\lambda z)),$$

where the constant matrix A_λ is given by $(\nu_{\lambda,1}, \ldots, \nu_{\lambda,n})$ and the orientation of the chain γ^*_λ is induced from the one of $\sqrt{-1}M^*$ through the canonical projection $E^* = M^* \times \sqrt{-1}M^* \to \sqrt{-1}M^*$. Then, as we see in Example 5.2.5, $\tau_\lambda := (0, \tau_{01,\lambda})$ satisfies the conditions in Lemma 5.2.2. Hence, by the definition of the boundary value map explained in Subsection 5.2, we have

$$\mathcal{IL}(f) = \sum_{\lambda\in\Lambda} b_{\Omega_\lambda}\left(\left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\gamma^*_\lambda} f(\zeta)e^{\zeta z}d\zeta\right) \otimes \nu_{\mathbb{D}_M} \in \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}{}')), \tag{7.11}$$

where $\Omega_\lambda := M\hat{\times}\sqrt{-1}\Gamma_\lambda$ with $\Gamma_\lambda := \bigcap_{k=1}^n\{y \in M;\ \langle y, \nu_{\lambda,k}\rangle > 0\}$.

Let $\Lambda = \{+1, -1\}$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^n$, we define

$$\begin{aligned}
\Gamma_\alpha &:= \{x = (x_1, \cdots, x_n) \in M;\ \alpha_k x_k > 0\ (k = 1, \cdots, n)\},\\
\Gamma^*_\alpha &:= \{\eta = (\eta_1, \cdots, \eta_n) \in M^*;\ \alpha_k \eta_k > 0\ (k = 1, \cdots, n)\}.
\end{aligned} \tag{7.12}$$

We denote by $+^n \in \Lambda^n$ (resp. $-^n \in \Lambda^n$) the multi-index in Λ^n whose entries are all $+1$ (resp. -1). Hence, Γ_{+^n} (resp. $\Gamma^*_{+^n}$) designates the first orthant of M (resp. M^*).

Let $G \subset M$ be an $\mathbb{R}_+$-conic proper closed convex subset and $a \in M$. Set $K = \overline{a + G} \subset \mathbb{D}_M$ and let $f \in e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(\mathrm{N}^*_{pc}(K)) = e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(\widehat{\ }(G^\circ) \cap E^*_\infty)$. We also assume that $G \setminus \{0\} \subset \Gamma_{+^n}$. Then f is holomorphic on $W_\psi \cap E^*$ given in (7.1) with $S = \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and $h(\xi) = a\xi$, and it satisfies (7.2) there. It follows from the assumption $G \setminus \{0\} \subset \Gamma_{+^n}$ that we can find $a^* = (a^*_1, \cdots, a^*_n) \in M^*$ such that the open subset W_ψ given in (7.1) satisfies

$$\overline{a^* + \Gamma^*_{+^n}} \subset W_\psi. \tag{7.13}$$

Because of this fact, we can take a specific real n-chain $\tilde{\gamma}^* \subset E^*$ defined below which enjoys some good properties:

$$\tilde{\gamma}^* := \left\{ \zeta = \xi + \sqrt{-1}\eta \in E^*;\, \eta \in M^* \setminus \{0\},\, \xi = a^* + \hat{\psi}(|\eta|) \left(\frac{|\eta_1|}{|\eta|}, \frac{|\eta_2|}{|\eta|}, \ldots, \frac{|\eta_n|}{|\eta|} \right) \right\},$$

where $\hat{\psi}(t)$ is a continuous infra-linear function on $[0, \infty)$ which satisfies $\hat{\psi}(0) = 0$ and $\tilde{\gamma}^* \subset W_\psi$. Note that the orientation of $\tilde{\gamma}^*$ is the same as that of $\sqrt{-1}M^*$. For $\alpha \in \Lambda^n$, we also define

$$\tilde{\gamma}^*_\alpha := \left\{ \zeta = \xi + \sqrt{-1}\eta \in E^*;\, \eta \in \Gamma^*_\alpha,\, \xi = a^* + \hat{\psi}(|\eta|) \left(\frac{|\eta_1|}{|\eta|}, \frac{|\eta_2|}{|\eta|}, \ldots, \frac{|\eta_n|}{|\eta|} \right) \right\}.$$

We can replace the chain γ^* of $\mathcal{IL}_\omega$ in Definition 7.0.3 with the above chain $\tilde{\gamma}^*$, which is guaranteed by the same proof as that in Lemma 7.0.5. Therefore, we have obtained

Lemma 7.1.4. Under the above situation, we can take the chain $\tilde{\gamma}^*$ as the chain of the Laplace inverse integral of f. In particular, we have

$$\mathcal{IL}(f) = \sum_{\alpha \in \Lambda^n} b_{\Omega_\alpha} \left(\left(\frac{\sqrt{-1}}{2\pi} \right)^n \int_{\tilde{\gamma}^*_\alpha} f(\zeta) e^{\zeta z} d\zeta \right) \otimes \nu_{\mathbb{D}_M} \in \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, \mathcal{V}_{\mathbb{D}_M}')), \tag{7.14}$$

where $\Omega_\alpha := M \hat{\times} \sqrt{-1}\Gamma_\alpha \subset \mathbb{D}_E$.

Note that each integral

$$h_\alpha(z) := \left(\frac{\sqrt{-1}}{2\pi} \right)^n \int_{\tilde{\gamma}^*_\alpha} f(\zeta) e^{\zeta z} d\zeta \tag{7.15}$$

belongs to $\mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega_\alpha)$. We will now explain an advantage of this expression: Set

$$\Omega := \widehat{\ }((\mathbb{C} \setminus \mathbb{R}_{\geq 0}) \times (\mathbb{C} \setminus \mathbb{R}_{\geq 0}) \times \cdots \times (\mathbb{C} \setminus \mathbb{R}_{\geq 0})) \quad \subset \mathbb{D}_E.$$

Proposition 7.1.5. For any $\alpha \in \Lambda^n$, the $\operatorname{sgn}(\alpha) h_\alpha(z) \in \mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega_\alpha)$ analytically extends to the same holomorphic function in $\mathscr{O}^{\exp}_{\mathbb{D}_E}(\Omega)$. Here we set $\operatorname{sgn}(\alpha) = \alpha_1 \alpha_2 \cdots \alpha_n$.

Proof. Let β be the subset in $\{1, \ldots, n\}$, and set

$$\begin{aligned} \Omega_{\alpha,\beta} &:= \Omega_\alpha \bigcap \widehat{\ }\{z \in E;\, \operatorname{Re} z_k < 0\ (k \in \beta)\} \\ &= \widehat{\ }\{z = x + \sqrt{-1}y \in E;\, x_k < 0\ (k \in \beta),\ \ \alpha_j y_j > 0\ (j = 1, 2, \ldots, n)\} \end{aligned}$$

and

$$\widetilde{\Omega_{\alpha,\beta}} := \widehat{\ }\{z = x + \sqrt{-1}y \in E;\, x_k < 0\ (k \in \beta),\ \ \alpha_j y_j > 0\ (j \notin \beta)\}.$$

Clearly we have

$$\Omega_{\alpha,\beta} \subset \widetilde{\Omega_{\alpha,\beta}}, \qquad \Omega = \bigcup_{\alpha \in \Lambda^n,\, \beta \subset \{1,2,\ldots,n\}} \widetilde{\Omega_{\alpha,\beta}}.$$

Let us define the continuous function $\tilde{\gamma}^*_{\alpha,\beta} : [0,1] \times \Gamma^*_\alpha \to E^*$ by

$$\tilde{\gamma}^*_{\alpha,\beta}(s, \eta) := \xi + \sqrt{-1}\tilde{\eta} \qquad (\eta \in \Gamma^*_\alpha,\, s \in [0,1]).$$

Here

$$\xi = a^* + \left(((1-\delta_{\beta,1}(s))\hat{\psi}(\eta) + \delta_{\beta,1}(s)|\eta|)\frac{|\eta_1|}{|\eta|},\ \dots, ((1-\delta_{\beta,n}(s))\hat{\psi}(\eta) + \delta_{\beta,n}(s)|\eta|)\frac{|\eta_n|}{|\eta|} \right)$$

and

$$\tilde{\eta} = \big((1-\delta_{\beta,1}(s))\eta_1, \dots, (1-\delta_{\beta,n}(s))\eta_n\big),$$

where $\delta_{\beta,k}(s) = s$ if $k \in \beta$ and $\delta_{\beta,k}(s) = 0$ otherwise. Since $\tilde{\gamma}^*_{\alpha,\beta}(0, \Gamma^*_\alpha) = \tilde{\gamma}^*_\alpha$ holds, we have

$$\partial\tilde{\gamma}^*_{\alpha,\beta}([0,1], \Gamma^*_\alpha) = -\tilde{\gamma}^*_\alpha + \tilde{\gamma}^*_{\alpha,\beta}(1, \Gamma^*_\alpha) - \tilde{\gamma}^*_{\alpha,\beta}([0,1], \partial\Gamma^*_\alpha).$$

Let z be a point in $\Omega_{\alpha,\beta}$. Then, as f is holomorphic, we have

$$0 = \int_{\tilde{\gamma}^*_{\alpha,\beta}([0,1],\Gamma^*_\alpha)} d(f(\zeta)e^{\zeta z}d\zeta) = \int_{\partial\tilde{\gamma}^*_{\alpha,\beta}([0,1],\Gamma^*_\alpha)} f(\zeta)e^{\zeta z}d\zeta,$$

which implies

$$\int_{\tilde{\gamma}^*_{\alpha,\beta}(1,\Gamma^*_\alpha)} f(\zeta)e^{\zeta z}d\zeta - \int_{\tilde{\gamma}^*_\alpha} f(\zeta)e^{\zeta z}d\zeta = \int_{\tilde{\gamma}^*_{\alpha,\beta}([0,1],\partial\Gamma^*_\alpha)} f(\zeta)e^{\zeta z}d\zeta.$$

Note that

$$\tilde{\gamma}^*_{\alpha,\beta}([0,1], \partial\Gamma^*_\alpha) = \bigcup_{k=1}^{n} \left(\tilde{\gamma}^*_{\alpha,\beta}([0,1], \partial\Gamma^*_\alpha) \cap \{\zeta_k = a^*_k\}\right)$$

holds. By noticing $d\zeta_k = 0$ on each real n-chain $\tilde{\gamma}^*_{\alpha,\beta}([0,1], \partial\Gamma^*_\alpha) \cap \{\zeta_k = a^*_k\}$, we get

$$\int_{\tilde{\gamma}^*_{\alpha,\beta}([0,1],\partial\Gamma^*_\alpha)} f(\zeta)e^{\zeta z}d\zeta = 0,$$

from which

$$\int_{\tilde{\gamma}^*_\alpha} f(\zeta)e^{\zeta z}d\zeta = \int_{\tilde{\gamma}^*_{\alpha,\beta}(1,\Gamma^*_\alpha)} f(\zeta)e^{\zeta z}d\zeta$$

follows. It is easy to see that the last integral belongs to $\mathscr{O}^{\exp}_{\mathbb{D}_E}(\widetilde{\Omega_{\alpha,\beta}})$. Hence, by taking arbitrary $\beta \subset \{1, \dots, n\}$, we see that $\operatorname{sgn}(\alpha)h_\alpha(z)$ analytically extends to $\bigcup_{\beta\subset\{1,\dots,n\}} \widetilde{\Omega_{\alpha,\beta}}$.

In particular, on $\widetilde{\Omega_{\alpha,\beta}}$ with $\beta = \{1, \dots, n\}$, i.e., ,

$$\widetilde{\Omega_{\alpha,\beta}} = \hat{}\{z = x + \sqrt{-1}y \in E;\ x_k < 0\ (k = 1, \dots, n)\},$$

$\operatorname{sgn}(\alpha)h_\alpha(z)$ coincides with the integration on the real domain

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{a^*+\Gamma^*_{+n}} f(\xi)e^{\xi z}d\xi,$$

which does not depend on the index $\alpha \in \Lambda^n$. Therefore, all the analytic extensions of $\operatorname{sgn}(\alpha)h_\alpha$ coincide on this domain, and thus, they form the holomorphic function of exponential type on the domain

$$\bigcup_{\alpha,\beta} \widetilde{\Omega_{\alpha,\beta}} = \Omega.$$

This completes the proof.

□

8 Laplace inversion formula

This section is devoted to proof for the Laplace inversion formula, that is, $\mathcal{L}$ and $\mathcal{IL}$ are mutually inverse.

Theorem 8.0.1. Let $G \subset M$ be an $\mathbb{R}_+$-conic proper closed convex subset and $a \in M$. Set $K = \overline{a + G} \subset \mathbb{D}_M$. Then the Laplace transformation

$$\mathcal{L} : \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})) \to e^{-a\zeta} \mathscr{O}^{\inf}_{E^*_\infty}(\mathrm{N}^*_{pc}(K))$$

and the inverse Laplace transformation

$$\mathcal{IL} : e^{-a\zeta} \mathscr{O}^{\inf}_{E^*_\infty}(\mathrm{N}^*_{pc}(K)) \to \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}))$$

are inverse to each other.

Remark 8.0.2. For K and G in the above theorem, as G is a cone,

$$\mathrm{N}^*_{pc}(K) = \mathrm{N}^*_{pc}(G) = \widehat{\ }(G^\circ) \cap E^*_\infty \tag{8.1}$$

holds, where G° is the dual open cone of G in E^*, that is,

$$G^\circ = \{\zeta \in E^*; \mathrm{Re}\,\langle \zeta, x\rangle > 0 \quad (\forall x \in G)\}.$$

Thanks to Corollary 6.2.7 and Lemmas 7.0.5 and 8.0.4, the following corollary immediately follows from Theorem 8.0.1:

Corollary 8.0.3. Let $K \subset \mathbb{D}_M$ be a regular closed subset satisfying that $K \cap M$ is convex and $\mathrm{N}^*_{pc}(K) \cap M^*_\infty$ is connected (in particular, non-empty). Then the Laplace transformation

$$\mathcal{L} : \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})) \to \mathscr{O}^{\inf - h_K}_{E^*_\infty}(\mathrm{N}^*_{pc}(K))$$

and the inverse Laplace transformation

$$\mathcal{IL} : \mathscr{O}^{\inf - h_K}_{E^*_\infty}(\mathrm{N}^*_{pc}(K)) \to \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}))$$

are inverse to each other.

Lemma 8.0.4. Let K be a closed subset in $\mathbb{D}_M$. Assume that K is regular and $K \cap M$ is convex and that $\mathrm{N}^*_{pc}(K)$ is non-empty. Then we have

$$K = \bigcap_{\xi \in \mathrm{N}^*_{pc}(K) \cap M^*_\infty} \overline{\{x \in M\,;\, \langle x,\, \xi\rangle \geq h_K(\xi)\}}.$$

Proof. It is enough to show that, for any $x_0 \in M$ with $x_0 \notin K$, there exists a hypersurface L in M passing through x_0 such that K and $\overline{L}$ are disjoint in $\mathbb{D}_M$.

Since $\mathrm{N}^*_{pc}(K)$ is not empty, we can take $\xi_0 \in \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and $r \in \mathbb{R}$ such that

$$K \subset \widehat{\ }\{x \in M\,;\, \langle x,\, \xi_0\rangle > r\}.$$

Set

$$L_{\xi_0} := \widehat{\ }\{x \in M\,;\, \langle x,\, \xi_0\rangle = r\}.$$

We may assume $x_0 \in \{x \in M;\, \langle x,\, \xi_0\rangle > r\}$ from the beginning.

Since $K \cap M$ is convex, we can find a hypersurface L which separates x_0 and K in M. The claim follows if $\overline{L}$ also separates them in $\mathbb{D}_M$. Hence we may assume that $\overline{L} \cap K \cap M_\infty$ is non-empty, from which we conclude that the both normal vectors of L are not in $\mathrm{N}^*_{pc}(K)$, and thus, we have $\dim(L \cap L_{\xi_0}) = n - 2$.

We can take the hypersurface $\tilde{L}$ in M which passes x_0 and $L \cap L_{\xi_0}$. Then the hypersurface $\tilde{L}$ has the required properties, which completes the proof. ∎

8.1 The proof for $\mathcal{L} \circ \mathcal{IL} = \mathrm{id}$.

Let $f \in e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(\mathrm{N}^*_{pc}(K)) = e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(\widehat{\ }(G^\circ) \cap E^*_\infty)$. By a coordinate transformation, we may assume that $a = 0$ and $G \subset \Gamma_{+^n} \cup \{0\}$ from the beginning (see (7.12) for the set Γ_{+^n}). Let $\Lambda = \{+1, -1\}$, and let $h_\alpha(z)$ ($\alpha \in \Lambda^n$) be a holomorphic function defined in (7.15). Then, by Lemma 7.1.4, we have

$$\mathcal{IL}(f) = \sum_{\alpha \in \Lambda^n} b_{\Omega_\alpha}(h_\alpha(z)) \otimes \nu_{\mathbb{D}_M}.$$

Note that $\mathrm{Supp}(\mathcal{IL}(f)) \subset \overline{G} \subset \widehat{\Gamma_{+^n}} \cup \{0\}$ hold. It follows from Proposition 7.1.5 that we can compute the Laplace transform of $\mathcal{IL}(f)$ by the formula given in Example 6.3.5. Hence we have

$$(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\alpha \in \Lambda^n} \mathrm{sgn}(\alpha) \int_{\gamma_\alpha} dz \int_{\gamma^*_\alpha} f(\zeta) e^{(\zeta - \tilde{\zeta})z} d\zeta.$$

Here we take $\epsilon > 0$ sufficiently small and $\gamma_\alpha \subset E$ is given by

$$\left\{ z = b + (B_\epsilon + \sqrt{-1}\epsilon A_\alpha)x \,;\, x \in \Gamma_{+^n} \right\},$$

where the diagonal matrix

$$A_\alpha = \begin{pmatrix} \alpha_1 & 0 & & & \\ 0 & \alpha_2 & 0 & & \\ & & \vdots & & \\ & & 0 & \alpha_{n-1} & 0 \\ & & & 0 & \alpha_n \end{pmatrix},$$

$b = -c(1, 1, \dots, 1) \in \Gamma_{-^n}$ with a sufficiently small $c > 0$ and B_ϵ is given in Example 6.3.5. The $\gamma^*_\alpha \subset E^*$ is given by

$$\left\{ \zeta = a^* + \xi(\delta I + \sqrt{-1}A_\alpha) \,;\, \xi \in \Gamma^*_{+^n} \right\},$$

where I is the identity matrix, $\epsilon > \delta > 0$ and $a^* = a(1, 1, \cdots, 1) \in \Gamma^*_{+^n}$ for a sufficiently large $a > 0$. Note that the orientation of γ_α and γ^*_α are determined by those of the parameter spaces Γ_{+^n} and $\Gamma^*_{+^n}$, respectively.

Remark 8.1.1. The above integral does not depend on the choice of $\epsilon > 0$ if it is sufficiently small, and we make ϵ tend to 0 later.

In what follows, we may assume that $\tilde{\zeta} \in E^*$ is in a sufficiently small open neighborhood of $a^* + \Gamma^*_{+^n}$ and that $|\tilde{\zeta}|$ is large enough. As a matter of fact, if we could show $(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = f(\tilde{\zeta})$ for such a $\tilde{\zeta}$, the claims follows from the unique continuation property of f.

When $z \in \gamma_\alpha$ and $\zeta \in \gamma^*_\alpha$, we have

$$\begin{aligned} \mathrm{Re}(\zeta - \tilde{\zeta})z &= -\mathrm{Re}\,\tilde{\zeta}z + \mathrm{Re}\,\zeta z \\ &= \left(\langle a^* - \mathrm{Re}\,\tilde{\zeta},\, b + B_\epsilon x \rangle + \epsilon \sum_{k=1}^n \alpha_k x_k \,\mathrm{Im}\,\tilde{\zeta}_k \right) + \delta\langle \xi,\, b \rangle + \Big(\delta\langle \xi,\, B_\epsilon x \rangle - \epsilon\langle \xi,\, x \rangle \Big). \end{aligned}$$

Note that, for $x \in \Gamma_{+^n}$ and $\xi \in \Gamma^*_{+^n}$, we have

$$\left(\delta\langle \xi,\, B_\epsilon x\rangle - \epsilon\langle \xi,\, x\rangle\right) \le -\min\{\epsilon - \delta, \epsilon\delta\}|x||\xi| \le 0.$$

Hence the above integration absolutely converges and, by the Fubini's theorem, we obtain

$$(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\alpha\in\Lambda^n} \operatorname{sgn}(\alpha) \int_{\gamma^*_\alpha} f(\zeta)d\zeta \int_{\gamma_\alpha} e^{(\zeta-\tilde{\zeta})z} dz.$$

Then, if ζ is quite near a^* and $\tilde{\zeta} \in E^*$ belongs to a sufficiently small open neighborhood of $a^* + \Gamma^*_{+^n}$ and if $|\tilde{\zeta}|$ is large enough, we get

$$\int_{\gamma_\alpha} e^{(\zeta-\tilde{\zeta})z} dz = \det(Q_{\alpha,\epsilon}) \int_{\Gamma_{+^n}} e^{(\zeta-\tilde{\zeta})(b+Q_{\alpha,\epsilon}x)} dx = \frac{\det(Q_{\alpha,\epsilon})\, e^{(\zeta-\tilde{\zeta})b}}{(\tilde{\zeta}-\zeta)Q_{\alpha,\epsilon}},$$

where

$$\frac{1}{(\tilde{\zeta}-\zeta)Q_{\alpha,\epsilon}} := \frac{1}{\prod_{k=1}^n e_k(\tilde{\zeta}-\zeta)Q_{\alpha,\epsilon}}.$$

Here e_k is the unit row vector whose k-th entry is 1 and

$$Q_{\alpha,\epsilon} = B_\epsilon + \sqrt{-1}\epsilon A_\alpha.$$

By uniqueness of the analytic continuation, the above formula holds at any point ζ in a neighborhood of the chain γ^*_α, and hence, we have

$$(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\alpha\in\Lambda^n} \operatorname{sgn}(\alpha) \det(Q_{\alpha,\epsilon}) \int_{\gamma^*_\alpha} \frac{f(\zeta)\, e^{(\zeta-\tilde{\zeta})b}}{(\tilde{\zeta}-\zeta)Q_{\alpha,\epsilon}} d\zeta.$$

Now if we could show that there exist $s > \delta$ and a complex open neighborhood $T \subset \mathbb{C}$ of $(0, s)$ such that the denominator of the integrand in the above integral does not vanish when $\zeta \in \gamma^*_\alpha$ and $\epsilon \in T$ (δ and other constants are fixed, where we do not keep the condition $\epsilon > \delta$ anymore), then the above integral becomes an analytic function of ϵ ($\epsilon > 0$), and thus, it turns out to be a constant function of ϵ due to Remark 8.1.1. Hence, by letting ϵ to 0, we have obtained

$$(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_{\alpha\in\Lambda^n} \operatorname{sgn}(\alpha) \int_{\gamma^*_\alpha} \frac{f(\zeta)\, e^{(\zeta-\tilde{\zeta})b}}{\zeta-\tilde{\zeta}} d\zeta$$

which is clearly equal to $f(\tilde{\zeta})$ by the Cauchy integral formula.

Let $g(\zeta, \eta)$ be the first element of the vector $\zeta Q_{\alpha,\epsilon} - \eta$, and let us show $g(\zeta, \eta) \neq 0$ for any $\zeta \in \gamma^*_\alpha$ and for any η contained in a sufficiently small neighborhood of the point $R(1, 1, \cdots, 1)Q_{\alpha,\epsilon}$ with a sufficiently large $R > 0$. Set $\epsilon = \epsilon' + \sqrt{-1}\epsilon''$ for a sufficiently small $\epsilon' > 0$ and $\epsilon'' \in \mathbb{R}$ with $|\epsilon''| < \delta\epsilon'/2$. The real part of $g(\zeta, \eta)$ is, for $\zeta = a^* + \xi(\delta E + \sqrt{-1}A_\alpha)$ with $\xi \in \overline{\Gamma_{+^n}}$,

$$(\delta - \epsilon' - \alpha_1\epsilon'')\xi_1 - ((\delta\epsilon' - \alpha_2\epsilon'')\xi_2 + \cdots + (\delta\epsilon' - \alpha_n\epsilon'')\xi_n) + (1 - (n-1)\epsilon' - \alpha_1\epsilon'')a - \operatorname{Re}\,\eta_1$$

and its imaginary part is

$$\alpha_1((1 + \epsilon' - \alpha_1\epsilon'')\xi_1 + (\epsilon' - (n-1)\alpha_1\epsilon'')a) - ((\alpha_2\epsilon' + \delta\epsilon'')\xi_2 + \cdots + (\alpha_n\epsilon' + \delta\epsilon'')\xi_n) - \operatorname{Im}\,\eta_1.$$

If $\operatorname{Re} g(\zeta, \eta) = 0$, then we have

$$(\delta\epsilon' - \alpha_2\epsilon'')\xi_2 + \cdots + (\delta\epsilon' - \alpha_n\epsilon'')\xi_n = (\delta - \epsilon' - \alpha_1\epsilon'')\xi_1 - (\operatorname{Re}\eta_1 - (1 - (n-1)\epsilon' - \alpha_1\epsilon'')a),$$

which gives the estimate

$$(\delta\epsilon' - |\epsilon''|)(\xi_2 + \cdots + \xi_n) \le (\delta - (\epsilon' - |\epsilon''|))\xi_1 - (\operatorname{Re}\eta_1 - a),$$

that is, we have obtained

$$(\epsilon' - \delta^{-1}|\epsilon''|)(\xi_2 + \cdots + \xi_n) \le (1 - \delta^{-1}(\epsilon' - |\epsilon''|))\xi_1 - \delta^{-1}(\operatorname{Re}\eta_1 - a).$$

Hence, when $\operatorname{Re} g(\zeta, \eta) = 0$, we get

$$\begin{aligned}|\operatorname{Im} g(\zeta, \eta)| &\ge (1 + \epsilon' - |\epsilon''|)\xi_1 - (\epsilon' + \delta|\epsilon''|)(\xi_2 + \cdots + \xi_n) - |\operatorname{Im}\eta_1| \\ &\ge \ell(\epsilon', \epsilon'')\xi_1 + \delta^{-1}\left(\frac{\epsilon' + \delta|\epsilon''|}{\epsilon' - \delta^{-1}|\epsilon''|}\right)(\operatorname{Re}\eta_1 - a) - |\operatorname{Im}\eta_1|,\end{aligned}$$

where

$$\ell(\epsilon', \epsilon'') = (1 + \epsilon' - |\epsilon''|) - \left(\frac{\epsilon' + \delta|\epsilon''|}{\epsilon' - \delta^{-1}|\epsilon''|}\right)\left(1 - \delta^{-1}(\epsilon' - |\epsilon''|)\right).$$

Note that, for each $\epsilon' > 0$, we have $\ell(\epsilon', \epsilon'') > 0$ if $|\epsilon''|$ is sufficiently small. In what follows, we consider the case for such an $\epsilon = \epsilon' + \sqrt{-1}\epsilon''$. When η is contained in a sufficiently small neighborhood of the point $R(1, 1, \cdots, 1)Q_{\alpha,\epsilon}$, we have

$$\operatorname{Re}\,\eta_1 \sim R - ((n-1)\epsilon' + \alpha_1\epsilon'')R, \qquad \operatorname{Im}\,\eta_1 \sim (\alpha_1\epsilon' - (n-1)\epsilon'')R.$$

Hence, if R is sufficiently large, $\operatorname{Im} g(\zeta, \eta)$ never becomes zero. This completes the proof.

8.2 The proof for $\mathcal{IL} \circ \mathcal{L} = \mathrm{id}$.

Let G be an $\mathbb{R}_+$-conic proper closed convex subset in M and $a \in M$. Set $K = \overline{a + G} \subset \mathbb{D}_M$. Then we take an open convex cone $V \subset \mathbb{D}_E$ containing K. Let $u \in \Gamma_K(\mathbb{D}_M;\, \mathscr{O}^{\exp}_{\mathbb{D}_E} \otimes \mathscr{V}^{\exp}_{\mathbb{D}_M})$ with a representative $\nu = (\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K')$. We will show $(\mathcal{IL} \circ \mathcal{L})(u) = u$. By a coordinate transformation, we may assume $a = 0$ and

$$\overline{G} \setminus \{0\} \subset \widehat{\Gamma_{+^n}} \subset \overline{\Gamma_{+^n}} \subset V$$

from the beginning. Then, it follows from Lemma 7.1.4 that we get

$$(\mathcal{IL} \circ \mathcal{L})(u) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_{\alpha \in \Lambda^n} b_{\Omega_\alpha}\left(\int_{\tilde{\gamma}^*_\alpha} \mathcal{L}(u)\, e^{\zeta\tilde{z}} d\zeta\right).$$

Set

$$g_\alpha(\tilde{z}) := \int_{\tilde{\gamma}^*_\alpha} \mathcal{L}(u)\, e^{\zeta\tilde{z}} d\zeta.$$

It follows from Proposition 7.1.5 that g_α extends to a holomorphic function on Ω of exponential type. Here

$$\Omega = \widehat{\ }((\mathbb{C} \setminus \mathbb{R}_{\ge 0}) \times (\mathbb{C} \setminus \mathbb{R}_{\ge 0}) \times \cdots \times (\mathbb{C} \setminus \mathbb{R}_{\ge 0})) \quad \subset \mathbb{D}_E.$$

We first consider $g_\alpha(\tilde z)$ at a point in $\Gamma_{-^n} \times \sqrt{-1}\Gamma_\alpha$. Let us take $\tilde z$ in $\Gamma_{-^n} \times \sqrt{-1}\Gamma_\alpha$ and fix it. Then, at this $\tilde z$, we can deform the n-chain $\tilde\gamma^*_\alpha$ to

$$\left\{\zeta = \xi + \sqrt{-1}\eta' \in E^*; \begin{array}{l}\eta'_k = \alpha_k \eta_k \ (k = 1, \dots, n), \\ \xi = a^* + \epsilon'\eta, \quad \eta \in \Gamma^*_{+^n}\end{array}\right\}$$

with $a^* \in \Gamma^*_{+^n}$ and $\epsilon' > 0$. Here the orientation of the modified chain $\tilde\gamma^*_\alpha$ is the same as the original one and we assume $|a^*|$ to be sufficiently large.

Now, since ζ runs in $\tilde\gamma^*_\alpha$, the real $2n$-chain D of the integration

$$\mathcal{L}(u)(\zeta) := \int_D e^{-z\zeta}\nu_1(z) - \int_{\partial D} e^{-z\zeta}\nu_{01}(z),$$

can be

$$D = \{z = x + \sqrt{-1}y;\, x \in b + \Gamma',\, |y| < \epsilon \operatorname{dist}(x,\, M \setminus (b + \Gamma'))\},$$

where $b = -\epsilon\,(1, \dots, 1)$ and $\Gamma' \subset M$ is an $\mathbb{R}_+$-conic open convex cone such that

$$\overline{G} \setminus \{0\} \subset \widehat{\Gamma'} \subset \overline{\Gamma'} \setminus \{0\} \subset \widehat{\Gamma_{+^n}}.$$

Note that, if $\zeta = \xi + \sqrt{-1}\eta \in \tilde\gamma^*_\alpha$ and $z = x + \sqrt{-1}y \in D$, we have

$$\operatorname{Re}(\tilde z - z)\zeta = (\operatorname{Re}\tilde z - x)a^* - \sum_{k=1}^n \alpha_k(\operatorname{Im}\tilde z_k)\eta_k + \left(\epsilon'(\operatorname{Re}\tilde z - x)\eta + \sum_{k=1}^n \alpha_k y_k \eta_k\right).$$

If $\epsilon > 0$ is sufficiently small and $\operatorname{Re}\tilde z \in b + \Gamma_{-^n}$, by noticing

$$|y| < \epsilon(|b| + |x|) \qquad (z = x + \sqrt{-1}y \in D),$$

we can easily see that, for any $z = x + \sqrt{-1}y \in D$ and $\zeta = \xi + \sqrt{-1}\eta \in \tilde\gamma^*_\alpha$,

$$\epsilon'(\operatorname{Re}\tilde z - x)\eta + \sum_{k=1}^n \alpha_k y_k \eta_k = \Big(\epsilon'(\operatorname{Re}\tilde z - x) + (\alpha_1 y_1, \dots, \alpha_n y_n)\Big)\eta \le 0$$

holds. Hence, the double integral in g_α absolutely converges and we can apply Fubini's theorem to g_α, from which we get

$$g_\alpha(\tilde z) = \int_D \nu_1(z) \int_{\tilde\gamma^*_\alpha} e^{(\tilde z - z)\zeta} d\zeta - \int_{\partial D} \nu_{01}(z) \int_{\tilde\gamma^*_\alpha} e^{(\tilde z - z)\zeta} d\zeta.$$

Now let us consider the integral $\displaystyle\int_{\tilde\gamma^*_\alpha} e^{(\tilde z - z)\zeta} d\zeta$. If $\operatorname{Re}(\tilde z - z) \in \Gamma_{-^n}$ and $|\operatorname{Im}(\tilde z - z)|$ is sufficiently small, then we can deform the n-chain to the one in M^* as was done in the proof of Proposition 7.1.5, we have

$$\int_{\tilde\gamma^*_\alpha} e^{(\tilde z - z)\zeta} d\zeta = \operatorname{sgn}(\alpha) \int_{a^* + \Gamma^*_{+^n}} e^{(\tilde z - z)\xi} d\xi = \operatorname{sgn}(\alpha) \frac{e^{(\tilde z - z)a^*}}{z - \tilde z},$$

where $\operatorname{sgn}(\alpha) = \alpha_1\alpha_2 \cdots \alpha_n$. Note that, by the unique continuation property,

$$\int_{\tilde\gamma^*_\alpha} e^{(\tilde z - z)\zeta} d\zeta = \operatorname{sgn}(\alpha) \frac{e^{(\tilde z - z)a^*}}{z - \tilde z}$$

holds at a point where the integral is defined. Summing up, we have obtained

$$g_\alpha(\tilde{z}) = \operatorname{sgn}(\alpha) \left(\int_D \frac{e^{(\tilde{z}-z)a^*}\nu_1(z)}{z-\tilde{z}} - \int_{\partial D} \frac{e^{(\tilde{z}-z)a^*}\nu_{01}(z)}{z-\tilde{z}} \right)$$

if $\tilde{z} \in \Gamma_{-^n} \times \sqrt{-1}\Gamma_\alpha$. By deforming D appropriately, we see that the integrals in the right-hand side converge on $M \times \sqrt{-1}\Gamma_\alpha$, and hence, the above equation also holds there. It follows from Theorem 6.4.3 that we have

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_{\alpha\in\Lambda^n} \operatorname{sgn}(\alpha)\, b_{\Omega_\alpha} \left(\int_D \frac{e^{(\tilde{z}-z)a^*}\nu_1(z)}{z-\tilde{z}} - \int_{\partial D} \frac{e^{(\tilde{z}-z)a^*}\nu_{01}(z)}{z-\tilde{z}} \right) = [\nu] = u.$$

This completes the proof.

9 Application to PDE with constant coefficients

Let $\mathfrak{R}$ be the polynomial ring $\mathbb{C}[\zeta_1, \cdots, \zeta_n]$ on E^* and $\mathfrak{D}$ the ring $\mathbb{C}[\partial_{x_1}, \cdots, \partial_{x_n}]$ of linear differential operators on M with constant coefficients. We denote by σ the principal symbol map from $\mathfrak{D}$ to $\mathfrak{R}$, that is,

$$\mathfrak{D} \ni P(\partial) = \sum c_\alpha \partial^\alpha \mapsto \sigma(P)(\zeta) = \sum_{|\alpha|=\mathrm{ord}(P)} c_\alpha \zeta^\alpha \in \mathfrak{R}.$$

For an $\mathfrak{D}$ module $\mathfrak{M} = \mathfrak{D}/\mathfrak{I}$ with the ideal $\mathfrak{I} \subset \mathfrak{D}$, we define the closed subset in E^*_∞

$$\mathrm{Char}_{E^*_\infty}(\mathfrak{M}) = \{\zeta \in E^*_\infty;\, \sigma(P)(\zeta) = 0 \quad (\forall\, P \in \mathfrak{I})\}.$$

Here we identify a point in E^*_∞ with a unit vector in E^*.

Recall that $\{f_1, \cdots, f_\ell\}$ ($f_k \in \mathfrak{R}$) is said to be a regular sequence over $\mathfrak{R}$ if and only if the conditions below are satisfied:

1. $(f_1, \cdots, f_\ell) \neq \mathfrak{R}$.
2. For any $k = 1, 2, \cdots, \ell$, the f_k is not a zero divisor on $\mathfrak{R}/(f_1, \cdots, f_{k-1})$.

The following theorem is fundamental in the theory of operational calculus: Let $P_1(\partial)$, $\cdots$, $P_\ell(\partial)$ be in $\mathfrak{D}$, and define the $\mathfrak{D}$ module

$$\mathfrak{M} = \mathfrak{D}/(P_1(\partial), \cdots, P_\ell(\partial)).$$

Theorem 9.0.1. Let K be a regular closed subset in $\mathbb{D}_M$. Assume that $K \cap M$ is convex and $\mathrm{N}^*_{pc}(K) \cap M^*_\infty$ is connected, and that $P_1(\zeta)$, $\cdots$, $P_\ell(\zeta)$ form a regular sequence over $\mathfrak{R}$. Then the condition

$$\mathrm{N}^*_{pc}(K) \cap \mathrm{Char}_{E^*_\infty}(\mathfrak{M}) = \emptyset$$

implies

$$\mathrm{Ext}^k_{\mathfrak{D}}(\mathfrak{M},\, \Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M})) = 0 \qquad (k = 0, 1).$$

Proof. Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}$ modules or a $\mathbb{Z}$ module itself and $s_i : \mathcal{F} \to \mathcal{F}$ ($i = 1, \cdots, \ell$) a morphism such that $s_i \circ s_j = s_j \circ s_i$ holds for $1 \le i, j \le \ell$. Then we denote by

$K(s_1, \cdots, s_\ell; \mathcal{F})$ the Koszul complex associated to $(s_1, \cdots, s_\ell)$ with coefficients in $\mathcal{F}$. That is,

$$0 \to \overset{\text{0-th degree}}{\mathcal{F} \otimes (\overset{0}{\wedge}\Lambda)} \xrightarrow{d} \mathcal{F} \otimes (\overset{1}{\wedge}\Lambda) \xrightarrow{d} \mathcal{F} \otimes (\overset{2}{\wedge}\Lambda) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{F} \otimes (\overset{\ell}{\wedge}\Lambda) \to 0,$$

where Λ is a free $\mathbb{Z}$ module of rank ℓ with basis e_1, e_2, $\cdots$, e_ℓ and

$$d(f \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{\ell} s_j(f) \otimes e_j \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Since $P_1(\zeta)$, $\cdots$, $P_\ell(\zeta)$ form a regular sequence, the complex $K(P_1(\partial), \cdots, P_\ell(\partial); \mathfrak{D})[\ell]$ is a free resolution of $\mathfrak{M}$ and we get

$$\mathbf{R}\mathrm{Hom}_{\mathfrak{D}}(\mathfrak{M}, \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M})) \simeq K(P_1(\partial), \cdots, P_\ell(\partial); \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M})).$$

Hence it follows from Corollary 8.0.3 that we have

$$\mathbf{R}\mathrm{Hom}_{\mathfrak{D}}(\mathfrak{M}, \Gamma_K(\mathbb{D}_M; \mathscr{B}^{\exp}_{\mathbb{D}_M})) \simeq K(P_1(\zeta), \cdots, P_\ell(\zeta); \Gamma(\mathrm{N}^*_{pc}(K); \mathscr{O}^{\inf - h_K}_{E^*_\infty})). \tag{9.1}$$

The lemma below is a key for the theorem:

Lemma 9.0.2. Let $\zeta^* \notin \mathrm{Char}_{E^*_\infty}(\mathfrak{M})$. Then the Koszul complex

$$K(P_1(\zeta), \cdots, P_\ell(\zeta); (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}) \tag{9.2}$$

is exact.

Proof. By the definition of $\mathrm{Char}_{E^*_\infty}(\mathfrak{M})$, we can find $h(\zeta)$ and $a_j(\zeta)$ ($j = 1, 2, \cdots, \ell$) in $\mathfrak{R}$ such that

$$\sigma(h)(\zeta^*) \neq 0, \qquad h(\zeta) = \sum_{j=1}^{\ell} a_j(\zeta) P_j(\zeta).$$

In particular, as $\sigma(h)(\zeta^*) \neq 0$ holds, h is also invertible in the germ $(\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}$ of the sheaf $\mathscr{O}^{\inf - h_K}_{E^*_\infty}$ at ζ^*. Set $\Lambda = \{1, 2, \cdots, \ell\}$ and let $s = \{s_k\}$ be a homotopy from $K(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*})$ to itself, where

$$s_k : K^{k+1}(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}) \to K^k(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*})$$

is given by

$$s_k(\sum_{\alpha \in \Lambda^{k+1}} f_\alpha(\zeta) e_\alpha) = \sum_{\beta \in \Lambda^k} \sum_{j=1}^{\ell} a_j(\zeta) f_{j\,\beta}(\zeta) e_\beta,$$

where $e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{k+1}}$ and $j\,\beta$ is a sequence such that β follows j. Then, by the simple computation, we can easily get the equality

$$s \circ d - d \circ s = h.$$

Here $h : K(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}) \to K(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*})$ is the morphism of complexes defined by

$$K^k(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}) \ni u \mapsto hu \in K^k(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}),$$

which is an isomorphism because $h(\zeta)$ is invertible on $(\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*}$. Therefore the isomorphism h is homotopic to zero, from which we conclude that the complex $K(P_1, \cdots, P_\ell; (\mathscr{O}^{\inf - h_K}_{E^*_\infty})_{\zeta^*})$ is quasi-isomorphic to zero. This completes the proof of the lemma. □

It follows from the lemma that the Koszul complex

$$K(P_1(\zeta), \cdots, P_\ell(\zeta);\ \mathscr{O}_{E_\infty^*}^{\inf - h_K}) \tag{9.3}$$

of sheaves is exact on $\mathrm{N}^*_{pc}(K)$ because of the condition $\mathrm{N}^*_{pc}(K) \cap \mathrm{Char}_{E_\infty^*}(\mathfrak{M}) = \emptyset$. Applying the left exact functor $\Gamma(\mathrm{N}^*_{pc}(K); \bullet)$ to the complex (9.3), we get a short exact sequence

$$0 \to \Gamma(\mathrm{N}^*_{pc}(K);\ \mathscr{O}_{E_\infty^*}^{\inf - h_K}) \to \Gamma(\mathrm{N}^*_{pc}(K);\ \mathscr{O}_{E_\infty^*}^{\inf - h_K}) \otimes (\overset{1}{\wedge}\Lambda) \to \Gamma(\mathrm{N}^*_{pc}(K);\ \mathscr{O}_{E_\infty^*}^{\inf - h_K}) \otimes (\overset{2}{\wedge}\Lambda).$$

Then, by noticing (9.1), the claim follows from the above short exact sequence. □

Corollary 9.0.3. Let $P(\partial) \in \mathfrak{D}$, and let K be a regular closed subset in $\mathbb{D}_M$ satisfying that $K \cap M$ is convex and $\mathrm{N}^*_{pc}(K) \cap M^*_\infty$ is connected. Then the morphism

$$\Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M}) \xrightarrow{P(\partial)\bullet} \Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M})$$

becomes isomorphic if $\sigma(P)(\zeta) \neq 0$ holds for any $\zeta \in \mathrm{N}^*_{pc}(K)$.

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