

A NOTE ON PURITY OF CRYSTALLINE LOCAL SYSTEMS

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ABSTRACT. In this short note, we prove a purity result for crystalline local systems on a smooth p -adic affine formal scheme. Our method is based on the prismatic description of crystalline local systems [DLMS24] (cf. [GR24]).

1. INTRODUCTION

Let K be a complete discrete valued field of mixed characteristic $(0, p)$ with the ring of integers \mathcal{O}_K and perfect residue field k . Denote $K_0 = W(k)[p^{-1}]$ and $G_K = \text{Gal}(\overline{K}/K)$ where \overline{K} is an algebraic closure of K . To any finite dimensional continuous \mathbf{Q}_p -representation V of G_K , Fontaine attached a K_0 -vector space $D_{\text{cris}}(V)$. More precisely, Fontaine introduced the crystalline period ring \mathbf{B}_{cris} equipped with a natural Frobenius and G_K -action, and considered $D_{\text{cris}}(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{cris}})^{G_K}$ ([Fon82], [Fon94]). We have $\dim_{K_0} D_{\text{cris}}(V) \leq \dim_{\mathbf{Q}_p} V$ in general, and V is called *crystalline* if the equality holds. The underlying motivation of this notion comes from its close connection to good reduction. If X is a proper smooth scheme over K with good reduction, i.e. if there is a proper smooth scheme $\mathcal{X}/\mathcal{O}_K$ such that $\mathcal{X} \times_{\mathcal{O}_K} K = X$, then the G_K -representation $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p)$ is crystalline and $D_{\text{cris}}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p)) \cong H_{\text{cris}}^i(\mathcal{X}_k/W(k))[p^{-1}]$.

When X is an abelian variety over K , the converse also holds: X has good reduction if and only if $H_{\text{ét}}^1(X_{\overline{K}}, \mathbf{Q}_p)$ is crystalline ([CI99], [Mok93]). However, for abelian schemes over regular bases of mixed characteristic, an interesting discrepancy occurs in terms of purity. Motivated by Grothendieck's work on Nagata–Zariski purity ([GR62]), we can ask the following question.

Question 1.1. Let $R = \mathcal{O}_K[[t_1, \dots, t_d]]$ (with $d \geq 1$), and let $\mathfrak{m} \subset R$ be the maximal ideal. Given any abelian scheme over $\text{Spec} R \setminus \{\mathfrak{m}\}$, does it extend uniquely to an abelian scheme over $\text{Spec} R$?

The answer to this question is positive if the ramification index $e = [K : K_0] \leq p - 1$, but is negative if $e \geq p$ ([VZ10]). We remark that when $e \leq p - 1$, the purity result implies the uniqueness of integral canonical models of Shimura varieties ([VZ10, Cor. 30]).

On the other hand, for *arbitrary* ramification index e , one can still study an analogous question on purity for geometric families of crystalline representations as follows. Faltings introduced the notion of crystalline p -adic étale local systems on the generic fiber of a smooth proper scheme over \mathcal{O}_K ([Fal88]). Furthermore, for certain affine schemes which include the cases we study in this paper, Brinon studied the foundation for crystalline local systems *à la* Fontaine by generalizing the construction of the crystalline period ring \mathbf{B}_{cris} ([Bri06], [Bri08]). We consider the small affine case, i.e.

when R is the p -adic completion of an étale algebra over $\mathcal{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$ such that $\mathrm{Spec}(R/\pi R)$ is connected. Denote $\mathcal{G}_R := \pi_1^{\text{ét}}(\mathrm{Spec} R[p^{-1}])$.

Let $\pi \in \mathcal{O}_K$ be a uniformizer, and let \mathcal{O}_L be the p -adic completion of $R_{(\pi)}$. Denote by $\mathcal{G}_{\mathcal{O}_L}$ the absolute Galois group of $L = \mathcal{O}_L[p^{-1}]$. Choose a geometric point of $\mathrm{Spec}(L)$, which gives a geometric point of $\mathrm{Spec}(R[p^{-1}])$ via the map $\mathrm{Spec}(L) \rightarrow \mathrm{Spec}(R[p^{-1}])$. By the change of paths for étale fundamental groups, we then have a continuous map of Galois groups $\mathcal{G}_{\mathcal{O}_L} \rightarrow \mathcal{G}_R$. For a finite dimensional continuous \mathbf{Q}_p -representation V of \mathcal{G}_R , we refer the reader to [Bri08, §8.2] for the definition of V being Hodge–Tate, de Rham, or crystalline. In this paper, we prove the the following purity statement.

Theorem 1.2. *Let R be the p -adic completion of an étale algebra over $\mathcal{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$, and let V be a finite dimensional continuous \mathbf{Q}_p -representation of \mathcal{G}_R . Then V is crystalline if and only if $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is crystalline.*

Note that the “only if” part of the above theorem follows directly from the definition.

Remark 1.3. In [Tsu, Thm. 5.4.8], Tsuji has already proved that if V is de Rham and $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is crystalline, then V is crystalline. Furthermore, the purity of de Rham representations is expected to hold; it is expected that a similar argument as in the proof of [LZ17, Thm. 1.5 (ii)] would imply that V is de Rham if and only if $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is de Rham. Another possible approach is based on the method in [Tsu11]. By [Tsu11, Thm. 9.1], V is Hodge–Tate if and only if $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is Hodge–Tate. It is expected that a similar argument as in *loc. cit.* can be used via the results in [AB10] to show V is de Rham if and only if $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is de Rham. Since any crystalline representation is de Rham, the expected purity of de Rham representations combined with the result of Tsuji [Tsu, Thm. 5.4.8] would imply Theorem 1.2.

Our method in this paper is completely different from the ones in the above remark. We employ the prismatic description of crystalline local systems given in [DLMS24] (cf. [GR24]).

Notation. Fix a prime p . Let k be a perfect field of characteristic p , and let K be a finite totally ramified extension of $K_0 := W(k)[p^{-1}]$ with ring of integers \mathcal{O}_K . Fix a uniformizer $\pi \in \mathcal{O}_K$, and let $E = E(u) \in W(k)[u]$ be the monic minimal polynomial of π .

For a ring A and a finitely generated ideal $J \subset A$, the J -adic completion of an A -module means the classical completion. Similarly, being J -adically complete or J -complete is in the classical sense. For a \mathbf{Z} -module M , we denote its p -adic completion by M_p^\wedge .

A p -adically completed étale map from a p -adically complete ring B refers to the p -adic completion of an étale map from B . Write $\mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ for the p -adic completion of the Laurent polynomial ring $\mathcal{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$, and similarly for $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$.

For an element a of a \mathbf{Q} -algebra A and $n \geq 0$, write $\gamma_n(a)$ for the element $\frac{a^n}{n!} \in A$.

ACKNOWLEDGEMENTS

I would like to thank Heng Du, Tong Liu, and Koji Shimizu for helpful discussions. I also thank the anonymous referees for many valuable suggestions to improve the paper.

2. ABSOLUTE PRISMATIC SITE & KISIN DESCENT DATUM

We first recall some of the main results in [DLMS24] on the equivalence between the category of crystalline local systems and the category of *Kisin descent data*. As in [DLMS24, Assumption 2.9], we consider the cases when the base ring is either small over \mathcal{O}_K or a complete discrete valuation ring.

Let R be a p -adically completed étale algebra over $\mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ for some $d \geq 0$ such that $\mathrm{Spec}(R/\pi R)$ is connected. There exists a subring $R_0 \subset R$ such that R_0 is p -adically completed étale over $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ and $R_0 \otimes_{W(k)} \mathcal{O}_K = R$ (see e.g. [GR24, Lem. 2.9]). Let $\varphi: R_0 \rightarrow R_0$ be the (unique) lift of Frobenius on R_0/pR_0 with $\varphi(T_i) = T_i^p$. Let \mathcal{O}_{L_0} be the p -adic completion of $(R_0)_{(p)}$ equipped with the Frobenius induced from φ on R_0 . Note that \mathcal{O}_{L_0} is a complete discrete valuation ring whose residue field has a finite p -basis given by $\{T_1, \dots, T_d\}$. We have a natural injective map $R_0 \rightarrow \mathcal{O}_{L_0}$ compatible with φ . This extends \mathcal{O}_K -linearly to $R \rightarrow \mathcal{O}_L := \mathcal{O}_{L_0} \otimes_{W(k)} \mathcal{O}_K$.

Assumption. In the following, we will assume the base ring S is either R or \mathcal{O}_L . Denote $S_0 = R_0$ (resp. $S_0 = \mathcal{O}_{L_0}$) when $S = R$ (resp. $S = \mathcal{O}_L$).

Definition 2.1 ([BS22, Def. 3.2]). A *bounded prism* is a pair (A, I) where A is a δ -ring (cf. [BS22, Def. 2.1]) and $I \subset A$ is an invertible ideal such that $p \in I + \varphi(I)A$, A/I has bounded p^∞ -torsion, and A is (p, I) -complete (see [BS22, Lem. 3.7 (1)]). Here, $\varphi: A \rightarrow A$ is given by $\varphi(x) = x^p + p\delta(x)$.

Definition 2.2 ([BS23, Def. 2.3]). The *absolute prismatic site* S_Δ of the p -adic formal scheme $\mathrm{Spf}S$ consists of the pairs $((A, I), \mathrm{Spf}A/I \rightarrow \mathrm{Spf}S)$ where (A, I) is a bounded prism and $\mathrm{Spf}A/I \rightarrow \mathrm{Spf}S$ is a morphism of p -adic formal schemes. For simplicity, we often omit the structure map $\mathrm{Spf}A/I \rightarrow \mathrm{Spf}S$ and write $(A, I) \in S_\Delta$. The morphisms are the opposite of morphisms of bounded prisms compatible with the structure maps to $\mathrm{Spf}S$. We equip S_Δ with the topology given by (p, I) -completely faithfully flat morphisms of bounded prisms $(A, I) \rightarrow (B, J)$.

An important object in S_Δ is the *Breuil–Kisin prism* given as follows. Denote $\mathfrak{S}_S = S_0[[u]]$ equipped with the Frobenius extending that on S_0 such that $\varphi(u) = u^p$. Then $(\mathfrak{S}_S, (E))$ is a bounded prism, and it is an object in S_Δ via $\mathfrak{S}_S/(E) \cong S$.

The self-product of $(\mathfrak{S}_S, (E))$ in S_Δ exists and is computed in [DLMS24, Ex. 3.4], which we briefly explain. Write $\mathfrak{S}_S \widehat{\otimes}_{\mathbf{Z}_p} \mathfrak{S}_S$ for the p -complete tensor product equipped with the induced \otimes -product Frobenius, and consider $d: \mathfrak{S}_S \widehat{\otimes}_{\mathbf{Z}_p} \mathfrak{S}_S \rightarrow S$ given by the composite $\mathfrak{S}_S \widehat{\otimes}_{\mathbf{Z}_p} \mathfrak{S}_S \rightarrow \mathfrak{S}_S \rightarrow \mathfrak{S}_S/(E) \cong S$ where the first map is the multiplication. Let J be the kernel of d , and consider

$$\mathfrak{S}_S^{(1)} := (\mathfrak{S}_S \widehat{\otimes}_{\mathbf{Z}_p} \mathfrak{S}_S) \left\{ \begin{array}{c} J \\ E \end{array} \right\}_\delta^\wedge.$$

Here the \mathfrak{S}_S -algebra structure of $\mathfrak{S}_S^{(1)}$ is given by $a \mapsto a \otimes 1$, and $\{\cdot\}_\delta^\wedge$ means adjoining elements in the category of derived (p, E) -complete simplicial δ - \mathfrak{S}_S -algebras. Note that E in $\{\frac{J}{E}\}_\delta^\wedge$ denotes $E \otimes 1$, but using $1 \otimes E$ yields the same $\mathfrak{S}_S^{(1)}$. We have $(\mathfrak{S}_S^{(1)}, (E)) \in S_\Delta$, and it is the self-product of $(\mathfrak{S}_S, (E))$ in S_Δ . Similarly, the self-triple-product $(\mathfrak{S}_S^{(2)}, (E))$ of the Breuil–Kisin prism exists in S_Δ . Write $p_i: \mathfrak{S}_S \rightarrow \mathfrak{S}_S^{(1)}$ with $i = 1, 2$ and $q_i: \mathfrak{S}_S \rightarrow \mathfrak{S}_S^{(2)}$ with $i = 1, 2, 3$ for the projection maps. Note that by the rigidity of maps of prisms ([BS22, Lem. 3.5]), we have $(p_1(E)) = (E) = (p_2(E))$ as ideals of $\mathfrak{S}_S^{(1)}$, and similarly $(q_i(E)) = (E)$ as ideals of $\mathfrak{S}_S^{(2)}$ for each $i = 1, 2, 3$.

The Breuil–Kisin prism covers the final object of $\mathrm{Shv}(S_\Delta)$, and thus a crystal on S_Δ can be described by a \mathfrak{S}_S -module with a descent datum involving the self-product and self-triple-product. For example, *completed prismatic F -crystals* on S_Δ given in [DLMS24, Def. 3.16] can be described in terms of *Kisin descent data* defined below ([DLMS24, Prop. 3.26]).

Definition 2.3 (cf. [DLMS24, Def. 3.14]).

- We say that a finite \mathfrak{S}_S -module N is *projective away from (p, E)* if N is p -torsion free, $N[p^{-1}]$ is projective over $\mathfrak{S}_S[p^{-1}]$, and $N[E^{-1}]_p^\wedge$ is projective over $\mathfrak{S}_S[E^{-1}]_p^\wedge$.
- We say a finite \mathfrak{S}_S -module N is *saturated* if N is torsion free and $N = N[p^{-1}] \cap N[E^{-1}]$.
- Let N be a \mathfrak{S}_S -module equipped with a φ -semi-linear endomorphism $\varphi_N: N \rightarrow N$. We say (N, φ_N) has *finite E -height* if $1 \otimes \varphi_N: \mathfrak{S}_S \otimes_{\varphi, \mathfrak{S}_S} N \rightarrow N$ is injective and its cokernel is killed by a power of E .

Remark 2.4. When $S = \mathcal{O}_L$, any finite $\mathfrak{S}_{\mathcal{O}_L}$ -module which is projective away from (p, E) and saturated is free over $\mathfrak{S}_{\mathcal{O}_L}$, since $\mathfrak{S}_{\mathcal{O}_L}$ is a regular local ring of dimension 2 (cf. [DLMS24, Rem. 3.18]).

Definition 2.5 ([DLMS24, Def. 3.25]). Let $\mathrm{DD}_{\mathfrak{S}_S}$ denote the category consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f)$ (called *Kisin descent datum*) where

- \mathfrak{M} is a finite \mathfrak{S}_S -module that is projective away from (p, E) and saturated;
- $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semi-linear endomorphism such that $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ has finite E -height;
- $f: \mathfrak{S}_S^{(1)} \otimes_{p_1, \mathfrak{S}_S} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}_S^{(1)} \otimes_{p_2, \mathfrak{S}_S} \mathfrak{M}$ is an isomorphism of $\mathfrak{S}_S^{(1)}$ -modules compatible with Frobenii and satisfies the cocycle condition over $\mathfrak{S}_S^{(2)}$ (i.e. if $p_{12}, p_{23}, p_{13}: \mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(2)}$ denote the projections, then $p_{23}^* f \circ p_{12}^* f = p_{13}^* f$).

The main input we will need is the following theorem proved in [DLMS24]. Recall that if a finite dimensional continuous \mathbf{Q}_p -representation V of $\mathcal{G}_S := \pi_1^{\text{ét}}(\mathrm{Spec} S[p^{-1}])$ is crystalline, then it is de Rham, and we can attach a $S[p^{-1}]$ -module $D_{\mathrm{dR}}(V)$ projective of rank equal to $\dim_{\mathbf{Q}_p} V$ ([Bri08, §8]). Then $D_{\mathrm{dR}}(V)$ is equipped with a decreasing exhaustive filtration by $S[p^{-1}]$ -submodules $\mathrm{Fil}^i D_{\mathrm{dR}}(V)$, and the Hodge–Tate weights of V are defined to be the integers i such that $\mathrm{Fil}^i D_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{i+1} D_{\mathrm{dR}}(V)$.

Theorem 2.6 ([DLMS24, Prop. 3.26, Thm. 3.29]). *The category $\text{DD}_{\mathfrak{S}_S}$ is naturally equivalent to the category of \mathbf{Z}_p -lattices of crystalline representations of \mathcal{G}_S with non-negative Hodge–Tate weights.*

3. PURITY OF CRYSTALLINE LOCAL SYSTEMS

We prove Theorem 1.2 in this section. Let $\mathcal{G}_{\mathcal{O}_L} \rightarrow \mathcal{G}_R$ be a map of Galois groups as in §1. Let V be a finite dimensional continuous \mathbf{Q}_p -representation of \mathcal{G}_R such that $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is crystalline. By applying a suitable power of Tate twist (i.e. twist by a power of the p -adic cyclotomic character), we may assume that the Hodge–Tate weights of $V|_{\mathcal{G}_{\mathcal{O}_L}}$ are non-negative. Let $T \subset V$ be a \mathcal{G}_R -stable \mathbf{Z}_p -lattice. By [BS23, Cor. 3.8], we can naturally associate to T an étale φ -module \mathcal{M} which is finite projective over $\mathfrak{S}_R[E^{-1}]_p^\wedge$ together with a $\mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge$ -linear isomorphism

$$f_{\text{ét}}: \mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge \otimes_{p_1, \mathfrak{S}_R[E^{-1}]_p^\wedge} \mathcal{M} \xrightarrow{\cong} \mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge \otimes_{p_2, \mathfrak{S}_R[E^{-1}]_p^\wedge} \mathcal{M},$$

which is compatible with φ and satisfies the cocycle condition over $\mathfrak{S}_R^{(2)}[E^{-1}]_p^\wedge$. Here, we associate \mathcal{M} and T contravariantly following the convention in [DLMS24] (see [DLMS24, §3.4]).

Note that the map $R_0 \rightarrow \mathcal{O}_{L_0}$ extends to $\mathfrak{S}_R \rightarrow \mathfrak{S}_{\mathcal{O}_L}$ by $u \mapsto u$, which is compatible with Frobenius. Since $V|_{\mathcal{G}_{\mathcal{O}_L}}$ is crystalline with non-negative Hodge–Tate weights, by Theorem 2.6 and [DLMS24, Prop. 3.27], there exists a Kisin descent datum $(\mathfrak{M}_L, \varphi_{\mathfrak{M}_L}, f_L) \in \text{DD}_{\mathfrak{S}_{\mathcal{O}_L}}$ over $\mathfrak{S}_{\mathcal{O}_L}$ such that we have a φ -compatible isomorphism $h: \mathfrak{M}_L \otimes_{\mathfrak{S}_{\mathcal{O}_L}} \mathfrak{S}_{\mathcal{O}_L}[E^{-1}]_p^\wedge \cong \mathcal{M} \otimes_{\mathfrak{S}_R[E^{-1}]_p^\wedge} \mathfrak{S}_{\mathcal{O}_L}[E^{-1}]_p^\wedge$ and the base change of the isomorphism

$$f_L: \mathfrak{S}_{\mathcal{O}_L}^{(1)} \otimes_{p_1, \mathfrak{S}_{\mathcal{O}_L}} \mathfrak{M}_L \xrightarrow{\cong} \mathfrak{S}_{\mathcal{O}_L}^{(1)} \otimes_{p_2, \mathfrak{S}_{\mathcal{O}_L}} \mathfrak{M}_L$$

to $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$ agrees with the base change of $f_{\text{ét}}$ to $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$ (with respect to h). Let $\mathcal{M}_L = \mathfrak{M}_L \otimes_{\mathfrak{S}_{\mathcal{O}_L}} \mathfrak{S}_{\mathcal{O}_L}[E^{-1}]_p^\wedge$, and regard \mathcal{M} as a $\mathfrak{S}_R[E^{-1}]_p^\wedge$ -submodule of \mathcal{M}_L via the isomorphism h .

Consider the \mathfrak{S} -module

$$\mathfrak{M} := \mathfrak{M}_L \cap \mathcal{M} \subset \mathcal{M}_L$$

equipped with the induced Frobenius $\varphi_{\mathfrak{M}}$. Note that \mathfrak{M} is torsion free. By [DLMS24, Prop. 4.20, 4.21], \mathfrak{M} is finite over \mathfrak{S}_R , saturated, and has finite E -height with respect to $\varphi_{\mathfrak{M}}$. Furthermore, by [DLMS24, Prop. 4.13, 4.26], \mathfrak{M} is projective away from (p, E) and we have natural φ -equivariant isomorphisms

$$\mathfrak{M} \otimes_{\mathfrak{S}_R} \mathfrak{S}_{\mathcal{O}_L} \cong \mathfrak{M}_L \quad \text{and} \quad \mathfrak{M} \otimes_{\mathfrak{S}_R} \mathfrak{S}_R[E^{-1}]_p^\wedge \cong \mathcal{M}.$$

We claim that $f_{\text{ét}}$ and f_L induce an isomorphism

$$f: \mathfrak{S}_R^{(1)} \otimes_{p_1, \mathfrak{S}_R} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}_R^{(1)} \otimes_{p_2, \mathfrak{S}_R} \mathfrak{M}$$

so that f is compatible with $f_{\text{ét}}$ and f_L . For this, we need some preliminary facts.

Lemma 3.1. *The natural map*

$$\mathfrak{S}_R^{(1)}/(p, E) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p, E)$$

is injective.

Proof. By [DL23, Prop. 2.2.8 (2), 4.1.3], we have

$$\mathfrak{S}_R^{(1)}/(E) \cong R[\gamma_i(z_j), i \geq 0, j = 0, \dots, d]_p^\wedge$$

and

$$\mathfrak{S}_{\mathcal{O}_L}^{(1)}/(E) \cong \mathcal{O}_L[\gamma_i(z_j), i \geq 0, j = 0, \dots, d]_p^\wedge$$

where z_0, \dots, z_d can be considered as variables. Thus,

$$\mathfrak{S}_R^{(1)}/(p, E) \cong R[\gamma_i(z_j), i \geq 0, j = 0, \dots, d]/(p)$$

and similarly for $\mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p, E)$. Since $R/(p) \rightarrow \mathcal{O}_L/(p)$ is injective, the map

$$\mathfrak{S}_R^{(1)}/(p, E) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p, E)$$

is injective. \square

Lemma 3.2 ([DLMS24, Cor. 3.6]). *Let $S = R$ or $S = \mathcal{O}_L$. Then $\mathfrak{S}_S^{(1)}$ is p -torsion free and E -torsion free. Furthermore,*

$$\mathfrak{S}_S^{(1)} = \mathfrak{S}_S^{(1)}[p^{-1}] \cap \mathfrak{S}_S^{(1)}[E^{-1}],$$

and $\mathfrak{S}_S^{(1)}[E^{-1}]$ is p -adically separated.

Lemma 3.3. *Let $S = R$ or $S = \mathcal{O}_L$. Then $\mathfrak{S}_S^{(1)}[E^{-1}]_p^\wedge$ is p -torsion free.*

Proof. By [DLMS24, Lem. 3.5], $p_1: \mathfrak{S}_S \rightarrow \mathfrak{S}_S^{(1)}$ is classically faithfully flat. So the induced map $\mathfrak{S}_S[E^{-1}]_p^\wedge \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]_p^\wedge$ is classically faithfully flat by [Sta, Tag 0912]. Since $\mathfrak{S}_S[E^{-1}]_p^\wedge$ is p -torsion free, the statement follows. \square

Lemma 3.4. *Let $S = R$ or $S = \mathcal{O}_L$. Then $\mathfrak{S}_S^{(1)}/(p)$ is E -adically complete.*

Proof. Note that $\mathfrak{S}_S^{(1)}$ is (p, E) -complete. Consider the exact sequence

$$(3.1) \quad 0 \rightarrow p\mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)}/(p) \rightarrow 0.$$

By Lemma 3.2, $\mathfrak{S}_S^{(1)}$ is p -torsion free and $\mathfrak{S}_S^{(1)}/(p)$ is E^n -torsion free for each $n \geq 1$. In particular, the induced sequence

$$0 \rightarrow p\mathfrak{S}_S^{(1)}/E^n p\mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)}/(E^n) \rightarrow \mathfrak{S}_S^{(1)}/(p, E^n) \rightarrow 0$$

is exact. By [Sta, Tag 03CA], the sequence (3.1) remains exact after E -completion. Since $p\mathfrak{S}_S^{(1)}$ is a free $\mathfrak{S}_S^{(1)}$ -module of rank 1, $p\mathfrak{S}_S^{(1)}$ is E -complete. Thus, $\mathfrak{S}_S^{(1)}/(p)$ is E -complete. \square

Lemma 3.5. *Let $S = R$ or $S = \mathcal{O}_L$. The natural maps*

$$\mathfrak{S}_S^{(1)}/(p) \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]/(p) \text{ and } \mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]_p^\wedge$$

are injective. Furthermore, the maps

$$\mathfrak{S}_R^{(1)}/(p) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p), \mathfrak{S}_R^{(1)} \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}, \text{ and } \mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$$

are injective.

Proof. By Lemma 3.2, $\{p, E\}$ form a regular sequence for $\mathfrak{S}_S^{(1)}$, and the maps $\mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]$ and $\mathfrak{S}_S^{(1)}[E^{-1}] \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]_p^\wedge$ are injective. Thus, the maps $\mathfrak{S}_S^{(1)}/(p) \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]/(p)$ and $\mathfrak{S}_S^{(1)} \rightarrow \mathfrak{S}_S^{(1)}[E^{-1}]_p^\wedge$ are injective.

Since $\mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p)$ is E -torsion free, we deduce from Lemma 3.1 inductively that the map $\mathfrak{S}_R^{(1)}/(p, E^n) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p, E^n)$ is injective for each $n \geq 1$. By taking the inverse limit over n giving the \bar{E} -adic completions and using Lemma 3.4, we have that the map $\mathfrak{S}_R^{(1)}/(p) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p)$ is injective. Similarly, since $\mathfrak{S}_{\mathcal{O}_L}^{(1)}$ is p -torsion free and $\mathfrak{S}_R^{(1)}$ and $\mathfrak{S}_{\mathcal{O}_L}^{(1)}$ are p -complete, it follows that $\mathfrak{S}_R^{(1)} \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}$ is injective. Furthermore, since $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]$ is p -torsion free and $\mathfrak{S}_R^{(1)}[E^{-1}]/(p) \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]/(p)$ is injective, the map $\mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$ is injective. \square

Proposition 3.6. *We have*

$$\mathfrak{S}_R^{(1)} = \mathfrak{S}_{\mathcal{O}_L}^{(1)} \cap \mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge$$

as subrings of $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$.

Proof. By Lemma 3.5, the map

$$\mathfrak{S}_R^{(1)}/(p) \rightarrow (\mathfrak{S}_{\mathcal{O}_L}^{(1)}/(p)) \cap (\mathfrak{S}_R^{(1)}[E^{-1}]/(p))$$

is injective, where the intersection is taken as subrings of $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]/(p)$. This map is also surjective by Lemma 3.1. Since $\mathfrak{S}_{\mathcal{O}_L}^{(1)}[E^{-1}]_p^\wedge$ is p -torsion free by Lemma 3.3 and $\mathfrak{S}_R^{(1)}$ is p -complete, it follows that the map $\mathfrak{S}_R^{(1)} \rightarrow \mathfrak{S}_{\mathcal{O}_L}^{(1)} \cap \mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge$ is surjective. \square

Now, since $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}_R[p^{-1}]$, we have

$$(\mathfrak{S}_{\mathcal{O}_L}^{(1)}[p^{-1}]_{\otimes_{p_i, \mathfrak{S}_R[p^{-1}]}} \mathfrak{M}[p^{-1}]) \cap (\mathfrak{S}_R^{(1)}[E^{-1}]_p^\wedge[p^{-1}]_{\otimes_{p_i, \mathfrak{S}_R[p^{-1}]}} \mathfrak{M}[p^{-1}]) \cong (\mathfrak{S}_R^{(1)}[p^{-1}]_{\otimes_{p_i, \mathfrak{S}_R[p^{-1}]}} \mathfrak{M}[p^{-1}])$$

for $i = 1, 2$ by Proposition 3.6. Thus, by [DLMS24, Lem. 4.10], $f_{\acute{e}t}$ and f_L induce a morphism

$$f: \mathfrak{S}_R^{(1)} \otimes_{p_1, \mathfrak{S}_R} \mathfrak{M} \rightarrow \mathfrak{S}_R^{(1)} \otimes_{p_2, \mathfrak{S}_R} \mathfrak{M}.$$

Furthermore, since $f_{\acute{e}t}$ and f_L are isomorphisms, it follows that f obtained as their intersection is an isomorphism. Since $f_{\acute{e}t}$ is compatible with Frobenius, so is f . It remains to show that f satisfies the cocycle condition over $\mathfrak{S}_R^{(2)}$.

Lemma 3.7. *For each $i = 1, 2, 3$, the natural map*

$$\mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M} \rightarrow \mathfrak{S}_R^{(2)}[E^{-1}]_p^\wedge \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}$$

is injective.

Proof. First note that $q_i: \mathfrak{S}_R \rightarrow \mathfrak{S}_R^{(2)}$ is classically faithfully flat by [DLMS24, Lem. 3.5]. So by the same argument as in [DLMS24, Cor. 3.6 Pf.], we deduce that $\mathfrak{S}_R^{(2)}$ is p -torsion free and E -torsion free, and $\mathfrak{S}_R^{(2)}[E^{-1}]$ is p -adically separated. In particular, the map $\mathfrak{S}_R^{(2)} \rightarrow \mathfrak{S}_R^{(2)}[E^{-1}]_p^\wedge$ is injective.

Furthermore, since $\mathfrak{M} \rightarrow \mathfrak{M}[p^{-1}]$ is injective, $\mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M} \rightarrow \mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}[p^{-1}]$ is injective. The map $\mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}[p^{-1}] \rightarrow \mathfrak{S}_R^{(2)} [E^{-1}]_p^\wedge \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}[p^{-1}]$ is injective since $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}_R[p^{-1}]$. Thus, the composite

$$\mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M} \rightarrow \mathfrak{S}_R^{(2)} [E^{-1}]_p^\wedge \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}[p^{-1}]$$

is injective. Since this map factors through $\mathfrak{S}_R^{(2)} \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M} \rightarrow \mathfrak{S}_R^{(2)} [E^{-1}]_p^\wedge \otimes_{q_i, \mathfrak{S}_R} \mathfrak{M}$, the statement follows. \square

Since $f_{\text{ét}}$ satisfies the cocycle condition over $\mathfrak{S}_R^{(2)} [E^{-1}]_p^\wedge$, we deduce from Lemma 3.7 that f satisfies the cocycle condition over $\mathfrak{S}_R^{(2)}$. By Theorem 2.6, $V = T[p^{-1}]$ is a crystalline representation of \mathcal{G}_R . This completes the proof of Theorem 1.2.

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