

Batalin-Vilkovisky structures on moduli spaces of flat connections

Anton Alekseev*, Florian Naef†, Ján Pulmann‡, Pavol Ševera§

Abstract

Let Σ be a compact oriented 2-manifold (possibly with boundary), and let \mathcal{G}_Σ be the linear span of free homotopy classes of closed oriented curves on Σ equipped with the Goldman Lie bracket $[\cdot, \cdot]_{\text{Goldman}}$ defined in terms of intersections of curves. A theorem of Goldman gives rise to a Lie homomorphism Φ^{even} from $(\mathcal{G}_\Sigma, [\cdot, \cdot]_{\text{Goldman}})$ to functions on the moduli space of flat connections $\mathcal{M}_\Sigma(G)$ for $G = U(N), GL(N)$, equipped with the Atiyah-Bott Poisson bracket.

The space \mathcal{G}_Σ also carries the Turaev Lie cobracket δ_{Turaev} defined in terms of self-intersections of curves. In this paper, we address the following natural question: which geometric structure on moduli spaces of flat connections corresponds to the Turaev cobracket?

We give a constructive answer to this question in the following context: for G a Lie supergroup with an odd invariant scalar product on its Lie superalgebra, and for nonempty $\partial\Sigma$, we show that the moduli space of flat connections $\mathcal{M}_\Sigma(G)$ carries a natural Batalin-Vilkovisky (BV) structure, given by an explicit combinatorial Fock-Rosly formula. Furthermore, for the queer Lie supergroup $G = Q(N)$, we define a BV-morphism $\Phi^{\text{odd}}: \wedge \mathcal{G}_\Sigma \rightarrow \text{Fun}(\mathcal{M}_\Sigma(Q(N)))$ which replaces the Goldman map, and which captures the information both on the Goldman bracket and on the Turaev cobracket. The map Φ^{odd} is constructed using the “odd trace” function on $Q(N)$.

*Section of Mathematics, University of Geneva, Rue du Conseil-Général 7-9, 1205 Geneva, Switzerland anton.alekseev@unige.ch

†School of Mathematics, Trinity College, Dublin 2, Ireland naeff@tcd.ie

‡School of Mathematics, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, U.K. Jan.Pulmann@ed.ac.uk

§No affiliation, pavol.severa@gmail.com

1 Introduction

Let G be a connected Lie group, \mathfrak{g} its Lie algebra, and Σ a connected oriented 2-dimensional manifold (possibly with boundary). We denote by

$$\mathcal{M}_\Sigma(G) = \text{Hom}(\pi_1(\Sigma), G)/G$$

the moduli space of flat connections on Σ . Usually, the action of G on the representation space $\text{Hom}(\pi_1(\Sigma), G)$ is not free, and it doesn't need to be proper. For this reason, the space $\mathcal{M}_\Sigma(G)$ is often singular, and there are various approaches to dealing with it. For the purpose of this introduction, we will be considering the smooth part of the moduli space.

Assume that \mathfrak{g} admits an invariant scalar product. Then, by the Atiyah-Bott Theorem [4], $\mathcal{M}_\Sigma(G)$ carries a canonical Poisson structure $\{\cdot, \cdot\}_{\text{Atiyah-Bott}}$. If Σ is closed, this Poisson structure is non-degenerate, and $\mathcal{M}_\Sigma(G)$ becomes a symplectic space.

In [14], Goldman discovered an extraordinary relation between the Atiyah-Bott Poisson structure on $\mathcal{M}_\Sigma(G)$ and topology of oriented closed curves on Σ . In more detail, let γ_1, γ_2 be two such curves on Σ which intersect transversally at a finite number of points. Denote their free homotopy classes by $|\gamma_1|, |\gamma_2|$, and define the Goldman bracket by formula

$$[|\gamma_1|, |\gamma_2|]_{\text{Goldman}} = \sum_{p \in \gamma_1 \cap \gamma_2} \epsilon_p |\gamma_1 \cdot_p \gamma_2|. \quad (1)$$

Here ϵ_p is the sign of the intersection (with respect to the orientation of Σ) and $\gamma_1 \cdot_p \gamma_2$ is the oriented connected sum of γ_1 and γ_2 at p .

Theorem 1 (Goldman [14]). *The bracket $[\cdot, \cdot]_{\text{Goldman}}$ is well defined, and it gives rise to a Lie bracket on the linear span \mathcal{G}_Σ of free homotopy classes of oriented closed curves on Σ .*

Furthermore, let $G = U(n)$ or $GL(n)$, and choose $\gamma \in \pi_1(\Sigma)$. The Goldman function on $\mathcal{M}_\Sigma(G)$, associated to γ , is defined by

$$\Phi_{|\gamma|}^{\text{even}}([\rho]) = \text{Tr } \rho(\gamma),$$

where $\rho \in \text{Hom}(\pi_1(\Sigma), G)$ is a lift of $[\rho] \in \mathcal{M}_\Sigma(G)$. The function $\Phi_{|\gamma|}^{\text{even}}$ is well defined, and it depends only on the free homotopy class of γ . The following theorem establishes a relation between Goldman brackets and Atiyah-Bott Poisson structures:

Theorem 2 (Goldman [14]).

$$\{\Phi_{|\gamma_1|}^{\text{even}}, \Phi_{|\gamma_2|}^{\text{even}}\}_{\text{Atiyah-Bott}} = \Phi_{[|\gamma_1|, |\gamma_2|]_{\text{Goldman}}}^{\text{even}}.$$

Theorem 2 can be restated in several ways. First, it says that the map

$$\Phi^{\text{even}}: (\mathcal{G}_\Sigma, [\cdot, \cdot]_{\text{Goldman}}) \rightarrow (\text{Fun}(\mathcal{M}_\Sigma(G)), \{\cdot, \cdot\}_{\text{Atiyah-Bott}})$$

is a Lie algebra homomorphism. Equivalently, the symmetric algebra $S\mathcal{G}_\Sigma$ is naturally a Poisson algebra with $[\cdot, \cdot]_{\text{Goldman}}$ extended to a Poisson bracket, and the map Φ^{even} extends to a homomorphism of Poisson algebras

$$\Phi^{\text{even}}: S\mathcal{G}_\Sigma \rightarrow \text{Fun}(\mathcal{M}_\Sigma(G)).$$

If one considers polynomial functions on the moduli space, this map is actually surjective for all N (see [21]).

Note that the class of a trivial loop $\bigcirc \in \mathcal{G}_\Sigma$ belongs to the center of the Lie algebra \mathcal{G}_Σ , and that the Goldman bracket descends to the quotient space

$$\mathcal{G}_\Sigma^{\text{red}} = \mathcal{G}_\Sigma / \mathbb{R} \bigcirc.$$

In [25], Turaev showed that the space \mathcal{G}_Σ carries a natural Lie cobracket

$$\delta_{\text{Turaev}}(|\gamma|) = \sum_{p \in \gamma \cap \gamma} \epsilon_p |\gamma'_p| \wedge |\gamma''_p|.$$

Here we assume that the curve γ has a finite number of transverse self-intersections which are denoted by $p \in \gamma \cap \gamma$. By resolving the oriented intersection at the point p , we obtain two closed oriented curves γ'_p and γ''_p . Similar to the definition of the Goldman bracket, ϵ_p is the sign of the intersection with respect to the orientation of Σ .

Theorem 3 (Turaev [25], Chas [9]). *The triple $(\mathcal{G}_\Sigma^{\text{red}}, [\cdot, \cdot]_{\text{Goldman}}, \delta_{\text{Turaev}})$ is an involutive Lie bialgebra. That is, δ_{Turaev} is well defined, it is a Lie cobracket and a Lie algebra 1-cocycle with respect to the Goldman bracket, and the composition map $[\cdot, \cdot]_{\text{Goldman}} \circ \delta_{\text{Turaev}}$ vanishes.*

In this paper, we address the following natural question:

Question: what is the geometric structure on the moduli space of flat connection which corresponds to the Turaev cobracket?

We start with the following standard wisdom, see e.g. [8, Sec. 5]:

Theorem 4. *Let $(\mathcal{G}, [\cdot, \cdot], \delta)$ be an involutive Lie bialgebra. Then, the exterior algebra $\wedge \mathcal{G}$ is naturally a Batalin-Vilkovisky (BV) algebra. That is, it carries a unique second order differential operator Δ such that*

$$\Delta^2 = 0, \quad \Delta(1) = 0, \quad \Delta(x) = \delta(x), \quad \Delta(x \wedge y) = \Delta(x) \wedge y - x \wedge \Delta(y) + [x, y]$$

for all $x, y \in \mathcal{G}$.

Our first result provides a refinement of the Atiyah-Bott Poisson structures in the case when a Lie group is replaced by a Lie supergroup, and an invariant scalar product on the Lie algebra is replaced by an odd invariant scalar product on the Lie superalgebra. In contrast to the even case, it is in general necessary to consider foliated surfaces.

Theorem 5. *Let G be a Lie supergroup with an odd invariant scalar product on its Lie superalgebra, and let Σ be an oriented 2-dimensional manifold with nonempty boundary, equipped with a foliation f . Then, the moduli space $\mathcal{M}_\Sigma(G)$ carries a natural BV structure Δ^f . Furthermore, if $\mathfrak{g} = \text{Lie}(G)$ is a unimodular Lie superalgebra, then Δ^f is independent of f .*

Next, we consider the case of the Lie supergroup $G = Q(n)$. Its Lie superalgebra $\mathfrak{q}(n)$ is unimodular and carries an odd invariant scalar product. Hence, moduli spaces $\mathcal{M}_\Sigma(Q(n))$ carry canonical BV structures Δ . Elements of $Q(n)$ are expressions of the form

$$u = u_{\text{even}} + \xi u_{\text{odd}},$$

where $u_{\text{even}}, u_{\text{odd}} \in \text{Mat}_n(\mathbb{R})$, and ξ is an odd element with $\xi^2 = 1$. We define an analogue of the Goldman map

$$\Phi^{\text{odd}}: |\gamma| \mapsto \Phi_{|\gamma|}^{\text{odd}}([\rho]) = \text{otr}(\rho(\gamma)),$$

where $\text{otr}(u) = \text{Tr}(u_{\text{odd}})$. Images of elements $|\gamma| \in \mathcal{G}_\Sigma$ are odd functions on the moduli space, and we extend the map Φ^{odd} to

$$\Phi^{\text{odd}}: \wedge \mathcal{G}_\Sigma^{\text{red}} \rightarrow \text{Fun}(\mathcal{M}_\Sigma(Q(n))).$$

Our second result is given by the following theorem:

Theorem 6. *The map Φ^{odd} is a morphism of BV algebras with respect to the BV structure on $\wedge \mathcal{G}_\Sigma$ defined by the Goldman-Turaev Lie bialgebra $\mathcal{G}_\Sigma^{\text{red}}$, and the BV structure on $\text{Fun}(\mathcal{M}_\Sigma(Q(n)))$ defined by Theorem 5.*

The supergroup $Q(n)$ comes with another odd function odet , which leads to an extension of the Goldman-Turaev Lie bialgebra to $\mathcal{G}_\Sigma^{\text{red}} \oplus H_1(\Sigma)$. The bracket between $\mathcal{G}_\Sigma^{\text{red}}$ and $H_1(\Sigma)$ is given by the intersection pairing, and the cobracket is extended by 0 to $H_1(\Sigma)$. An extension of the map Φ^{odd} , sending a 1-cycle $[a]$ to the function

$$\rho \mapsto \text{odet} \rho(a),$$

is a well defined morphism of BV algebras.

Theorem 6 is an analogue of the Goldman's result for $G = U(N), \text{GL}(N)$, and it gives a constructive answer to the main question addressed in the paper.

In order to prove these results, we use the technique introduced by Fock-Rosly [12]. In more detail, we choose a finite set $V \subset \partial\Sigma$ and consider a subgroupoid $\Pi_1(\Sigma, V)$ with base points V . The set of representations

$$\mathcal{M}_{\Sigma, V}(G) = \text{Hom}(\Pi_1(\Sigma, V)G),$$

has the property that¹

$$\mathcal{M}_{\Sigma}(G) \cong \mathcal{M}_{\Sigma, V}(G)/G^V.$$

In contrast to $\mathcal{M}_{\Sigma}(G)$, the space $\mathcal{M}_{\Sigma, V}(G)$ admits easy descriptions given a choice of an embedded graph $\Gamma \subset \Sigma$ with the following properties: the set of vertices of Γ is V , and the surface Σ retracts to Γ . Then, $\mathcal{M}_{\Sigma, V}(G) \cong G^E$, where E is the set of edges of Γ .

In this context, Fock and Rosly [12] defined a bivector $\{\cdot, \cdot\}_{\text{Fock-Rosly}}$ on $\mathcal{M}_{\Sigma, V}(G)$ given by an explicit combinatorial formula. In more detail, let $s \in \mathfrak{g} \otimes \mathfrak{g}$ be the canonical element with respect to the invariant scalar product on \mathfrak{g} . Then,

$$\pi_{\text{Fock-Rosly}}^{\Gamma} = \frac{1}{2} \sum_{v \in V} \sum_{a < b \in S(v)} s_{ab}. \quad (2)$$

Here $S(v)$ is the star of the vertex v which consists of half-edges with endpoint v , $a < b$ refers to the order of elements of $S(v)$ induced by the orientation of Σ , and s_{ab} is a bidifferential operator acting on pairs of functions on G^E by differentiating the copies of G corresponding to the half-edges a and b . The following theorem summarises several results:

Theorem 7 (Fock-Rosly [12], Massuyeau-Turaev [18], Nie [19], Li-Bland-Ševěra [7]). *Bivectors defined on $\mathcal{M}_{\Sigma, V}(G)$ by different choices of the graph Γ coincide with each other. The bivector $\pi_{\text{Fock-Rosly}}$ descends to $\mathcal{M}_{\Sigma}(G)$ giving $\{\cdot, \cdot\}_{\text{Atiyah-Bott}}$.*

It is worth noting that $\pi_{\text{Fock-Rosly}}$ is not a Poisson bivector. Instead, it has a controllable defect in the Jacobi identity. Such bivectors are called quasi-Poisson bivectors [1, 2], and they have many properties similar to Poisson bivectors.

In the case when G is a Lie supergroup with an odd invariant scalar product on its Lie superalgebra \mathfrak{g} , we introduce a canonical element $t \in \mathfrak{g} \otimes \mathfrak{g}$. Assuming that \mathfrak{g} is unimodular, we define a second order differential operator on $\mathcal{M}_{\Sigma, V}(G) \cong G^E$:

$$\Delta^{\Gamma} = \frac{1}{2} \sum_{v \in V} \sum_{a < b \in S(v)} t_{ab}. \quad (3)$$

In comparison to equation (2), there are two differences: first, t is an odd element while s is even. Second, we view s_{ab} as a bidifferential operator acting

¹By G^V , we mean a collection of elements of G indexed by V , i.e. a morphism of sets $V \rightarrow G$.

on a pair of functions on G^E while we consider t_{ab} as a second order differential operator acting on one function. Despite these differences, properties of Δ^Γ resemble those of the Fock-Rosly bivector:

Theorem 8. *Second order differential operators defined on $\mathcal{M}_{\Sigma,V}(G)$ by different choices of the graph Γ coincide with each other. The operator Δ descends to a BV operator on $\mathcal{M}_\Sigma(G)$.*

Again, the operator Δ is not a BV operator. Instead, its square is non-vanishing in a controllable way giving rise to a notion of quasi-BV operators. To simplify the presentation, Theorem 8 is stated for the case of unimodular Lie superalgebras. A more general statement can be found in the body of the paper.

The structure of the paper is as follows: in Section 2, we recall the Fock-Rosly construction. In Section 3, we describe an analogous construction of the quasi-BV operator Δ for supergroups. In Section 4, we give an alternative, topological construction of Δ , via intersections of curves. In Section 5, we consider moduli spaces for the Lie supergroup $Q(n)$ and we establish a relation between Δ and the Goldman-Turaev Lie bialgebra. Appendix A is devoted to skeletons and foliations on surfaces with boundary. Appendix B contains information on Lie algebras and Lie superalgebras with an (even or odd) invariant scalar product. Appendix C provides a Hopf algebra viewpoint on the fusion operation for quasi-BV spaces.

Acknowledgements

We are indebted to V. Serganova for fruitful discussions, and we would like to thank E. Getzler and V. Turaev for their interest in our work. Research of AA was supported in part by the grants 182767, 208235, 200400 and by the National Center for Competence in Research (NCCR) SwissMAP of the Swiss National Science Foundation. Research of FN was supported in part by the Marie Skłodowska-Curie action, grant agreement no. 896370. Research of JP was supported by the Postdoc.Mobility grant 203065 of the SNSF.

2 Quasi-Poisson and quasi-BV structures

In this section, we recall a description of the Poisson structure on the moduli space of flat connections on a surface, due to Fock and Rosly [12]. We start with some useful results on surfaces.

2.1 Surfaces

2.1.1 Skeletons

Let Σ be an oriented, compact surface with boundary and $\{p_1, \dots, p_n\} = V \subset \partial\Sigma$ a finite, non-empty subset of *marked points*. Recall that $\Pi_1(\Sigma, V)$, the fundamental groupoid with base V , is the full subgroupoid of the fundamental groupoid $\Pi_1(\Sigma)$ on objects $V \in \Sigma$. In other words, the set of objects of $\Pi_1(\Sigma, V)$ is V , while morphisms $p_i \rightarrow p_j$ are homotopy classes of paths from p_i to p_j .

By a graph, we mean a 1-dimensional CW complex Γ . The sets of vertices and edges of Γ are denoted $\text{Vert } \Gamma$ and $\text{Edges } \Gamma$. For each vertex $p \in \text{Vert } \Gamma$, we denote by $\text{he}(p)$ the set of half-edges of p .

Definition 1. *A skeleton of Σ is a topological embedding $\Gamma \hookrightarrow \Sigma$ of an oriented graph Γ such that*

1. *restricted to each edge $e \in \text{Edges } \Gamma$, the injection $e \hookrightarrow \Sigma$ is a smooth embedding of manifolds with boundary,*
2. *the image of $\text{Vert } \Gamma$ equals V , with no other intersection of the image of Γ with $\partial\Sigma$,*
3. *Σ deformation retracts to the image of Γ .*

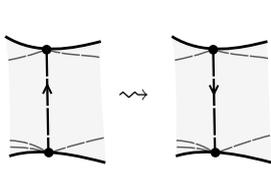
The edges of such skeleton then freely generate the fundamental groupoid $\Pi_1(\Sigma, V)$, under composition and inversion.

Remark 1. *Skeletons are closely related to ciliated graphs of [12]. Indeed, to any skeleton we can associate a ciliated graph using the orientation of the surface, with the cilium pointing outside of the surface. The thickening of such graph is then homeomorphic to the original surface.*

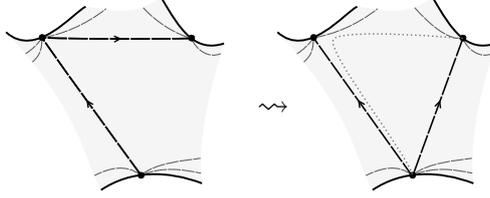
Remark 2. *An alternative approach to describing the fundamental groupoid $\Pi_1(\Sigma, V)$ and thus the moduli space of flat connections, would be to use triangulations, or their dual uni-trivalent graphs. This approach would result in slightly more complicated computations in Sections 2.2 and 3.2, so we have chosen to use skeletons.*

It is easy to see that one can always find a skeleton. Moreover, any two skeletons can be connected by isotopy, edge reversions and *slides*. A slide is a move of a half-edge along a neighboring edge – see Figure 1.

Proposition 1. *Any two skeletons $\Gamma, \Gamma' \subset \Sigma$ are connected by a finite sequence of isotopies, edge reversions and slides.*



(a) Reversing an edge.



(b) Sliding the top edge along the left edge. Note that some of the marked vertices could be identified.

Figure 1: The moves between skeletons. The solid line is the boundary $\partial\Sigma$.

This fact seems to be known among experts, see the work of Bene [5, Theorem 5.3] (where Σ has one boundary component and one marked point) and Jackson [15, Corollary 6.21] (where each boundary component has one marked point). We present a proof of this version of the claim in Appendix A.1.

2.1.2 Foliations

For Batalin-Vilkovisky structures on moduli spaces, we will need surfaces equipped with a 1-dimensional foliation. Since Σ has a boundary, we will consider foliations in the following sense:

Definition 2. *By a foliation of a surface with boundary, we mean a decomposition of Σ into subsets which can be extended, in each local chart $U \xrightarrow{\cong} V \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$, to a smooth 1-dimensional foliation of an open subset of \mathbb{R}^2 containing V .*

If marked points $V \subset \partial\Sigma$ are chosen, a foliation is moreover required to be tangent to the boundary at each marked point. A homotopy of foliations is required to preserve this tangency.

Note that we don't put any requirements on the foliation away from V , i.e. it can be tangent to the boundary also away from V . For an example and classification of such foliations, see Appendix A.2.

We can use a foliation to measure the number of turns of a path connecting points of V .

Definition 3. *Let γ be an immersed path connecting two points of V , transverse to the boundary at its endpoints. Then, we define $\text{rot}_\gamma \in \frac{1}{2}\mathbb{Z}$ as the number of counter-clockwise turns the foliation takes with respect to the path.*

Concretely, for generic γ , there is a contribution of $\pm\frac{1}{2}$ for each time γ becomes tangent to the foliation, with $-\frac{1}{2}$ if the turn is compatible with the orientation of the surface, as on Figure 2. Alternatively, we can pick a metric on the surface such that γ is perpendicular to the foliation at its endpoints. Then the angle between $\dot{\gamma}$ and the foliation gives a closed loop in \mathbb{RP}^1 ; the rotation number is one half of the homotopy class of this map, where we take as the generator the counter-clockwise turn.



(a) Contribution of $-\frac{1}{2}$ to the rotation number

(b) Contribution of $\frac{1}{2}$ to the rotation number

Figure 2: Rotation number. The surface is oriented counter-clockwise, as depicted.

The number of rotations rot_γ is an invariant of regular homotopy of γ . Moreover, it depends only on the homotopy class of the compatible foliation. It satisfies

$$\text{rot}_{\gamma^{-1}} = -\text{rot}_\gamma,$$

where γ^{-1} is the path γ with reversed orientation.

Given two composable regular homotopy classes of paths γ_1, γ_2 , there are two ways to compose them, depending on the order of their half-edges (see Figure 5). The corresponding rotation numbers of the composition $\gamma_2\gamma_1$ differ by 1, as shown on Figure 3.



(a) $\text{rot}_{\gamma_2\gamma_1} = \text{rot}_{\gamma_1} + \text{rot}_{\gamma_2} - \frac{1}{2}$

(b) $\text{rot}_{\gamma_2\gamma_1} = \text{rot}_{\gamma_1} + \text{rot}_{\gamma_2} + \frac{1}{2}$

Figure 3: Rotation number of a composition

2.2 The Poisson bivector of Fock and Rosly

Let us now recall the construction of Fock and Rosly [12], of a Poisson structure on the moduli space of flat connections on a surface.

As before, let Σ be an oriented, compact surface with boundary, and V a non-empty, finite set of marked points belonging to $\partial\Sigma$. Let G be a connected Lie group, with a Lie algebra \mathfrak{g} . We will study $\mathcal{M}_{\Sigma,V}(G)$, the space of principal G -bundles with a flat connection and a chosen trivialization at V , modulo isomorphisms. Concretely, we will describe the moduli space as the space of all groupoid homomorphisms from $\Pi_1(\Sigma, V)$ to G

$$\mathcal{M}_{\Sigma,V}(G) = \text{Hom}(\Pi_1(\Sigma, V), G).$$

Given a flat connection on Σ , we get an element of $\text{Hom}(\Pi_1(\Sigma, V), G)$ assigning holonomy $\text{hol}(\gamma) \in G$ to any path γ in Σ between any two points of V . This gives an isomorphism between the moduli space of flat bundles and $\text{Hom}(\Pi_1(\Sigma, V), G)$.

If we choose a skeleton Γ of Σ with N edges $\gamma_1, \dots, \gamma_N$, we get an isomorphism

$$\Psi_\Gamma: G^{\times N} \xrightarrow{\sim} \mathcal{M}_{\Sigma,V}(G)$$

given by specifying the holonomies $(\text{hol}(\gamma_1), \dots, \text{hol}(\gamma_N)) \in G^{\times N}$. Note that in our convention, composition of two paths $\gamma_1\gamma_2$ is a path first traversing γ_2 and then γ_1 so that $\text{hol}(\gamma_1\gamma_2) = \text{hol}(\gamma_1)\text{hol}(\gamma_2)$.

On the moduli space of flat connections, there are $|V|$ pairwise-commuting G (and \mathfrak{g}) actions ρ_p , coming from gauge transformations at points of V . They correspond to left/right multiplication of the holonomies of paths incident at that vertex. For example, for $V = \{p\}$, the G -action is

$$(\text{hol}(\gamma_1), \dots, \text{hol}(\gamma_N)) \mapsto (g \text{hol}(\gamma_1)g^{-1}, \dots, g \text{hol}(\gamma_N)g^{-1}).$$

To define the bivector field of Fock and Rosly on the moduli space, we need to choose a skeleton Γ on Σ . Let us denote by $\text{he}(p)$ the set of half-edges incident at $p \in V$, which is linearly ordered using the orientation of Σ , see Figure 5.

If Γ has edges $\gamma_1, \dots, \gamma_N$, we can define an action of $G^{\text{he}(p)}$ on the moduli space $\mathcal{M}_{\Sigma,V}(G) \cong G^{\times N}$ as follows. If a is a half-edge of an edge γ_i , and $g \in G$, we define $(g)_a: G^{\times N} \rightarrow G^{\times N}$ by

$$\begin{aligned} (g)_a &= 1_G \times \dots \times L_g \times \dots \times 1_G & \text{if } a \text{ arrives at } p, \\ (g)_a &= 1_G \times \dots \times R_{g^{-1}} \times \dots \times 1_G & \text{if } a \text{ leaves } p, \end{aligned} \tag{4}$$

where L_g or $R_{g^{-1}}$ act on the i th factor of $G^{\times N}$, i.e. the factor corresponding to $\text{hol}(\gamma_i)$. For $x \in \mathfrak{g}$, we denote the induced Lie algebra action $(x)_a$, i.e. $(x)_a = -x^R$ or x^L for incoming/outgoing half-edge a . Note that the action ρ_p is the product/sum of these actions over all half-edges incident at p , for example $\rho_p(x) = \sum_{a \in \text{he}(p)} (x)_a$.



Figure 4: Lie group and Lie algebra action associated to half-edges.

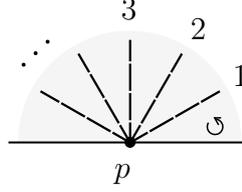


Figure 5: Orientation of the surface gives a cyclic order on half-edges at p . Using the boundary, we can pick the first half-edge.

Let us now also assume that on \mathfrak{g} , there's a nondegenerate, invariant symmetric pairing, with inverse $s \in (\text{Sym}^2 \mathfrak{g})^{\mathfrak{g}}$. If e_i is a basis of \mathfrak{g} , let us write² $s = s^{ij} e_i \otimes e_j$. Recall that the Cartan element $\phi \in \wedge^3 \mathfrak{g}$ is defined as

$$\phi(\alpha, \beta, \gamma) = \frac{1}{24} \alpha([s^{\#}(\beta), s^{\#}(\gamma)]) \quad \text{for } \alpha, \beta, \gamma \in \mathfrak{g}^*,$$

where $s^{\#}: \mathfrak{g}^* \rightarrow \mathfrak{g}$ comes from the non-degenerate pairing on \mathfrak{g} .

Let us define a bivector on $G^{\times N}$

$$\pi_{\text{FR}}^{\Gamma} := \sum_{p \in V} \sum_{\substack{a, b \in \text{he}(p) \\ a < b}} \frac{1}{2} s^{ij} (e_i)_a \wedge (e_j)_b \quad (5)$$

where we use the linear order of half-edges from Figure 5. Using the isomorphism $\Psi_{\Gamma}: G^{\times N} \xrightarrow{\sim} \mathcal{M}_{\Sigma, V}(G)$ we get a bivector field $(\Psi_{\Gamma})_* \pi_{\text{FR}}^{\Gamma}$ on the moduli space $\mathcal{M}_{\Sigma, V}(G)$. The following theorem then follows from the work of Fock and Rosly [12], see also [2, 18, 19, 7] [18], Nie [19], Li-Bland-Ševera [7].

Remark 3. To define π^{Γ} , we don't have to start with an invariant pairing; a non-necessarily-invertible element $s \in (\text{Sym}^2(\mathfrak{g}))^{\mathfrak{g}}$ is sufficient. Theorem 9 below holds in this case as well (as noticed in e.g. [7]). See also Remark 8.

Theorem 9. The bivector $\pi_{\text{FR}} := (\Psi_{\Gamma})_* \pi_{\text{FR}}^{\Gamma}$ on $\mathcal{M}_{\Sigma, V}(G)$ does not depend on the choice of the skeleton Γ and is invariant under the G -action ρ_p on $\mathcal{M}_{\Sigma, V}(G)$ for each $p \in V$.

Moreover,

$$[\pi_{\text{FR}}, \pi_{\text{FR}}]/2 = \sum_{p \in V} \rho_p(\phi)$$

where ρ acts as a trivector field on the moduli space, acting using the \mathfrak{g} -action ρ_p . In other words, π_{FR} is a \mathfrak{g}^V -quasi-Poisson bivector on the moduli space [1, 7].

²We use the Einstein summation convention throughout the paper.

Remark 4. Fock and Rosly also fix a classical r -matrix, i.e. an element $\Lambda \in \bigwedge^2 \mathfrak{g}$ satisfying $[\Lambda, \Lambda]/2 = -\phi$. Then, the bivector $\pi_{\text{FR}} + \sum_{p \in V} \rho_p(\Lambda)$ is Poisson.

A different way to obtain a Poisson structure is to look at the character variety $\mathcal{M}_{\Sigma, V}(G)/G^V$, i.e. the moduli space of flat G -connections on Σ , with no marked points.

We will now reprove this theorem, since a similar proof will be used in the Section 3.2.

2.3 Proof of Theorem 9

Denote

$$\mathfrak{g}^{\text{he}(p)} = \bigoplus_{a \in \text{he}(p)} \mathfrak{g}$$

the Lie algebra acting at each marked point. Let $\iota_a: \mathfrak{g} \rightarrow \mathfrak{g}^{\text{he}(p)}$ be the inclusion associated with half-edge a . Recall that the Lie algebra actions $x \mapsto (x)_a$ extend to a morphism from $\bigwedge \mathfrak{g}^{\text{he}(p)}$ to multivector fields on $G^{\times N}$, compatible with the wedge product and the Schouten brackets³. For example, the element $\iota_a(x) \in \mathfrak{g}^{\text{he}(p)} \subset \bigwedge \mathfrak{g}^{\text{he}(p)}$ is sent to the vector field $(x)_a \in \mathfrak{X}(G^{\times N})$.

Moreover, if a, b are two half-edges, we can define $\tilde{s}_{ab} = s^{ij} \iota_a(e_i) \wedge \iota_b(e_j)$, so that the bivector π_{FR}^Γ is given by the image of

$$\sum_{p \in V} \sum_{\substack{a, b \in \text{he}(p) \\ a < b}} \frac{1}{2} \tilde{s}_{ab} \in \bigwedge \left(\bigoplus_{p \in V} \mathfrak{g}^{\text{he}(p)} \right).$$

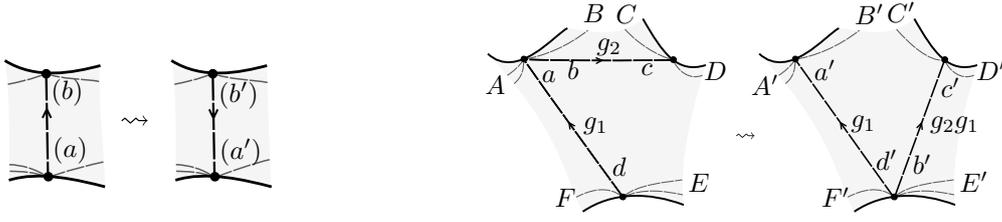
Similarly, we define $\tilde{\phi}_{abc} = \frac{1}{24} f^{ijk} \iota_a(e_i) \wedge \iota_b(e_j) \wedge \iota_c(e_k)$, where $\phi = \frac{1}{24} f^{ijk} e_i \wedge e_j \wedge e_k \in \bigwedge^3 \mathfrak{g}$ uses the antisymmetric Cartan tensor $f^{ijk} = f_{xy}^i s^{xj} s^{yk}$. These elements satisfy a version of the Drinfeld-Kohno relations under the Schouten bracket, see Proposition 15 in the Appendix B.1. Now we can prove the theorem, mostly on the level of $\bigwedge \mathfrak{g}^{\text{he}(p)}$.

Proof of Theorem 9. From the invariance of the inner product it follows that $[(x)_a + (x)_b, \tilde{s}_{ab}] = 0$ for each a, b at p , and thus s_{ab} is invariant.

To prove $[\pi_{\text{FR}}^\Gamma, \pi_{\text{FR}}^\Gamma]/2 = \sum_p \rho(\phi)_p$, we first look at each vertex p separately. The action of ϕ at a vertex p is given by the action of

$$\sum_{a, b, c \in \text{he}(p)} \tilde{\phi}_{abc} = \sum_{a \in \text{he}(p)} \tilde{\phi}_{aaa} + 3 \sum_{a, b \in \text{he}(p), a < b} \left(\tilde{\phi}_{aab} + \tilde{\phi}_{abb} \right) + 6 \sum_{a, b, c \in \text{he}(p), a < b < c} \tilde{\phi}_{abc}.$$

³i.e. a morphism of Gerstenhaber algebras



(a) Half-edge actions a and b get mapped to a' and b' .
 (b) The capital letters can correspond to more half-edges, the action x_A , for $x \in \mathfrak{g}$, is given as a sum of individual half-edge actions.

Figure 6: Choice of names for half-edges. The orientation of the surface is counter-clockwise.

Using Proposition 15, we get that the term $[\frac{1}{2}\tilde{s}_{ab}, \frac{1}{2}\tilde{s}_{ab}]/2$ will cancel the term $3(\tilde{\phi}_{aab} + \tilde{\phi}_{abb})$ and that $6\tilde{\phi}_{abc}$ cancels with

$$[\frac{1}{2}\tilde{s}_{ab}, \frac{1}{2}\tilde{s}_{bc}] + [\frac{1}{2}\tilde{s}_{ab}, \frac{1}{2}\tilde{s}_{ac}] + [\frac{1}{2}\tilde{s}_{bc}, \frac{1}{2}\tilde{s}_{ac}] = -6\phi_{abc} + 6\phi_{bac} + 6\phi_{bca}.$$

Finally, for a path γ with half-edges a and b , one has $x_a = -(\text{Ad}_{\text{hol}_\gamma} x)_b$ and using the invariance of ϕ , the term ϕ_{aaa} thus cancels with ϕ_{bbb} .

To show that π_{FR} is independent of Γ , it is enough to show that the diffeomorphism $\Phi = \Psi_{\Gamma'}^{-1}\Psi_\Gamma$ sends π_{FR}^Γ to $\pi_{\text{FR}}^{\Gamma'}$. Moreover, we just need to check this on edge reversions and slides of skeletons, c.f. Proposition 1. The isomorphism $\Phi: G^{\text{edges}(\Gamma)} \rightarrow G^{\text{edges}(\Gamma')}$, corresponding to the change of edge orientation, satisfies

$$\Phi_*((x)_a) = (x)_{a'} \quad \text{and} \quad \Phi_*((x)_b) = (x)_{b'},$$

with labels for half-edge as in Figure 6a. This is because $\text{Inv}: g \mapsto g^{-1}$ satisfies $\text{Inv}_*(x^L) = -x^R$. Thus, Φ_* relates the two bivector fields associated to Γ and Γ' , because $\Phi_*(s_{ab}) = s_{a'b'}$ and the two bivector fields are equal term-by-term.

For the case of the slide move of Figure 1b, we denote the half-edge as on Figure 6b. The diffeomorphism $\Phi: G^{\text{edges}(\Gamma)} \rightarrow G^{\text{edges}(\Gamma')}$ is in this case (on the relevant holonomies) given as $(g_1, g_2) \mapsto (g_1, g_2g_1)$. Thus, on vector fields it gives

$$\begin{aligned} \Phi_*((x)_a) &= (x)_{a'} + (\text{Ad}_{g_2}(x))_{c'} \\ \Phi_*((x)_b) &= -(\text{Ad}_{g_2} x)_{c'} = (\text{Ad}_{g_1^{-1}} x)_{b'} \\ \Phi_*((x)_c) &= (x)_{c'} \\ \Phi_*((x)_d) &= (x)_{d'} + (x)_{b'} \\ \Phi_*((x)_X) &= (x)_{X'} \quad \text{for any uppercase } X \end{aligned}$$

The relevant part (dropping terms containing only s_{XY} for X, Y uppercase) of the quasi-Poisson bivector field for the original graph is

$$s_{A(a+b)} + s_{(a+b)B} + s_{ab} + s_{Cc} + s_{cD} + s_{Ed} + s_{dF}$$

and under Φ_* , it gets mapped to

$$s_{A'a'} + s_{a'B'} + s_{(a'+\text{Ad}_{g_2} c')(-\text{Ad}_{g_2} c')} + s_{C'c'} + s_{c'D'} + s_{E'(b'+d')} + s_{(b'+d')F'}.$$

Using $(x)_{a'} = -(\text{Ad}_{g_1}^{-1}(x))_{d'}$ and properties of s , we get

$$s_{(a'+\text{Ad}_{g_2} c')(-\text{Ad}_{g_2} c')} = s_{a'(\text{Ad}_{g_1}^{-1} b')} = s_{(\text{Ad}_{g_1} a')b'} = -s_{d'b'} = s_{b'd'}. \quad (6)$$

which gives the quasi-Poisson bivector field $\pi_{\text{FR}}^{\Gamma'}$. Note the term $s_{\text{Ad}_{g_2} c', -\text{Ad}_{g_2} c'}$ which is zero; it will be non-trivial in the definition of the quasi-BV operator.

We also need to consider a case where some of the marked points on Figure 1 coincide. In other words, it might happen that one of the two edges involved in a slide move is a loop. Instead of repeating the above calculation, we recall the so-called fusion procedure.

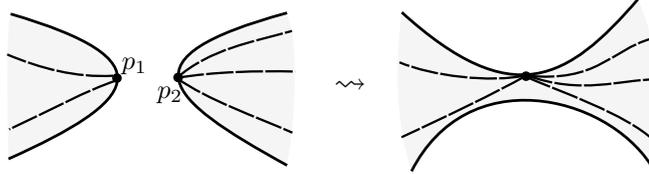


Figure 7: Fusion at vertices p_1 and p_2 . Orientation of Σ determines the position of the new point.

Fusion of a surface at two points p_1 and p_2 is given by a corner-connected sum of the two surfaces, with the two marked points replaced by one point as on Figure 7. If we start with a surface with a skeleton, the fused surface also has a natural skeleton, and conversely any skeleton with a loop can be obtained by fusion which creates that loop. This becomes evident when the surface is seen as a fattening of its skeleton, as on Figure 8.

As manifolds, the two moduli spaces are the same, however, the bivector on the fused surface has an additional term:

Proposition 2. *For a surface with a skeleton $\Gamma \subset \Sigma$, the quasi-Poisson structure on the surface given by fusion at two marked points p_1 and p_2 is*

$$\pi_{\text{FR}}^{\Gamma_{\text{fused}}} = \pi_{\text{FR}}^{\Gamma} + \frac{1}{2} s^{ij} \rho_{p_1}(e_i) \wedge \rho_{p_2}(e_j),$$

where ρ_p denotes the \mathfrak{g} -action at the vertex p , i.e. $\rho_p(x) = \sum_{a \in \text{he}(p)} (x)_a$.

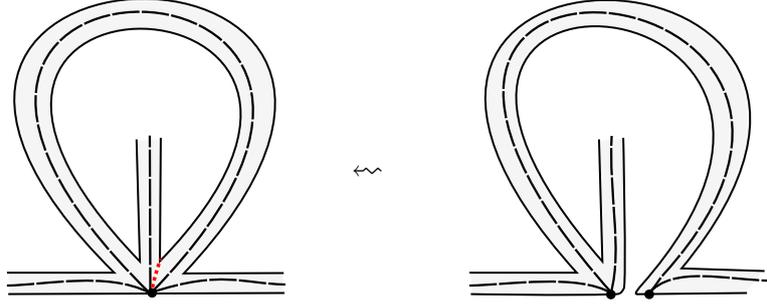


Figure 8: A loop in a skeleton can be obtained via fusion.

If Γ and Γ' are two skeletons related by slide not involving loops, we know the corresponding bivector fields $(\Psi_\Gamma)_* \pi_{\text{FR}}^\Gamma$ and $(\Psi_{\Gamma'})_* \pi_{\text{FR}}^{\Gamma'}$ are equal for the unfused surface. However, the fusion term

$$\frac{1}{2} s^{ij} \rho_{p_1}(e_i) \wedge \rho_{p_2}(e_j)$$

is independent of Γ . Thus, the bivector fields $(\Psi_{\Gamma_{\text{fused}}})_* \pi_{\text{FR}}^{\Gamma_{\text{fused}}}$ and $\pi_{\text{FR}}^{\Gamma'_{\text{fused}}}$ are also equal, where now Γ_{fused} and Γ'_{fused} are related by a slide involving a loop. \square

2.4 Goldman Lie algebra

If we choose an Ad-invariant function f on G , there is a distinguished set of functions $f_{|\gamma|}: [\rho] \mapsto f(\rho(\gamma))$ on the moduli space $\mathcal{M}_\Sigma(G)$, where $|\gamma|$ is a free homotopy class of loops⁴ on Σ and γ its arbitrarily chosen representative in $\pi_1(\Sigma)$. In many cases, the Poisson bracket of these functions was described by Goldman [14]; let us recall the case of $GL_n(\mathbb{R})$ with $f = \text{Tr}$.

Denote by $\mathcal{G}_\Sigma = \mathbb{R}\pi_1^{\text{free}}(\Sigma)$ the vector space generated by free homotopy classes of loops on Σ , and by $[-, -]_G$ the Goldman Lie bracket on \mathcal{G}_Σ , given by resolving intersections of loops (see [14, Section 3.13] or Section 5.1 below for details).

Theorem 10. *[[14, Section 3.13]] Let $G = GL_n(\mathbb{R})$ and let*

$$\Phi^{\text{even}}: \mathcal{G}_\Sigma \rightarrow C^\infty(\mathcal{M}_\Sigma(GL(n))), \quad \Phi_{|\gamma|}^{\text{even}}([\rho]) := \text{Tr}_{|\gamma|}([\rho]) = \text{Tr}(\rho(\gamma)).$$

Then Φ^{even} is well defined and

$$\Phi_{[|\gamma_1|, |\gamma_2|]_G}^{\text{even}} = \{\Phi_{|\gamma_1|}^{\text{even}}, \Phi_{|\gamma_2|}^{\text{even}}\}_{\text{FR}},$$

i.e. Φ^{even} is a map of Lie algebras.

⁴A free homotopy class a loop is a map $S^1 \rightarrow \Sigma$, with two such maps identified if they can be extended to a map from the cylinder $S^1 \times [0, 1]$. Equivalently, $\pi_1^{\text{free}}(\Sigma)$ is the set of conjugacy classes of $\pi_1(\Sigma)$, and furthermore $\mathcal{G}_\Sigma \cong \mathbb{R}\pi_1(\Sigma)/[\mathbb{R}\pi_1(\Sigma), \mathbb{R}\pi_1(\Sigma)]$.

We will describe a slight generalization of Goldman's theorem. Our motivation is to also capture the determinant, and we will extend the Goldman Lie algebra by the first homology $H_1(\Sigma)$ to achieve this.

Definition 4. On $\mathcal{G}_\Sigma \oplus H_1(\Sigma, \mathbb{R})$, extend the Goldman Lie bracket by

$$\begin{aligned} [|\gamma|, a]_G &:= \langle [\gamma], a \rangle |\gamma|, \\ [a, b]_G &:= \langle a, b \rangle \circ, \end{aligned}$$

where $|\gamma_{1,2}| \in \pi_1^{\text{free}}(\Sigma)$, $a, b \in H_1(\Sigma, \mathbb{R})$, $\langle a, b \rangle$ is the intersection pairing on $H_1(\Sigma, \mathbb{R})$, $[\gamma] \in H_1(\Sigma, \mathbb{R})$ is the homology class given by the free homotopy class $|\gamma|$ and $\circ \in \mathcal{G}_\Sigma$ is the homotopy class of the constant loop.

It is straightforward to check that the above bracket on $\mathcal{G}_\Sigma \oplus H_1(\Sigma, \mathbb{R})$ satisfies the Jacobi identity. Let us now also extend the map $\Phi^{\text{even}}: \mathcal{G}_\Sigma \rightarrow C^\infty(\mathcal{M}_\Sigma(G))$.

Definition 5. Let $G = GL_n(\mathbb{R})^+$, the connected component of the identity. Define $\tilde{\Phi}^{\text{even}}: \mathcal{G}_\Sigma \oplus H_1(\Sigma, \mathbb{R}) \rightarrow C^\infty(\mathcal{M}_\Sigma(G))$ by

$$\begin{aligned} \tilde{\Phi}_\gamma^{\text{even}}([\rho]) &= \text{Tr}(\rho(\gamma)), \\ \tilde{\Phi}_a^{\text{even}}([\rho]) &= \log \det(\rho(\gamma_a)), \end{aligned}$$

where $|\gamma| \in \pi_1^{\text{free}}(\Sigma)$ and $\gamma_a \in \pi_1(\Sigma)$ is a representative of $a \in H_1(\Sigma) \cong \pi_1(\Sigma)^{\text{ab}}$.

Proposition 3. The map $\tilde{\Phi}^{\text{even}}$ is well defined.

Proof. We need to check that two different representatives γ_a, γ'_a of a give the same function. This follows from the fact that $\gamma_a = \gamma'_a C$, where C is a product of commutators, and the determinant of a commutator is equal to 1.

We can extend $\tilde{\Phi}^{\text{even}}$ from $H_1(\Sigma, \mathbb{Z})$ to $H_1(\Sigma, \mathbb{R})$, since $\tilde{\Phi}^{\text{even}}$ is a map of abelian groups. This follows from the fact that $\gamma_a \gamma_b$ is a representative for $a + b$, and $\log \det \text{hol}_{\gamma_a \gamma_b} = \log \det \text{hol}_{\gamma_a} \log \det \text{hol}_{\gamma_b}$. \square

Theorem 11. The map $\tilde{\Phi}^{\text{even}}: \mathcal{G}_\Sigma \oplus H_1(\Sigma, \mathbb{R}) \rightarrow C^\infty(\mathcal{M}_\Sigma(G))$ is a morphism of Lie algebras.

Proof. Instead of Equation (5), it is more convenient to use a description of the Poisson structure on $\mathcal{M}_\Sigma(G)$ as in [14, p. 265, *Product formula*]. Namely, if f, f' are two Ad-invariant functions on G , then⁵

$$\{f|_{\gamma_1}, f'|_{\gamma_2}\}_{\text{FR}} = \sum_{p \in \gamma_1 \cap \gamma_2} \pm s^{ij} \partial_{t_1}|_0 f((A_1)_p e^{t_1 e_i}) \partial_{t_2}|_0 f'((A_2)_p e^{t_2 e_j}),$$

⁵See also [12, Prop. 4.3], [16, Comment 18], [19, Theorem 2.5] and [7, Prop. 4] for various versions of this claim; we prove a similar statement in the odd case in Theorem 13. It is not difficult to check that our conventions for π_{FR} do indeed match that for the product formula above, by considering e.g. the 1-punctured torus.

where the sum is over all (transverse double) intersections of γ_1 and γ_2 , and $(A_1)_p, (A_2)_p$ are holonomies along γ_1, γ_2 starting at p . The sign is given by the orientation of $((\dot{\gamma}_1)_p, (\dot{\gamma}_2)_p)$ relative to the orientation of Σ .

Let us only do the calculation for the simpler case of $GL_n(\mathbb{R})$. The basis is given by elementary matrices $E_{(\alpha\beta)}, \alpha, \beta = 1 \dots n$, and $s = \sum_{\alpha, \beta} E_{(\alpha\beta)} \otimes E_{(\beta\alpha)}$. If $f = \text{Tr}$, we have

$$\partial_t|_0 \text{Tr}(A_p e^{tE_{(\alpha\beta)}}) = \text{Tr}(A_p E_{(\alpha\beta)}) = (A_p)_{\beta\alpha},$$

while for $f = \log \det$, we have

$$\partial_t|_0 \log \det(A_p e^{tE_{(\alpha\beta)}}) = \partial_t|_0 \log \det(e^{tE_{(\alpha\beta)}}) = \text{Tr}(E_{(\alpha\beta)}) = \delta_{\alpha, \beta}. \quad (7)$$

Thus, using the product formula, we get

$$\begin{aligned} \{\text{Tr}_{|\gamma|}, \log \det_{|\gamma_a|}\}_{\text{FR}} &= \sum_{p \in \gamma \cap \gamma_a} \pm (A_p)_{\beta\alpha} \delta_{\beta, \alpha} \\ &= \sum_{p \in \gamma \cap \gamma_a} \pm \text{Tr}_{|\gamma|} = \langle [\gamma], a \rangle \text{Tr}_{|\gamma|} \end{aligned}$$

and

$$\{\log \det_{|\gamma_a|}, \log \det_{|\gamma_b|}\}_{\text{FR}} = \sum_{p \in \gamma_a \cap \gamma_b} \pm \delta_{\alpha, \beta} \delta_{\beta, \alpha} = \langle a, b \rangle n = \langle a, b \rangle \text{Tr}_{\bigcirc}.$$

□

Remark 5. *For simplicity, we described the Goldman theorem as stated in [14], without marked points on Σ . See [18] and [19] for a version with marked points.*

Remark 6. *One can consider the group $GL_n(\mathbb{C})$ as well. Then, seen as a real group, the above theorem holds with $f = 2 \text{Re Tr}$ and with Re log det instead of $\log \det$.*

Alternatively, one can define a holomorphic bivector field as in Equation (5), using left-invariant holomorphic vector fields. Taking $f = \text{Tr}$ and replacing $GL_n(\mathbb{C})$ with its universal cover to define $\log \det$, the above theorem then holds as well.

3 BV operators on moduli spaces

In this section, we prove an analogue of Theorem 9 for Lie supergroups equipped with an odd pairing on their Lie algebras.

3.1 Lie superalgebras with an odd pairing

To get a Batalin-Vilkovisky structure on the moduli space, we will use a Lie superalgebra \mathfrak{g} with an odd invariant pairing.

Definition 6. *If \mathfrak{g} is a Lie superalgebra, an odd invariant pairing is a graded-symmetric, non-degenerate odd map $\langle \cdot, \cdot \rangle: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{k}$ satisfying*

$$\langle [x, y], z \rangle + (-1)^{|x||y|} \langle y, [x, z] \rangle = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

A Lie superalgebra with such pairing is called odd metric. This pairing defines a \mathfrak{g} -equivariant isomorphism⁶ $t^\flat: \Pi \otimes \mathfrak{g} \rightarrow \mathfrak{g}^*$ by $t^\flat(\Pi \otimes x)(y) = \langle x, y \rangle$, whose inverse we will denote t^\sharp .

The (odd version of the) Cartan element is defined⁷ as

$$\phi(\alpha, \beta, \gamma) = (-1)^{|\beta|+1} \frac{1}{24} \alpha([t^\sharp \beta, t^\sharp \gamma]),$$

where $\alpha, \beta, \gamma \in \mathfrak{g}^*$.

Finally, define an odd element $\nu \in \mathfrak{g}$ by

$$\langle \nu, x \rangle = \text{str}_{\mathfrak{g}} \text{ad}_x, \quad \forall x \in \mathfrak{g}.$$

Recall that a Lie (super)algebra is called unimodular if $\text{ad}_x: \mathfrak{g} \rightarrow \mathfrak{g}$ is traceless for all $x \in \mathfrak{g}$. By the previous definition, a unimodularity of \mathfrak{g} is equivalent to vanishing of ν . For more details on t , ϕ and their coordinate expressions, see Appendix B.2

3.2 BV structure on the moduli of flat connections

Let us now turn our attentions to moduli spaces of flat G connections, with G a supergroup (see [26] and [10]).

Definition 7. *Let Σ be a compact, oriented surface with boundary and $V \subset \partial\Sigma$ a finite, non-empty set of marked points. For a supergroup G , the moduli space of flat G -connections on (Σ, V) is the supermanifold*

$$\mathcal{M}_{\Sigma, V}(G) = \text{Hom}_{SGrpd}(\Pi_1(\Sigma, V), G). \quad (8)$$

There is a natural action of G^V on this space, let us denote by $\rho_p(g)$ and $\rho_p(x)$ the Lie group and Lie algebra actions of the p th factor. In this section, we will define a second order differential operator on $\mathcal{M}_{\Sigma, V}(G)$ that will turn this space into a so-called \mathfrak{g}^V -quasi BV manifold.

⁶ Π denotes the one-dimensional super vector space $\mathbb{R}^{0|1}$ concentrated in the odd degree.

⁷The sign $(-1)^\beta$ is because the odd map t^\sharp passes through it.

Definition 8. A supermanifold M with an action of an odd metric algebra \mathfrak{g} is called \mathfrak{g} -quasi-BV if it is equipped with an odd, second-order, \mathfrak{g} -invariant differential operator Δ satisfying

$$\Delta(1) = 0 \quad \text{and} \quad \Delta^2 = \phi,$$

where ϕ acts on M as an element of $U\mathfrak{g}$.

If we choose a skeleton of Γ with edges $\gamma_1, \dots, \gamma_N$, we get an isomorphism

$$\Psi_\Gamma: G^{\times N} \xrightarrow{\sim} \mathcal{M}_{\Sigma, V}(G).$$

This way, the abstract space of all maps of groupoids gets a concrete description, which we will mostly use.⁸

As in Section 3.2, can define, for any half-edge a , an action of the Lie algebra \mathfrak{g} , denoted by $(x)_a$ for $x \in \mathfrak{g}$, by the equation (4), i.e. left-invariant or minus of the right invariant vector field on the corresponding component of G^V .

To define the quasi BV operator, we also need to fix a foliation of Σ as in Definition 3, i.e. we require that the foliation is tangent to the boundary at the marked points. Let us also choose Γ such that its edges γ_i are transverse to the foliation at the marked points.

If \mathfrak{g} has odd invariant pairing as in Definition 6, denote by t^{ij} the matrix inverse to the matrix of the pairing $t_{ij} = \langle e_i, e_j \rangle$ for a homogeneous basis e_i of \mathfrak{g} . Then we define

$$\Delta^\Gamma = \sum_{p \in V} \sum_{\substack{a, b \in \text{he}(p) \\ a < b}} \frac{1}{2} (-1)^{|e_i|} t^{ij} (e_i)_a (e_j)_b + \sum_{\gamma \in \text{Edges } \Gamma} \frac{1}{2} \text{rot}_\gamma \cdot (\nu)_{a_\gamma} \quad (9)$$

where a_γ is the outgoing half-edge of γ .

Remark 7. If \mathfrak{g} is unimodular, the second term of Δ^Γ is zero and we don't need the foliation of the surface. However, if \mathfrak{g} is not unimodular, just the first term of Δ^Γ would not be invariant under slide moves, as we will see below.

Remark 8. As in the even case (see Remark 3), the definition of Δ^Γ and Theorem 12 below are valid also in the case one starts with \mathfrak{g} and an element $t = (-1)^{|e_i|} t^{ij} e_i \wedge e_j \in (\wedge^2 \mathfrak{g})^\mathfrak{g}$, i.e. t doesn't have to be the inverse of an odd, nondegenerate pairing on \mathfrak{g} .

⁸The functor from supermanifolds to sets, given by

$$X \mapsto \text{Hom}_{\text{Groupoid}}(\Pi_1(\Sigma, V), \text{Hom}_{\text{SuperMfld}}(X, G))$$

is representable by each of these moduli spaces constructed using Γ . This gives an alternative definition of the moduli space $\mathcal{M}_{\Sigma, V}(G)$.

Theorem 12. *The operator $\Delta := (\Psi_\Gamma)_* \Delta^\Gamma$ on $\mathcal{M}_{\Sigma, V}(G)$ is independent of the choice of Γ compatible with the foliation and does not change under homotopy of the foliation.*

Futhermore, Δ satisfies all the properties of Definition 8 and thus equips $\mathcal{M}_{\Sigma, V}(G)$ with a structure of a \mathfrak{g}^V -quasi BV manifold.

The proof will be similar to the proof in Section 2.3. Now, instead of multivector fields, we work with differential operators, and thus the Gerstenhaber algebra $\bigwedge \mathfrak{g}^{\text{he}(p)}$ will be replaced by the associative algebra $U\mathfrak{g}^{\text{he}(p)}$.

Proof. Let us define $\tilde{t}_{ab} \in U\mathfrak{g}^{\text{he}(p)}$ by $(-1)^{|e_i|t^{ij}} \iota_a(e_i) \iota_b(e_j)$, where $\iota_a: \mathfrak{g} \rightarrow \mathfrak{g}^{\text{he}(p)}$ is the inclusion into the a th copy. Similarly, $\tilde{\phi}_{abc} = \phi^{xyz} \iota_a(e_x) \iota_b(e_y) \iota_c(e_z)$ and $\tilde{\nu}_a = \iota_a(\nu)$ (see Proposition 16 for the definition of ϕ^{xyz}). The operator Δ^Γ can be then written as the action of

$$\Delta^\Gamma = \sum_{p \in V} \sum_{\substack{a, b \in \text{he}(p) \\ a < b}} \frac{1}{2} \tilde{t}_{ab} + \sum_{\gamma \in \text{Edges } \Gamma} \frac{1}{2} \text{rot}_\gamma \cdot \tilde{\nu}_{a_\gamma}$$

These elements have properties similar to those of \tilde{s}_{ab} and $\tilde{\phi}_{abc}$ in the even case, which are collected in Proposition 17 in Appendix B.2. Using this result, we can just follow the proof of Theorem 9 without many modifications. For example, the invariance of Δ w.r.t. the action of \mathfrak{g}^V follows from the invariance of \tilde{t} and $\tilde{\nu}$. Since ν is central, it does not enter into the calculation of $\Delta^2 = [\Delta, \Delta]/2$ and we can repeat the arguments of Section 2.3 verbatim.

The invariance of Δ under edge inversion is as before, using $\text{Inv}_*((x)^L) = -(x)^R$. The additional term $\nu_a = \nu^L$, is sent to $\text{Inv}_*(\nu^L) = -\nu^R = -\nu^L$, since ν is G -invariant. This minus sign is canceled by $\text{rot}_{\gamma^{-1}} = -\text{rot}_\gamma$.

For the slide, the formulas expressing the action of $\Phi = \Psi_{\Gamma'}^{-1} \Psi_\Gamma$ on the vector fields are still correct. However, the term $-\frac{1}{2} t_{\text{Ad}_{g_2}(c'), \text{Ad}_{g_2}(c')}$ from equation (6) is not zero, but gives $-\tilde{\nu}_{c'}/4 = \tilde{\nu}_{b'}/4$. This counters the discrepancy between

$$\Phi_* \left(\frac{\text{rot}_{\gamma_1}}{2} \nu_d + \frac{\text{rot}_{\gamma_2}}{2} \nu_b \right) = \frac{\text{rot}_{\gamma_1}}{2} \nu_{d'} + \frac{\text{rot}_{\gamma_1}}{2} \nu_{b'} + \frac{\text{rot}_{\gamma_2}}{2} \nu_{b'}$$

and

$$\frac{\text{rot}_{\gamma_1}}{2} \nu_{d'} + \frac{\text{rot}_{\gamma_2 \gamma_1}}{2} \nu_{b'} = \frac{\text{rot}_{\gamma_1}}{2} \nu_{d'} + \frac{\text{rot}_{\gamma_1} + \text{rot}_{\gamma_2} + \frac{1}{2}}{2} \nu_{b'}$$

where γ_i is the path with holonomy g_i in the Figure 6b.

Finally, the fusion works as before. The foliation on the fused surface is extended as on Figure 9. The paths acquire no additional rotation with respect to this new foliation, so we again have

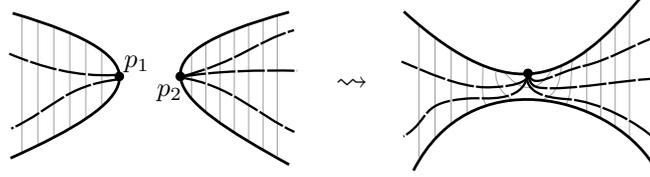


Figure 9: Fusion of a foliated surface at vertices p_1 and p_2 .

Proposition 4. For a surface with a skeleton $\Gamma \subset \Sigma$, the quasi BV structure on the surface given by fusion at two marked points p_1 and p_2 is

$$\Delta^{\Gamma_{\text{fused}}} = \Delta^{\Gamma} + \frac{1}{2}(-1)^{|e_i|} t^{ij} \rho_{p_1}(e_i) \rho_{p_2}(e_j),$$

where ρ_p denotes the \mathfrak{g} -action at the vertex p , i.e. $\rho_p(x) = \sum_{a \in \text{he}(p)} (x)_a$.

Then if we have a loop in Γ , we can always see the surface as a fusion of a different surface, where the loop is split. Moreover, the surface retracts to a thickening of its skeleton, on which the foliation is as on Figure 10 (see Appendix A.2). Thus, the foliation on the fused surface can be obtained from the foliation of the unfused surface by deformation, with rotation numbers unchanged.

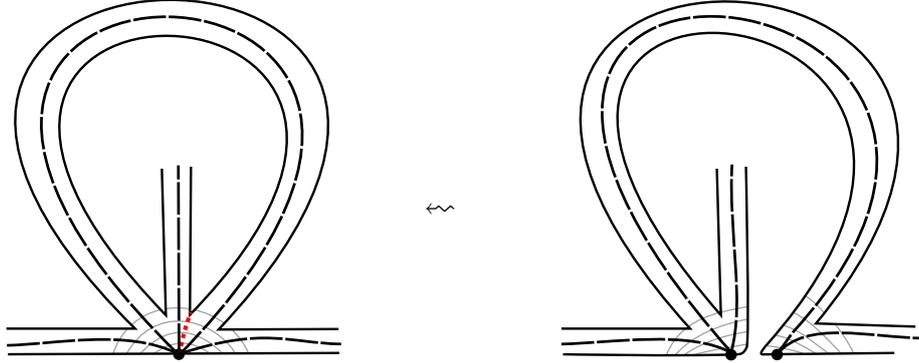


Figure 10: A loop in a skeleton of a foliated surface can be obtained via fusion.

If a slide move contains a loop, we have that operators $\Delta^{\Gamma_{\text{unfused}}}$ and $\Delta^{\Gamma'_{\text{unfused}}}$ give the same Δ on the moduli space of the unfused surface, and the additional term from fusion does not depend on Γ . \square

For more details on fusion of quasi-BV manifolds, see Appendix C.

4 Topological interpretation of Δ

In this section, we give a topological interpretation of the operator Δ , in terms of chords at intersections of loops on the surface.

4.1 Curves and chords

We start by introducing a class of functions on the moduli space, given by evaluating functions on G on holonomies along paths in Σ and their derivatives using *chords*. Such functions appeared before in [3], we present a version adapted to supergroups.

First, the case without chords is simply given by assigning holonomies to paths in Σ . For any path γ on Σ connecting two points of V , we have a map $\text{hol}_\gamma: \mathcal{M}_{\Sigma,V}(G) \rightarrow G$ giving the holonomy of along γ . This map, by definition, depends only on the homotopy class of γ fixing the endpoints.

Definition 9. *Let $\gamma_1, \dots, \gamma_k$ be k paths between points of V . Then we define*

$$\text{hol}_{\gamma_1, \dots, \gamma_k}: \mathcal{O}(G^{\times k}) \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma,V}(G))$$

as the pullback along the map $\mathcal{M}_{\Sigma,V}(G) \rightarrow G^{\times k}$ given by the product of maps $\text{hol}_{\gamma_i}: \mathcal{M}_{\Sigma,V}(G) \rightarrow G$.

This map only depends on the homotopy class of the paths⁹ γ_i . If we choose a skeleton, with N edges, we get a map $\mathcal{O}(G^{\times k}) \rightarrow \mathcal{O}(G^{\times N})$, as illustrated on the following example.

Example 1. *If γ is the boundary loop and e_1, e_2 are the two generators of the fundamental group as on the Figure 11a, then we have $\gamma = e_2^{-1}e_1^{-1}e_2e_1$ and thus the map hol_γ is given by the diagram on Figure 11b.*

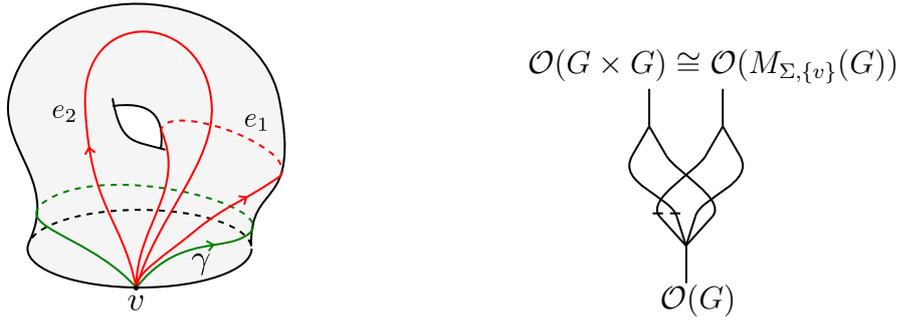
There are two ways to read this diagram, dual to each other. From top to bottom, it is built from structure maps of a (super)group, i.e. the two strands correspond to the two copies of G , they are followed by two diagonal maps $G \rightarrow G \times G$ and the group multiplication in the correct order, with bars signifying inverses.

From bottom to top, it can be seen as a diagram in the symmetric monoidal category of vector superspaces. It is built from structure maps of the (super) Hopf algebra $\mathcal{O}(G)$, i.e. an iterated coproduct, followed by the antipode (the bar), symmetry and the product of the Hopf algebra. Denoting the coproduct, antipode, symmetry and product by \square, S, τ and m , this can be written as

$$(m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (S^{\otimes 2} \otimes 1^{\otimes 2}) \circ \square^{(4)}: \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G),$$

where $\square^{(4)} = (\square \otimes 1^{\otimes 1}) \circ (\square \otimes 1) \circ \square$.

⁹By such homotopy we mean a continuous map $[0, 1] \times ([0, 1]^k) \rightarrow \Sigma$ fixing the endpoints of the k paths.



(a) A punctured torus with a loop. (b) The corresponding map hol_γ .

Figure 11: Interpreting loops on a surface as functions on the moduli space.

Let us now assume that the Lie algebra \mathfrak{g} of G is odd metric. Then we can define functions assigned to paths with one chord, by acting with $t \in \mathfrak{g} \otimes \mathfrak{g}$ at the chord endpoints on the outgoing half-edges.

Definition 10. Let $\gamma_1, \dots, \gamma_k$ be as before and let us choose a path δ on Σ connecting two points on $\gamma_1 \sqcup \dots \sqcup \gamma_k$, a so-called chord.

Then we define $\text{hol}_{\gamma_1, \dots, \gamma_k}^\delta : \Pi \otimes \mathcal{O}(G^{\times k}) \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma, V}(G))$ as follows: First, we deform the paths so that the endpoints of δ are in V , as on Figure 12



Figure 12: Moving endpoints of a chord to a marked point.

This gives us a map $\text{hol}' : \mathcal{M}_{\Sigma, V}(G) \rightarrow G^{\times(k+3)}$, because of the additional holonomy along the chord and the two subdivisions. In the case when the chord connects 2 different paths, we define $\text{hol}_{\gamma_1, \dots, \gamma_k}^\delta$ as on Figure 13.

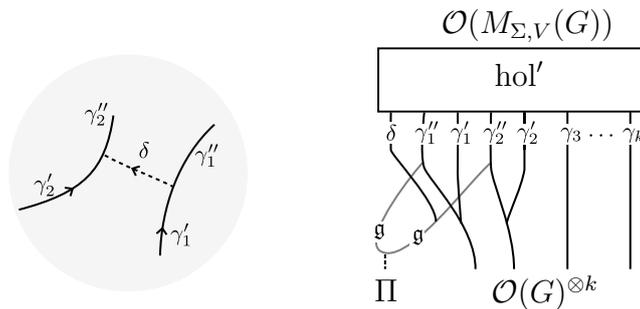


Figure 13: Definition of a chord connecting two paths.

If the chord lies on one path, we define $\text{hol}_{\gamma_1, \dots, \gamma_k}^\delta$ as on Figure 14. Finally, we require that changing the orientation of the chord is equivalent to multiplication by -1 .

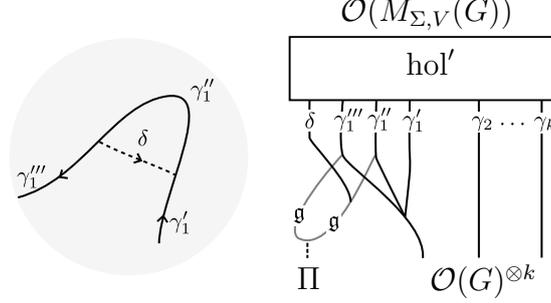


Figure 14: Definition of a chord connecting two points on one path. Oppositely oriented chord would result in -1 times the above diagram.

The symbol Ψ is the odd tensor $(-1)^{|e_i|t^{ij}} e_i \otimes e_j: \Pi \rightarrow \mathfrak{g} \otimes \mathfrak{g}$. We use the adjoint action $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathcal{O}(G)$ to (co)act by the holonomy along δ . The Lie algebra action $\mathfrak{g} \otimes \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ is by left-invariant vector fields.

A useful rule of thumb is that we act on the half-edge of γ leaving the chord endpoint. Intuitively, one of the endpoints of the chord is acted on by the holonomy along the chord. Specifically, note that the chord is a path lying on Σ , and chords with different homotopy classes will act differently.

Remark 9. In the case of usual Lie algebra, one would write the first map, evaluated at $f_1 \otimes f_2 \in \mathcal{O}(G \times G)$, as

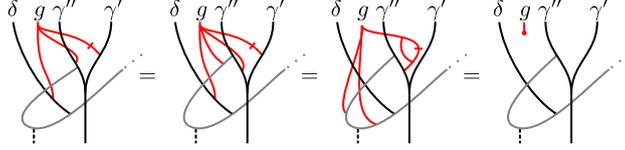
$$\sum_{i,j} s^{ij} \frac{\partial}{\partial t_1} \Big|_{t_1=0} f_1(\gamma_1'' e^{t_1 e_i} \gamma_1') \frac{\partial}{\partial t_2} \Big|_{t_2=0} f_2(\gamma_2'' e^{t_2 \text{Ad}_\delta(e_j)} \gamma_2').$$

where we use δ or γ to denote the holonomies along the respective loops

In the graded case, the diagram should be read from the bottom to top, as a diagram in the category of supervector spaces. The grey lines represent the Lie algebra \mathfrak{g} , the black lines the super Hopf algebra $\mathcal{O}(G)$, and the dotted line represents $\Pi = \mathbb{R}^{1|0}$. The adjoint action $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathcal{O}(G)$ would correspond in the even case to the map sending $x \in \mathfrak{g}$ to the \mathfrak{g} -valued function $g \mapsto \text{Ad}_g x$.

Proposition 5. The map $\text{hol}_{\gamma_1, \dots, \gamma_k}^\delta$ only depends on the homotopy classes of the curves (the endpoints of the chord only move along the path they belong to), and is independent of how we deformed the endpoints of the chord to V . Changing the orientation of the chord multiplies the corresponding hol function by -1 .

Proof. Choosing a different path for a chord endpoint, possibly ending at a different marked vertex, multiplies the holonomy along δ by a well-defined holonomy $g \in G$ (computed from the new vertex to the old vertex). Let us treat only the case when the starting point of the chord is moved, since the case of the endpoint follows by reversing the direction. The new holonomy along the moved chord is given by g followed by the holonomy along the old chord, i.e. $\delta \mapsto \delta g$. The two holonomies γ' , γ'' change to $g^{-1}\gamma'$ and $\gamma''g$. All of these actions cancel out due to the invariance of t , i.e.



where the additional holonomy g is highlighted in red.

The dependence on orientation of the chord follows from the invariance and antisymmetry of t . \square

Moreover, we can construct such functions assigned to homotopy classes of free loops, provided they are

Moreover, we can get a function on $\mathcal{M}_{\Sigma, V}(G)$ from a loop in Σ , provided that it is assigned a function invariant under conjugation by G .

Proposition 6. *Restrict $\text{hol}_{\gamma_1, \dots, \gamma_k}^{\delta}$ be the subspace of functions on $G^{\times k}$ which are invariant under conjugation of the i th factor. Then this map depend only on the free homotopy class of γ_i , i.e. it is independent on $v \in V$ and of the representative in conjugacy classes in $\pi_1(\Sigma, v)$.*

Proof. Changing how the loop γ_i is deformed to a based loop (also possibly changing the base point) conjugates the holonomy, under which the function f is invariant. The case when there is a chord on γ_i is analogous; the case when both endpoints lie on γ_i is slightly more involved, and follows from a suitable version of the equation

$$f_i(d e^{te_i} c b e^{t' \text{Ad}_{\delta}(e_j)} a) = f_i(b e^{t' \text{Ad}_{\delta}(e_j)} a d e^{te_i} c)$$

with the holonomies a, b, c, d as on Figure 15. \square

4.2 Quasi-BV structure on holonomies

We can now express the action of the BV operator on these functions coming from chords. We will from now on assume that all the families of paths have a finite number of transverse double (possibly self-) intersections and no other intersections. We also assume that the curves intersect the boundary only at their endpoints, which lie in V , and that they become tangent to the foliation in a finite number of points only.

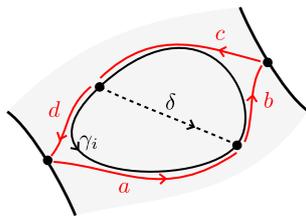


Figure 15: Notation for holonomies for conjugation invariance.

Definition 11. For any collection $\gamma_1, \dots, \gamma_n$ of paths on Σ , let p be an intersection or self-intersection point. Let us place a chord connecting the two segments near p as on Figure 16a for an intersection in the interior of Σ and as on Figure 16b for an intersection at the boundary.

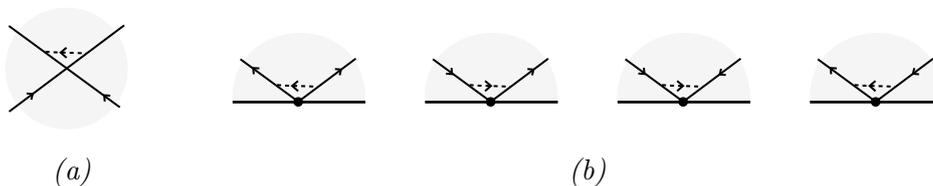


Figure 16: Rules for adding chords at intersections in the interior and at the boundary. The surface orientation is counter-clockwise.

Define a function on the moduli space given by the sum over all intersection points

$$\sum_p \lambda_p \text{hol}_{\gamma_1, \dots, \gamma_k}^{\delta_p} + \frac{1}{2} \text{hol}_{\gamma_1, \dots, \gamma_k} \circ \sum_k \text{rot}_{\gamma_i} \nu^{L,i} \quad (10)$$

where λ_p is 1 if the intersection is in the interior of the surface and $1/2$ if the intersection is at the boundary. The second sum is over all paths, acting with the vector $\nu \in \mathfrak{g}$ as a left-invariant vector field on the i th copy of G in $G^{\times k}$.

The second sum can be seen as adding a rot_{γ_i} -multiple of a short chord on each path γ , as we will see it the proof of the following proposition.

Proposition 7. The above function does not change if we move the individual paths γ_i by homotopy.

Proof. We need to check invariance with respect to the three Reidemeister moves (see [19, Proof of Prop. 2.11]).

RI: At the intersection, we get a *short chord*:



From Definition 10, such chord acts by $(-1)^{|e_i|t^{ij}}(e_i)^L(e_j)^L = \nu^L/2$, see also Proposition 17. This cancels with $-\nu^L/2$ coming from the full clockwise turn the path undertakes.

RII: We get a cancellation

which holds by the invariance and antisymmetry of the chord. Other possible cases follow since changing the orientation of any of the paths multiplies the whole equation by -1 . If one of the intersection points is at the boundary, the situation is analogous.

RIII: The following identity holds term by term. Again, changing the orientation of any of the paths multiplies the terms where a chord lies on that path by -1 .

□

Theorem 13. Let $\gamma_1, \dots, \gamma_n$ be as before. Then

$$\Delta \circ \text{hol}_{\gamma_1, \dots, \gamma_k} = \sum_p \lambda_p \text{hol}_{\gamma_1, \dots, \gamma_k}^{\delta_p} + \frac{1}{2} \text{hol}_{\gamma_1, \dots, \gamma_k} \sum_k \text{rot}_{\gamma_i} \nu^{L,i}. \quad (11)$$

where the right hand side is defined in Definition 11

The formula (11) completely determines the quasi BV operator, since we can get any function on the moduli space via holonomies.

Corollary 1. The quasi BV operator acting on $\mathcal{M}_{\Sigma, V}(G)/G_v$ is equal to the quasi BV operator on $\mathcal{M}_{\Sigma, V \setminus \{v\}}(G)$. Specifically, there is a canonical BV operator on $\mathcal{O}(\mathcal{M}_{\Sigma}(G))$, i.e. on the $G^{\times V}$ -invariant functions in $\mathcal{O}(\mathcal{M}_{\Sigma, V}(G))$. The choice of V is arbitrary. In general the foliation must be deformed to be compatible with V , which is always possible, see Figure 17.

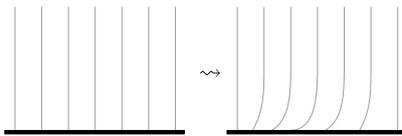


Figure 17: Deforming a foliation to be compatible with a new marked point.

Proof of Theorem 13. Both sides of Equation (11) are maps

$$\mathcal{O}(G^k) \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma,V}(G)).$$

The left hand side, evaluated on a tensor product of k functions on G , is equal to the BV operator from Theorem 12 acting on the function on the moduli space given by evaluating these k functions on the k holonomies along $\gamma_1, \dots, \gamma_k$.

The right hand side is equal to a similar function given by the holonomies, *together with one chord for each intersection of γ_i* (plus the term containing rotations of γ_i .)

Let us describe the strategy of the proof. We will start with the left hand side of (11). We have an explicit formula (9) for the quasi-BV operator, once we choose a skeleton Γ . If we deform the paths γ_i to intersect only near the vertices of Γ (the marked points), the action of the first term of (9) can be rewritten in terms of chords from Definition 10. These chords will act on any pair of path segments meeting at a marked point. Analyzing the possible positions of such pairs of path segments, we will show that if the paths don't intersect, these chords will cancel each other; if the paths do intersect, we recover the intersection formula (10).

Let us start with $\Delta \circ \text{hol}_{\gamma_1, \dots, \gamma_k}$. For a chosen skeleton Γ , $\text{hol}_{\gamma_1, \dots, \gamma_k}$ is given (from bottom to top) by applying the iterated coproduct on each component of $\mathcal{O}(G^{\times k})$, then a permutation of these factors, and finally by multiplying together factors corresponding to the same edge of Γ .

Both sides of equation (11) are homotopy invariant. Thus, the above morphism can be visualized by retracting the surface, and with it the paths γ_i , to Γ . Moreover, we can arrange all the intersections to take place in a small neighborhood of the vertices of V and such that each pair of paths going through the vertex neighborhood intersects at most once in this neighborhood, as on Figure 18.

The operator Δ in $\Delta \circ \text{hol}_{\gamma_1, \dots, \gamma_k}$ (defined in (9), ignoring the rotation term and signs for a moment) acts on pairs of half-edges of Γ . By Leibniz rule, we get at each vertex a sum

$$\sum_{i < j, i, j \text{ not in the same half-edge of } \Gamma} \frac{1}{2} \text{hol}_{\gamma_1, \dots, \gamma_k}^{i \rightarrow j}$$

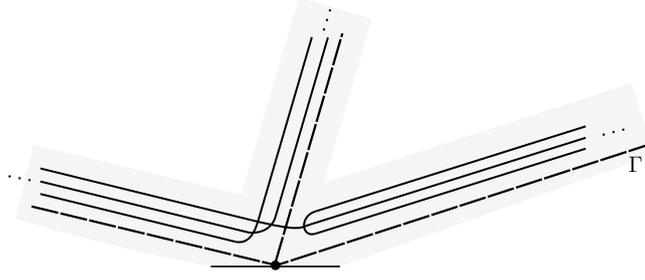


Figure 18: Paths intersecting only near a marked point v .

over all half-edges of the part of $\gamma_1, \dots, \gamma_k$ incident to the vertex. The chord $i \rightarrow j$ connects the two half-edges i and j and comes from the term \tilde{t}_{ab} of Δ .

We can remove the condition that i and j do not follow the same half-edge of Γ . Indeed, the above sum is equal to

$$\sum_{i < j} \text{hol}_{\gamma_1, \dots, \gamma_k}^{i \rightarrow j},$$

where the added terms cancel with terms from neighboring vertices by the invariance and antisymmetry of t .

The sum above contains chords connecting half-edges of paths close to marked points. Every chord connects either two consecutive half-edges in a path (i.e. lies on one path segment), or connects two different segments of (possibly the same) path going through the marked point. Splitting the above sum, we get

$$\sum_{e_1 \neq e_2} \frac{1}{2} (\text{1 to 4 chords between these two segments}) + \frac{1}{2} \sum_e \text{hol}_{\gamma_1, \dots, \gamma_k}^e. \quad (12)$$

The first term is a sum over all pairs of segments of paths γ_i going through v , and for each such pair, we collect all chord that connect them. In the second term, we get a “short” chord placed on the segment e (at the marked point), which equals the action of $\pm\nu/2$. Together with the rotation part of Δ , it combines to give the second term of the RHS of (11), since at each vertex the path undergoes an extra $1/2$ turn in addition to the rotation along edges of Γ .

Signs: If a half-edge γ' arrives at the vertex v , the element $x \in \mathfrak{g}$ coming from the chord \tilde{t}_{ab} acts as x^{-R} on that holonomy; for an outgoing edge γ'' , x acts by x^L , see the Figure 1a. However, for a function of the product $\gamma''\gamma'$, the equality $x^{L''} = x^{R'}$ holds. We can thus move every action to act by left invariant vector fields on the outgoing half-edge, as in Definition 10, with a minus sign for each chord acting on an incoming edge. Let us now apply this rule.

In the case where the chord lies on one path segment, we get a $\frac{1}{2}$ times the short chord, as picture on Figure 19. The orientation of the chord is given by

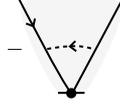


Figure 19: A chord lying on one path segment from the second sum in (12).

the order of half-edges, as in equation (9), with the minus sign as discussed above. This gives an action of $-\nu/4$, which is consistent with the counter-clockwise half-turn, as on Figure 2.

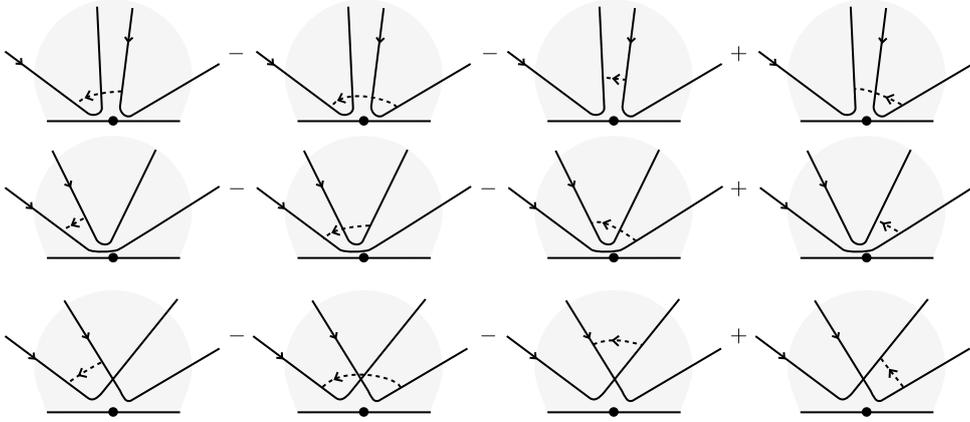


Figure 20: Case with two traversing paths.

Next, we treat the sums over pairs of segments from Equation (12) case by case. If two path segments meet near the marked point v , none, one or both of them start on end at v . In all cases, we will show that we recover the intersection rule from Definition 11:

1. If neither of the paths starts nor ends at the marked point, there are 4 chords. There are three different ways to connect 4 half-edges to 2 path segments, two in which the paths don't intersect and one in which they do. These four possibilities are shown on Figure 20.

The first two lines, without an intersection of the segments, vanish. In the last line, the first two terms cancel, but the other two chords add up. This corresponds to the chord that comes from two paths intersecting from Definition 11. The direction is correct (chord leaves the first outgoing half-edge) and the sum of two chords cancels with the factor $\frac{1}{2}$ in Equation (12).

2. If one path starts or ends at v , we get the three cases shown on Figure 21. As before, we get a non-zero contribution only in the last case, which, after multiplying by $\frac{1}{2}$ from (12), has the correct factor of $+1$.

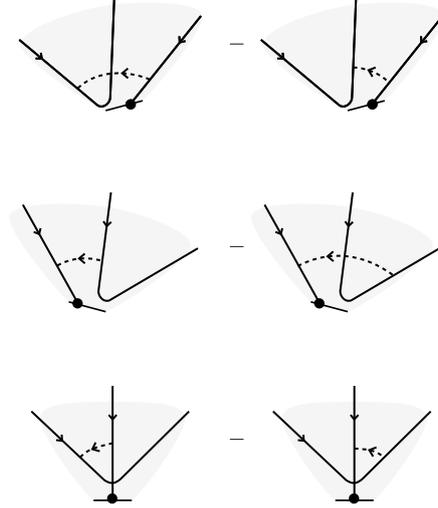


Figure 21: Case with one path ending at the marked point.

3. If both paths start or end at v , there is only one term, shown on Figure 22. Together with the factor $\frac{1}{2}$ from (12), we get an agreement with Definition 11.

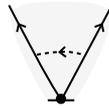


Figure 22: Case with both paths ending at the marked point.

The remaining cases, in which the paths are oriented differently, follow from the above calculations: changing an orientation of one of the paths changes signs on both sides of the equation. \square

4.3 Another formula for the quasi-BV operator

We will now present two more formulas for the quasi-BV operator in terms of the surface. Their role is to make the role of the foliation clearer. Concretely, in the formula (10), the term containing the rotation numbers rot_γ has to be added by hand to the sum over all intersections. If we instead consider intersection of the collection of paths $\gamma_1, \dots, \gamma_k$ and *the same collection, shifted in*

the direction of the foliation, we will obtain the rotation numbers automatically. We will first consider a case of a general foliation, and then a simpler situation of an orientable foliation (see Appendix A.2).

We will shift the paths γ_i in the direction of the foliation. For each path γ , there are two possible choices for the direction of this shift.

Definition 12. Let $\gamma_1, \dots, \gamma_k$ as before. Choose a direction of a small shift for each γ_i such that the shifted and unshifted paths all intersect transversally in double points. If a path γ_i intersects a shifted path γ_j^{sh} in p , the point p has a preimage on the original path γ_j , let us call it p_0 . See Figure 23, with the shifted path shown in red.

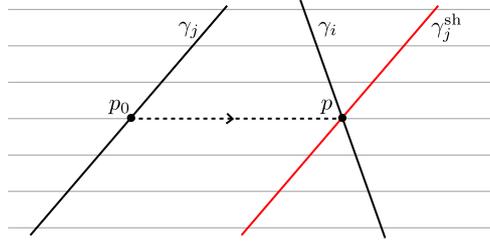


Figure 23: Rule for chords at the intersection of a shifted (red) and an unshifted (black) path

Then, we define a function on the moduli space by averaging over all possible choices of the directions of shifts, and for each choice by summing over all intersections of shifted and unshifted paths, with a chord going from p_0 to p

$$\frac{1}{2^k} \sum_{2^k \text{ possible shifts}} \sum_{p \in \gamma \cap \gamma^{\text{sh}}} \frac{1}{2} \alpha_p \text{hol}_{\gamma_1, \dots, \gamma_k}^{p_0 \rightarrow p}. \quad (13)$$

where the sign α_p is +1 iff the half-edges leaving p , ordered (shifted, unshifted), are compatible with the orientation of the surface.

Proposition 8. The function (13) from the above definition is equal to $\Delta \circ \text{hol}_{\gamma_1, \dots, \gamma_k}$.

Proof. An intersection point p occurs either where two path segments intersect, or when the path becomes tangent to the foliation. For brevity, let us denote the choice of directions of the shifts by C .

In the first case, if the intersection happens away from the boundary, we get (see Figure 24), for each C , two chords at the intersection, with all the 2 or 4 possible shifts giving the same answer.

Together with the factor $\frac{1}{2}$ from Equation (13), we get an agreement with Definition 11.

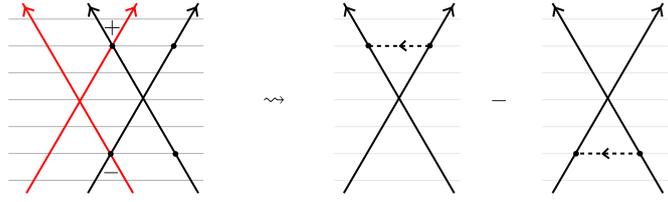


Figure 24: Definition 12 applied near an intersection in the interior of Σ .

If the intersection happens on the boundary, it is either an intersection of two different paths, or a self-intersection. In the first case, the four cases on Figure 25 appear for different C .

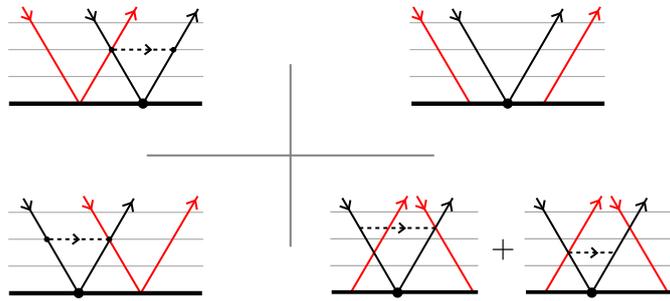


Figure 25: Definition 12 applied near an intersection at the boundary of Σ .

Here, we already oriented the chord to absorb the possible sign α_p ; the terms also carry a factor $\frac{1}{2}$.

If the segments meeting at the boundary belong to the same path, there are two cases to distinguish: If the foliation makes an odd number of turns along the path, we get the left column of the above figure, and for an even number of turns, we get the right column of the above figure.

Finally, if the path becomes tangent to the foliation, there is a contribution only in the cases if it looks like a local extremum, see Figure 26.

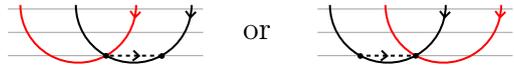


Figure 26: Definition 12 when a path becomes tangent to the foliation.

This short chord, together with the factor $\frac{1}{2}$, acts by $\nu/4$, as expected (compare with the proof of Proposition 7)

Since all of these cases have the same frequency among all possible C , we see that the average number of chords is $\frac{1}{2}$ in all three cases, which recovers the factor λ_p from Definition 11. As before, the cases with different orientations of the intersecting segments follow from these, since changing an orientation multiplies the term by -1 . \square

However, the foliation is orientable¹⁰, i.e. we can consistently choose a direction of the shift, we can remove the symmetrization from Equation (13)

Definition 13. Let $\gamma_1, \dots, \gamma_k$ be as in Definition 12 and assume that the foliation of Σ is orientable. Choose one such orientation, shift all paths γ along this vector field and define, as before, a function by summing over all intersections of the shifted and unshifted paths.

$$\sum_{p \in \gamma \cap \gamma^{\text{sh}}} \frac{1}{2} \alpha_p \text{hol}_{\gamma_1, \dots, \gamma_k}^{p_0 \rightarrow p}. \quad (14)$$

The sign α_p and the chord direction is as in Definition 12.

Proposition 9. The function (14) from the above definition is equal to $\Delta \circ \text{hol}_{\gamma_1, \dots, \gamma_k}$. Specifically, it does not depend on the sign of the orientation of the foliation.

Proof. The proof is similar to the proof of Proposition 8, only the foliation is now oriented, which excludes the cases in right column, when the intersection happens at the boundary. The remaining cases recover Definition 11 without averaging. \square

5 Goldman-Turaev Lie bialgebra and $Q(n)$

In this section, we specialize to $G = Q(n)$, the queer Lie supergroup. This will allow us to relate the Goldman-Turaev Lie bialgebra with the BV operator on the moduli space of flat connections, extending the correspondence of the Goldman bracket and the Atiyah-Bott Poisson structure [14].

5.1 The Lie supergroup $Q(n)$

Let us recall the definition of the queer Lie supergroup $Q(n)$ (see [17, §1.8] for more details).

Definition 14. For $n \geq 1$, define the following associative algebra

$$q_{\text{as}}(n) = \text{Mat}_n(\mathbb{R}) \otimes \mathbb{R}[\xi]/(\xi^2 - 1)$$

This algebra is \mathbb{Z}_2 -graded by setting ξ to be odd. The odd function otr on $q_{\text{as}}(n)$ is defined by

$$\text{otr}(X + \xi Y) = \text{Tr } Y$$

and is cyclically symmetric¹¹ $\text{otr}(A_1 A_2) = \text{otr}(A_2 A_1)$.

¹⁰See Appendix A.2.

¹¹The usual Koszul sign $(-1)^{|A_1||A_2|}$ is equal to $+1$, since A_1 and A_2 have opposite parity.

We define $Q(n)$, to be the Lie supergroup of invertible elements associated to $q_{\text{as}}(n)$. Its Lie superalgebra, denoted $\mathfrak{q}(n)$, is the space $q_{\text{as}}(n)$ with the bracket given by the graded commutator. The odd trace makes $\mathfrak{q}(n)$ into an odd metric Lie algebra, with pairing given by $A_1 \otimes A_2 \mapsto \text{otr}(A_1 A_2)$.

The Lie superalgebra $\mathfrak{q}(n)$ is unimodular, see Proposition 11 and Remark 10.

5.2 Goldman-Turaev Lie bialgebra

Any collection of k loops γ_i on Σ gives a function on the moduli space $\mathcal{M}_{\Sigma, V}(Q(n))$, by taking a product of the odd traces of holonomies along γ_i . Our goal is now to study the action of the BV operator Δ on such functions, to which end we need to recall the Goldman-Turaev Lie bialgebra.

Recall from Section 2.4 that $\mathcal{G}_{\Sigma} = \mathbb{R}\pi_1^{\text{free}}$ is the \mathbb{R} -vector space generated by homotopy classes of free loops in an oriented surface Σ . The following two operations were defined by Goldman and Turaev [14, 25].

Definition 15. Let γ_1, γ_2 be two immersed loops on Σ representing their classes $|\gamma_1|, |\gamma_2| \in \mathcal{G}_{\Sigma}$ with transversal double intersections. Their Goldman bracket is given by a sum over their intersections

$$[|\gamma_1|, |\gamma_2|]_{\text{G}} = \sum_{p \in \gamma_1 \cap \gamma_2} \beta_p \left[\begin{array}{c} \text{Diagram 1: Two intersecting loops with arrows} \\ \rightsquigarrow \\ \text{Diagram 2: Two separate loops with arrows} \end{array} \right],$$

where we modify the loops only in a small disc around p , connecting them into one loop. The sign β_p is +1 iff the two tangent vectors $(\dot{\gamma}_1, \dot{\gamma}_2)$ at p agree with the orientation of Σ .

The Turaev cobracket of $|\gamma_1|$ is defined as a sum over all self-intersections of γ_1

$$\delta_{\text{T}}|\gamma_1| = \sum_{p \text{ self-intersection of } \gamma_1} \left[\begin{array}{c} \text{Diagram 1: Self-intersecting loop with arrow} \\ \rightsquigarrow \\ \text{Diagram 2: Two separate loops with arrows} \end{array} \right],$$

where we see the resulting two loops as lying in $\mathcal{G}_{\Sigma} \wedge \mathcal{G}_{\Sigma}$, with the first loop being the one starting to the right (this is fixed by the orientation of the surface).

Goldman proved [14] that $(\mathcal{G}_{\Sigma}, [\cdot, \cdot]_{\text{G}})$ is a well-defined Lie algebra. Moreover, the constant loop is in the center of $[\cdot, \cdot]_{\text{G}}$ and on the quotient $\mathcal{G}_{\Sigma}^{\text{red}} = \mathcal{G}_{\Sigma}/\mathbb{R}\mathcal{O}$, the above bracket and cobracket give a well-defined Lie bialgebra by a result of Turaev [25]. Moreover, $(\mathcal{G}_{\Sigma}^{\text{red}}, [\cdot, \cdot]_{\text{G}}, \delta_{\text{T}})$ is involutive, i.e. $[\cdot, \cdot]_{\text{G}} \circ \delta_{\text{T}} = 0$, by a result of Chas [9]. Therefore, one can define a BV operator on the commutative superalgebra $\wedge \mathcal{G}_{\Sigma}^{\text{red}}$ using the following standard result.

Proposition 10 ([8, Sec. 5]). *Let $(\mathcal{G}, [\cdot, \cdot], \delta)$ be an involutive Lie bialgebra. Define an operator $\Delta^{[\cdot, \cdot], \delta}$ on the commutative superalgebra $\wedge \mathcal{G}$ by*

$$\begin{aligned} \Delta^{[\cdot, \cdot], \delta}(x_1, \dots, x_n) &= \sum_{i < j} (-1)^{i+j+1} [x_i, x_j] x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_n \\ &\quad + \sum_i (-1)^{i-1} x_1 \dots \delta(x_i) \dots x_n, \\ \Delta^{[\cdot, \cdot], \delta}(1) &= 0, \end{aligned}$$

where $x_i \in \mathcal{G}$ and we omit the symbol \wedge .

Then $\Delta^{[\cdot, \cdot], \delta}$ makes $\wedge \mathcal{G}$ into a Batalin-Vilkovisky algebra¹².

5.3 The odd Goldman map

As we mentioned above, collection of loops $\gamma_1, \dots, \gamma_k$ on Σ defines a function on $\mathcal{M}_{\Sigma, V}(Q(n))$ by taking the product of odd traces of all the holonomies. Since this function depends on the order of the loops γ_i only up to the sign of a permutation, we get a map $\Phi^{\text{odd}}: \wedge \mathcal{G}_{\Sigma}^{\text{red}} \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma, V}(Q(n)))$. We will now show that this map intertwines the natural BV operators on both sides, defined in Proposition 10 and Theorem 12, respectively.

Theorem 14. *Let $\Phi^{\text{odd}}: \wedge \mathcal{G}_{\Sigma}^{\text{red}} \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma, V}(Q(n)))$ be the algebra map defined by sending the generators γ to $\text{hol}_{\gamma}(\text{otr})$. Then*

$$\Delta \circ \Phi^{\text{odd}} = \Phi^{\text{odd}} \circ \Delta^{[\cdot, \cdot]_{\mathcal{G}}, 2\delta_{\mathbb{T}}}, \quad (15)$$

i.e. Φ^{odd} is a map of (quasi-)BV algebras.

Note that in order to get an agreement, we need to use $2\delta_{\mathbb{T}}$ as a cobracket on $\mathcal{G}_{\Sigma}^{\text{red}}$ in Proposition 10.

Proof. It will be simpler to consider, instead of the supergroup $Q(n)$, a more general unimodular Lie supergroup G obtained as the supergroup of invertible elements of an associative superalgebra A with an invariant, non-degenerate odd trace otr . Similarly to $\mathfrak{q}(n)$, let us denote by $\{e_i\}$ a basis of A and by $\{\phi^i\}$ the dual basis of A^* . Let us also introduce the structure constants c_{ij}^k by $e_i e_j = c_{ij}^k e_k$, cyclically-symmetric coefficients $t_{i_1 \dots i_n} = \text{otr } e_{i_1} \dots e_{i_n}$ and the inverse of the pairing $t = (-1)^{|e_i|} t^{ij} e_i \otimes e_j \in A \otimes A$ with $t^{ij} t_{jk} = \delta_k^i$. In our conventions for supergroups, we have for the coproduct $\square \phi^i = (-1)^{|\phi^j| |\phi^k|} c_{jk}^i \phi^j \otimes \phi^k$ and for the left action of A , seen as a Lie algebra of G , $(e_a)^{\text{L}} \phi^i = (-1)^{|e_a|} c_{ja}^i \phi^j$.

¹²There are two conventions for a definition of a BV algebra used in literature, either with $\Delta(xy) = \Delta(x)y + (-1)^{|x|} x \Delta(y) + \{x, y\}$ or with $\Delta(xy) = \Delta(x)y + (-1)^{|x|} x \Delta(y) + (-1)^{|x|} \{x, y\}$. The bracket $\{\cdot, \cdot\}$ is then either graded-symmetric or satisfies $\{y, x\} = (-1)^{(|x|+1)(|y|+1)+1} \{x, y\}$, respectively. We use the first convention.

Both sides of Equation (15) are a sum over all (possibly self-) intersections of loops; we will prove the equality (15) term-by-term. Moreover, we can permute both sides such that the BV operators act on the first two loops for the case of an intersection, or the first loop in the case of a self-intersection. Let us treat these cases separately.



(a) A chord at an intersection of two loops

(b) The corresponding function on the moduli space

Figure 27: The term of the LHS of (15) corresponding to an intersection of two loops.



(a) A resolution of intersection from the Goldman bracket

(b) The corresponding function on the moduli space

Figure 28: The term of the RHS of (15) corresponding to an intersection of two loops.

intersection of two loops: Using Theorem 13, we get on the LHS of (15) the chord diagram as shown on Figure 27a. Using Definition 10, this term is equal to the function on Figure 27b. The RHS of (15) is given by the Goldman bracket from Definition 15, i.e. the holonomy of the loop on Figure 28a. This loop is (up to cyclic permutation) equal to $\gamma'_1 \gamma''_1 \gamma'_2 \gamma''_2$ which gives the function on Figure 28b.

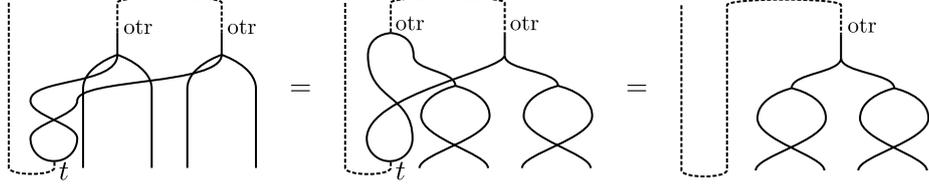
Our goal is to prove the equality of the two function on Figures 27b and 28b. Let us consider the parts of the diagrams below the box marked hol' , which can be seen as odd elements of $(A^*)^{\otimes 4} \subset \mathcal{O}(G)^{\otimes 4}$. Concretely, from Figure 27b we get

$$(-1)^{|e_a|+|e_b|} t^{ab} t_{ij} t_{kl} (e_a)^L \phi^i \otimes \phi^j \otimes (e_b)^L \phi^k \otimes \phi^l,$$

while from Figure 28b we get

$$t_{jilk} \phi^i \otimes \phi^j \otimes \phi^k \otimes \phi^l.$$

The equality of these tensors can be proven directly using the invariance of the odd trace. Alternatively, we can see both sides as maps $A^{\otimes 4} \rightarrow \Pi^{\otimes 3} \cong \Pi$, and prove the identity diagrammatically (taking care with signs), getting



To obtain the left-most diagram, we use the fact that acting by $(e_a)^L$ on a linear function corresponds to right-multiplication by e_a . Then, the first equality follows from the invariance of the odd trace, while the second equality follows from cancellation of the pairing $e_i \otimes e_j \mapsto t_{ij} = \text{otr}(e_i e_j)$ and $t = (-1)^{|e_i|} t^{ij} e_i \otimes e_j$.



(a) A chord at a self-intersection.

(b) The corresponding function on the moduli space, from Definition 11.

Figure 29: The term of the LHS of (15) corresponding to a self-intersection.

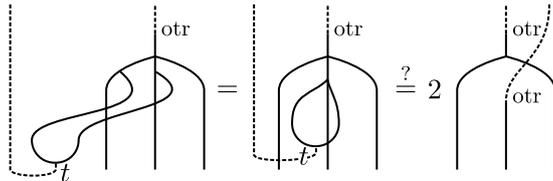
self-intersection: Here, the situation is analogous, with the loops and corresponding functions shown on Figures 29 and 30. These give the following elements of $(A^*)^{\otimes 3}$:

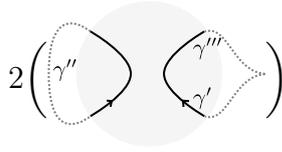
$$(-1)^{|e_a|+|e_b||\phi^i|+|\phi^j|} t^{ab} t_{ijk} (e_a)^L \phi^i \otimes (e_b)^L \phi^j \otimes \phi^k$$

and

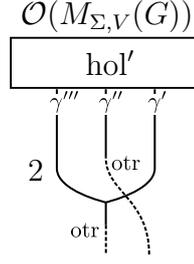
$$(-1)^{|\phi^k|} t_{ikt_j} \phi^i \otimes \phi^j \otimes \phi^k$$

respectively. Again, one can proceed in coordinates, or diagrammatically:





(a) A resolution of intersection from the Turaev cobracket. The factor 2 is introduced in (15). The loop on the right is the first one in the wedge product.



(b) The corresponding function on the moduli space.

Figure 30: The term of the RHS of (15) corresponding to a self-intersection.

The last equality does not hold in general, but for $G = Q(n)$ follows from the identity

$$-\frac{1}{2} \left(\text{diagram of a loop with a self-intersection} \right) = \left(\text{diagram of a vertical line with a unit and a dashed line with a loop} \right), \quad (16)$$

which is proven in Proposition 11. □

Proposition 11. *In the algebra $q_{\text{as}}(n)$, the identity (16) holds.*

Proof. In the notation of the proof of Theorem 14, the identity reads

$$-\frac{1}{2} (-1)^{|e_a|+|e_b|+|e_i|} t^{ab} c_{aib}^j = t_i u^j,$$

where $u^j e_j \in q_{\text{as}}(n)$ is the unit of the algebra and $e_a e_i e_b = c_{aib}^j e_j$ are the structure constants of the iterated product.

For $q_{\text{as}}(1)$, we have $t = 1 \otimes \xi - \xi \otimes 1$, and the identity holds since

$$\begin{aligned} -\frac{1}{2} (-1 \cdot \xi \cdot \xi - (-1)^{1 \cdot 1} \xi \cdot \xi \cdot 1) &= 1 \stackrel{\check{}}{=} \text{otr}(\xi)1, \\ -\frac{1}{2} (1 \cdot 1 \cdot \xi - \xi \cdot 1 \cdot 1) &= 0 \stackrel{\check{}}{=} \text{otr}(1)1. \end{aligned}$$

The algebra $q_{\text{as}}(n)$ is obtained by tensoring $q_{\text{as}}(1)$ with the algebra $\text{Mat}_n(\mathbb{R})$, which satisfies $\sum_{ab} S^{ab} E_a X E_b = \text{Tr}(X) 1_{n \times n}$ where E_a is a basis of $\text{Mat}_n(\mathbb{R})$ and S^{ab} the inverse to $\text{Tr}(E_a E_b)$. This is true because a basis E_a consists of elementary matrices, with $a = (\alpha\beta)$ being a pair of indices. Then the matrix product $S^{ab} E_a X E_b$ is a matrix with $X_{\beta\beta}$ on the position (α, α) and zeros

elsewhere, and summing over all α and β gives the identity matrix times the trace of X .

The tensor product of two such algebras again satisfies (16), which is immediate diagrammatically

Here $q_{\text{as}}(1)$ is represented by the solid line and $\text{Mat}_n(\mathbb{R})$ by the dash-dotted line. \square

Remark 10. *Given an associative superalgebra where the identity (16) holds, its commutator Lie superalgebra is unimodular. This can be seen by precomposing the identity (16) with the unit.*

5.4 Odd determinants and $H_1(\Sigma)$

Let us extend the Lie bialgebra structure from $\mathcal{G}_\Sigma^{\text{red}}$ to $\mathcal{G}_\Sigma^{\text{red}} \oplus H_1(\Sigma, \mathbb{R})$, as in Section 2.4.

Definition 16. *On $\mathcal{G}_\Sigma^{\text{red}} \oplus H_1(\Sigma)$, define a Lie bracket $[\cdot, \cdot]_G$ as in Definition 4, with $\circ = 0$. The cobracket δ_Γ is extended from δ_Γ by 0 on $H_1(\Sigma)$.*

It is easy to check that one obtains again an involutive Lie bialgebra, and thus $\wedge(\mathcal{G}_\Sigma^{\text{red}} \oplus H_1(\Sigma, \mathbb{R}))$ becomes a BV algebra using Proposition 10.

The role of the function $\log \det$ from Section 2.4 will be played by the following function:

Definition 17. *The odd determinant $\text{odet}: Q(n) \rightarrow \Pi$ is the odd function on $Q(n)$ defined by*

$$\text{odet}(X + \xi Y) = \sum_{j \geq 1 \text{ odd}} \frac{\text{Tr}((X^{-1}Y)^j)}{j}. \quad (17)$$

The odd determinant satisfies

$$\text{odet}(GH) = \text{odet}(G) + \text{odet}(H), \quad (18)$$

and thus is invariant [17, Theorem 1.8.5].

Proposition 12. *Define an algebra map*

$$\tilde{\Phi}^{\text{odd}}: \wedge(\mathcal{G}_\Sigma^{\text{red}} \oplus H_1(\Sigma, \mathbb{R})) \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma, V}(Q(n))),$$

by extending Φ^{odd} via

$$\tilde{\Phi}^{\text{odd}}(a) = \text{hol}_{\gamma_a}(\text{odet}),$$

where $\gamma_a \in \pi_1(\Sigma)$ represents $a \in H_1(\Sigma) \cong \pi_1(\Sigma)^{\text{ab}}$. Then $\tilde{\Phi}^{\text{odd}}$ is a map of (quasi-)BV algebras, with respect to Δ on $\mathcal{O}(\mathcal{M}_{\Sigma, V}(Q(n)))$ and $\Delta^{[\cdot, \cdot]_{\text{G}}, 2\delta_{\text{T}}}$ on $\wedge(\mathcal{G}_{\Sigma}^{\text{red}} \oplus H_1(\Sigma, \mathbb{R}))$.

Proof. Let us start by stating the odd analogue¹³ of Equation (7): an element $x \in \mathfrak{q}(n) \cong T_e Q(n)$, i.e. a derivative at the group identity $e \in Q(n)$, satisfies

$$x(\text{odet}) = -\text{otr } x. \quad (19)$$

This is easily seen from Definition 17: only the term $j = 1$ of the sum (17) is linear in the odd coordinate, and can have a non-zero contribution when evaluated at $e \in Q(n)$. The sign comes from our convention for the isomorphism $\mathfrak{q}(n) \cong T_e Q(n)$; we identify $x \in \mathfrak{q}(n)$ with the derivation at e given as $\phi^i \mapsto (-1)^{|\phi^i|} \phi^i(x)$.

Equations (18) and (19) imply, for the left-invariant action of $x \in \mathfrak{q}(n)$

$$x^{\text{L}} \text{odet} = -\text{otr } x, \quad (20)$$

i.e. a constant function on $Q(n)$. Similarly, again using (18), we get

Now, we can prove that the extended map $\tilde{\Phi}^{\text{odd}}$ is a map of quasi-BV algebras. There are three new cases to consider, containing loops γ_a for $a \in H_1(\Sigma, \mathbb{R})$

self-intersection of a loop γ_a : The cobracket is extended by zero to the first homology. Similarly, a chord acting on a triple coproduct of odet is zero, since at least one leg of the chord will differentiate the constant function.

an intersection of two loops γ_a, γ_b : The Goldman bracket is extended to $H_1(\Sigma, \mathbb{R})$ by zero. On the other hand, the chord acting on two functions odet will give a function proportional to $(-1)^{|e_i|} t^{ij} \text{otr}(e_i) \text{otr}(e_j)$. Since either e_i or e_j are even, their odd trace is 0 and the function corresponding to the chord also vanishes.

an intersection of γ_a with γ : This is the only nonzero term, let us analyze the two terms similarly as in the proof of Theorem 14. The chord and the corresponding function is shown on Figure 31. The extended Goldman

¹³Equation (7) says, in other terms, that the Lie algebra morphism $\text{Tr}: \mathfrak{gl}(n) \rightarrow \mathbb{R}$ is the differential of the Lie group morphism $\log \det: GL_n(\mathbb{R})^+ \rightarrow \mathbb{R}$. Similarly, (19) says that the Lie superalgebra morphism $\text{otr}: \mathfrak{q}(n) \rightarrow \mathbb{R}$ is the differential of the Lie supergroup morphism $\text{odet}: Q(n) \rightarrow \mathbb{R}$.

bracket $[\gamma, a]$ contributes just γ , for a positive intersection. Using (20) in Figure 31b, we get the following element of $\mathcal{O}(Q(n))^{\otimes 4}$

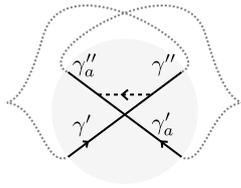
$$(-1)^{|e_a|+|e_b|+1} t^{ab} t_{ij}(e_a)^L \phi^i \otimes \phi^j \otimes \text{otr}(e_b) \otimes 1, \quad (21)$$

where $1 \in \mathcal{O}(Q(n))$ is the constant function equal to 1. If we choose a basis¹⁴ of $q_{\text{as}}(n)$ as $E_{(\alpha\beta)}$ and $\xi E_{(\alpha\beta)}$, the element t can be written as

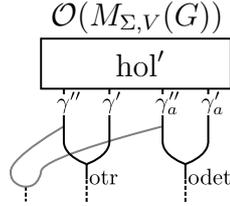
$$t = \sum_{\alpha, \beta=1}^n E_{(\alpha\beta)} \otimes \xi E_{(\beta\alpha)} - \xi E_{(\alpha\beta)} \otimes E_{(\beta\alpha)}.$$

Then, in Equation (21), only the first term of the above sum contributes, and only when $\alpha = \beta$, i.e. we get the left-invariant action of the identity matrix $\sum_{\alpha} E_{(\alpha\alpha)}$, which acts trivially

$$t_{ij}(\sum_{\alpha} E_{(\alpha\alpha)})^L \phi^i \otimes \phi^j \otimes 1 \otimes 1 = t_{ij} \phi^i \otimes \phi^j \otimes 1 \otimes 1.$$



(a) A chord at an intersection of two loops γ and γ_a



(b) The corresponding function on the moduli space

Figure 31: The term corresponding to an intersection of two loops γ and γ_a .

□

5.5 Surjectivity of the map $\tilde{\Phi}^{\text{odd}}$

In this section, we investigate which (algebraic) functions on the moduli space without marked points are in the image of the maps Φ^{odd} and $\tilde{\Phi}^{\text{odd}}$.

Let us briefly recall the even case. The even analogue of Φ^{odd} is the extension of Φ^{even} (defined in Theorem 10) to an algebra map $\text{Sym } \mathcal{G}_{\Sigma} \rightarrow \mathcal{O}(\mathcal{M}_{\Sigma}(GL(n)))$. The image of this extension is generated by traces of arbitrary holonomies. Choosing a set of generators of $\pi_1(\Sigma)$ with holonomies denoted by A_1, \dots, A_N , this image is generated by traces of monomials in $A_1^{\pm 1}, \dots, A_N^{\pm 1}$.

¹⁴Recall that $E_{(\alpha\beta)}$ are the elementary matrices.

By [21], any function on the space $\text{Mat}_n(\mathbb{R})^{\times N}$ polynomial in the matrix entries and invariant under simultaneous conjugation by $GL(n)$ is a product of traces of monomials in the matrices.

If we restrict to the space $GL(n)^{\times N}/GL(n)$, the algebraic functions we consider are polynomials in the matrix entries and the inverses of the determinants of A_i , invariant under simultaneous conjugation. This algebra is equal to the image of the extension of Φ^{even} : this follows from the fact that the additional generators $\det A_i^{-1}$ can be written as a polynomial in traces of powers of A_i^{-1} .

Let us now turn to the odd case. The image of Φ^{odd} is generated by odd traces of arbitrary holonomies, while the image of $\tilde{\Phi}^{\text{odd}}$ has additional generators for odd determinants of holonomies.

A natural class of algebraic functions on the moduli space $\mathcal{M}_\Sigma(GL(n)) \cong Q(n)^{\times N}/Q(n)$ is given by invariant polynomials of matrix entries and inverses of determinants of their even parts. It was proven by Berele [6] that all functions on $q_{\text{as}}(n)^{\times N}$ polynomial in entries and invariant under simultaneous conjugation by $Q(n)$ are products of odd traces of monomials of matrices¹⁵. However, restricting to invertible matrices, already for $N = n = 1$ we see that not all algebraic functions are coming from odd traces. Indeed, denoting by a and da even and the odd coordinate

$$A = a + \xi da \in Q(1),$$

we get $\text{otr } A^k = ka^{k-1}da$, and we need to include the odd determinant $\text{odet } A = a^{-1}da$, which is also invariant. Thus, we are led to the following conjecture.

Conjecture 1. *The algebra of all algebraic functions on*

$$\mathcal{M}_\Sigma(Q(n)) \cong (Q(n)^{\times N})/Q(n)$$

is generated by odd traces of products of matrices and their inverses, and by the odd determinants of the N matrices.

Proposition 13. *For $n = 1$, the conjecture is true.*

Proof. On $Q(1)^{\times N}$, we denote the even coordinates a_i and the odd coordinates da_i . It is enough to look at invariants under $q(1)$, since odd traces and determinants are already $Q(1)$ invariant. The even part of $q(1)$ acts trivially, the odd part acts via

$$[\xi, a + \xi da] = 2da.$$

In other words, on $Q(1)^{\times N}$, ξ acts as the odd vector field

$$2 \sum_i da_i \frac{\partial}{\partial a_i}$$

¹⁵Invariants under conjugation by $Q(n)$ were also studied by Sergeev [23, 22] and Shander [24]

Invariant functions on $Q(1)^{\times N}$ are thus the same thing as closed differential forms on the complement of coordinate hyperplanes of \mathbb{R}^N , polynomial in the coordinates and their inverses. We will freely go between function on $Q(1)^{\times N}$ and such differential forms.

The odd trace of the i th matrix $A_i = a_i + \xi da_i$ is equal to da_i , its odd determinant equals da_i/a_i . More generally, let f be a non-commutative polynomial in N variables, then

$$f(A_1, \dots, A_N) = \alpha + \xi d\alpha$$

for some function $\alpha = f(a_1, \dots, a_N) +$ forms of degree ≥ 2 . This is true because it holds for the matrices A_i and remains true for product of such matrices of that form

$$(\alpha_1 + \xi d\alpha_1)(\alpha_2 + \xi d\alpha_2) = (\alpha_1\alpha_2 + d\alpha_1 d\alpha_2) + \xi(\alpha_1 d\alpha_2 + d\alpha_1 \alpha_2).$$

The cohomology of the algebra of non-constant algebraic functions on $Q(1)$ with respect to d is one-dimensional, generated by da/a . Thus, any $q(1)$ invariant function on $Q(1)^{\times N}$ can be written as a product of functions da_i/a_i plus an exact term. The term da_i/a_i is equal to the product of odd determinants, we thus need to treat exact forms.

Let $\beta = \beta^{(1)} + \beta^{(2)} + \dots$ be an exact form with $\beta^{(i)}$ an i -form. We can always write $\beta^{(k)} = d(\sum_{|I|=k-1} \beta_I da_I) = \sum_{|I|=k-1} d(\beta_I) da_I$ for some functions β_I , $I = (I_1, \dots, I_{k-1})$ is a multi-index of length $k-1$. Then, if $\beta^{(i)}$ vanish for $i < k$, the k -th form component of

$$\beta - \text{otr}(\beta_I(A)) \text{otr}(A^{I_1}) \dots \text{otr}(A^{I_{k-1}})$$

vanishes. Here $\text{otr}(\beta_I(A))$ is obtained by replacing a_i by A_i , whose 1-form part does not depend on the chosen order.

Working order by order, we finally write any invariant function as a product of odd determinants plus sum of products of odd traces. \square

Remark 11. *In the even case, the kernels of the maps Φ^{even} are non-empty for every n ; however their intersection over $n \in \mathbb{N}$ is empty, as shown by Etingof [11]. It would be interesting to study an analogous question for $Q(n)$.*

A Skeletons and foliations

A.1 Skeletons

Proof of Proposition 1. We start by replacing the skeleton by a homotopy equivalent graph embedded in Σ . Specifically, we arbitrarily resolve each marked point p , with valence $n(p)$, to a binary tree with a root at p and $n(p)$ leaves. This way, the skeleton Γ is transformed into a uni-trivalent graph $\tilde{\Gamma}$ in Σ , dual to a triangulation. In general, let us consider a uni-trivalent graph $\tilde{\Gamma}$ in the surface such that

1. the univalent vertices are mapped bijectively to V ,
2. the trivalent vertices are mapped to the interior of Σ , and
3. Σ deformation retracts to $\tilde{\Gamma}$.

To go back from such uni-trivalent graph to a (possibly different) skeleton, we need to choose a subset T of edges of $\tilde{\Gamma}$, let's call them *red*, such that their complement $\tilde{\Gamma} \setminus T$ is a spanning forest¹⁶, with one tree rooted at each marked point. Then, contracting the trees to their roots, the red edges become a skeleton of the surface.

We will now proceed in two steps. First, we will prove that for a fixed uni-trivalent graph, any two choices of red edges give skeletons that can be related by slides. Then, we will use a result of Penner [20] relating different uni-trivalent graphs via flips.

With two choices T, U of the sets of red edges on the same uni-trivalent graph $\tilde{\Gamma}$, let us denote the two complementary forests by $(T_{p_1}, \dots, T_{p_n})$ and $(U_{p_1}, \dots, U_{p_n})$, where we enumerated their trees by their roots. One of the red edges e_0 in T must be in U_p for some $p \in V$ (otherwise $T = U$). Let us consider the union of e_0 with the forest $\tilde{\Gamma} \setminus T$. This subgraph will not be a forest anymore, but this additional edge¹⁷

1. either created a loop γ in a tree T_{p_i} , or
2. it connected T_{p_i} with some other tree T_{p_j} , creating a path between γ between p_i and p_j .

In both cases, the path γ has to contain an edge e_1 that belongs to U . This implies that $e_1 \neq e_0$, as $e_0 \in U_p \subset \tilde{\Gamma} \setminus U$. Thus, considering $T' := T \setminus \{e_0\} \cup$

¹⁶Recall that a spanning forest of a graph is a subgraph consisting of a disjoint union of trees that contains all vertices of the graph.

¹⁷Spanning tree is equivalently characterized by the property that addition of any edge creates a single loop \llbracket . For spanning forests, the result we use follows by joining all the roots of the trees to a new vertex, creating a spanning tree of this enlarged graph. Then, adding an edge creates a loop in this spanning tree, which might or might not pass through the new vertex, giving the two possibilities.

$\{e_1\}$, we obtain a new red set, i.e. a set of edges such that $\tilde{\Gamma} \setminus T'$ is a forest rooted at V .

Repeating this move, we transform T to U , since each such move subtracts 2 from the finite number $\sum_i |T_{p_i} \triangle U_{p_i}|$.

For the skeletons corresponding to the forests, this move corresponds to the slide, along e_0 , of the red half-edges between the removed red edge e_0 and the added red edge e_1 , as encountered along γ . On Figure 32, we show the case when γ is a loop. The case when the added edge connects two trees is analogous.

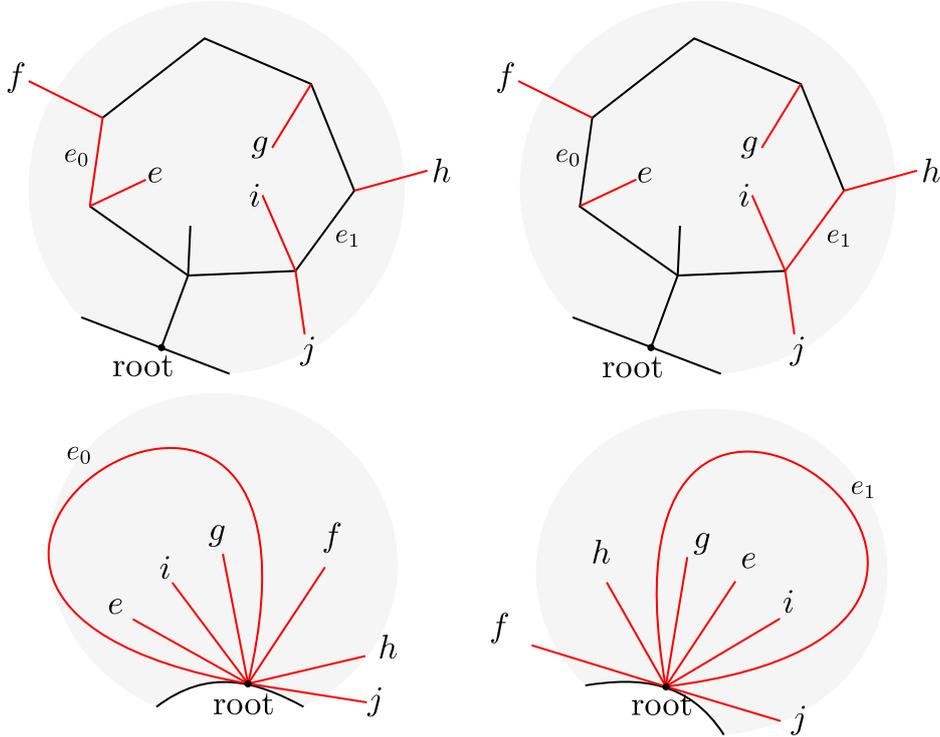


Figure 32: Replacing the red edge e_0 with e_1 . On the top, the red edges and the black forest are shown. On the bottom, the forest is contracted to show the corresponding skeletons. The half-edges g, f and h undergo a slide along e_0 , because they are between e_0 and e_1 .

Now, let us turn to relating different uni-trivalent graphs. We will use the fact that any two uni-trivalent graphs as above are related by a sequence of flips, i.e. moves as on Figure 33.

For surfaces with marked points on the boundary, this claim follows from the work of Penner [20]. There, the boundary components of Σ without marked points are represented as punctures, and the univalent vertices of the uni-trivalent graph are also allowed to lie on these punctures, in addition to the marked points. Any two such *Penner graphs* are then related by

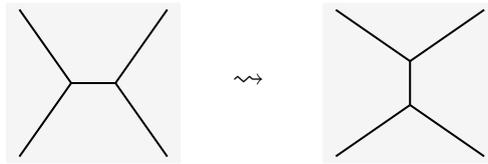


Figure 33: Flip move. The edge in the middle has to connect two distinct trivalent vertices.

a sequence of isotopies, flips and so-called quasi-flips [20, Definition 5.5.7, Corollary 5.5.10], which are moves involving an edge ending on a puncture, depicted on Figure 34.



Figure 34: Quasi-flip. The edge in the middle has to connect a trivalent vertex and a puncture.

To get our kind of a uni-trivalent graph from such Penner graph, we can replace the punctured vertex by a “tadpole” graph, as on Figure 35.

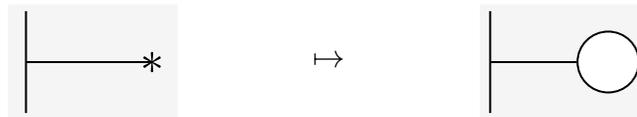


Figure 35: Replacing punctured vertices with trivalent graphs.

A general uni-trivalent graph might not be isotopic to one coming from a Penner graph, as the loop around a boundary component without marked points might contain more than one trivalent vertex. However, using flips, we can transform it to such graph. Then, the quasi-flip can be implemented using two ordinary flips, see Figure 36. Thus, any two uni-trivalent graphs we consider are connected via flips.

Finally, let us explain how the flips interact with the skeletons. For a flip along an edge, we can find a spanning forest containing such edge, by adding this edge to the forest and removing another edge from a newly formed loop or path connecting marked points. If this new loop or path contains only the one edge, this edge is not valid for flips.

However, flip along an edge contained in the spanning forest has no effect on the corresponding skeleton. Thus, we can relate any two skeletons, replaced by uni-trivalent graphs with red edges, by moving red edges and flips along non-red edges. Since moving of red edges corresponds to slides and flips along non-red edges don't change the skeleton, the proposition is proven. \square

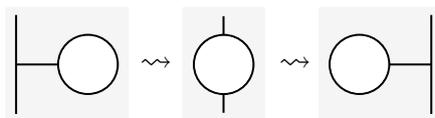


Figure 36: A quasi-flip can be implemented as a sequence of two flips

A.2 Foliations

An example of a foliated surface, with multiple paths and their rotation numbers, is shown below in Figure 37. Now, we classify foliations on surfaces with

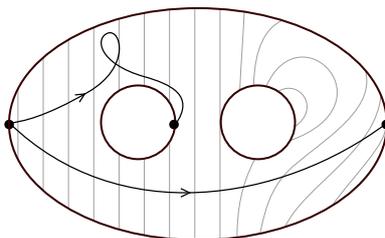
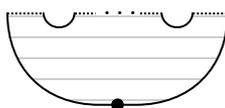


Figure 37: A foliation of a sphere with three punctures. The top path has rotation number $-\frac{1}{2}$, the bottom path 0.

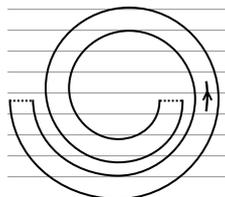
boundary and marked points.

Proposition 14. *If Γ is a skeleton of Σ , then a homotopy class of foliations is uniquely specified by choosing the rotation number for each edge of Γ . Conversely, for any choice of half-integers for edges of Γ , there exists a foliation, with these rotation numbers.*

Proof. existence: Realize each vertex of the skeleton as the following foliated coupon:



For any $r \in \frac{1}{2}\mathbb{Z}$, construct a strip with r rotations of the foliation by embedding it in a horizontally-foliated plane as a spiral; the case of $r = -3/2$ is shown below:



Finally, glue together these strips and the coupons, which is possible as the foliations agree on the dotted intervals.

uniqueness: If we fix one foliation and a metric on Σ , foliations are in bijection with based maps $\Sigma/V \rightarrow S^1$. Homotopy classes of these maps are classified by specifying the degree of the map along each edge of Γ , as $\Sigma/V \sim \Gamma/V$. \square

Let us remark that 1-dimensional foliations are in 1-1 correspondence with 1-dimensional distributions in $T\Sigma$, tangent to $\partial\Sigma$ at V . This follows from the Frobenius theorem. If there exists a nowhere-vanishing section of such distribution, then the corresponding foliation is called **orientable**. Not all foliations are orientable, i.e. it is not always possible to choose a non-vanishing vector field tangent to the foliation; the foliation on Figure 37 is such non-orientable example.

Orientable foliations also arise from framings such that the first component of the framing is tangent to $\partial\Sigma$ at V . To each orientable foliation, there is unique-up-to homotopy such framing compatible with the orientation of Σ .

B Lie algebras with a pairing

B.1 Ordinary Lie algebras with a pairing

Let \mathfrak{g} be an ordinary Lie algebra with a symmetric, invariant pairing. Let $\text{he}(p)$ be an ordered set.

Recall from Section 2.3 the elements

$$\tilde{s}_{ab} = s^{ij} \iota_a(e_i) \wedge \iota_b(e_j) \in \wedge^2 \mathfrak{g}^{\text{he}(p)}$$

and

$$\tilde{\phi}_{abc} = \frac{1}{24} f^{ijk} \iota_a(e_i) \wedge \iota_b(e_j) \wedge \iota_c(e_k) \in \wedge^3 \mathfrak{g}^{\text{he}(p)}.$$

Let us now collect some useful properties of these two elements of $\wedge \mathfrak{g}^{\text{he}(p)}$.

Proposition 15 (Properties of s). *We have $\tilde{s}_{ab} = -\tilde{s}_{ba}$ and $\tilde{\phi}_{abc}$ is symmetric w.r.t. permutation of its labels. For $a \neq b$*

$$[\tilde{s}_{ab}, \tilde{s}_{ab}] = 24(\tilde{\phi}_{aab} + \tilde{\phi}_{abb}),$$

and for a, b and c different,

$$[\tilde{s}_{ab}, \tilde{s}_{bc}] = -24\tilde{\phi}_{abc}.$$

Finally, for a, b, c and d all distinct, $[\tilde{s}_{ab}, \tilde{s}_{cd}] = 0$.

Proof. The symmetry properties follow from (anti-)symmetry of s^{ij} and ϕ^{ijk} . Next,

$$\begin{aligned} [\tilde{s}_{ab}, \tilde{s}_{ab}] &= t^{ij} t^{kl} (\iota_a([e_i, e_k]) \wedge \iota_b(e_j) \wedge \iota_b(e_l) + \iota_a(e_i) \wedge \iota_a(e_k) \wedge \iota_b([e_j, e_l])) \\ &= f^{mjl} \iota_a(e_m) \wedge \iota_b(e_j) \wedge \iota_b(e_l) + f^{nik} \iota_a(e_i) \wedge \iota_a(e_k) \wedge \iota_b(e_n) \\ &= 24\tilde{\phi}_{abb} + 24\tilde{\phi}_{aab}. \end{aligned}$$

For one common index, we have

$$\begin{aligned} [\tilde{s}_{ab}, \tilde{s}_{bc}] &= t^{ij} t^{kl} \iota_a(e_i) \wedge \iota_b([e_j, e_k]) \wedge \iota_c(e_l) \\ &= f^{nil} \iota_a(e_i) \wedge \iota_b(e_n) \wedge \iota_c(e_l) = -24\tilde{\phi}_{abc}. \end{aligned}$$

□

B.2 Lie superalgebras with an odd pairing

Let us denote by e_i a homogeneous basis of an odd metric Lie algebra \mathfrak{g} as in Definition 6. Denote $t_{ij} = \langle t_i, t_j \rangle$ and $[e_i, e_j] = f_{ij}^k e_k$. Then we can express ϕ and ν from Definition 6 in coordinates and also define an invariant element in $\wedge^2 \mathfrak{g}$ inverse to the pairing on \mathfrak{g} .

Proposition 16.

1. The Cartan element ϕ is invariant and graded-symmetric and thus defines an element $\phi \in (\text{Sym}^3 \mathfrak{g})^{\mathfrak{g}}$. In coordinates, we have

$$\phi = \phi^{xyz} e_x e_y e_z = \frac{(-1)^{|e_y|}}{24} t^{xj} f_{jk}^y t^{kz} e_x e_y e_z, \quad (22)$$

where t^{ij} is the matrix inverse to t_{ij}

2. The element $(-1)^{|e_i|} t^{ij} e_i \wedge e_j \in \wedge^2 \mathfrak{g}$ is \mathfrak{g} -invariant.
3. The element $(-1)^{|e_i|} t^{ij} f_{ij}^k e_k \in \mathfrak{g}$ is equal to ν and is in the center of \mathfrak{g} .

Proof. 1. From the definition of ϕ , we have

$$\phi(\alpha, \gamma, \beta) = (-1)^{(|\gamma|+1)(|\beta|+1)+1+|\gamma|-|\beta|} \phi(\alpha, \beta, \gamma) = (-1)^{|\beta||\gamma|} \phi(\alpha, \beta, \gamma),$$

From the invariance of the pairing we have for any $x, y, z \in \mathfrak{g}$

$$\langle x, [y, z] \rangle = -(-1)^{|x||y|} \langle [y, x], z \rangle = \langle z, [x, y] \rangle.$$

Thus, $\phi(\alpha, \beta, \gamma) \propto (-1)^{|\beta|} \langle t^\# \alpha, [t^\# \beta, t^\# \gamma] \rangle$ satisfies

$$\phi(\beta, \gamma, \alpha) = (-1)^{|\alpha|+|\beta|} \phi(\alpha, \beta, \gamma) = (-1)^{|\gamma|} \phi(\alpha, \beta, \gamma)$$

which means ϕ is symmetric (since $|\alpha| + |\beta| + |\gamma| \stackrel{\text{mod } 2}{=} 0$ for the result to be non-zero).

The invariance follows from this as

$$\begin{aligned} & \langle [w, x], [y, z] \rangle + (-1)^{|w||x|} \langle x, [w, [y, z]] \rangle \\ &= \langle [w, x], [y, z] \rangle + (-1)^{|w||x|} \langle [y, z], [x, w] \rangle. \end{aligned}$$

Now just use that $\phi(\text{ad}_x(\alpha \otimes \beta \otimes \gamma))$ is (up to a numerical factor) equal to $\langle \cdot, [\cdot, \cdot] \rangle$ evaluated on $\text{ad}_x(t^\# \alpha \otimes t^\# \beta \otimes t^\# \gamma)$

Evaluating ϕ on three basis elements of \mathfrak{g}^* , we get

$$\begin{aligned} \phi(e^x, e^y, e^z) &= (-1)^{|e_y||e_x|} \phi(e^y, e^x, e^z) \\ &= (-1)^{|e_x|+|e_y||e_x|+1} \frac{1}{24} e^y([t^{xi} e_i, t^{zj} e_j]) = (-1)^{|e_x|+|e_y||e_x|+1} \frac{1}{24} t^{xi} f_{ij}^y t^{jz}, \end{aligned}$$

where we used that $t^{ij} = (-1)^{|e_i||e_j|} t^{ji} = t^{ji}$. This corresponds to the formula $\phi = (-1)^{|e_y|+1} \frac{1}{24} t^{xj} f_{jk}^y t^{kz} e_x e_y e_z$, since pairing this with e^x, e^y, e^z gives

$$(-1)^{|e_y|+1+|e_x|(|e_y|+|e_z|)+|e_y||e_z|} \frac{1}{24} t^{xj} f_{jk}^y t^{kz}$$

and these two signs are equal, since $|e_x| + |e_y| + |e_z| = 0$. Note that the sign $(-1)^{|e_x|(|e_y|+|e_z|)+|e_y||e_z|}$ is the Koszul sign from the pairing of $e_x \otimes e_y \otimes e_z$ with $e^x \otimes e^y \otimes e^z$.

- It follows from a direct calculation, using the symmetry of ϕ , that

$$\text{ad}_{e_k}(-1)^{|e_i|} t^{ij} e_i \wedge e_j = (-1)^{|e_k|} 48 t_{kc} \phi^{cxy} e_x \wedge e_y = 0$$

Alternatively, see Remark 12

- By definition, ν is equal to $\nu^i e_i = (-1)^{|e_k|} f_{lk}^k t^{li} e_i$. From the symmetry of ϕ^{xyz} , we get $(-1)^{|e_k|} t^{il} f_{lx}^k = (-1)^{|e_k|} f_{xi}^l t^{lk}$, which gives

$$(-1)^{|e_k|} t^{il} f_{lk}^k = (-1)^{|e_k|} f_{kl}^i t^{lk}.$$

Because the Lie bracket commutes with the action of \mathfrak{g} , we get that $\nu = (-1)^{|e_i|} t^{ij} [e_i, e_j]$ is invariant by the previous point. \square

Recall that Π is the vector space \mathbb{R}^{01} and $\Pi \mathfrak{g} := \Pi \otimes \mathfrak{g}$. The implicit Koszul sign from commuting with Π allows us to state the above result more invariantly.

Remark 12. One can use the isomorphism $\mathfrak{g}^* \cong \Pi \mathfrak{g}$ and the décalage isomorphism $\bigwedge^n(\Pi \mathfrak{g}) \cong \text{Sym}^n(\mathfrak{g})$ to clarify the previous proposition. The map $x \otimes y \otimes z \mapsto \langle x, [y, z] \rangle$ is graded antisymmetric, and thus can be seen as an element of $\bigwedge^3(\mathfrak{g})$. Using the following sequence of \mathfrak{g} -equivariant isomorphisms,

$$\Pi \otimes \bigwedge^3(\mathfrak{g}^*) \cong \Pi \otimes \bigwedge^3(\Pi \mathfrak{g}) \cong \text{Sym}^3(\mathfrak{g}),$$

this element is mapped to $-\phi$. Similarly, the odd pairing lives in $\text{Sym}^2(\mathfrak{g}^*) \cong \text{Sym}^2(\Pi\mathfrak{g}) \cong \bigwedge^2(\mathfrak{g})$; $t_{ij}e^ie^j$ gets sent to $(-1)^{|e_i|}t^{ij}e_i \wedge e_j$, on which we can apply the Lie bracket to get ν .

Note that we can map ϕ to $U\mathfrak{g}$ using the symmetrization map. Explicitly, ϕ as an element of $U\mathfrak{g}$ is equal to $\phi^{xyz}e_xe_ye_z = \frac{(-1)^{|e_y|}}{24}t^{xj}f_{jk}^y t^{kz}e_xe_ye_z$.

Let us also prove the odd analogue of Proposition 17 for elements $\tilde{\nu}$, \tilde{t} and $\tilde{\phi}$, defined in the proof of Theorem 12. The only added feature is that \tilde{t}_{aa} is not zero.

Proposition 17 (Properties of t). *We have $\tilde{t}_{ab} = -\tilde{t}_{ba}$ for $a \neq b$ and $\tilde{t}_{aa} = \iota_a(\nu)/2$. The element $\tilde{\phi}_{abc}$ is symmetric w.r.t. permutation of its labels. Both are invariant under the diagonal action of \mathfrak{g} .*

For $a \neq b$

$$[\tilde{t}_{ab}, \tilde{t}_{ab}] = 24(\tilde{\phi}_{aab} + \tilde{\phi}_{abb}),$$

and for a, b and c different,

$$[\tilde{t}_{ab}, \tilde{t}_{bc}] = -24\tilde{\phi}_{abc}.$$

Finally, for a, b, c and d all distinct, $[\tilde{t}_{ab}, \tilde{t}_{cd}] = 0$.

Proof of Proposition 17. For \tilde{t}_{aa} , we get

$$\begin{aligned} \tilde{t}_{aa} &= (-1)^{|e_i|}t^{ij}\iota_a(e_i)\iota_a(e_j) \\ &= (-1)^{|e_i|}\frac{1}{2}t^{ij}(\iota_a(e_i)\iota_a(e_j) - \iota_a(e_j)\iota_a(e_i)) \\ &= \frac{1}{2}\iota_a\left((-1)^{|e_i|}t^{ij}f_{ij}^ke_k\right). \end{aligned}$$

Let us calculate only $[\tilde{t}_{ab}, \tilde{t}_{bc}]$:

$$\begin{aligned} [\tilde{t}_{ab}, \tilde{t}_{bc}] &= (-1)^{|e_i|+|e_k|}t^{ij}t^{kl}\iota_a(e_i)\iota_b([e_j, e_k])\iota_c(e_l) \\ &= (-1)^{|e_i|+|e_j|+|e_m|}t^{ij}f_{jk}^m t^{kl}\iota_a(e_i)\iota_b(e_m)\iota_c(e_l) \\ &= -24\phi^{iml}\iota_a(e_i)\iota_b(e_m)\iota_c(e_l). \end{aligned}$$

The calculation for $[\tilde{t}_{ab}, \tilde{t}_{ab}]$ is analogous. \square

Remark 13. *The elements s/t and ϕ satisfy similar relations, as shown in Proposition 15 and 17. This can be seen as having a morphism to $\bigwedge \mathfrak{g}^{\text{he}(p)}$ or $U\mathfrak{g}^{\text{he}(p)}$ from a super analogue of the Drinfeld-Kohno Lie algebra, which we will call $\hat{\mathfrak{p}}_{\text{odd}}(n)$.*

The algebra $\hat{\mathfrak{p}}_{\text{odd}}(n)$ is generated by odd elements $\{t_{ab}\}_{a,b \in \{1, \dots, n\}}$ and even elements $\{\phi_{aaa}\}_{a \in \{1, \dots, n\}}$, where t_{aa}, ϕ_{aaa} are central and t_{ab} satisfy $t_{ba} = -t_{ab}$ for $a \neq b$ and

$$[t_{ab}, t_{ac} + t_{bc}] = 0, \quad [t_{ab}, t_{cd}] = 0$$

for a, b, c or a, b, c, d all distinct. The elements $\phi_{aab} + \phi_{abb}$ and ϕ_{abc} can be defined as the commutators of t 's; the relation $[t_{ab}, t_{ac} + t_{bc}] = 0$ tells us that ϕ_{abc} is symmetric.

A \mathbb{Z} -graded version of this algebra, without the central elements, appears in the study of rational cohomology of the little n -discs operad for odd n , see [13, Part II, Section 14.1.1 and Theorem 14.1.14]

C Hopf-like algebras governing the fusion of quasi-BV structures

We can characterize \mathfrak{g} -quasi-BV manifolds as being manifolds with action of an algebra $\mathcal{H}^{\mathfrak{g}}$ by differential operators, which we define below. Then, the fusion of quasi-BV manifolds is captured by a coproduct-like structure on these algebras.

Definition 18. Let (\mathfrak{g}, t) be a quadratic Lie algebra. Define

$$\mathcal{H}^{\mathfrak{g}} \equiv U\mathfrak{g} \otimes U(\mathbb{k}\Delta),$$

where the odd generator Δ (graded) commutes with the Lie algebra \mathfrak{g} and satisfies

$$\Delta^2 \equiv \frac{1}{6} f^{ijk} e_i e_j e_k \in \text{Sym}^3(\mathfrak{g}) \subset U\mathfrak{g}.$$

Here $f^{ijk} = (-1)^{k+jk} f_{ab}^k t^{ai} t^{bj}$ is graded symmetric, so one can write the image of $f^{ijk} e_i e_j e_k$ in $U\mathfrak{g}$ as just $f^{ijk} e_i e_j e_k$.

The coproduct is defined as

$$\begin{aligned} \tilde{\square}(\xi) &= \xi \otimes 1 + 1 \otimes \xi, \quad \text{where } \xi \in \mathfrak{g}, \\ \tilde{\square}(\Delta) &= \Delta \otimes 1 + 1 \otimes \Delta + (-1)^i t^{ij} e_i \otimes e_j. \end{aligned}$$

The antipode turns out to be the regular antipode for $\xi \in \mathfrak{g}$ and

$$S(\Delta) = -\Delta - \nu.$$

Finally, let us introduce an additional filtration on $\mathcal{H}^{\mathfrak{g}}$, which is the usual filtration on $U\mathfrak{g}$ and where Δ increases filtration degree by 2.

On a supermanifold M with \mathfrak{g} -action, Δ should correspond to a quasi-BV Delta:

Proposition 18. The above definition makes $\mathcal{H}^{\mathfrak{g}}$ into a Hopf algebra.

A \mathfrak{g} -quasi-BV structure on M is equivalently given by an algebra map $\mathcal{H}^{\mathfrak{g}} \rightarrow \text{DiffOp}(M)$ which preserves parity, filtration and sends ξ and Δ to operators annihilating the constant function $1 \in \mathcal{O}(M)$.

We will denote the image of Δ by Δ again.

Proof. This means that elements of \mathfrak{g} are mapped to vector fields and Δ is mapped to an odd second-order differential operator, such that M has an action of \mathfrak{g} and \mathfrak{g} -invariant operator Δ whose square is given by the action by $\phi \in U\mathfrak{g}$. \square

C.1 Fusion

We will now introduce two algebra maps between these Hopf algebras. For moduli spaces of flat connections, these maps will be

Definition 19. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_{1,2}$ be odd metric Lie superalgebras. Let

$$\square: \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}} \rightarrow \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}}$$

be the algebra map defined by sending Δ to $\Delta + (-1)^{e_i} t_{\mathfrak{g}}^{ij} e_i \otimes e_j$, where the second term is an element of $\mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g})^{\otimes 2} \cong U(\mathfrak{g} \oplus \mathfrak{g})$, and by sending an element $\xi \in \mathfrak{g}$ to $(\xi, \xi) \in \mathfrak{g} \oplus \mathfrak{g}$.

Let

$$i: \mathcal{H}^{\mathfrak{h}_1 \oplus \mathfrak{h}_2} \rightarrow \mathcal{H}^{\mathfrak{h}_1} \otimes \mathcal{H}^{\mathfrak{h}_2}$$

be the algebra map defined by sending Δ to $\Delta \otimes 1 + 1 \otimes \Delta$ and by sending $\eta_1 \in \mathfrak{h}_1$ to $\eta_1 \otimes 1$ and $\eta_2 \in \mathfrak{h}_2$ to $1 \otimes \eta_2$.

A fusion at \mathfrak{g} of a $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$ -quasi-BV manifold M is defined by the composition

$$\mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}} \xrightarrow{\square} \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \rightarrow \text{DiffOp}(M).$$

A fusion at \mathfrak{g} of a $\mathfrak{g} \oplus \mathfrak{h}_1$ -quasi-BV manifold and $\mathfrak{g} \oplus \mathfrak{h}_2$ -quasi-BV manifold is defined by the composition

$$\mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2} \xrightarrow{i \circ \square} \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}_1} \otimes \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}_2} \rightarrow \text{DiffOp}(M) \otimes \text{DiffOp}(N) \rightarrow \text{DiffOp}(M \times N).$$

Proposition 19. The maps \square and i are well defined. The fusion is associative in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}} & \xrightarrow{\square} & \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \\ \downarrow \square & & \downarrow \square_2 \\ \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} & \xrightarrow{\square_1} & \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \end{array}$$

where in $\square_{1,2}: \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \rightarrow \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}}$, the index specifies on which \mathfrak{g} we act.

Proof. For the maps \square and i , the only nontrivial identity $(\square(\Delta))^2 = \square(\Delta^2)$. This is a calculation analogous to the one in the proof of Proposition 18

Both legs of the associativity diagram act as a triple coproduct on \mathfrak{g} , identity on \mathfrak{h} and on Δ , they give

$$\begin{aligned} \Delta \mapsto & \Delta \otimes 1 \otimes 1 + 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta \\ & + (-1)^{|e_i|} t_{\mathfrak{g}}^{ij} (e_i \otimes e_j \otimes 1 + e_i \otimes 1 \otimes e_j + 1 \otimes e_i \otimes e_j). \end{aligned}$$

□

Remark 14. For $\mathfrak{h}_1 = \mathfrak{h}_2 = 0$, the composition $i \circ \square$ is the coproduct $\tilde{\square}$ on $\mathcal{H}^{\mathfrak{g}}$.

Finally, we give a topological interpretation to the maps \square and i .

Proposition 20. Let $\mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \rightarrow \text{DiffOp}(M_{\Sigma, \{p, p', \dots\}}(G))$ be the quasi-BV structure from Theorem 12, with the two \mathfrak{g} actions corresponding to the two points p, p' . Then the $\mathfrak{g} \oplus \mathfrak{h}$ -quasi-BV structure on the surface given by fusion of p, p' into p'' is given by the composition

$$\mathcal{H}^{\mathfrak{g} \oplus \mathfrak{h}} \xrightarrow{\square} \mathcal{H}^{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} \rightarrow \text{DiffOp}(M_{\Sigma, \{p'', \dots\}}(G)) \cong \text{DiffOp}(M_{\Sigma, \{p, p', \dots\}}(G)).$$

Similarly, if Σ_1, V_1 and Σ_2, V_2 are two surfaces with corresponding $\mathfrak{g}^{V_1, 2}$ -quasi-BV structures, then the quasi-BV structure on $(\Sigma_1 \sqcup \Sigma_2, V_1 \sqcup V_2)$ is given by the composition

$$\begin{aligned} \mathcal{H}^{\mathfrak{g}^{V_1} \oplus \mathfrak{g}^{V_2}} & \xrightarrow{i} \mathcal{H}^{\mathfrak{g}^{V_1}} \otimes \mathcal{H}^{\mathfrak{g}^{V_2}} \\ & \rightarrow \text{DiffOp}(M_{\Sigma_1, V_1}(G) \times M_{\Sigma_2, V_2}(G)) \cong \text{DiffOp}(M_{\Sigma_1 \sqcup \Sigma_2, V_1 \sqcup V_2}(G)). \end{aligned}$$

Proof. The additional term from Proposition 4 comes from $\square(\Delta) = \Delta + (-1)^{|e_i|} t_{\mathfrak{g}}^{ij} e_i \otimes e_j$. For the disjoint union of surfaces, the quasi-BV operators are simply added together. □

Bibliography

- [1] Anton Alekseev and Yvette Kosmann-Schwarzbach. “Manin Pairs and Moment Maps”. *Journal of Differential Geometry* 56.1 (Sept. 2000). ISSN: 0022-040X. DOI: [10.4310/jdg/1090347528](https://doi.org/10.4310/jdg/1090347528). arXiv: [math/9909176](https://arxiv.org/abs/math/9909176) [math.DG].
- [2] Anton Alekseev, Yvette Kosmann-Schwarzbach, and Eckhard Meinrenken. “Quasi-Poisson Manifolds”. *Canadian Journal of Mathematics* 54.1 (Feb. 2002), pp. 3–29. ISSN: 0008-414X, 1496-4279. DOI: [10.4153/CJM-2002-001-5](https://doi.org/10.4153/CJM-2002-001-5). arXiv: [math/0006168](https://arxiv.org/abs/math/0006168) [math.DG].

- [3] Jørgen E. Andersen, Josef Mattes, and Nicolai Reshetikhin. “The poisson structure on the moduli space of flat connections and chord diagrams”. *Topology* 35.4 (Oct. 1996), pp. 1069–1083. ISSN: 0040-9383. DOI: [10.1016/0040-9383\(95\)00059-3](https://doi.org/10.1016/0040-9383(95)00059-3).
- [4] Michael F. Atiyah and Raoul Bott. “The Yang-Mills Equations over Riemann Surfaces”. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 308.1505 (1983), pp. 523–615. ISSN: 0080-4614. URL: <https://www.jstor.org/stable/37156>.
- [5] Alex J. Bene. “A Chord Diagrammatic Presentation of the Mapping Class Group of a Once Bordered Surface”. *Geometriae Dedicata* 144.1 (June 2009), pp. 171–190. ISSN: 1572-9168. DOI: [10.1007/s10711-009-9395-9](https://doi.org/10.1007/s10711-009-9395-9). arXiv: [0802.2747](https://arxiv.org/abs/0802.2747) [math.GT].
- [6] Allan Berele. “Invariant theory for matrices over the Grassmann algebra”. *Advances in Mathematics* 237 (Apr. 2013), pp. 33–61. ISSN: 0001-8708. DOI: [10.1016/j.aim.2012.12.021](https://doi.org/10.1016/j.aim.2012.12.021).
- [7] David Li-Bland and Pavol Ševera. “Moduli Spaces for Quilted Surfaces and Poisson Structures”. *Documenta Mathematica* 20 (2015), pp. 1071–1135. arXiv: [1212.2097](https://arxiv.org/abs/1212.2097) [math.DG].
- [8] Ricardo Campos, Sergei Merkulov, and Thomas Willwacher. “The Frobenius properad is Koszul”. *Duke Mathematical Journal* 165.15 (Oct. 2016). ISSN: 0012-7094. DOI: [10.1215/00127094-3645116](https://doi.org/10.1215/00127094-3645116). arXiv: [1402.4048](https://arxiv.org/abs/1402.4048) [math.QA].
- [9] Moira Chas. “Combinatorial Lie bialgebras of curves on surfaces”. *Topology* 43.3 (May 2004), pp. 543–568. ISSN: 0040-9383. DOI: [10.1016/j.top.2003.03.003](https://doi.org/10.1016/j.top.2003.03.003). arXiv: [math/0105178](https://arxiv.org/abs/math/0105178) [math.GT].
- [10] Pierre Deligne et al., eds. *Quantum fields and strings: A course for mathematicians. Vol. 1, 2*. 1999. ISBN: 978-0-8218-2012-4.
- [11] Pavel Etingof. “Casimirs of the Goldman Lie algebra of a closed surface”. *International Mathematics Research Notices* (Jan. 2006). ISSN: 1687-0247. DOI: [10.1155/imrn/2006/24894](https://doi.org/10.1155/imrn/2006/24894). arXiv: [math/0407505](https://arxiv.org/abs/math/0407505) [math.QA]. URL: <http://dx.doi.org/10.1155/IMRN/2006/24894>.
- [12] Vladimir V. Fock and Alexei A. Rosly. “Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix”. *Translations of the American Mathematical Society-Series 2* 191 (Feb. 1999). arXiv: [math/9802054](https://arxiv.org/abs/math/9802054), pp. 67–86.

- [13] Benoit Fresse. *Homotopy of Operads and Grothendieck-Teichmüller Groups*. Mathematical Surveys and Monographs. American Mathematical Soc., May 2017. ISBN: 9781470434823.
- [14] William M. Goldman. “Invariant functions on Lie groups and Hamiltonian flows of surface group representations”. *Inventiones mathematicae* 85.2 (June 1986), pp. 263–302. ISSN: 1432-1297. DOI: [10.1007/BF01389091](https://doi.org/10.1007/BF01389091).
- [15] Adele Jackson. “Presentations of the Mapping Class Group”. MA thesis. Australian National University, 2018. URL: <https://tqft.net/web/research/students/AdeleJackson/thesis.pdf>.
- [16] Sean Lawton. “Poisson geometry of $SL(3, \mathbb{C})$ -character varieties relative to a surface with boundary”. *Transactions of the American Mathematical Society* 361.5 (Dec. 2008), pp. 2397–2429. ISSN: 1088-6850. DOI: [10.1090/s0002-9947-08-04777-6](https://doi.org/10.1090/s0002-9947-08-04777-6).
- [17] Dimitry Leites et al. *Seminar on supersymmetry. Vol. 1. Algebra and Calculus: Main chapters*. Moscow Center of Continuous Mathematical Education (MCCME), 2011. URL: <https://staff.math.su.se/mleites/books/2011-sos1.pdf>.
- [18] Gwénaél Massuyeau and Vladimir G. Turaev. “Quasi-Poisson Structures on Representation Spaces of Surfaces”. *International Mathematics Research Notices* 2014.1 (Oct. 2012), pp. 1–64. ISSN: 1073-7928. DOI: [10.1093/imrn/rns215](https://doi.org/10.1093/imrn/rns215). arXiv: [1205.4898](https://arxiv.org/abs/1205.4898) [math.GT].
- [19] Xin Nie. “The quasi-Poisson Goldman formula”. *Journal of Geometry and Physics* 74 (Dec. 2013), pp. 1–17. ISSN: 0393-0440. DOI: [10.1016/j.geomphys.2013.06.010](https://doi.org/10.1016/j.geomphys.2013.06.010). arXiv: [1301.5231](https://arxiv.org/abs/1301.5231) [math.DG].
- [20] Robert C. Penner. *Decorated Teichmüller theory*. Vol. 1. The QGM Master Class Series. European Mathematical Society, 2012. ISBN: 978-3-03719-075-3.
- [21] Claudio Procesi. “The invariant theory of $n \times n$ matrices”. *Advances in Mathematics* 19.3 (1976), pp. 306–381. ISSN: 0001-8708. DOI: [https://doi.org/10.1016/0001-8708\(76\)90027-X](https://doi.org/10.1016/0001-8708(76)90027-X).
- [22] Alexander Sergeev. “An analog of the classical invariant theory for Lie superalgebras. I”. *Michigan Mathematical Journal* 49.1 (June 2001). ISSN: 0026-2285. DOI: [10.1307/mmj/1008719038](https://doi.org/10.1307/mmj/1008719038). arXiv: [math/9810113](https://arxiv.org/abs/math/9810113) [math.RT].

- [23] Alexander Sergeev. “The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{G}(n, m)$ and $Q(n)$ ”. English. *Math. USSR, Sb.* 51 (1985), pp. 419–427. ISSN: 0025-5734. DOI: [10.1070/SM1985v051n02ABEH002867](https://doi.org/10.1070/SM1985v051n02ABEH002867).
- [24] Vladimir Shander. *Invariant functions on supermatrices*. 1998. arXiv: [math/9810112](https://arxiv.org/abs/math/9810112) [[math.RT](#)].
- [25] Vladimir G. Turaev. “Skein quantization of Poisson algebras of loops on surfaces”. *Annales scientifiques de l’École Normale Supérieure* 24.6 (1991), pp. 635–704. DOI: [10.24033/asens.1639](https://doi.org/10.24033/asens.1639).
- [26] Veeravalli S. Varadarajan. *Supersymmetry for Mathematicians: An Introduction*. Vol. 11. Courant Lecture Notes. American Mathematical Society, July 2004. ISBN: 9780821835746. DOI: [10.1090/cln/011](https://doi.org/10.1090/cln/011).