

Dynamic programming principle for stochastic optimal control problem under degenerate G -expectation

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October 19, 2022

Abstract. In this paper, we study a stochastic optimal control problem under degenerate G -expectation. By using implied partition method, we show that the approximation result for admissible controls still hold. Based on this result, we prove that the value function is deterministic, and obtain the dynamic programming principle. Furthermore, we prove that the value function is the unique viscosity solution to the related HJB equation under degenerate case.

Key words. G -expectation, Dynamic programming principle, Hamilton-Jacobi-Bellman equation, Stochastic optimal control

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

It is well-known that many economic and financial problems involve volatility uncertainty (see [7, 8]), which is characterized by a family of nondominated probability measures. In this case, this kind of problems cannot be modeled within a probability space framework. So we need a new framework to deal with it. Motivated by the study of this problem, Peng [20, 21] established the theory of G -expectation $\hat{\mathbb{E}}[\cdot]$. The G -Brownian motion $B = (B^1, \dots, B^d)^T$ and Itô's integral with respect to B were constructed. Moreover, the theory of stochastic differential equation driven by G -Brownian motion (G -SDE) has been established.

Stochastic optimal control problems have important applications in economy and finance, such as the utility maximization problems in finance. The dynamic programming principle (DPP), originated by Bellman in the 1950s, is a powerful tool to solve stochastic optimal control problems. Under the probability space framework, DPP and the related Hamilton-Jacobi-Bellman (HJB) equation have been intensively studied by a lot of researchers for various kinds of stochastic optimal control problems (see books [19, 26] and the references therein).

Under the G -expectation framework, Hu and Ji [11] first investigated the stochastic recursive optimal control problem under non-degenerate G , and obtained the related DPP and HJB equation. For the application of DPP and HJB, Fouque et al. [9] first studied the portfolio selection with ambiguous correlation and

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stochastic volatilities (see [10, 24] for further research results). In addition, for the different formulation and method in studying problems with volatility uncertainty in finance, we refer the readers to [5, 6, 18, 23, 25] and the references therein.

In this paper, we study the following stochastic control system under degenerate G :

$$\begin{cases} dX_s^{t,x,u} = b(s, X_s^{t,x,u}, u_s)ds + h_{ij}(s, X_s^{t,x,u}, u_s)d\langle B^i, B^j \rangle_s + \sigma(s, X_s^{t,x,u}, u_s)dB_s, \\ X_t^{t,x,u} = x, \end{cases} \quad (1.1)$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$, the control domain U is a given nonempty compact set of \mathbb{R}^m , and the set of admissible controls $(u_s)_{s \in [t, T]}$ is denoted by $\mathcal{U}[t, T] = M_G^2(t, T; U)$. The value function is defined by

$$V(t, x) := \operatorname{ess\,inf}_{u \in \mathcal{U}[t, T]} \widehat{\mathbb{E}}_t \left[\Phi(X_T^{t,x,u}) + \int_t^T f(s, X_s^{t,x,u}, u_s)ds + \int_t^T g_{ij}(s, X_s^{t,x,u}, u_s)d\langle B^i, B^j \rangle_s \right]. \quad (1.2)$$

Under the non-degenerate G , Hu, Wang and Zheng [16] proved that $I_{[c, c']}(B_t^i) \in L_G^2(\Omega_t)$ for each $c < c'$ and $i \leq d$. Based on this result, Hu and Ji [11] showed that $\mathcal{U}[t, T]$ contains enough simple and non-trivial admissible controls. Furthermore, this kind of controls is dense in $\mathcal{U}[t, T]$ under the norm in $M_G^2(t, T; \mathbb{R}^m)$ (see Lemma 13 in [11]), which is the key point to prove that the value function $V(\cdot, \cdot)$ satisfies the DPP. But under the degenerate G , if B^i is degenerate, then $I_{[c, c']}(B_t^i) \notin L_G^2(\Omega_t)$ for each $c < c'$ and $t > 0$ (see Theorem 4.1 in [17]), which is completely different from the non-degenerate case. Therefore, a natural question is whether the above control problem (1.1)-(1.2) under degenerate G is well-posed.

In order to overcome this difficulty, we need to assume that there exists a non-degenerate B^{i^*} . By using implied partition method which was proposed in [11] to find optimal control, we obtain that $u \in \mathcal{U}[t, T]$ can be approximated by a sequence of $u^k \in \mathcal{U}[t, T]$, $k \geq 1$, under the norm in $M_G^2(t, T; \mathbb{R}^m)$ (see Lemma 4.4). Based on this result, we prove that the value function $V(\cdot, \cdot)$ is deterministic and satisfies the DPP. Furthermore, we show that $V(\cdot, \cdot)$ is the unique viscosity solution to the related HJB equation under degenerate case.

This paper is organized as follows. In Section 2, we recall some basic notions and results of G -expectation. The formulation of our stochastic optimal control problem under degenerate G is given in Section 3. In Section 4, we prove that the value function $V(\cdot, \cdot)$ is deterministic, and obtain the DPP. In Section 5, we show that $V(\cdot, \cdot)$ is the unique viscosity solution to the related second-order fully nonlinear HJB equation under degenerate case.

2 Preliminaries

We recall some basic notions and results of G -expectation. The readers may refer to [12, 13, 22] for more details.

Let $T > 0$ be fixed and let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ be the space of \mathbb{R}^d -valued continuous functions on $[0, T]$ with $\omega_0 = 0$. The canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0, T]$. For each given $t \in [0, T]$, set

$$\operatorname{Lip}(\Omega_t) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}}) : N \geq 1, t_1 < \dots < t_N \leq t, \varphi \in C_{b, \operatorname{Lip}}(\mathbb{R}^{d \times N})\},$$

where $C_{b, \operatorname{Lip}}(\mathbb{R}^{d \times N})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d \times N}$. It is easy to verify that

$$\operatorname{Lip}(\Omega_t) := \{\phi(B_{t_1}, B_{t_2}, \dots, B_{t_N}) : N \geq 1, t_1 < \dots < t_N \leq t, \phi \in C_{b, \operatorname{Lip}}(\mathbb{R}^{d \times N})\}.$$

Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function, where \mathbb{S}_d denotes the set of $d \times d$ symmetric matrices. Then there exists a bounded and convex set $\Sigma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \text{tr}[A\gamma] \text{ for } A \in \mathbb{S}_d,$$

where \mathbb{S}_d^+ denotes the set of $d \times d$ nonnegative matrices. If there exists a $\delta > 0$ such that $\gamma \geq \delta I_d$ for any $\gamma \in \Sigma$, G is called non-degenerate. Otherwise, G is called degenerate. In particular, if $d = 1$, then

$$G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-) \text{ for } a \in \mathbb{R},$$

where $\bar{\sigma}^2 = \sup \Sigma$ and $\underline{\sigma}^2 = \inf \Sigma \geq 0$. Under this case, G is degenerate iff $\underline{\sigma}^2 = 0$.

Peng [20, 21] constructed the G -expectation $\hat{\mathbb{E}} : Lip(\Omega_T) \rightarrow \mathbb{R}$ and the conditional G -expectation $\hat{\mathbb{E}}_t : Lip(\Omega_T) \rightarrow Lip(\Omega_t)$ as follows:

- (i) For $s \leq t \leq T$ and $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, define $\hat{\mathbb{E}}[\varphi(B_t - B_s)] = u(t - s, 0)$, where u is the viscosity solution (see [3]) of the following G -heat equation:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x).$$

- (ii) For $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}}) \in Lip(\Omega_T)$, define

$$\hat{\mathbb{E}}_{t_i}[X] = \varphi_i(B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}) \text{ for } i = N - 1, \dots, 1 \text{ and } \hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi_1(B_{t_1})],$$

where $\varphi_{N-1}(x_1, \dots, x_{N-1}) := \hat{\mathbb{E}}[\varphi(x_1, \dots, x_{N-1}, B_{t_N} - B_{t_{N-1}})]$ and

$$\varphi_i(x_1, \dots, x_i) := \hat{\mathbb{E}}[\varphi_{i+1}(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i})] \text{ for } i = N - 2, \dots, 1.$$

The space $(\Omega_T, Lip(\Omega_T), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t \in [0, T]})$ is a consistent sublinear expectation space, where $\hat{\mathbb{E}}_0 = \hat{\mathbb{E}}$. The canonical process $(B_t)_{t \in [0, T]}$ is called the G -Brownian motion under $\hat{\mathbb{E}}$.

For each $t \in [0, T]$, denote by $L_G^p(\Omega_t)$ the completion of $Lip(\Omega_t)$ under the norm $\|X\|_{L_G^p} := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$. $\hat{\mathbb{E}}_t$ can be continuously extended to $L_G^1(\Omega_T)$ under the norm $\|\cdot\|_{L_G^1}$.

Theorem 2.1 ([4, 15]) *There exists a weakly compact set of probability measures \mathcal{P} on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for all } X \in L_G^1(\Omega_T).$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

For this \mathcal{P} , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for } A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds ‘‘quasi-surely’’ (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s.

Definition 2.2 Let $M_G^0(0, T)$ be the space of simple processes in the following form: for each $N \in \mathbb{N}$ and $0 = t_0 < \dots < t_N = T$,

$$\eta_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(t),$$

where $\xi_i \in Lip(\Omega_{t_i})$ for $i = 0, 1, \dots, N-1$.

Denote by $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norm $\|\eta\|_{M_G^p} := \left(\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] \right)^{1/p}$ for $p \geq 1$. For each $\eta^i \in M_G^2(0, T)$, $i = 1, \dots, d$, denote $\eta = (\eta^1, \dots, \eta^d)^T \in M_G^2(0, T; \mathbb{R}^d)$, the G -Itô integral $\int_0^T \eta_t^T dB_t$ is well defined.

3 Formulation of the control problem

Let U be a given nonempty compact set of \mathbb{R}^m . For each $t \in [0, T]$, we denote by

$$\mathcal{U}[t, T] := \{u : u \in M_G^2(t, T; \mathbb{R}^m) \text{ with values in } U\}$$

the set of admissible controls on $[t, T]$.

In the following, we use Einstein summation convention. For each given $t \in [0, T]$, $\xi \in L_G^2(\Omega_t; \mathbb{R}^n) = \{(\xi_1, \dots, \xi_n)^T : \xi_i \in L_G^2(\Omega_t), i \leq n\}$ and $u \in \mathcal{U}[t, T]$, we consider the following G -SDE:

$$\begin{cases} dX_s^{t, \xi, u} = b(s, X_s^{t, \xi, u}, u_s) ds + h_{ij}(s, X_s^{t, \xi, u}, u_s) d\langle B^i, B^j \rangle_s + \sigma(s, X_s^{t, \xi, u}, u_s) dB_s, \\ X_t^{t, \xi, u} = \xi, \end{cases} \quad (3.1)$$

where $s \in [t, T]$, $\langle B \rangle = (\langle B^i, B^j \rangle)_{i, j=1}^d$ is the quadratic variation of B . The cost function is defined by

$$J(t, \xi, u) = \hat{\mathbb{E}}_t \left[\Phi(X_T^{t, \xi, u}) + \int_t^T f(s, X_s^{t, \xi, u}, u_s) ds + \int_t^T g_{ij}(s, X_s^{t, \xi, u}, u_s) d\langle B^i, B^j \rangle_s \right]. \quad (3.2)$$

Suppose that $b, h_{ij} : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$, $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$, $f, g_{ij} : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following conditions:

(H1) There exists a constant $L > 0$ such that for any $(s, x, v), (s, x', v') \in [0, T] \times \mathbb{R}^n \times U$,

$$\begin{aligned} & |b(s, x, v) - b(s, x', v')| + |h_{ij}(s, x, v) - h_{ij}(s, x', v')| + |\sigma(s, x, v) - \sigma(s, x', v')| \\ & \leq L(|x - x'| + |v - v'|), \\ & |f(s, x, v) - f(s, x', v')| + |g_{ij}(s, x, v) - g_{ij}(s, x', v')| + |\Phi(x) - \Phi(x')| \\ & \leq L[(1 + |x| + |x'|)|x - x'| + |v - v'|]; \end{aligned}$$

(H2) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$; $b, h_{ij}, \sigma, f, g_{ij}$ are continuous in s .

We have the following theorems.

Theorem 3.1 ([22]) *Let Assumptions (H1) and (H2) hold. Then, for each $\xi \in L_G^2(\Omega_t; \mathbb{R}^n)$ and $u \in \mathcal{U}[t, T]$, there exists a unique solution $X \in M_G^2(t, T; \mathbb{R}^n)$ for the G-SDE (3.1).*

Theorem 3.2 ([11, 22]) *Let Assumptions (H1) and (H2) hold, and let $\xi, \xi' \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \geq 2$ and $u, v \in \mathcal{U}[t, T]$. Then, for each $\delta \in [0, T - t]$, we have*

$$\begin{aligned} \hat{\mathbb{E}}_t[|X_{t+\delta}^{t, \xi, u} - X_{t+\delta}^{t, \xi', v}|^2] &\leq C(|\xi - \xi'|^2 + \hat{\mathbb{E}}_t[\int_t^{t+\delta} |u_s - v_s|^2 ds]), \\ \hat{\mathbb{E}}_t[|X_{t+\delta}^{t, \xi, u}|^p] &\leq C(1 + |\xi|^p), \\ \hat{\mathbb{E}}_t \left[\sup_{s \in [t, t+\delta]} |X_s^{t, \xi, u} - \xi|^p \right] &\leq C(1 + |\xi|^p) \delta^{p/2}, \end{aligned}$$

where $C > 0$ depends on $T, \bar{\sigma}^2 = \sup\{|\gamma| : \gamma \in \Sigma\}, p$ and L .

Our stochastic optimal control problem is to find $u \in \mathcal{U}[t, T]$ which minimizes the cost function $J(t, x, u)$ for each given $t \in [0, T]$ and $x \in \mathbb{R}^n$. For this purpose, we need the following definition of essential infimum of $\{J(t, x, u) : u \in \mathcal{U}[t, T]\}$.

Definition 3.3 ([11]) *The essential infimum of $\{J(t, x, u) : u \in \mathcal{U}[t, T]\}$, denoted by $\operatorname{ess\,inf}_{u \in \mathcal{U}[t, T]} J(t, x, u)$, is a random variable $\zeta \in L_G^2(\Omega_t)$ satisfying:*

- (i) *for any $u \in \mathcal{U}[t, T]$, $\zeta \leq J(t, x, u)$ q.s.;*
- (ii) *if η is a random variable satisfying $\eta \leq J(t, x, u)$ q.s. for any $u \in \mathcal{U}[t, T]$, then $\zeta \geq \eta$ q.s.*

For each $(t, x) \in [0, T] \times \mathbb{R}^n$, we define the value function

$$V(t, x) := \operatorname{ess\,inf}_{u \in \mathcal{U}[t, T]} J(t, x, u). \quad (3.3)$$

In this paper, we consider the degenerate d -dimensional G -Brownian motion $B = (B^1, \dots, B^d)^T$ with $d \geq 2$. For each given $\beta \in \mathbb{R}^d$, by Proposition 3.1.5 in [22], we know that $B^\beta := \beta^T B$ is a 1-dimensional G_β -Brownian motion with

$$G_\beta(a) = G(\beta\beta^T)a^+ + G(-\beta\beta^T)a^- \text{ for } a \in \mathbb{R}. \quad (3.4)$$

In particular, B^i is a 1-dimensional G_i -Brownian motion with

$$G_i(a) = \frac{1}{2}(\bar{\sigma}_i^2 a^+ - \underline{\sigma}_i^2 a^-) \text{ for } a \in \mathbb{R},$$

where $\bar{\sigma}_i^2 = 2G(e_i e_i^T)$, $\underline{\sigma}_i^2 = -2G(-e_i e_i^T)$, $\{e_i : i \leq d\}$ is the standard basis of \mathbb{R}^d . If $\underline{\sigma}_i^2 = 0$ for any $i \leq d$, then $I_{[a, b]}(B_i^i) \notin L_G^1(\Omega_t)$ for any $i \leq d$ and non-empty interval $[a, b]$ by Theorem 4.1 in [17]. Under the case $U = \{0, 1\}$, $\mathcal{U}[t, T]$ contains only deterministic controls, which causes our control problem (3.3) to be ill-posed. Thus we need the following assumption:

(H3) *There exists an $i^* \leq d$ such that $\underline{\sigma}_{i^*}^2 = -2G(-e_{i^*} e_{i^*}^T) = \inf_{\gamma \in \Sigma} \{\gamma_{i^* i^*}\} > 0$, where $\gamma = (\gamma_{ij})_{i, j=1}^d$.*

Remark 3.4 The assumption (H3) implies that B^{i^*} is a 1-dimensional non-degenerate G_{i^*} -Brownian motion. It is clear that

$$G((a_{ij})_{i,j=1}^d) = \frac{1}{2} \sum_{i=1}^{d-1} \bar{\sigma}_i^2 a_{ii}^+ + \frac{1}{2} (\bar{\sigma}_d^2 a_{dd}^+ - \underline{\sigma}_d^2 a_{dd}^-)$$

satisfies (H3), where $\bar{\sigma}_i^2 > 0$ for $i \leq d-1$ and $\bar{\sigma}_d^2 \geq \underline{\sigma}_d^2 > 0$, and B^d is a 1-dimensional non-degenerate G_d -Brownian motion.

In the following we will prove that $V(\cdot, \cdot)$ is deterministic. Furthermore, we will obtain the dynamic programming principle and the related fully nonlinear HJB equation under degenerate case.

4 Dynamic programming principle

We use the following notations: for each given $0 \leq t \leq s \leq T$,

$$Lip(\Omega_s^t) := \{\varphi(B_{t_1} - B_t, \dots, B_{t_N} - B_t) : N \geq 1, t_1, \dots, t_N \in [t, s], \varphi \in Cb.Lip(\mathbb{R}^{d \times N})\};$$

$$L_G^p(\Omega_s^t) := \{\text{the completion of } Lip(\Omega_s^t) \text{ under the norm } \|\cdot\|_{L_G^p}, p \geq 1\};$$

$$M_G^{0,t}(t, T) := \{\eta_s = \sum_{k=0}^{N-1} \xi_k I_{[t_k, t_{k+1})}(s) : t = t_0 < \dots < t_N = T, \xi_k \in Lip(\Omega_{t_k}^t)\};$$

$$M_G^p(t, T) := \{\text{the completion of } M_G^{0,t}(t, T) \text{ under the norm } \|\cdot\|_{M_G^p}, p \geq 1\};$$

$$\mathcal{U}^t[t, T] := \{u : u \in M_G^{2,t}(t, T; \mathbb{R}^m) \text{ with values in } U\};$$

$$\mathbb{U}[t, T] := \{u = \sum_{k=1}^N I_{A_k} u^k : N \geq 1, u^k \in \mathcal{U}^t[t, T], I_{A_k} \in L_G^2(\Omega_t), (A_k)_{k=1}^N \text{ is a partition of } \Omega\}.$$

For simplicity, the constant C will change from line to line in the following. In order to prove that $V(\cdot, \cdot)$ is deterministic, we need the following lemmas.

Lemma 4.1 Let Assumption (H3) hold. Then there exists a constant $\lambda > 0$ such that $B^i + \lambda B^{i^*}$ is non-degenerate for $i \leq d$ and $i \neq i^*$.

Proof. By (3.4), $B^i + \lambda B^{i^*}$ is non-degenerate if and only if

$$-2G(-(e_i + \lambda e_{i^*})(e_i + \lambda e_{i^*})^T) = \inf_{\gamma \in \Sigma} \{\gamma_{ii} + 2\lambda \gamma_{ii^*} + \lambda^2 \gamma_{i^*i^*}\} > 0.$$

Since Σ is bounded, we know $\alpha := \sup_{\gamma \in \Sigma} |\gamma| < \infty$. Taking $\lambda = (2\alpha + 1)(\underline{\sigma}_{i^*}^2)^{-1}$ and noting that $\gamma_{ii} \geq 0$, we obtain

$$\inf_{\gamma \in \Sigma} \{\gamma_{ii} + 2\lambda \gamma_{ii^*} + \lambda^2 \gamma_{i^*i^*}\} \geq \lambda^2 \underline{\sigma}_{i^*}^2 - 2\lambda \alpha = \lambda > 0.$$

Thus $B^i + \lambda B^{i^*}$ is non-degenerate for each $i \neq i^*$. \square

Lemma 4.2 Let Assumption (H3) hold and let $\xi \in L_G^2(\Omega_s)$ with fixed $s \in [t, T]$. Then there exists a sequence $\xi^k = \sum_{j=1}^{N_k} \sum_{l=1}^{\bar{N}_k} x_{jl}^k I_{A_j^k} I_{\bar{A}_l^k}$, $k \geq 1$, such that

$$\lim_{k \rightarrow \infty} \hat{\mathbb{E}} [|\xi - \xi^k|^2] = 0,$$

where $x_{jl}^k \in \mathbb{R}$, $I_{A_j^k} \in L_G^2(\Omega_t)$, $I_{\bar{A}_l^k} \in L_G^2(\Omega_s^t)$, $j \leq N_k$, $l \leq \bar{N}_k$, $k \geq 1$, $(A_j^k)_{j=1}^{N_k}$ is a $\mathcal{B}(\Omega_t)$ -partition of Ω , and $(\bar{A}_l^k)_{l=1}^{\bar{N}_k}$ is a $\mathcal{B}(\Omega_s^t)$ -partition of Ω .

Proof. Since $L_G^2(\Omega_s)$ is the completion of $Lip(\Omega_s)$ under the norm $\|\cdot\|_{L_G^2}$, we only need to prove the case

$$\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}}), \quad (4.1)$$

where $N \geq 1$, $0 < t_1 < \dots < t_N \leq s$, $t_i = t$ for some $i \leq N$, $\varphi \in C_{b,Lip}(\mathbb{R}^{d \times N})$. In the following, we only prove the case

$$\xi = \varphi(B_t, B_s - B_t)$$

for simplicity. The proof for (4.1) is similar.

By Lemma 4.1, there exists a constant $\lambda > 0$ such that $B^i + \lambda B^{i*}$ is non-degenerate for each $i \neq i^*$. Set

$$B_r^\lambda = (B_r^1 + \lambda B_r^{i^*}, \dots, B_r^{i^*-1} + \lambda B_r^{i^*}, B_r^{i^*}, B_r^{i^*+1} + \lambda B_r^{i^*}, \dots, B_r^d + \lambda B_r^{i^*})^T.$$

It follows from Theorem 3.20 in [16] that

$$I_{\{B_t^\lambda \in [c, c']\}} \in L_G^2(\Omega_t) \text{ and } I_{\{B_s^\lambda - B_t^\lambda \in [c, c']\}} \in L_G^2(\Omega_s^t) \quad (4.2)$$

for any $c = (c_1, \dots, c_d)^T$, $\bar{c} = (\bar{c}_1, \dots, \bar{c}_d)^T \in \mathbb{R}^d$ with $c < c'$. For each $k \geq 1$, it is easy to find a finite number of disjoint intervals $[c^{j,k}, \bar{c}^{j,k}]$, $j = 1, \dots, N_k - 1$, such that $|\bar{c}^{j,k} - c^{j,k}| < k^{-1}$ and $[-ke, ke] = \cup_{j \leq N_k - 1} [c^{j,k}, \bar{c}^{j,k}]$ with $e = [1, \dots, 1]^T \in \mathbb{R}^d$. Define

$$A_j^k = \{B_t^\lambda \in [c^{j,k}, \bar{c}^{j,k}]\}, \bar{A}_j^k = \{B_s^\lambda - B_t^\lambda \in [c^{j,k}, \bar{c}^{j,k}]\} \text{ for } j \leq N_k - 1,$$

and $A_{N_k}^k = \Omega \setminus \cup_{j \leq N_k - 1} A_j^k$, $\bar{A}_{N_k}^k = \Omega \setminus \cup_{j \leq N_k - 1} \bar{A}_j^k$. It is easy to verify that $(A_j^k)_{j=1}^{N_k}$ is a $\mathcal{B}(\Omega_t)$ -partition of Ω , and $(\bar{A}_j^k)_{j=1}^{N_k}$ is a $\mathcal{B}(\Omega_s^t)$ -partition of Ω . By (4.1), we know that $I_{A_j^k} \in L_G^2(\Omega_t)$ and $I_{\bar{A}_j^k} \in L_G^2(\Omega_s^t)$ for $j \leq N_k$. Set

$$\xi^k = \sum_{j=1}^{N_k} \sum_{l=1}^{N_k} \varphi(\tilde{c}^{j,k}, \tilde{c}^{l,k}) I_{A_j^k} I_{\bar{A}_l^k}, \quad (4.3)$$

where $\tilde{c}^{j,k} = (c_1^{j,k} - \lambda c_{i^*}^{j,k}, \dots, c_{i^*-1}^{j,k} - \lambda c_{i^*}^{j,k}, c_{i^*}^{j,k}, c_{i^*+1}^{j,k} - \lambda c_{i^*}^{j,k}, \dots, c_d^{j,k} - \lambda c_{i^*}^{j,k})^T$ for $j \leq N_k - 1$ and $\tilde{c}^{N_k,k} = 0$. For $j, l \leq N_k - 1$, one can check that

$$\begin{aligned} |\xi - \xi^k| I_{A_j^k} I_{\bar{A}_l^k} &\leq L_\varphi (|B_t - \tilde{c}^{j,k}| + |B_s - B_t - \tilde{c}^{l,k}|) I_{A_j^k} I_{\bar{A}_l^k} \\ &\leq CL_\varphi (|B_t^\lambda - c^{j,k}| + |B_s^\lambda - B_t^\lambda - c^{l,k}|) I_{A_j^k} I_{\bar{A}_l^k}, \end{aligned}$$

where L_φ is the Lipschitz constant of φ , the constant $C > 0$ depends on λ and d . Thus, we obtain

$$\begin{aligned} |\xi - \xi^k| &\leq CL_\varphi \frac{2}{k} + 2M_\varphi (I_{A_{N_k}^k} + I_{\bar{A}_{N_k}^k}) \\ &\leq CL_\varphi \frac{2}{k} + \frac{2M_\varphi}{k} (|B_t^\lambda| + |B_s^\lambda - B_t^\lambda|), \end{aligned}$$

where M_φ is the bound of φ . From this, we can get $\hat{\mathbb{E}} [|\xi - \xi^k|^2] \leq Ck^{-2}$, which implies the desired result.

□

Remark 4.3 In the above proof, the partition of B^i is deduced by the partition of $B^i + \lambda B^{i^*}$ and B^{i^*} . So, the random variable ξ^k defined in (4.3) is called an implied partition of ξ .

Lemma 4.4 Let Assumption (H3) hold and let $u \in \mathcal{U}[t, T]$ be given. Then there exists a sequence $(u^k)_{k \geq 1}$ in $\mathbb{U}[t, T]$ such that

$$\lim_{k \rightarrow \infty} \hat{\mathbb{E}} \left[\int_t^T |u_s - u_s^k|^2 ds \right] = 0.$$

Proof. Since $u \in M_G^2(t, T; \mathbb{R}^m)$, there exists a sequence

$$\bar{u}_s^k = \sum_{i=0}^{N-1} \xi_i^k I_{[t_i^k, t_{i+1}^k)}(s), \quad t = t_0^k < \dots < t_N^k = T, \quad \xi_i^k \in Lip(\Omega_{t_i^k}; \mathbb{R}^m), \quad k \geq 1,$$

such that $\hat{\mathbb{E}} \left[\int_t^T |u_s - \bar{u}_s^k|^2 ds \right] \rightarrow 0$ as $k \rightarrow \infty$. Note that $I_{AB} = I_A I_B \in L_G^2(\Omega_t)$ if $I_A, I_B \in L_G^2(\Omega_t)$, then, by Lemma 4.2, we can find

$$\bar{\xi}_i^k = \sum_{j=1}^{N_k} \sum_{l=1}^{\bar{N}_{i,k}} x_{jl}^{i,k} I_{A_j^k} I_{\bar{A}_l^{i,k}} \quad \text{for } i = 0, \dots, N-1,$$

such that $x_{jl}^{i,k} \in \mathbb{R}^m$, $I_{A_j^k} \in L_G^2(\Omega_t)$, $I_{\bar{A}_l^{i,k}} \in L_G^2(\Omega_{t_i^k}^t)$, $j \leq N_k$, $l \leq \bar{N}_{i,k}$, $i \leq N-1$, $(A_j^k)_{j=1}^{N_k}$ is a $\mathcal{B}(\Omega_t)$ -partition of Ω , $(\bar{A}_l^{i,k})_{l=1}^{\bar{N}_{i,k}}$ is a $\mathcal{B}(\Omega_{t_i^k}^t)$ -partition of Ω and

$$\hat{\mathbb{E}} [|\xi_i^k - \bar{\xi}_i^k|^2] < \frac{1}{k} \quad \text{for } i = 0, \dots, N-1.$$

Set $\tilde{u}_s^k = \sum_{i=0}^{N-1} \bar{\xi}_i^k I_{[t_i^k, t_{i+1}^k)}(s)$. Then we have

$$\begin{aligned} \hat{\mathbb{E}} \left[\int_t^T |u_s - \tilde{u}_s^k|^2 ds \right] &\leq 2\hat{\mathbb{E}} \left[\int_t^T |u_s - \bar{u}_s^k|^2 ds \right] + 2 \left[\int_t^T |\bar{u}_s^k - \tilde{u}_s^k|^2 ds \right] \\ &\leq 2\hat{\mathbb{E}} \left[\int_t^T |u_s - \bar{u}_s^k|^2 ds \right] + 2 \sum_{i=0}^{N-1} \hat{\mathbb{E}} [|\xi_i^k - \bar{\xi}_i^k|^2] (t_{i+1}^k - t_i^k) \\ &\leq 2\hat{\mathbb{E}} \left[\int_t^T |u_s - \bar{u}_s^k|^2 ds \right] + \frac{2(T-t)}{k} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Since U is a nonempty compact set of \mathbb{R}^m , for each $x_{jl}^{i,k}$, there exists a $\tilde{x}_{jl}^{i,k}$ such that

$$|x_{jl}^{i,k} - \tilde{x}_{jl}^{i,k}| = \inf\{|x_{jl}^{i,k} - x| : x \in U\}.$$

Set

$$\tilde{\xi}_i^k = \sum_{j=1}^{N_k} \sum_{l=1}^{\bar{N}_{i,k}} \tilde{x}_{jl}^{i,k} I_{A_j^k} I_{\bar{A}_l^{i,k}} \quad \text{for } i = 0, \dots, N-1,$$

and

$$u_s^k = \sum_{i=0}^{N-1} \tilde{\xi}_i^k I_{[t_i^k, t_{i+1}^k)}(s).$$

It is easy to check that $u^k \in \mathcal{U}[t, T]$. Since $u_s \in U$, we know $|\tilde{u}_s^k - u_s^k| \leq |\tilde{u}_s^k - u_s|$. Thus

$$\begin{aligned} \hat{\mathbb{E}} \left[\int_t^T |u_s - u_s^k|^2 ds \right] &\leq 2\hat{\mathbb{E}} \left[\int_t^T |u_s - \tilde{u}_s^k|^2 ds \right] + 2\hat{\mathbb{E}} \left[\int_t^T |\tilde{u}_s^k - u_s^k|^2 ds \right] \\ &\leq 4\hat{\mathbb{E}} \left[\int_t^T |u_s - \tilde{u}_s^k|^2 ds \right] \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, which implies the desired result. \square

Theorem 4.5 *Let Assumptions (H1)-(H3) hold. Then the value function $V(t, x)$ exists for each $(t, x) \in [0, T] \times \mathbb{R}^n$ and*

$$V(t, x) := \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v). \quad (4.4)$$

Proof. For each $v \in \mathcal{U}^t[t, T]$, it is easy to deduce $X_s^{t,x,v} \in L_G^p(\Omega_s^t)$ for any $p \geq 2$, which implies that $J(t, x, v)$ is a constant. Since $\mathcal{U}^t[t, T] \subset \mathcal{U}[t, T]$, by Definition 3.3, we only need to prove that, for any fixed $u \in \mathcal{U}[t, T]$,

$$J(t, x, u) \geq \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v), \quad \text{q.s.} \quad (4.5)$$

By Lemma 4.4, there exists a sequence $u_s^k = \sum_{j=1}^{N_k} I_{A_j^k} v_s^{j,k}$, $k \geq 1$, such that $I_{A_j^k} \in L_G^2(\Omega_t)$, $v^{j,k} \in \mathcal{U}^t[t, T]$, $(A_j^k)_{j=1}^{N_k}$ is a partition of Ω and

$$\hat{\mathbb{E}} \left[\int_t^T |u_s^k - u_s|^2 ds \right] \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (4.6)$$

It is easy to check that $X_s^{t,x,u^k} = \sum_{j=1}^{N_k} I_{A_j^k} X_s^{t,x,v^{j,k}}$ for $s \in [t, T]$. Thus

$$\begin{aligned} J(t, x, u^k) &= \hat{\mathbb{E}}_t \left[\Phi(X_T^{t,x,u^k}) + \int_t^T f(s, X_s^{t,x,u^k}, u_s^k) ds + \int_t^T g_{ij}(s, X_s^{t,x,u^k}, u_s^k) d\langle B^i, B^j \rangle_s \right] \\ &= \sum_{j=1}^{N_k} I_{A_j^k} \hat{\mathbb{E}}_t \left[\Phi(X_T^{t,x,v^{j,k}}) + \int_t^T f(s, X_s^{t,x,v^{j,k}}, v_s^{j,k}) ds + \int_t^T g_{ij}(s, X_s^{t,x,v^{j,k}}, v_s^{j,k}) d\langle B^i, B^j \rangle_s \right] \\ &= \sum_{j=1}^{N_k} I_{A_j^k} J(t, x, v^{j,k}) \geq \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v). \end{aligned}$$

It follows from (H1) and Hölder's inequality that

$$\begin{aligned} &\hat{\mathbb{E}} [|J(t, x, u^k) - J(t, x, u)|] \\ &\leq C \left\{ \left(1 + \left(\sup_{s \in [t, T]} \hat{\mathbb{E}} [|X_s^{t,x,u^k}|^2 + |X_s^{t,x,u}|^2] \right)^{1/2} \right) \left(\sup_{s \in [t, T]} \hat{\mathbb{E}} [|X_s^{t,x,u^k} - X_s^{t,x,u}|^2] \right)^{1/2} \right. \\ &\quad \left. + \left(\hat{\mathbb{E}} \left[\int_t^T |u_s^k - u_s|^2 ds \right] \right)^{1/2} \right\}, \end{aligned}$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . By Theorem 3.2 and the above inequality, we obtain

$$\hat{\mathbb{E}} [|J(t, x, u^k) - J(t, x, u)|] \leq C(1 + |x|) \left(\hat{\mathbb{E}} \left[\int_t^T |u_s^k - u_s|^2 ds \right] \right)^{1/2}, \quad (4.7)$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . Combining (4.6) and (4.7), we get

$$\hat{\mathbb{E}} [|J(t, x, u^k) - J(t, x, u)|] \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (4.8)$$

Since $J(t, x, u^k) \geq \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v)$, we have

$$\hat{\mathbb{E}} \left[\left(J(t, x, u^k) - \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v) \right)^- \right] = 0. \quad (4.9)$$

By (4.8) and (4.9), we obtain

$$\hat{\mathbb{E}} \left[\left(J(t, x, u) - \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v) \right)^- \right] = 0,$$

which implies $J(t, x, u) \geq \inf_{v \in \mathcal{U}^t[t, T]} J(t, x, v)$, q.s. Thus we obtain (4.4). \square

Now we use (4.4) to study the properties of $V(\cdot, \cdot)$ in x .

Proposition 4.6 *Let Assumptions (H1)-(H3) hold. Then there exists a constant $C > 0$ depending on T , $\bar{\sigma}^2$ and L such that*

$$|V(t, x) - V(t, x')| \leq C(1 + |x| + |x'|)|x - x'| \text{ and } |V(t, x)| \leq C(1 + |x|^2)$$

for $t \in [0, T]$, $x, x' \in \mathbb{R}^n$.

Proof. For each given $v \in \mathcal{U}^t[t, T]$, similar to the proof of (4.7), we can get

$$\begin{aligned} & |J(t, x, v) - J(t, x', v)| \\ & \leq C \left(1 + \left(\sup_{s \in [t, T]} \hat{\mathbb{E}} [|X_s^{t, x, v}|^2 + |X_s^{t, x', v}|^2] \right)^{1/2} \right) \left(\sup_{s \in [t, T]} \hat{\mathbb{E}} [|X_s^{t, x, v} - X_s^{t, x', v}|^2] \right)^{1/2}. \end{aligned}$$

By Theorem 3.2 and the above inequality, we have

$$|J(t, x, v) - J(t, x', v)| \leq C(1 + |x| + |x'|)|x - x'|,$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . Thus, by (4.4), we obtain

$$|V(t, x) - V(t, x')| \leq \sup_{v \in \mathcal{U}^t[t, T]} |J(t, x, v) - J(t, x', v)| \leq C(1 + |x| + |x'|)|x - x'|.$$

Note that U is compact, then we can deduce

$$|J(t, x, v)| \leq C \left(1 + \sup_{s \in [t, T]} \hat{\mathbb{E}} [|X_s^{t, x, v}|^2] \right),$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . By Theorem 3.2, we obtain $|V(t, x)| \leq C(1 + |x|^2)$. \square

The following theorem is the dynamic programming principle for control problem (3.3).

Theorem 4.7 *Let Assumptions (H1)-(H3) hold. Then, for each $t < T$, $\delta \leq T - t$, $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} V(t, x) &= \operatorname{ess\,inf}_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}}_t \left[V(t + \delta, X_{t+\delta}^{t, x, u}) + \int_t^{t+\delta} f(s, X_s^{t, x, u}, u_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t, x, u}, u_s) d\langle B^i, B^j \rangle_s \right] \\ &= \inf_{v \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[V(t + \delta, X_{t+\delta}^{t, x, v}) + \int_t^{t+\delta} f(s, X_s^{t, x, v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t, x, v}, v_s) d\langle B^i, B^j \rangle_s \right]. \end{aligned}$$

Proof. Similar to the proof of Theorem 4.5, we have

$$\begin{aligned} & \operatorname{ess\,inf}_{u \in \mathcal{U}[t, t+\delta]} \widehat{\mathbb{E}}_t \left[V(t+\delta, X_{t+\delta}^{t,x,u}) + \int_t^{t+\delta} f(s, X_s^{t,x,u}, u_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s \right] \\ &= \inf_{v \in \mathcal{U}^t[t, t+\delta]} \widehat{\mathbb{E}} \left[V(t+\delta, X_{t+\delta}^{t,x,v}) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right]. \end{aligned}$$

For each fixed $v \in \mathcal{U}^t[t, T]$, we assert that

$$V(t+\delta, X_{t+\delta}^{t,x,v}) \leq \widehat{\mathbb{E}}_{t+\delta} \left[\Phi(X_T^{t,x,v}) + \int_{t+\delta}^T f(s, X_s^{t,x,v}, v_s) ds + \int_{t+\delta}^T g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right]. \quad (4.10)$$

Since

$$\begin{aligned} X_s^{t,x,v} &= X_{t+\delta}^{t,x,v} + \int_{t+\delta}^s b(r, X_r^{t,x,v}, v_r) dr + \int_{t+\delta}^s h_{ij}(r, X_r^{t,x,v}, v_r) d\langle B^i, B^j \rangle_r \\ &\quad + \int_{t+\delta}^s \sigma(r, X_r^{t,x,v}, v_r) dB_r, \quad s \in [t+\delta, T], \end{aligned}$$

we have $X_s^{t,x,v} = X_s^{t+\delta, X_{t+\delta}^{t,x,v}, v}$ for $s \in [t+\delta, T]$. Thus inequality (4.10) is equivalent to

$$V(t+\delta, X_{t+\delta}^{t,x,v}) \leq J(t+\delta, X_{t+\delta}^{t,x,v}, v).$$

By Lemma 4.2, there exists a sequence $\xi^k = \sum_{j=1}^{N_k} x_j^k I_{A_j^k}$, $k \geq 1$, such that

$$\lim_{k \rightarrow \infty} \widehat{\mathbb{E}} [|X_{t+\delta}^{t,x,v} - \xi^k|^2] = 0, \quad (4.11)$$

where $x_j^k \in \mathbb{R}^n$, $I_{A_j^k} \in L_G^2(\Omega_{t+\delta})$, $j \leq N_k$, $(A_j^k)_{i=1}^{N_k}$ is a $\mathcal{B}(\Omega_{t+\delta})$ -partition of Ω . It is easy to verify that

$$X_s^{t+\delta, \xi^k, v} = \sum_{j=1}^{N_k} X_s^{t+\delta, x_j^k, v} I_{A_j^k} \text{ for } s \in [t+\delta, T].$$

Thus, by the definition of $V(t+\delta, x_j^k)$, we obtain

$$\begin{aligned} & J(t+\delta, \xi^k, v) \\ &= \widehat{\mathbb{E}}_{t+\delta} \left[\Phi(X_T^{t+\delta, \xi^k, v}) + \int_{t+\delta}^T f(s, X_s^{t+\delta, \xi^k, v}, v_s) ds + \int_{t+\delta}^T g_{ij}(s, X_s^{t+\delta, \xi^k, v}, v_s) d\langle B^i, B^j \rangle_s \right] \\ &= \sum_{j=1}^{N_k} I_{A_j^k} \widehat{\mathbb{E}}_{t+\delta} \left[\Phi(X_T^{t+\delta, x_j^k, v}) + \int_{t+\delta}^T f(s, X_s^{t+\delta, x_j^k, v}, v_s) ds + \int_{t+\delta}^T g_{ij}(s, X_s^{t+\delta, x_j^k, v}, v_s) d\langle B^i, B^j \rangle_s \right] \\ &\geq \sum_{j=1}^{N_k} I_{A_j^k} V(t+\delta, x_j^k), \end{aligned}$$

which implies

$$J(t+\delta, \xi^k, v) \geq V(t+\delta, \xi^k). \quad (4.12)$$

Similar to the proof of (4.7), we can get

$$\widehat{\mathbb{E}} [|J(t+\delta, \xi^k, v) - J(t+\delta, X_{t+\delta}^{t,x,v}, v)|] \leq C \left(1 + (\widehat{\mathbb{E}} [|\xi^k|^2 + |X_{t+\delta}^{t,x,v}|^2])^{1/2} \right) (\widehat{\mathbb{E}} [|X_{t+\delta}^{t,x,v} - \xi^k|^2])^{1/2}, \quad (4.13)$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . By Proposition 4.6, we have

$$\hat{\mathbb{E}} [|V(t + \delta, \xi^k) - V(t + \delta, X_{t+\delta}^{t,x,v})|] \leq C \left(1 + (\hat{\mathbb{E}} [|\xi^k|^2 + |X_{t+\delta}^{t,x,v}|^2])^{1/2} \right) (\hat{\mathbb{E}} [|X_{t+\delta}^{t,x,v} - \xi^k|^2])^{1/2}, \quad (4.14)$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . Thus, by (4.11), (4.13) and (4.14), we obtain (4.10) by taking $k \rightarrow \infty$ in (4.12). From (4.10), we can easily deduce

$$J(t, x, v) \geq \hat{\mathbb{E}} \left[V(t + \delta, X_{t+\delta}^{t,x,v}) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right],$$

which implies

$$V(t, x) \geq \inf_{v \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[V(t + \delta, X_{t+\delta}^{t,x,v}) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right].$$

Now we prove the inequality in the opposite direction. For each given $v \in \mathcal{U}^t[t, t + \delta]$, we only need to prove

$$V(t, x) \leq \hat{\mathbb{E}} \left[V(t + \delta, X_{t+\delta}^{t,x,v}) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right]. \quad (4.15)$$

Let ξ^k be defined in (4.11). By Theorem 4.5, for each x_j^k , there exists a $v^{j,k} \in \mathcal{U}^t[t + \delta, T]$ such that

$$|V(t + \delta, x_j^k) - J(t + \delta, x_j^k, v^{j,k})| \leq k^{-1}. \quad (4.16)$$

Set $\tilde{v}_s^k = \sum_{j=1}^{N_k} I_{A_j^k} v_s^{j,k} I_{[t+\delta, T]}(s)$, it is easy to verify that

$$J(t + \delta, \xi^k, \tilde{v}^k) = \sum_{j=1}^{N_k} I_{A_j^k} J(t + \delta, x_j^k, v^{j,k}) \text{ and } V(t + \delta, \xi^k) = \sum_{j=1}^{N_k} I_{A_j^k} V(t + \delta, x_j^k).$$

Thus we obtain

$$|V(t + \delta, \xi^k) - J(t + \delta, \xi^k, \tilde{v}^k)| \leq k^{-1} \quad (4.17)$$

by (4.16). Set $v_s^k = v_s I_{[t, t+\delta)}(s) + \tilde{v}_s^k I_{[t+\delta, T]}(s)$. Then

$$V(t, x) \leq \hat{\mathbb{E}} \left[J(t + \delta, X_{t+\delta}^{t,x,v}, \tilde{v}^k) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right]. \quad (4.18)$$

By (4.11), (4.13), (4.14) and (4.17), we obtain (4.15) by taking $k \rightarrow \infty$ in (4.18). Thus we obtain the dynamic programming principle. \square

By using the dynamic programming principle, we study the properties of $V(\cdot, \cdot)$ in t .

Proposition 4.8 *Let Assumptions (H1)-(H3) hold. Then, for each $t < T$, $\delta \leq T - t$ and $x \in \mathbb{R}^n$, we have*

$$|V(t, x) - V(t + \delta, x)| \leq C(1 + |x|^2)\sqrt{\delta},$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L .

Proof. By Theorem 4.7, we have

$$V(t, x) = \inf_{v \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[V(t + \delta, X_{t+\delta}^{t,x,v}) + \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right].$$

By Proposition 4.6, Theorem 3.2 and Hölder's inequality, we get

$$\begin{aligned} & \hat{\mathbb{E}} [|V(t + \delta, X_{t+\delta}^{t,x,v}) - V(t + \delta, x)|] \\ & \leq C(1 + |x| + (\hat{\mathbb{E}} [|X_{t+\delta}^{t,x,v}|^2])^{1/2})(\hat{\mathbb{E}} [|X_{t+\delta}^{t,x,v} - x|^2])^{1/2} \\ & \leq C(1 + |x|^2)\sqrt{\delta}, \end{aligned}$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . It follows from (H1) and Theorem 3.2 that

$$\begin{aligned} & \hat{\mathbb{E}} \left[\left| \int_t^{t+\delta} f(s, X_s^{t,x,v}, v_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,v}, v_s) d\langle B^i, B^j \rangle_s \right| \right] \\ & \leq C \int_t^{t+\delta} (1 + \hat{\mathbb{E}} [|X_s^{t,x,v}|^2]) ds \\ & \leq C(1 + |x|^2)\delta, \end{aligned}$$

where $C > 0$ depends on T , $\bar{\sigma}^2$ and L . Thus we obtain the desired result. \square

5 HJB equation

In this section, we show that the value function $V(\cdot, \cdot)$ satisfies the following HJB equation:

$$\begin{cases} \partial_t V(t, x) + \inf_{v \in U} H(t, x, \partial_x V(t, x), \partial_{xx}^2 V(t, x), v) = 0, \\ V(T, x) = \Phi(x), \quad x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

where $(t, x, p, A, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}_n \times U$,

$$H(t, x, p, A, v) = G(F(t, x, p, A, v)) + \langle p, b(t, x, v) \rangle + f(t, x, v),$$

$$F_{ij}(t, x, p, A, v) = (\sigma^T(t, x, v)A\sigma(t, x, v))_{ij} + 2\langle p, h_{ij}(t, x, v) \rangle + 2g_{ij}(t, x, v).$$

Definition 5.1 ([3]) *A function $V(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity subsolution (resp. supersolution) to (5.1) if $V(T, x) \leq \Phi(x)$ (resp. $V(T, x) \geq \Phi(x)$) for each $x \in \mathbb{R}^n$, and for each given $(t, x) \in [0, T] \times \mathbb{R}^n$, $\phi \in C_{Lip}^{1,2}([0, T] \times \mathbb{R}^n)$ such that $\phi(t, x) = V(t, x)$ and $\phi \geq V$ (resp. $\phi \leq V$) on $[0, T] \times \mathbb{R}^n$, we have*

$$\partial_t \phi(t, x) + \inf_{v \in U} H(t, x, \partial_x \phi(t, x), \partial_{xx}^2 \phi(t, x), v) \geq 0 \quad (\text{resp. } \leq 0).$$

A function $V(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity solution to (5.1) if it is both a viscosity subsolution and a viscosity supersolution to (5.1).

Remark 5.2 $C_{Lip}^{1,2}([0, T] \times \mathbb{R}^n)$ denotes the set of real-valued functions that are continuously differentiable up to the first order (resp. second order) in t -variable (resp. x -variable) and whose derivatives are Lipschitz functions.

Theorem 5.3 *Let Assumptions (H1)-(H3) hold. Then the value function $V(\cdot, \cdot)$ is the unique viscosity solution to the HJB equation (5.1).*

Proof. By Propositions 4.6 and 4.8, we have $V(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$. Now, we prove that $V(\cdot, \cdot)$ is a viscosity subsolution to (5.1).

For each fixed $(t, x) \in [0, T] \times \mathbb{R}^n$, $\phi \in C_{Lip}^{1,2}([0, T] \times \mathbb{R}^n)$ such that $\phi(t, x) = V(t, x)$ and $\phi \geq V$, by Theorem 4.7, we deduce that, for $\delta \leq T - t$,

$$\phi(t, x) \leq \inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\phi(t + \delta, X_{t+\delta}^{t,x,u}) + \int_t^{t+\delta} f(s, X_s^{t,x,u}, u_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s \right].$$

Applying Itô's formula to $\phi(s, X_s^{t,x,u})$ on $[t, t + \delta]$, we get

$$\begin{aligned} & \hat{\mathbb{E}} \left[\phi(t + \delta, X_{t+\delta}^{t,x,u}) - \phi(t, x) + \int_t^{t+\delta} f(s, X_s^{t,x,u}, u_s) ds + \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s \right] \\ &= \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_1(s, X_s^{t,x,u}, u_s) ds + \int_t^{t+\delta} \Lambda_2^{ij}(s, X_s^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s \right], \end{aligned}$$

where

$$\begin{aligned} \Lambda_1(s, x, v) &= \partial_t \phi(s, x) + \langle b(s, x, v), \partial_x \phi(s, x) \rangle + f(s, x, v), \\ \Lambda_2^{ij}(s, x, v) &= \frac{1}{2} F_{ij}(s, x, \partial_x \phi(s, x), \partial_{xx}^2 \phi(s, x), v). \end{aligned}$$

Thus we obtain

$$\inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_1(s, X_s^{t,x,u}, u_s) ds + \int_t^{t+\delta} \Lambda_2^{ij}(s, X_s^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s \right] \geq 0. \quad (5.2)$$

Since $\phi \in C_{Lip}^{1,2}([0, T] \times \mathbb{R}^n)$, we have

$$\begin{aligned} & |\partial_t \phi(s, X_s^{t,x,u}) - \partial_t \phi(s, x)| + |\partial_x \phi(s, X_s^{t,x,u}) - \partial_x \phi(s, x)| \\ &+ |\partial_{xx}^2 \phi(s, X_s^{t,x,u}) - \partial_{xx}^2 \phi(s, x)| \leq C |X_s^{t,x,u} - x|, \end{aligned}$$

where $C > 0$ depends on ϕ . Then, by (H1), we get

$$\begin{aligned} & |\Lambda_1(s, X_s^{t,x,u}, u_s) - \Lambda_1(s, x, u_s)| + |\Lambda_2^{ij}(s, X_s^{t,x,u}, u_s) - \Lambda_2^{ij}(s, x, u_s)| \\ & \leq C(1 + |x|^2 + |X_s^{t,x,u}|^2) |X_s^{t,x,u} - x|, \end{aligned}$$

where $C > 0$ depends on L and ϕ . By Theorem 3.2 and Hölder's inequality, we obtain

$$\begin{aligned} & \hat{\mathbb{E}} \left[\int_t^{t+\delta} (|\Lambda_1(s, X_s^{t,x,u}, u_s) - \Lambda_1(s, x, u_s)| + |\Lambda_2^{ij}(s, X_s^{t,x,u}, u_s) - \Lambda_2^{ij}(s, x, u_s)|) ds \right] \\ & \leq C \int_t^{t+\delta} (1 + |x|^2 + (\hat{\mathbb{E}}[|X_s^{t,x,u}|^4])^{1/2}) (\hat{\mathbb{E}}[|X_s^{t,x,u} - x|^2])^{1/2} ds \\ & \leq C(1 + |x|^3) \delta^{3/2}, \end{aligned}$$

where $C > 0$ depends on T , $\bar{\sigma}^2$, L and ϕ . Thus we have

$$\inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_1(s, x, u_s) ds + \int_t^{t+\delta} \Lambda_2^{ij}(s, x, u_s) d\langle B^i, B^j \rangle_s \right] \geq -C(1 + |x|^3) \delta^{3/2}. \quad (5.3)$$

Set

$$\Lambda(s, x) = \inf_{v \in U} \{ \Lambda_1(s, x, v) + 2G((\Lambda_2^{ij}(s, x, v))_{i,j=1}^d) \}. \quad (5.4)$$

Then, by Proposition 4.1.4 in [22], we get

$$\begin{aligned} & \inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_1(s, x, u_s) ds + \int_t^{t+\delta} \Lambda_2^{ij}(s, x, u_s) d\langle B^i, B^j \rangle_s \right] \\ & \geq \int_t^{t+\delta} \Lambda(s, x) ds + \inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_2^{ij}(s, x, u_s) d\langle B^i, B^j \rangle_s - 2 \int_t^{t+\delta} G((\Lambda_2^{ij}(s, x, u_s))_{i,j=1}^d) ds \right] \\ & = \int_t^{t+\delta} \Lambda(s, x) ds. \end{aligned}$$

By measurable selection theorem, there exists a deterministic control $u^* \in \mathcal{U}^t[t, t + \delta]$ such that

$$\int_t^{t+\delta} \Lambda(s, x) ds = \int_t^{t+\delta} [\Lambda_1(s, x, u_s^*) + 2G((\Lambda_2^{ij}(s, x, u_s^*))_{i,j=1}^d)] ds,$$

which implies

$$\begin{aligned} & \inf_{u \in \mathcal{U}^t[t, t+\delta]} \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_1(s, x, u_s) ds + \int_t^{t+\delta} \Lambda_2^{ij}(s, x, u_s) d\langle B^i, B^j \rangle_s \right] \\ & \leq \int_t^{t+\delta} \Lambda(s, x) ds + \hat{\mathbb{E}} \left[\int_t^{t+\delta} \Lambda_2^{ij}(s, x, u_s^*) d\langle B^i, B^j \rangle_s - 2 \int_t^{t+\delta} G((\Lambda_2^{ij}(s, x, u_s^*))_{i,j=1}^d) ds \right] \\ & = \int_t^{t+\delta} \Lambda(s, x) ds. \end{aligned}$$

Thus, by (5.3), we obtain

$$\int_t^{t+\delta} \Lambda(s, x) ds \geq -C(1 + |x|^3) \delta^{3/2}, \quad (5.5)$$

where $C > 0$ depends on T , $\bar{\sigma}^2$, L and ϕ . It is easy to check that $\Lambda(s, x)$ defined in (5.4) is continuous in s .

Thus, by (5.5), we get

$$\Lambda(t, x) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_t^{t+\delta} \Lambda(s, x) ds \geq 0,$$

which implies that $V(\cdot, \cdot)$ is a viscosity subsolution to (5.1). Similarly, we can show that $V(\cdot, \cdot)$ is a viscosity supersolution to (5.1). Thus $V(\cdot, \cdot)$ is a viscosity solution to (5.1). The uniqueness is due to Theorem 3.5 in [1] (see also [2, 14]). \square

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