Nonnegative solutions of the heat equation in a cylindrical domain and Widder's theorem

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Abstract

It is shown every nonnegative solution of the heat equation in a bounded cylindrical domain has an integral representation in terms of a trace triple consisting of a bottom trace, a corner trace and a lateral trace on its parabolic boundary. Conversely this trace triple uniquely determines the solution.

Keywords: heat equation, integral representation formula, initial trace, lateral trace, trace triple

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1 Introduction

The study of the initial and lateral traces of nonnegative solutions of the heat equation was initiated by D.V. Widder. In a series of papers starting with [HW], [W1], [W2], D.V. Widder, P. Hartman and A. Winter solved the problem in the one dimensional case for solutions of the heat equation in an infinite rod, a half infinite rod and a finite rod. A complete treatment can be found in the book [W3]. In the case of an infinite rod, its n-dimensional version becomes the existence of the initial trace for nonnegative solutions of the heat equation in $\mathbb{R}^n$. This problem was solved and generalized to nonnegative weak solutions of second order uniformly parabolic equations in divergence form by D.G. Aronson in [A].

For the case of a finite rod, in Theorem 6 of Chapter VIII of [W2] Widder showed that every nonnegative solution of the heat equation in the domain $(0,\pi)\times(0,T)$ has an integral representation in terms of three trace measures on the parabolic boundary of the domain. These traces consist of nonnegative measures α on $(0,\pi)\times\{0\}$ and β, γ, on $\{0\}\times[0,T)$ and $\{\pi\}\times[0,T)$ respectively. It was pointed out in [W3] that $\alpha((\varepsilon,\pi/2)\times\{0\})$ and $\alpha((\pi/2,\pi-\varepsilon)\times\{0\})$ could be ∞ as $\varepsilon\to 0$ but on the other hand both $\beta(\{0\}\times(0,T_1))$ and $\gamma(\{\pi\}\times(0,T_1))$ are finite for any $0<T_1<T$. The higher dimensional extension of a finite rod is a bounded cylindrical domain in $\mathbb{R}^n, n\ge 2$. For a non-negative solution in such domain with zero boundary data, the existence of an initial trace consisting of a bottom and a corner ones was established K.M. Hui in [H].

Recently there is a lot of study of initial traces of non-negative solutions of various parabolic equations. For example similar initial trace problem for the positive solutions of the semilinear heat equation

$$u_t=\Delta u-u^q$$

in C^2 domain $\Omega\subset\mathbb{R}^n$ with compact boundary where $q>1$ was studied by M. Marcus and L. Véron [MV4]. The boundary trace problem for the corresponding elliptic problem was also studied by M. Marcus and L. Véron in [MV1], [MV2], [MV3]. Recently K. Hisa, K. Ishige and J. Takahashi [HIT] proved the existence and uniqueness of initial traces of non-negative solutions to the following semilinear heat equation,

$$u_t=\Delta u+u^p$$

on a half space of $\mathbb{R}^N$ under the zero Dirichlet boundary condition where $p>1$.

The initial trace problem for the porous medium equation was studied by Aronson and L.A. Caffarelli [AC], K.S. Chou and Y.C. Kwong [CK1], [CK2], B.E.J. Dahlberg and C.E. Kenig [DK], etc. The initial trace problem for the parabolic p-laplace equation and the doubly nonlinear parabolic equation were studied by E. DiBenedetto and M.A. Herrero [DiH1], [DiH2], and K. Ishige and J. Kinnunen [Is], [IsK], etc.

Note that in [W2] only solutions of one dimensional heat equation is studied and in [A] no measure initial data was considered. On the other hand in this paper we study nonnegative solutions of the heat equation in a bounded cylindrical domain in $\mathbb{R}^n$ for any $n \geq 1$ which may not be zero along their lateral boundary. It will be shown that such solution has an integral representation in terms of a trace triple consisting of a bottom trace, a corner trace and a lateral trace on its parabolic boundary. Conversely this trace triple uniquely determines the solution.

To make things precise, we introduce the following notations and definitions. Let $Q_T = \Omega \times (0, T)$ where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $T > 0$. For any $x \in \Omega$, let $\delta(x) = \text{dist}(x, \partial\Omega)$ be the distance of x from the boundary $\partial\Omega$ of Ω. Let $C_0^\infty(\overline{\Omega})$ be the space of all smooth functions in $\overline{\Omega}$ vanishing on $\partial\Omega$ and

$$L^1(\Omega, \delta) = \left\{ f \in L^1_{loc}(\Omega) : \int_\Omega |f(x)|\delta(x)\, dx < \infty \right\}. \tag{1.1}$$

We denote by

- $M(\Omega, \delta)$ the collection of all nonnegative Radon measures μ on Ω satisfying

$$\int_\Omega \delta(x)\, d\mu(x) < \infty \ ,$$

- $M(\partial\Omega)$ the collection of all nonnegative Radon measures on $\partial\Omega$. Note that since every Radon measure is finite on compact sets, hence all measures in this collection are finite.

- $M_s(\partial\Omega \times (0, T))$ the collection of all nonnegative Radon measures ν on $\partial\Omega \times (0, T))$ satisfying $\nu(\partial\Omega \times (0, T_1)) < \infty$ for all $T_1 \in (0, T)$.

Let u be a classical nonnegative solution of the heat equation in Q_T. We say that a pair of measures $(\mu, \lambda) \in M(\Omega, \delta) \times M(\partial\Omega)$ is the *initial trace* of u if for any $\varphi \in C_0^\infty(\overline{\Omega})$,

$$\lim_{t \to 0^+} \int_\Omega \varphi(x) u(x, t)\, dx = \int_\Omega \varphi\, d\mu + \int_{\partial\Omega} \frac{\partial \varphi}{\partial N}\, d\lambda \ ,$$

where $\partial/\partial N$ is the derivative with respect to the unit inner normal N. On the other hand we say that a measure $\nu \in M_s(\partial\Omega \times (0,T))$ is the *lateral trace* of u if for any $h \in C_c(\partial\Omega \times (0,T))$,

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\partial\Omega_\varepsilon} \tilde{h}(x,t)u(x,t)\, d\sigma(x)dt = \iint_{\partial\Omega\times(0,T)} h\, d\nu \ ,$$

where $\Omega_\varepsilon = \{x \in \Omega :\ \delta(x) > \varepsilon\}$ and $\tilde{h}$ is any continuous extension of h in a tubular neighborhood of $\partial\Omega \times (0,T)$ vanishing near $t = 0, T$.

Our main result is

Theorem 1.1. *Let u be a nonnegative classical solution of the heat equation in Q_T. There exists $(\mu,\lambda,\nu) \in M(\Omega,\delta) \times M(\partial\Omega) \times M_s(\partial\Omega \times (0,T))$ such that (μ,λ) is the initial trace and ν is the lateral trace for u. Moreover,*

$$\begin{aligned} u(x,t) &= \int_\Omega G(x,t;y,0)\, d\mu(y) + \int_{\partial\Omega} \frac{\partial G}{\partial N_y}(x,t;y,0)\, d\lambda(y) \\ &\quad + \iint_{\partial\Omega\times(0,t)} \frac{\partial G}{\partial N_y}(x,t;y,s)\, d\nu(y,s) \quad \forall (x,t) \in Q_T, \end{aligned} \tag{1.2}$$

where $\partial/\partial N_y$ is the derivative with respect to the unit inner normal N_y at $y \in \partial\Omega$ and $G(x,t;y,s)$ is the Green kernel of the heat equation in $\Omega \times \mathbb{R}$.

Conversely, given any $(\mu,\lambda,\nu) \in M(\Omega,\delta) \times M(\partial\Omega) \times M_s(\partial\Omega \times (0,T))$, (1.2) *gives a classical solution of the heat equation in Q_T whose trace triple is equal to (μ,λ,ν).*

Along a different vein, integral representation formulas for nonnegative solutions of the heat equation in bounded domains in term of the so-called kernel function were given by J.T. Kemper in [K]. His results were extended to nonnegative solutions of uniformly parabolic divergence equations by E.B. Fabes, N. Garofalo and S. Salsa in [FGS]. One may consult [M] by M. Murata for more recent results in this direction.

For the related generalized porous medium equation, existence and uniqueness of a pair of initial traces for its nonnegative solution of the initial Dirichlet problem in a bounded smooth cylindrical domain was proved by B.E.J. Dahlberg and C.E. Kenig in [DK]. Results on non-zero lateral traces for solutions of the porous medium equation were obtained by K.S. Chou and Y.C. Kwong in [CK1] and [CK2].

The plan of the paper is as follows. In section 2 we will prove the existence of initial trace for nonnegative solution of the heat equation in bounded smooth cylindrical domain. In section 3 we will prove the existence of the lateral trace and the representation formula (1.2) for the solution. We will also prove Theorem 1.1 in section 3.

2 Existence of initial trace

We will assume that Ω is a smooth bounded domain in $\mathbb{R}^n$ and u is a nonnegative solution of the heat equation in the cylindrical domain Q_T for some constant $T > 0$ for the rest of the paper. We will fix a sufficiently small $\varepsilon_0 > 0$ such that the boundary of the subdomain

$$\Omega_\varepsilon = \{x \in \Omega : \ \text{dist}\,(x, \partial\Omega) > \varepsilon\}, \qquad \varepsilon \in [0, \varepsilon_0]$$

is smooth and for each $x \in \Omega \setminus \Omega_{\varepsilon_0}$, there is a unique $z = z(x) \in \partial\Omega$ satisfying $x = z + \delta(x)N_z$ where $\delta(x)$ is the distance from x to $\partial\Omega$ and N_z is the unit inner normal of $\partial\Omega$ at z. The map $x \mapsto (z, \delta(x)) \in \partial\Omega \times (0, \varepsilon_0]$ forms a diffeomorphism from $\Omega \setminus \Omega_{\varepsilon_0}$ to $\partial\Omega \times (0, \varepsilon_0]$. We will extend the distance function δ on $\Omega \setminus \Omega_{\varepsilon_0}$ to a smooth positive function $\overline{\delta}$ on Ω and fix it throughout this paper. Apparently such choice of $\overline{\delta}$ does not alter $M(\Omega, \delta)$.

We first recall some basic properties of the Green kernel $G(x, t; y, s)$ of the heat equation (cf. [C], [I]). Note that the Green kernel $G(x, t; y, s)$ for the heat equation in $\Omega \times \mathbb{R}$ exists and is a continuous function in

$$\{(x, t, y, s) : \ x, y \in \overline{\Omega}, -\infty < s < t < \infty\}$$

which is smooth in its interior such that

- for each $(y, s) \in \Omega \times \mathbb{R}, G(\cdot, \cdot; y, s) > 0$ satisfies the heat equation in $\Omega \times (s, \infty)$, vanishes on $\partial\Omega \times (s, \infty)$ and satisfies

  $$\lim_{t \searrow s} G(x, t; y, s) = \delta_y \quad \forall s \in \mathbb{R}$$

  in the distribution sense where δ_y is the delta mass at y.

- for each $(x, t) \in \Omega \times \mathbb{R}, G(x, t; \cdot, \cdot)$ satisfies the backward heat equation in $\Omega \times (-\infty, t)$ and $G(x, t; y, s) = 0$, $\frac{\partial G_\varepsilon}{\partial N_y}(x, t; y, s) > 0$ for all $(y, s) \in \partial\Omega \times (-\infty, t)$.

- and if we let $\Omega_1 \subset \Omega_2$ and G_1, G_2, be the Green kernel of the heat equation with respect to the cylindrical domains $\Omega_1 \times \mathbb{R}$ and $\Omega_2 \times \mathbb{R}$ respectively, then

  $$G_1(x, t; y, s) \le G_2(x, t; y, s) \quad \forall x, y \in \Omega_1, s < t.$$

The proof of our main theorem Theorem 1.1 will be accomplished in several lemmas. First of all, by the integral representation formula for solutions of the heat equation in cylindrical domain (Theorem 5 of Chapter VII of [C]), u admits the following integral representation, namely, for any $(x,t)\in\Omega_\varepsilon\times(s,T), \varepsilon\in(0,\varepsilon_0)$,

$$u(x,t)=\int_{\Omega_\varepsilon} G_\varepsilon(x,t;y,s)u(y,s)\,dy+\int_s^t\int_{\partial\Omega_\varepsilon}\frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau)u(y,\tau)\,d\sigma(y)\,d\tau\ , \tag{2.1}$$

where G_ε is the Green kernel for the heat equation in $\Omega_\varepsilon\times\mathbb{R}$ and $\partial/\partial N_y$ is the derivative with respect to the unit inner normal N_y at $y\in\partial\Omega_\varepsilon$. Note that both $G_\varepsilon(x,t;y,s)$ and $\frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,s)$ are positive.

Since both terms on the right hand side of (2.1) are nonnegative, we have, for all $x\in\Omega_\varepsilon,\ 0<s<t<T,\ 0<\varepsilon\le\varepsilon_0$,

$$\int_{\Omega_\varepsilon} G_\varepsilon(x,t;y,s)u(y,s)\,dy\le u(x,t)\ , \tag{2.2}$$

and

$$\int_s^t\int_{\partial\Omega_\varepsilon}\frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau)u(y,\tau)\,d\sigma(y)\,d\tau\le u(x,t)\ . \tag{2.3}$$

Since $G_\varepsilon(x,t;\cdot,\cdot)\uparrow G(x,t;\cdot,\cdot)$ as $\varepsilon\to0$, letting $\varepsilon\to0$ in (2.2), by the monotone convergence theorem, we have

$$\int_\Omega G(x,t;y,s)u(y,s)\,dy\le u(x,t)\quad \forall x\in\Omega,\ 0<s<t<T\ . \tag{2.4}$$

Lemma 2.1. *For any $T_1\in(0,T)$, we have*

$$\sup_{0<t\le T_1}\int_\Omega u(x,t)\delta(x)\,dx<\infty.$$

Proof. We fix some $x_0\in\Omega$ and $T_2\in(T_1,T)$. By (2.4), we have

$$\int_\Omega G(x_0,T_2;w,s)u(w,s)\,dw\le u(x_0,T_2)\quad \forall\ 0<s<T_2. \tag{2.5}$$

Now for any $(w,s) \in (\Omega \setminus \Omega_{\varepsilon_0}) \times (0,T_1]$, we have

$$
\begin{aligned}
&G(x_0,T_2;w,s)\\
=&G(x_0,T_2;z(w)+\delta(w)N_{z(w)},s) - G(x_0,T_2;z(w),s)\\
=&\int_0^1 \frac{\partial G}{\partial a}(x_0,T_2;z(w)+a\delta(w)N_{z(w)},s)\,da\\
=&\left(\sum_{j=1}^n N_j(z(w))\int_0^1 \frac{\partial G}{\partial y_j}(x_0,T_2;z(w)+a\delta(w)N_{z(w)},s)\,da\right)\delta(w) \qquad (2.6)
\end{aligned}
$$

where $N_z = (N_1(z),\cdots,N_n(z))$ is the unit inner normal at $z \in \partial\Omega$. Since $\partial G/\partial N_y(x_0,T_2;y,s)$ is positive for $y \in \partial\Omega$ and $s < T_2$ and also uniformly continuous for $s \in (0,T_1]$, there exist constants $0 < \varepsilon_1 < \varepsilon_0$ and $c_1 > 0$ such that

$$
c_1 \le \sum_{j=1}^n N_j(z(w))\frac{\partial G}{\partial y_j}(x_0,T_2;z(w)+a\delta(w)N_{z(w)},s) \le \frac{1}{c_1} \qquad (2.7)
$$

holds for any $(w,s) \in (\Omega \setminus \Omega_{\varepsilon_1}) \times (0,T_1]$. Therefore by (2.6) and (2.7), we have

$$
c_1\delta(w) \le G(x_0,T_2;w,s) \le c_1^{-1}\delta(w) \quad \forall y \in \Omega \setminus \Omega_{\varepsilon_1},\ 0 < s \le T_1. \qquad (2.8)
$$

Since both $\delta(w)$ and $G(x_0,T_2;w,s)$ are positive and uniformly bounded above and below by some positive constants in $\overline{\Omega}_{\varepsilon_1}$, by (2.5) and (2.8) the lemma follows. □

Lemma 2.2. *For any $T_1 \in (0,T)$, we have*

$$
\sup_{0<\varepsilon\le\varepsilon_0}\int_0^{T_1}\int_{\partial\Omega_\varepsilon} u(x,\tau)\,d\sigma(x)\,d\tau < \infty.
$$

Proof. Let $x_0 \in \Omega_{\varepsilon_0}$, $T_2 \in (T_1,T)$ and $0 < \varepsilon \le \varepsilon_0$. Putting $x = x_0$ and $t = T_2$ in (2.3) and letting $s \to 0$, by the monotone convergence theorem, we have

$$
\int_0^{T_1}\int_{\partial\Omega_\varepsilon}\frac{\partial G_\varepsilon}{\partial N_y}(x_0,T_2;y,\tau)u(y,\tau)\,d\sigma(y)\,d\tau \le u(x_0,T_2)\ . \qquad (2.9)
$$

We now claim that there exists a constant $c_1 > 0$ such that

$$
\frac{\partial G_\varepsilon}{\partial N_y}(x_0,T_2;y,\tau) \ge c_1\ , \quad \forall y \in \partial\Omega_\varepsilon,\ 0 \le \tau \le T_1,\ 0 < \varepsilon \le \varepsilon_0.
$$

Suppose the claim does not hold. Then there exist sequences $\{\varepsilon_i\}_{i=1}^\infty \subset (0,\varepsilon_0]$, $\{\tau_i\}_{i=1}^\infty \subset [0,T_1]$, $\{y_i\}_{i=1}^\infty$, such that $y_i \in \partial\Omega_{\varepsilon_i}$ for any $i \in \mathbb{Z}^+$ and

$$
\frac{\partial G_{\varepsilon_i}}{\partial N_{y_i}}(x_0,T_2;y_i,\tau_i) \to 0\ , \quad \text{as } i \to \infty.
$$

Then there exists $\overline{\varepsilon}_0 \in [0, \varepsilon_0]$, $y_0 \in \partial\Omega_{\overline{\varepsilon}_0}$, $\tau_0 \in [0, T_1]$ and subsequences of $\{\varepsilon_i\}$, $\{y_j\}$, $\{\tau_i\}$, which we may assume without loss of generality to be the sequences themselves such that $\varepsilon_i \to \overline{\varepsilon}_0$, $y_j \to y_0$ and $\tau_i \to \tau_0$ as $i \to \infty$. It follows that

$$\frac{\partial G_{\overline{\varepsilon}_0}}{\partial N_{y_0}}(x_0, T_2; y_0, \tau_0) = 0.$$

On the other hand since $\partial G_{\overline{\varepsilon}_0}/\partial N_{y_0}(x_0, T_2; y, \tau)$ is positive for any $y \in \partial\Omega_{\overline{\varepsilon}_0}$ and $\tau \in [0, T_1]$, contradiction arises and our claim holds. By (2.9) and the claim, the lemma follows.

□

Lemma 2.3. *For any $T_1 \in (0, T)$, we have*

$$\int_0^{T_1} \int_\Omega u(x,t)\,dx\,dt < \infty.$$

Proof. For any $0 < \varepsilon < \varepsilon_0$, let φ_ε be the solution of

$$\begin{cases} -\Delta\varphi = 1 & \text{in } \Omega_\varepsilon \\ \varphi(x) = 0 & \forall x \in \partial\Omega_\varepsilon. \end{cases} \tag{2.10}$$

According to elliptic theory (cf. [GT], [Wi]), by decreasing ε_0 if necessary, there exists a constant $C_2 > 0$ such that

$$C_2 \le \frac{\partial\varphi_\varepsilon}{\partial N_y}(y) \le C_2^{-1} \quad \forall y \in \partial\Omega_\varepsilon, \varepsilon \in (0, \varepsilon_0). \tag{2.11}$$

Moreover

$$\varphi_\varepsilon(x) \le C\delta_\varepsilon(x) \le C\delta(x) \quad \forall x \in \Omega_\varepsilon \tag{2.12}$$

for some constant $C > 0$ where $\delta_\varepsilon(x) = \text{dist}(x, \partial\Omega_\varepsilon)$. Multiplying the heat equation by φ_ε and integrating over $\Omega_\varepsilon \times (t, T_1)$, we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} u(x, T_1)\varphi_\varepsilon(x)\,dx - \int_{\Omega_\varepsilon} u(x,t)\varphi_\varepsilon(x)\,dx \\ =&\int_t^{T_1} \int_{\Omega_\varepsilon} u\Delta\varphi_\varepsilon\,dx\,dt + \int_t^{T_1} \int_{\partial\Omega_\varepsilon} u\frac{\partial\varphi_\varepsilon}{\partial N}\,d\sigma\,dt. \end{aligned} \tag{2.13}$$

By (2.10), (2.11), (2.12), (2.13) and Lemma 2.1, we have

$$C_2 \int_t^{T_1} \int_{\partial\Omega_\varepsilon} u(x,t)\,d\sigma(x)\,dt \le C_3 + \int_t^{T_1} \int_{\Omega_\varepsilon} u(x,t)\,dx\,dt \tag{2.14}$$

for some constant $C_3 > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$. Now using the coarea formula [EG], we have

$$-\frac{d}{d\varepsilon}\int_{\Omega_\varepsilon} g\,dx = \int_{\partial\Omega_\varepsilon} \frac{g}{|\nabla\delta(x)|}\,d\sigma(x)$$

where

$$g(x) = \int_t^{T_1} u(x,\tau)\,d\tau$$

and noting that

$$|\nabla\delta(x)| \geq C_4 \quad \forall x \in \Omega \setminus \Omega_{\varepsilon_0}$$

for some constant $C_4 > 0$, by (2.14) we get

$$-\frac{d}{d\varepsilon}G(\varepsilon) \leq C_5 G(\varepsilon) + C_5 \tag{2.15}$$

for some constant $C_5 > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ where

$$G(\varepsilon) = \int_t^T \int_{\Omega_\varepsilon} u\,dx\,dt.$$

Integrating (2.15) from ε to ε_0, we have

$$\int_t^T \int_{\Omega_\varepsilon} u\,dx\,dt \leq e^{C_5(\varepsilon_0-\varepsilon)}\int_t^T \int_{\Omega_{\varepsilon_0}} u\,dx\,dt + e^{C_5(\varepsilon_0-\varepsilon)}. \tag{2.16}$$

Letting first $\varepsilon \to 0$ and then $t \to 0$ in (2.16), the lemma follows. □

Lemma 2.4. *There exists a Radon measure $\nu \in M_s(\partial\Omega \times (0,T))$ satisfying*

$$\lim_{\varepsilon\to 0^+}\int_0^{T_1}\int_{\partial\Omega_\varepsilon} u(x,\tau)\tilde{h}(x,\tau)\,d\sigma(x)\,d\tau = \iint_{\partial\Omega\times(0,T_1)} h\,d\nu \quad \forall 0 < T_1 < T \tag{2.17}$$

for any bounded continuous function h on $\partial\Omega \times (0,T)$ where $\tilde{h}$ is any bounded continuous extension of h in a tubular neighbourhood of $\partial\Omega \times (0,T)$. Moreover the measure ν is uniquely given by

$$\iint_{\partial\Omega\times(0,T)} h\,d\nu = \int_0^T \int_\Omega u\left(\overline{h}\Delta\overline{\delta} + 2\nabla\overline{h}\cdot\nabla\overline{\delta} + \overline{\delta}\Delta\overline{h} + \overline{\delta}\overline{h}_t\right)\,dx\,dt \tag{2.18}$$

for any $h \in C_c(\partial\Omega \times (0,T))$ where $\overline{h}$ is a continuous extension of h to $\overline{\Omega} \times (0,T)$ vanishing near $t = 0, T$, such that $\overline{h}$ is constant along each inner normal direction of $\partial\Omega$ in $(\overline{\Omega} \setminus \Omega_{\varepsilon_0}) \times (0,T)$.

Proof. For every bounded continuous function h on $\partial\Omega \times (0,T)$, we extend it to a bounded continuous function $\overline{h}$ on $(\overline{\Omega} \setminus \Omega_{\varepsilon_0}) \times (0,T)$ such that $\overline{h}$ is constant along each inner normal direction of $\partial\Omega$. In view of Lemma 2.2, it suffices to prove the lemma by taking $\tilde{h}$ to be $\overline{h}$.

First we fix $T_1 \in (0,T)$. For each $\varepsilon \in (0,\varepsilon_0]$, define a linear functional Λ_ε on $C_c(\partial\Omega \times (0,T_1))$ by

$$\Lambda_\varepsilon h = \int_0^{T_1} \int_{\partial\Omega_\varepsilon} \overline{h}(x,\tau)u(x,\tau)\, d\sigma(x)\, d\tau \quad \forall h \in C_c(\partial\Omega \times (0,T_1)).$$

By Lemma 2.2, we have

$$|\Lambda_\varepsilon h| \le C\|h\|_{L^\infty} \quad \forall h \in C_c(\partial\Omega \times (0,T_1)).$$

Therefore by the Riesz representation theorem (Theorem 7.2.8 of [Co]), there exists a Radon measure ν_ε on $\partial\Omega \times (0,T_1)$ such that

$$\begin{aligned} &\Lambda_\varepsilon h = \iint_{\partial\Omega\times(0,T)} h\, d\nu_\varepsilon \quad \forall h \in C_c(\partial\Omega \times (0,T_1)) \\ \Rightarrow \quad &\int_0^{T_1} \int_{\partial\Omega_\varepsilon} \overline{h}(x,\tau)u(x,\tau)\, d\sigma(x)\, d\tau = \iint_{\partial\Omega\times(0,T)} h\, d\nu_\varepsilon \quad \forall h \in C_c(\partial\Omega \times (0,T_1)). \end{aligned} \tag{2.19}$$

We now choose a sequence of monotone increasing functions $\{h_i\}_{i=1}^\infty \subset C_c(\partial\Omega \times (0,T_1))$, $0 \le h_i \le 1$ for any $i \in \mathbb{Z}^+$, satisfying

$$h_i(x,t) = 1 \quad \forall x \in \partial\Omega, \tfrac{1}{i} \le t \le T_1 - \tfrac{1}{i}, i \in \mathbb{Z}^+.$$

Putting $h = h_i$ in (2.19) and letting $i \to \infty$, by Lemma 2.2, we have

$$\sup_{0<\varepsilon\le\varepsilon_0} \nu_\varepsilon(\partial\Omega \times (0,T_1)) < \infty. \tag{2.20}$$

The measure ν_ε depends on T_1. However, by letting $T_1 \nearrow T$ it is clear that we could define a Radon measure, still denoted by ν_ε in $M_s(\partial\Omega \times (0,T))$ so that (2.20) remains valid for any $0 < T_1 < T$ and

$$\iint_{\partial\Omega\times(0,T)} h\, d\nu_\varepsilon = \int_0^T \int_{\partial\Omega_\varepsilon} \overline{h}(x,t)u(x,t)\, d\sigma(x)\, dt \quad \forall h \in C_c(\partial\Omega \times (0,T)). \tag{2.21}$$

By (2.20) and weak compactness any sequence $\{\varepsilon_j\}_{j=1}^\infty \subset (0,\varepsilon_0)$, $\varepsilon_j \to 0$ as $j \to \infty$, has a subsequence which we may assume without loss of generality to be the sequence itself that converges weakly to ν as $j \to \infty$ for some $\nu \in M_s(\partial\Omega \times (0,T))$. That is

$$\lim_{j\to\infty} \iint_{\partial\Omega\times(0,T)} h\, d\nu_{\varepsilon_j} = \iint_{\partial\Omega\times(0,T)} h\, d\nu \quad \forall h \in C_c(\partial\Omega \times (0,T))$$

and

$$\nu(\partial\Omega \times (0,T_1)) \le \sup_{0<\varepsilon\le\varepsilon_0} \nu_\varepsilon(\partial\Omega \times (0,T_1)) < \infty \quad \forall 0 < T_1 < T.$$

We now claim that the limit ν is unique. In order to prove this claim we extend $\overline{h}$ to a bounded continuous function on $\overline{\Omega} \times (0,T)$ vanishing near $t = 0, T$. For any $\varepsilon \in (0,\varepsilon_0)$, let $\eta_\varepsilon(x,t) = (\overline{\delta}(x) - \varepsilon)\overline{h}(x,t)$. For any $\varepsilon \in (0,\varepsilon_0)$, multiplying the heat equation for u by η_ε and integrating over $\Omega_\varepsilon \times (0,T)$, by integration by parts, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} u\left[\overline{h}\Delta\overline{\delta}(x) + 2\nabla\overline{h}\cdot\nabla\overline{\delta}(x) + (\overline{\delta}(x)-\varepsilon)\Delta\overline{h} + (\overline{\delta}(x)-\varepsilon)\overline{h}_t\right] dx\, dt \\
= &\; -\int_0^T \int_{\partial\Omega_\varepsilon} u(x,t)\overline{h}(x,t)\frac{\partial\overline{\delta}}{\partial N}(x)\, d\sigma(x)\, dt \\
= &\; \int_0^T \int_{\partial\Omega_\varepsilon} u(x,t)\overline{h}(x,t)\, d\sigma(x)\, dt
\end{aligned} \tag{2.22}$$

where $\partial/\partial N$ is the derivative with respect to the unit inner normal N at $\partial\Omega_\varepsilon$. Since

$$\overline{h}\Delta\overline{\delta}(x) = \nabla\overline{h}\cdot\nabla\overline{\delta}(x) = 0 \quad \forall x \in \Omega \setminus \Omega_{\varepsilon_0},$$

by (2.22), we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\varepsilon_0}} \left(u\overline{h}\Delta\overline{\delta}(x) + 2u\nabla\overline{h}\cdot\nabla\overline{\delta}(x)\right) dx\, dt \\
&\qquad + \int_0^T \int_{\Omega_\varepsilon} u\left[(\overline{\delta}(x)-\varepsilon)\Delta\overline{h} + (\overline{\delta}(x)-\varepsilon)\overline{h}_t\right] dx\, dt \\
= &\int_0^T \int_{\partial\Omega_\varepsilon} u(x,t)\overline{h}(x,t)\, d\sigma(x)\, dt \quad \forall 0 < \varepsilon < \varepsilon_0.
\end{aligned} \tag{2.23}$$

Putting $\varepsilon = \varepsilon_j$ in (2.23) and letting $j \to \infty$, by Lemma 2.2 and the Lebesgue dominated convergence theorem, we get (2.18). Hence the measure ν is uniquely determined by (2.18) and the claim holds. Since the sequence $\{\varepsilon_j\}_{j=1}^\infty$ is arbitrary, ν_ε converges weakly to ν as $\varepsilon \to 0$. This together with (2.21) implies that ν is the lateral trace of u.

Finally, since ν is finite on $\partial\Omega \times (0,T_1)$ for any $0 < T_1 < T$, by an approximation argument, one can show that (2.17) holds not only for any $h \in C_c(\partial\Omega \times (0,T))$ but also for any bounded continuous functions on $\partial\Omega \times (0,T)$. $\square$

Lemma 2.5. *There exists* $(\mu,\lambda)\in M(\Omega,\delta)\times M(\partial\Omega)$ *such that for any* $\eta\in C_0^\infty(\overline{\Omega})$, *we have*

$$\lim_{t\to 0^+}\int_\Omega u(x,t)\eta(x)\,dx=\int_\Omega \eta\,d\mu+\int_{\partial\Omega}\frac{\partial\eta}{\partial N}\,d\lambda \tag{2.24}$$

where $\partial/\partial N$ *is the derivative with respect to the unit inner normal* N *at* $\partial\Omega$.

Proof. For any $t\in(0,T)$, $0<\varepsilon\le\varepsilon_0$, let

$$w_\varepsilon(x,t):=G_\varepsilon(u(\cdot,t))(x):=\int_{\Omega_\varepsilon}G_\varepsilon(x,y)u(y,t)\,dy\quad\forall x\in\overline{\Omega}_\varepsilon$$

and

$$w(x,t):=G(u(\cdot,t))(x):=\int_\Omega G(x,y)u(y,t)\,dy\quad\forall x\in\Omega \tag{2.25}$$

be the Green potential of $u(\cdot,t)$ with respect to the domain Ω_ε and Ω respectively where $G(x,y)$, $G_\varepsilon(x,y)$, are the Green functions for the Laplacian $-\Delta$ on Ω, Ω_ε, respectively. Then $w_\varepsilon\ge 0$ on Ω_ε. By elliptic regularity theory [GT], for each $0<t<T$, w_ε is a classical solution of

$$\begin{cases}-\Delta w_\varepsilon=u(\cdot,t) & \text{in }\Omega_\varepsilon\\ \quad w_\varepsilon=0 & \text{on }\partial\Omega_\varepsilon\end{cases}$$

and $w(\cdot,t)$ is a classical solution of

$$-\Delta w(\cdot,t)=u(\cdot,t)\quad\text{ in }\Omega. \tag{2.26}$$

Now by Theorem 2.3 of [Wi] for any $x\in\Omega$, there exists a constant $C>0$ such that

$$G(x,y)\le C\delta(y)|x-y|^{1-n}\quad\forall x,y\in\Omega. \tag{2.27}$$

Since for any $x,y\in\Omega$, $G_\varepsilon(x,y)$ increases to $G(x,y)$ as $\varepsilon\to 0$, by (2.27), Lemma 2.1 and the Lebesgue dominated convergence theorem, $w_\varepsilon(x,t)$ increases to $w(x,t)$ as $\varepsilon\to 0$ for any $x\in\Omega$, $0<t<T$. Hence by (2.25), (2.27) and a direct computation, we get

$$\|w(\cdot,t)\|_{L^1(\Omega)}\le C\int_\Omega u(x,t)\delta(x)\,dx\quad\forall 0<t<T \tag{2.28}$$

for some constant $C>0$. Now

$$w_{\varepsilon,t}=G_\varepsilon(u_t)=G_\varepsilon(\Delta u)=-u(x,t)+\int_{\partial\Omega_\varepsilon}\frac{\partial G_\varepsilon}{\partial N_y}(x,y)u(y,t)\,d\sigma(y) \tag{2.29}$$

Integrating (2.29) over (t_1,t_2), $0<t_1<t_2<T_1<T$, we have

$$\begin{aligned} & w_\varepsilon(x,t_2) + \int_{t_2}^{T_1}\int_{\partial\Omega_\varepsilon} \frac{\partial G_\varepsilon}{\partial N_y}(x,y)u(y,\tau)\,d\sigma(y)\,d\tau \\ \le\ & w_\varepsilon(x,t_1) + \int_{t_1}^{T_1}\int_{\partial\Omega_\varepsilon} \frac{\partial G_\varepsilon}{\partial N_y}(x,y)u(y,\tau)\,d\sigma(y)\,d\tau\ . \end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\begin{aligned} & w(x,t_2) + \iint_{\partial\Omega\times(t_2,T_1)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau) \\ \le & w(x,t_1) + \iint_{\partial\Omega\times(t_1,T_1)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau) \quad \forall 0 < t_1 < t_2 < T_1 < T. \end{aligned}$$

Let $0 < T_1 < T$. This inequality shows that for each $x \in \Omega$, the function

$$H(x,t) := w(x,t) + \iint_{\partial\Omega\times(t,T_1)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau)$$

is decreasing in t. Hence for each $x \in \Omega$, $H^*(x) := \lim_{t\to 0} H(x,t)$ exists. Since

$$\iint_{\partial\Omega\times(0,T_1)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau)$$

exists and is finite, we conclude that

$$w^*(x) := \lim_{t\to 0^+} w(x,t)$$

exists. Letting $t \to 0$ in (2.28), by Lemma 2.1, $w^* \in L^1(\Omega)$.

Sinice by Theorem 2.3 of [Wi] there exists a constant $C > 0$ such that

$$|\nabla_y G(x,y)| \le C|x-y|^{1-n} \quad \forall x,y \in \Omega,$$

we have

$$\int_\Omega \left(\iint_{\partial\Omega\times(0,t)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau) \right) dx \le C\nu(\partial\Omega\times(0,t)) < \infty \quad \forall 0 < t < T.$$

Hence we have

$$\begin{aligned} & \int_\Omega |w^*(x) - w(x,t)|\,dx \\ \le & \int_\Omega |H^*(x) - H(x,t)|\,dx + \int_\Omega \left(\iint_{\partial\Omega\times(0,t)} \frac{\partial G}{\partial N_y}(x,y)\,d\nu(y,\tau) \right) dx \\ \le & \int_\Omega |H^*(x) - H(x,t)|\,dx + C\nu(\partial\Omega\times(0,t)). \end{aligned}$$

Letting $t \to 0$, we have

$$\lim_{t\to 0} \int_\Omega |w^*(x) - w(x,t)|\, dx = 0. \tag{2.30}$$

By (2.26) for each $0 < t < T$, $w(\cdot, t)$ is superharmonic in Ω. Hence w^* is superharmonic in Ω.

The rest of the proof follows from the arguments based on the proof of theorem 7 in [DK]. First of all, by (2.25), (2.30), Lemma 2.1 and the Fubini Theorem, for any $\eta \in C_0^\infty(\overline{\Omega})$, we have

$$\begin{aligned}\int_\Omega u(x,t)\eta(x)\, dx &= -\int_\Omega \left(\int_\Omega G(x,y)\Delta\eta(y)\, dy \right) u(x,t)\, dx \\ &= -\int_\Omega \left(\int_\Omega G(x,y) u(x,t)\, dx \right) \Delta\eta(y)\, dy \\ &= -\int_\Omega w(x,t)\Delta\eta(x)\, dx \\ &\to -\int_\Omega w^*\Delta\eta\, dx \qquad \text{as } t \to 0. \end{aligned} \tag{2.31}$$

On the other hand by the Riesz representation theorem for superharmonic functions [He], there exists $\mu \in M(\Omega, \delta)$ and a nonnegative harmonic function h in Ω such that

$$w^*(x) = \int_\Omega G(x,y)\, d\mu + h(x) \quad \forall x \in \Omega, \tag{2.32}$$

where $G(x,y)$ is the Green function for the Laplacian $-\Delta$ in Ω. Applying Martin representation theorem [He] to h, there exists $\lambda \in M(\partial\Omega)$ such that

$$h(x) = \int_{\partial\Omega} \frac{\partial G}{\partial N_y}(x,y)\, d\lambda(y) \quad \forall x \in \Omega. \tag{2.33}$$

By (2.31),(2.32), (2.33) and an argument similar to the proof of Theorem 7 of [DK] we get (2.24) and the lemma follows.

□

3 Existence of the lateral trace and the representation formula

In this section we will prove Theorem 1.1. The existence of lateral trace and initial trace have been established in Lemma 2.4 and Lemma 2.5 respectively. It remains to prove (1.2).

Lemma 3.1. *For any $x \in \Omega$, $0 < s < t < T$, we have*

$$\lim_{\varepsilon \to 0} \int_s^t \int_{\partial\Omega_\varepsilon} \frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau) u(y,\tau)\, d\sigma(y)\, d\tau = \iint_{\partial\Omega \times (s,t)} \frac{\partial G}{\partial N_y}(x,t;y,\tau)\, d\nu(y,\tau).$$

Proof. Let $x \in \Omega$ and $0 < \delta_1 < (t-s)/3$. We choose $0 < \varepsilon_1 < \min(\varepsilon_0, \delta(x)/2)$ and let $0 < \varepsilon < \varepsilon_1$. Then we have

$$\begin{aligned}
&\left| \int_s^t \int_{\partial\Omega_\varepsilon} \frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau) u(y,\tau)\, d\sigma(y)\, d\tau - \iint_{\partial\Omega \times (s,t)} \frac{\partial G}{\partial N_y}(x,t;y,\tau) d\nu(y,\tau) \right| \\
\le & \int_s^{t-\delta_1} \int_{\partial\Omega_\varepsilon} \left| \frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau) - \frac{\partial G}{\partial N_y}(x,t;y,\tau) \right| u(y,\tau)\, d\sigma(y)\, d\tau \\
+ & \left| \int_s^{t-\delta_1} \int_{\partial\Omega_\varepsilon} \frac{\partial G}{\partial N_y}(x,t;y,\tau) u(y,\tau)\, d\sigma(y) d\tau - \iint_{\partial\Omega \times (s,t-\delta_1)} \frac{\partial G}{\partial N_y}(x,t;y,\tau)\, d\nu(y,\tau) \right| \\
+ & \int_{t-\delta_1}^t \int_{\partial\Omega_\varepsilon} \frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau) u(y,\tau)\, d\sigma(y) d\tau + \iint_{\partial\Omega \times (t-\delta_1,t)} \frac{\partial G}{\partial N_y}(x,t;y,\tau)\, d\nu(y,\tau) \\
\equiv & I_1 + I_2 + I_3 + I_4. \qquad (3.1)
\end{aligned}$$

Since by Lemma 2.2,

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \nu_\varepsilon(\partial\Omega \times (0,t)) < \infty, \qquad (3.2)$$

we have

$$\begin{aligned}
I_1 \le & \nu_\varepsilon(\partial\Omega \times (s, t-\delta_1)) \max_{\substack{s \le \tau \le t-\delta_1 \\ y \in \partial\Omega_\varepsilon}} |\nabla_y (G - G_\varepsilon)(x,t;y,\tau)| \\
\le & C \max_{\substack{s \le \tau \le t-\delta_1 \\ y \in \partial\Omega_\varepsilon}} |\nabla_y (G - G_\varepsilon)(x,t;y,\tau)| \\
\to & 0 \qquad \text{as } \varepsilon \to 0. \qquad (3.3)
\end{aligned}$$

Next,

$$
\begin{aligned}
I_2 \leq & \left| \int_s^{t-\delta_1} \int_{\partial\Omega_\varepsilon} \frac{\partial G}{\partial N_y}(x,t;y,\tau) u(y,\tau)\, d\sigma(y)\, d\tau \right. \\
& \left. - \int_s^{t-\delta_1} \int_{\partial\Omega_\varepsilon} \frac{\partial G}{\partial N_{z(y)}}(x,t;z(y),\tau) u(y,\tau)\, d\sigma(y)\, d\tau \right| \\
& + \left| \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu_\varepsilon(z,\tau) - \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu(z,\tau) \right| \\
\leq & \nu_\varepsilon(\partial\Omega\times(s,t-\delta_1)) \max_{\substack{s\leq\tau\leq t-\delta_1 \\ y\in\partial\Omega_\varepsilon}} \left| \left(\frac{\partial G}{\partial N_y}(x,t;y,\tau) - \frac{\partial G}{\partial N_{z(y)}}(x,t;z(y),\tau) \right) \right| \\
& + \left| \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu_\varepsilon(z,\tau) - \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu(z,\tau) \right| \\
\leq & C \max_{\substack{s\leq\tau\leq t-\delta_1 \\ y\in\partial\Omega_\varepsilon}} \left| \left(\frac{\partial G}{\partial N_y}(x,t;y,\tau) - \frac{\partial G}{\partial N_{z(y)}}(x,t;z(y),\tau) \right) \right| \\
& + \left| \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu_\varepsilon(z,\tau) - \iint_{\partial\Omega\times(s,t-\delta_1)} \frac{\partial G}{\partial N_z}(x,t;z,\tau)\, d\nu(z,\tau) \right|. \\
\to & 0 \qquad \text{as } \varepsilon \to 0. \qquad (3.4)
\end{aligned}
$$

Finally, since for any $y \in \partial\Omega_\varepsilon$ we have $|x-y| \geq \delta(x)/2$ which together with the result of [H] implies that

$$
\frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau) \leq \frac{C_1}{(t-\tau)^{\frac{n+1}{2}}} e^{-\frac{C_2\delta^2(x)}{t-\tau}} \quad , \quad \forall y \in \partial\Omega_\varepsilon, 0<\tau<t\ , 0<\varepsilon<\varepsilon_1 \qquad (3.5)
$$

and

$$
\frac{\partial G}{\partial N_y}(x,t;y,\tau) \leq \frac{C_1}{(t-\tau)^{\frac{n+1}{2}}} e^{-\frac{C_2\delta^2(x)}{t-\tau}} \quad , \quad \forall y \in \partial\Omega, 0<\tau<t\ , \qquad (3.6)
$$

for some positive constants C_1 and C_2. Therefore by (3.2), (3.5) and (3.6), given any small $\varepsilon' > 0$, we can choose δ_1 sufficiently small such that

$$
\begin{aligned}
I_3 + I_4 \leq & \iint_{\partial\Omega_\varepsilon\times(t-\delta_1,t)} \frac{C_1 e^{-\frac{C_2\delta^2(x)}{t-\tau}}}{(t-\tau)^{\frac{n+1}{2}}}\, d\nu_\varepsilon(y,\tau) + \iint_{\partial\Omega\times(t-\delta_1,t)} \frac{C_1 e^{-\frac{C_2\delta^2(x)}{t-\tau}}}{(t-\tau)^{\frac{n+1}{2}}}\, d\nu(y,\tau) \\
\leq & C a_0 \varepsilon', \qquad (3.7)
\end{aligned}
$$

for some constant $C > 0$ where

$$
a_0 = \nu(\partial\Omega\times(0,t)) + \sup_{\varepsilon\in(0,\varepsilon_0)} \nu_\varepsilon(\partial\Omega\times(0,t)) < \infty.
$$

By (3.1), (3.3), (3.4) and (3.7), we have

$$\limsup_{\varepsilon\to 0}\left|\int_s^t\int_{\partial\Omega_\varepsilon}\frac{\partial G_\varepsilon}{\partial N_y}(x,t;y,\tau)u(y,\tau)\,d\sigma(y)\,d\tau-\iint_{\partial\Omega\times(s,t)}\frac{\partial G}{\partial N_y}(x,t;y,\tau)d\nu(y,\tau)\right|$$
$$\le Ca_0\varepsilon'. \tag{3.8}$$

Letting $\varepsilon'\to 0$ in (3.8) and the lemma follows. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: First of all, by letting $\varepsilon\to 0$ in (2.1), with the help of Lemma 3.1, we have for any $x\in\Omega$ and $0<\tau<t<T$,

$$\begin{aligned}u(x,t)=&\int_\Omega G(x,t;y,\tau)u(y,\tau)\,dy+\iint_{\partial\Omega\times(\tau,t)}\frac{\partial G}{\partial N_y}(x,t;y,s)\,d\nu(y,s)\\=&\int_\Omega G(x,t;y,0)u(y,\tau)\,dy+\int_\Omega\left(G(x,t;w,\tau)-G(x,t;w,0)\right)u(w,\tau)\,dw\\&+\iint_{\partial\Omega\times(\tau,t)}\frac{\partial G}{\partial N_y}(x,t;y,s)\,d\nu(y,s)\\=&J_1+J_2+J_3.\end{aligned}\tag{3.9}$$

By Lemma 2.5, we have

$$\lim_{\tau\to 0}J_1=\int_\Omega G(x,t;y,0)\,d\mu(y)+\int_{\partial\Omega}\frac{\partial G}{\partial N_y}(x,t;y,0)\,d\lambda(y)\ . \tag{3.10}$$

By the monotone convergence theorem, we have

$$\lim_{\tau\to 0}J_3=\iint_{\partial\Omega\times(0,t)}\frac{\partial G}{\partial N_y}(x,t;y,s)\,d\nu(y,s)\ . \tag{3.11}$$

Finally for any $x \in \Omega$, $0 < \tau < \frac{t}{2} < t < T$, we have

$$
\begin{aligned}
J_2 = &\left| \int_\Omega u(w,\tau) \int_0^\tau G_s(x,t;w,s)\,ds\,dw \right| \\
\le &\left| \int_{\overline{\Omega}_{\varepsilon_0}} u(w,\tau) \int_0^\tau G_s(x,t;w,s)\,ds\,dw \right| \\
&\quad + \left| \int_{\Omega \setminus \overline{\Omega}_{\varepsilon_0}} u(w,\tau) \int_0^1 \int_0^\tau \frac{dG_s}{da}(x,t;z(w)+a\delta(w)N_{z(w)},s)\,ds\,da\,dw \right| \\
\le &C\tau \int_{\overline{\Omega}_{\varepsilon_0}} u(w,\tau)\delta(w)\,dw \\
&+ \left| \int_\Omega u(w,\tau)\delta(w) \int_0^1 \int_0^\tau \sum_{j=1}^n N_j(z(w)) \frac{\partial G_s}{\partial y_j}(x,t;z(w)+a\delta(w)N_{z(w)},s)\,ds\,da\,dw \right| \\
\le &C\tau \int_\Omega u(w,\tau)\delta(w)\,dw \\
\to &0 \quad \text{as } \tau \to 0
\end{aligned} \tag{3.12}
$$

where $N_z = (N_1(z), \cdots, N_n(z))$ is the unit inner normal on $\partial\Omega$ for any $z \in \partial\Omega$. By (3.9), (3.10), (3.11) and (3.12), (1.2) follows.

To prove the converse part of the theorem, given $(\mu, \lambda, \nu) \in M(\Omega, \delta) \times M(\partial\Omega) \times M_s(\partial\Omega \times (0,T))$, let u be given by (1.2) and

$$
\begin{cases}
v_1(x,t) = \displaystyle\int_\Omega G(x,t;y,0)\,d\mu(y) + \int_{\partial\Omega} \frac{\partial G}{\partial N_y}(x,t;y,0)\,d\lambda(y) \quad \forall x \in \overline{\Omega}, t > 0, \\
v_2(x,t) = \displaystyle\iint_{\partial\Omega \times (0,t)} \frac{\partial G}{\partial N_y}(x,t;y,s)\,d\nu(y,s) \quad \forall x \in \Omega, 0 < t < T
\end{cases}
$$

where $\partial/\partial N_y$ is the derivative with respect to the unit inner normal N_y at $y \in \partial\Omega$. Then

$$
u(x,t) = v_1(x,t) + v_2(x,t) \quad \forall (x,t) \in Q_T.
$$

By the results of [H] and standard parabolic theory ([F], [LSU]), $v_1 \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ satisfies

$$
\begin{cases}
v_{1,t} = \Delta v_1 & \text{in } \Omega \times (0,\infty) \\
v_1 = 0 & \text{on } \partial\Omega \times (0,\infty)
\end{cases}
$$

and

$$
\lim_{t \to 0} \int_\Omega v_1 \eta\,dx = \int_\Omega \eta\,d\mu + \int_{\partial\Omega} \frac{\partial \eta}{\partial N}\,d\lambda \quad \forall \eta \in C_0^\infty(\overline{\Omega}).
$$

Hence v_1 has initial traces (μ, λ) and zero lateral trace. Thus it suffices to prove that v_2 satisfies the heat equation in Q_T, has initial traces $(0,0)$ and lateral trace ν.

We choose a sequence ν_j where $d\nu_j = \varphi_j\, d\sigma\, dt, \varphi_j \in C(\partial\Omega \times [0,T])$, such that

$$\lim_{j\to\infty} \int_{\partial\Omega\times(0,T_1)} h\, d\nu_j = \int_{\partial\Omega\times(0,T_1)} h\, d\nu \quad \forall h \in C(\partial\Omega \times [0,T_1]), 0 < T_1 < T.$$

Let

$$v_{2,j}(x,t) = \iint_{\partial\Omega\times(0,t)} \frac{\partial G}{\partial N_y}(x,t;y,s)\varphi_j\, d\sigma ds \quad \forall x \in \Omega, 0 < t < T, j \in \mathbb{Z}^+. \tag{3.13}$$

By standard parabolic theory each $v_{2,j} \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ is a classical solution of the heat equation in Q_T with initial traces $(0,0)$ and lateral value φ_j. Hence $v_{2,j}$ has $(0,0,\nu_j)$ as its trace triple for any $j \in \mathbb{Z}^+$. Moreover, for each $(x,t) \in Q_T$, $v_{2,j}(x,t)$ converges to $v_2(x,t)$ as $j \to \infty$ and for each $t \in (0,T)$, $v_{2,j}(\cdot,t)$ converges to $v_2(\cdot,t)$ in $L^1(\Omega)$ as $j \to \infty$. Also for each $T_1 \in (0,T)$ the sequence $\{\|v_{2,j}\|_{L^1(Q_{T_1})}\}_{j=1}^{\infty}$ are uniformly bounded . Without loss of generality we may assume that $v_{2,j}$ converges weakly to v_2 in each Q_{T_1} as $j \to \infty$ for any $0 < T_1 < T$. Since $v_{2,j}$ satisfies the heat equation in Q_T, we have

$$\begin{aligned} &\int_0^T \int_\Omega v_{2,j}(\varphi_t + \Delta\varphi)\, dx\, dt = 0\ , \quad \forall \varphi \in C_c^\infty(Q_T), j \in \mathbb{Z}^+ \\ \Rightarrow \quad &\int_0^T \int_\Omega v_2(\varphi_t + \Delta\varphi)\, dx\, dt = 0\ , \quad \forall \varphi \in C_c^\infty(Q_T) \quad \text{ as } j \to \infty. \end{aligned}$$

Hence v_2 is a weak solution of the heat equation. Thus by standard by parabolic theory [LSU] u is a classical solution of the heat equation in Q_T.

Next, let ν' be the lateral trace of v_2. Then by Lemma 2.4 for any $h \in C_c(\partial\Omega \times (0,T))$, we have

$$\iint_{\partial\Omega\times(0,T)} h\, d\nu' = \int_0^T \int_\Omega v_2 \left[\overline{h}\Delta\overline{\delta} + 2\nabla\overline{h} \cdot \nabla\overline{\delta} + \overline{\delta}\Delta\overline{h} + \overline{\delta}\overline{h}_t\right]\, dx\, dt \tag{3.14}$$

where $\overline{h}$ is a continuous extension of h to $\overline{\Omega} \times (0,T)$ vanishing near $t = 0, T$, such that $\overline{h}$ is constant along each inner normal direction of $\partial\Omega$ in $(\overline{\Omega} \setminus \Omega_{\varepsilon_0}) \times (0,T)$.

On the other hand, since the lateral trace of $v_{2,j}$ is ν_j, by putting $u = v_{2,j}$ and $\nu = \nu_j$ in (2.18) and letting $j \to \infty$, we get for any $h \in C_c(\partial\Omega \times (0,T))$, ν satisfies (2.18) with $u = v_2$ snd $\overline{h}$ being a continuous extension of h to $\overline{\Omega} \times (0,T)$ vanishing near $t = 0, T$, such that $\overline{h}$ is constant along each inner normal direction of $\partial\Omega$ in $(\overline{\Omega} \setminus \Omega_{\varepsilon_0}) \times (0,T)$. Hence by (2.18) and (3.14), $\nu' \equiv \nu$.

Multiplying the heat equation satisfied by $v_{2,j}$ with some $\eta \in C_0^\infty(\overline{\Omega})$, by integration by parts, we have

$$
\begin{aligned}
& \int_\Omega v_{2,j}(x,t)\eta(x)\,dx = \int_0^t \int_\Omega v_{2,j}\Delta\eta\,dx\,ds + \int_0^t \int_{\partial\Omega} \varphi_j \frac{\partial\eta}{\partial N}\,d\sigma\,dt \\
\Rightarrow\quad & \int_\Omega v_2(x,t)\eta(x)\,dx = \int_0^t \int_\Omega v_2\Delta\eta\,dx\,ds + \iint_{\partial\Omega\times(0,t)} \frac{\partial\eta}{\partial N}\,d\nu \qquad \text{as } j\to\infty \\
\Rightarrow\quad & \lim_{t\to 0}\int_\Omega v_2(x,t)\eta(x)\,dx = 0.
\end{aligned}
$$

Hence the initial trace of v_2 is $(0,0)$ and the theorem follows. □

References

[A] D.G. Aronson, *Non-negative solutions of linear parabolic equations*, Annali della Scuola Normale Superiore di Pisa 22 (1968), no. 4, 607–694.

[AC] D.G. Aronson and L.A. Caffarelli, *The initial trace of a solution of the porous medium equation*, Trans. Amer. Math. Soc. 280 (1983), 351–366.

[C] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, U.S.A., 1984.

[CK1] K.S. Chou and Y.C. Kwong, *The trace triple for nonnegative solutions of generalized porous medium equations*, Calulus of Variation and PDE's 58 (2019), Article number: 23.

[CK2] K.S. Chou and Y.C. Kwong, *Nonnegative solutions of the porous medium equation with continuous lateral boundary data*, preprint 2019.

[Co] D.L. Cohn, Measure theory, Second edition, Birkhäuser advanced texts Basler Lehrbucher, Springer, New York 2013.

[DK] B.E.J. Dahlberg and C.E. Kenig, *Non-negative solution of the initial-Dirichlet problem for generalized porous medium equations in cylinders*, J. Amer. Math. Soc. 1 (1988), no. 2, 401–412.

[DiH1] E. DiBenedetto and M.A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. Amer. Soc. 314 (1989), 187–224.

[DiH2] E. DiBenedetto and M.A. Herrero, *Nonnegative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when* $1 < p < 2$, Arch. Rational Mech. Anal. 111(1990), 225–290.

[EG] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, CRC Press, Boca Raton, 2015.

[FGS] E.B. Fabes, N. Garofalo and S. Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. 30 (1986), no. 4, 536-565.

[F] A. Friedman, *Partial Differential Equations of Parabolic Type*, Dover Publications, Mineola, N.Y., U.S.A., 2008.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, Heidelberg 2001.

[He] L.L. Helms, *Introduction to potential theory*, Wiley-Interscience, 1969.

[HIT] K. Hisa, K. Ishige and J. Takahashi, *Initial traces and solvability for a semilinear heat equation on a half space of* $\mathbb{R}^N$, Trans. Amer. Math. Soc. electronically published on May 9, 2023, https://doi.org/10.1090/tran/8922.

[H] K.M. Hui, *A Fatou theorem for the solution of the heat equation at the corner points of a cylinder*, Trans. Amer. Math. Soc. 333 (1992), no 2, 607-642.

[HW] P. Hartman and A. Winter, *On the solutions of the equation of heat conduction*, Amer. J. Math. 72 (1950), 367–395.

[Is] K. Ishige, *On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation*, SIAM J. Math. Anal.27 (1996), 1235–1260.

[IsK] K. Ishige and J. Kinnunen, *Intial trace for a doubly nonlinear parabolic equation*, J. Evol. Equ. 11 (2011), 943–957.

[I] S. Ito, *Diffusion Equations*, Translations of Math. Mono vol. 114, Amer. Math. Soc., Providence, 1992.

[K] J.T. Kemper, *Temperatures in several variables: kernel functions, representations and parabolic boundary values*, Trans. Amer. Math. Soc., 167 (1972), 243-262.

[LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uraltceva, *Linear and quasilinear equations of parabolic type,* Transl. Math. Mono. vol. 23, Amer. Math. Soc., Providence, R.I., U.S.A., 1968.

[MV1] M. Marcus and L. Véron, *Trace aux bord des solutions positives d'équations elliptiques et paraboliques non linéaires: résultats d'existence et d'unicité*, C.R. Acad. Sci. Paris 323, Ser. I (1996), 603–608.

[MV2] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rat. Mech. Anal. 144 (1998), 201–231.

[MV3] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. 77 (1998), 481–524

[MV4] M. Marcus and L. Véron, *Initial trace of positive solutions of some nonlinear parabolic equations*, Commun. Partial Differential Equations 24 (1999), no. 7&8, 1445-1499.

[M] M. Murata, *Integral representation of nonnegative solutions for parabolic equations and elliptic Martin boundaries*, J. Functional Analysis 245 (2007), 177-212.

[W1] D.V. Widder, *Positive temperatures on an infintie rod*, Trans. Amer. Math. Soc 55 (1944), 85-95.

[W2] D.V. Widder, *Positive temperatures on a semi-infinite rod*, Trans. Amer. Math. Soc. 75 (1953), 510–525.

[W3] D.V. Widder, *The Heat Equation*, Academic Press, NY 1975.

[Wi] K.O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. 21 (1968), 17-37.