Rerouting Planar Curves and Disjoint Paths^{*}

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Abstract

In this paper, we consider a transformation of k disjoint paths in a graph. For a graph and a pair of k disjoint paths \mathcal{P} and \mathcal{Q} connecting the same set of terminal pairs, we aim to determine whether \mathcal{P} can be transformed to \mathcal{Q} by repeatedly replacing one path with another path so that the intermediates are also k disjoint paths. The problem is called DISJOINT PATHS RECONFIGURATION. We first show that DISJOINT PATHS RECONFIGURATION is PSPACE-complete even when k = 2. On the other hand, we prove that, when the graph is embedded on a plane and all paths in \mathcal{P} and \mathcal{Q} connect the boundaries of two faces, DISJOINT PATHS RECONFIGURATION can be solved in polynomial time. The algorithm is based on a topological characterization for rerouting curves on a plane using the algebraic intersection number. We also consider a transformation of disjoint s-t paths as a variant. We show that the disjoint s-t paths reconfiguration problem in planar graphs can be determined in polynomial time, while the problem is PSPACE-complete in general.

1 Introduction

1.1 Disjoint Paths and Reconfiguration

The disjoint paths problem is a classical and important problem in algorithmic graph theory and combinatorial optimization. In the problem, the input consists of a graph G = (V, E) and 2kdistinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, called *terminals*, and the task is to find k vertex-disjoint paths P_1, \ldots, P_k such that P_i connects s_i and t_i for $i = 1, \ldots, k$ if they exist. A tuple $\mathcal{P} =$ (P_1, \ldots, P_k) of paths satisfying this condition is called a *linkage*. The disjoint paths problem has attracted attention since 1980s because of its practical applications to transportation networks, network routing [52], and VLSI-layout [21, 33]. When the number k of terminal pairs is part of the input, the disjoint paths problem was shown to be NP-hard by Karp [28], and it remains NP-hard even for planar graphs [34]. For the case of k = 2, polynomial-time algorithms were presented in [50, 51, 56], while the directed variant was shown to be NP-hard [20]. Later, for

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the case when the graph is undirected and k is a fixed constant, Robertson and Seymour [47] gave a polynomial-time algorithm based on the graph minor theory, which is one of the biggest achievements in this area. Although the setting of the disjoint paths problem is quite simple and easy to understand, a deep theory in discrete mathematics is required to solve the problem, which is a reason why this problem has attracted attention in the theoretical study of algorithms.

An interesting special case is the problem in planar graphs. In the early stages of the study of the disjoint paths problem, for the case when G is embedded on a plane and all the terminals are on one face or two faces, polynomial-time algorithms were given in [45, 53, 54]. In the series of graph minor papers, the disjoint paths problem on a plane or on a fixed surface was solved for fixed k [46]. For the planar case, faster algorithms were presented in [1, 43, 44]. The directed variant of the problem can be solved in polynomial time if the input digraph is planar and k is a fixed constant [14, 49].

In this paper, we consider a transformation of linkages in a graph. Roughly, in a transformation, we pick up one path among the k paths in a linkage, and replace it with another path to obtain a new linkage. To give a formal definition, suppose that G is a graph and $s_1, \ldots, s_k, t_1, \ldots, t_k$ are distinct terminals. For two linkages $\mathcal{P} = (P_1, \ldots, P_k)$ and $\mathcal{Q} = (Q_1, \ldots, Q_k)$, we say that \mathcal{P} is adjacent to \mathcal{Q} if there exists $i \in \{1, \ldots, k\}$ such that $P_j = Q_j$ for $j \in \{1, \ldots, k\} \setminus \{i\}$ and $P_i \neq Q_i$. We say that a sequence $\langle \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\ell \rangle$ of linkages is a reconfiguration sequence from \mathcal{P}_1 to \mathcal{P}_ℓ if \mathcal{P}_i and \mathcal{P}_{i+1} are adjacent for $i = 1, \ldots, \ell - 1$. If such a sequence exists, we say that \mathcal{P}_1 is reconfigurable to \mathcal{P}_ℓ . In this paper, we focus on the following reconfiguration problem, which we call DISJOINT PATHS RECONFIGURATION.

DISJOINT PATHS RECONFIGURATION **Input.** A graph G = (V, E), distinct terminals $s_1, \ldots, s_k, t_1, \ldots, t_k$, and two linkages \mathcal{P} and \mathcal{Q} . **Question.** Is \mathcal{P} reconfigurable to \mathcal{Q} ?

The problem can be regarded as the problem of deciding the reachability between linkages via rerouting paths. Such a problem falls in the area of *combinatorial reconfiguration*; see Section 1.3 for prior work on combinatorial reconfiguration. Note that DISJOINT PATHS RECONFIGURATION is a decision problem that just returns "YES" or "NO" and does not necessarily find a reconfiguration sequence when the answer is YES.

Although our study is motivated by theoretical interest in the literature of combinatorial reconfiguration, the problem can model rerouting problem in a telecommunication network as follows. Suppose that a linkage represents routing in a telecommunication network, and we want to modify linkage \mathcal{P} to another linkage \mathcal{Q} which is better than \mathcal{P} in some sense. If we can change only one path in a step in the network for some technical reasons, and we have to keep a linkage in the modification process, then this situation is modeled as DISJOINT PATHS RECONFIGURATION.

We also study a special case of the disjoint paths problem when $s_1 = \cdots = s_k$ and $t_1 = \cdots = t_k$, which we call the *disjoint s-t paths problem*. In the problem, for a graph and two terminals s and t, we seek for k internally vertex-disjoint (or edge-disjoint) paths connecting s and t. It is well-known that the disjoint s-t paths problem can be solved in polynomial time. The study of disjoint s-t paths was originated from Menger's min-max theorem [37] and the max-flow algorithm by Ford and Fulkerson [19]. Faster algorithms for finding maximum disjoint s-t paths or a maximum s-t flow have been actively studied in particular for planar graphs; see e.g. [16, 27, 29, 59].

In the same way as DISJOINT PATHS RECONFIGURATION, we consider a reconfiguration of internally vertex-disjoint s-t paths. Let G = (V, E) be a graph with two distinct terminals s

and t. We say that a set $\mathcal{P} = \{P_1, \ldots, P_k\}$ of k paths in G is an s-t linkage if P_1, \ldots, P_k are internally vertex-disjoint s-t paths. Note that \mathcal{P} is not a tuple but a set, that is, we ignore the ordering of the paths in \mathcal{P} . We say that s-t linkages \mathcal{P} and \mathcal{Q} are adjacent if $\mathcal{Q} = (\mathcal{P} \setminus P) \cup \{Q\}$ for some s-t paths P and Q with $P \neq Q$. We define the reconfigurability of s-t linkages in the same way as linkages. We consider the following problem.

DISJOINT s-t PATHS RECONFIGURATION **Input.** A graph G = (V, E), distinct terminals s and t, and two s-t linkages \mathcal{P} and \mathcal{Q} . **Question.** Is \mathcal{P} reconfigurable to \mathcal{Q} ?

1.2 Our Contributions

Since finding disjoint s-t paths is an easy combinatorial optimization problem, one may expect that DISJOINT s-t PATHS RECONFIGURATION is also tractable. However, this is not indeed the case. We show that DISJOINT s-t PATHS RECONFIGURATION is PSPACE-hard even when k = 2.

Theorem 1.1. The DISJOINT s-t PATHS RECONFIGURATION is PSPACE-complete even when k = 2 and the maximum degree of G is four.

Note that DISJOINT *s*-*t* PATHS RECONFIGURATION can be easily reduced to DISJOINT PATHS RECONFIGURATION by splitting each of *s* and *t* into *k* terminals. Thus, this theorem implies the PSPACE-hardness of DISJOINT PATHS RECONFIGURATION with k = 2.

In this paper, we mainly focus on the problems in planar graphs. To better understand DIS-JOINT PATHS RECONFIGURATION in planar graphs, we show a topological necessary condition.

Topological conditions play important roles in the disjoint paths problem. If there exist disjoint paths connecting terminal pairs in a graph embedded on a surface Σ , then obviously there must exist disjoint curves on Σ connecting them. For example, when terminals s_1, s_2, t_1 and t_2 lie on the outer face F in a plane graph G in this order, there exist no disjoint curves connecting the terminal pairs in the disk $\Sigma = \mathbb{R}^2 \setminus F$, and hence we can conclude that G contains no disjoint paths. Such a topological condition is used to design polynomial-time algorithms for the disjoint paths problem with k = 2 [50, 51, 56], and to deal with the problem on a disk or a cylinder [45]. When Σ is a plane (or a sphere), we can always connect terminal pairs by disjoint curves on Σ , and hence nothing is derived from the above argument. Indeed, Robertson and Seymour [46] showed that if the input graph is embedded on a surface and the terminals are mutually "far apart," then desired disjoint paths always exist.

In contrast, as we will show below in Theorem 1.2, there exists a topological necessary condition for the reconfigurability of disjoint paths. Thus, even when the terminals are mutually far apart, the reconfiguration of disjoint paths is not always possible. This shows a difference between the disjoint paths problem and DISJOINT PATHS RECONFIGURATION.

In order to formally discuss the topological necessary condition, we consider the reconfiguration of curves on a surface. Suppose that Σ is a surface and let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be distinct points on Σ . By abuse of notation, we say that $\mathcal{P} = (P_1, \ldots, P_k)$ is a *linkage* if it is a collection of disjoint simple curves on Σ such that P_i connects s_i and t_i . We also define the adjacency and reconfiguration sequences for linkages on Σ in the same way as linkages in a graph. Then, the reconfigurability between two linkages on a plane can be characterized with a word w_j associated to Q_j which is an element of the free group F_k generated by x_1, \ldots, x_k as follows; see Section 3 for the definition of w_j .

Theorem 1.2. Let $\mathcal{P} = (P_1, \ldots, P_k)$ and $\mathcal{Q} = (Q_1, \ldots, Q_k)$ be linkages on a plane (or a sphere). Then, \mathcal{P} is reconfigurable to \mathcal{Q} if and only if $w_j \in \langle x_j \rangle$ for any $j \in \{1, \ldots, k\}$, where $\langle x_j \rangle$ denotes the subgroup generated by x_j .



Figure 1: (Left) An example on the plane where (P_1, P_2) is not reconfigurable to (Q_1, Q_2) . (Right) An example in a graph where the condition in Theorem 1.2 holds but (P_1, P_2) is not reconfigurable to (Q_1, Q_2) .

See Figure 1 (left) for an example. It is worth noting that, if k = 2 and Σ is a connected orientable closed surface of genus $g \ge 1$, then such a topological necessary condition does not exist, i.e., the reconfiguration is always possible; see Appendix A.

For a graph embedded on a plane, we can identify paths and curves. Then, Theorem 1.2 gives a topological necessary condition for DISJOINT PATHS RECONFIGURATION in planar graphs. However, the converse does not necessarily hold: even when the condition in Theorem 1.2 holds, an instance of DISJOINT PATHS RECONFIGURATION may have no reconfiguration sequence. See Figure 1 (right) for a simple example. The polynomial solvability of DISJOINT PATHS RECONFIGURATION in planar graphs is open even for the case of k = 2.

With the aid of the topological necessary condition, we design polynomial-time algorithms for special cases, in which all the terminals are on a single face (called *one-face instances*), or s_1, \ldots, s_k are on some face and t_1, \ldots, t_k are on another face (called *two-face instances*). Note that one/two-face instances have attracted attention in the disjoint paths problem [45, 53, 54], in the multicommodity flow problem [39, 40], and in the shortest disjoint paths problem [8, 13, 15, 31]. We show that any one-face instance of DISJOINT PATHS RECONFIGURATION has a reconfiguration sequence (Proposition 4.1). Moreover, we prove a topological characterization for two-face instances of DISJOINT PATHS RECONFIGURATION (Theorem 4.2), which leads to a polynomial-time algorithm in this case.

Theorem 1.3. When the instances are restricted to two-face instances, DISJOINT PATHS RE-CONFIGURATION can be solved in polynomial time.

Based on this theorem, we give a polynomial-time algorithm for DISJOINT s-t PATHS RE-CONFIGURATION in planar graphs.

Theorem 1.4. There is a polynomial-time algorithm for DISJOINT s-t PATHS RECONFIGURA-TION in planar graphs.

Note that the number k of paths in Theorems 1.3 and 1.4 can be part of the input.

It is well known that G has an s-t linkage of size k if and only if G has no s-t separator of size k - 1 (Menger's theorem). The characterization for two-face instances (Theorem 4.2) implies the following theorem, which is interesting in the sense that one extra s-t connectivity is sufficient to guarantee the existence of a reconfiguration sequence.

Theorem 1.5. Let G = (V, E) be a planar graph with distinct vertices s and t, and let \mathcal{P} and \mathcal{Q} be s-t linkages. If there is no s-t separator of size k, then \mathcal{P} is reconfigurable to \mathcal{Q} .

As mentioned above, the polynomial solvability of DISJOINT PATHS RECONFIGURATION in planar graphs is open even for the case of k = 2. On the other hand, when k is not bounded, DISJOINT PATHS RECONFIGURATION is PSPACE-complete as the next theorem shows.

Theorem 1.6. The DISJOINT PATHS RECONFIGURATION is PSPACE-complete when the graph G is planar and of bounded bandwidth.

Here, we recall the definition of the bandwidth of a graph. Let G = (V, E) be an undirected graph. Consider an injective map $\pi: V \to \mathbb{Z}$. Then, the *bandwidth* of π is defined as $\max\{|\pi(u) - \pi(v)| \mid \{u, v\} \in E\}$. The *bandwidth* of G is defined as the minimum bandwidth of all injective maps $\pi: V \to \mathbb{Z}$.

1.3 Related Work

Combinatorial reconfiguration is an emerging field in discrete mathematics and theoretical computer science. In typical problems of combinatorial reconfiguration, we consider two discrete structures, and ask whether one can be transformed to the other by a sequence of local changes. See surveys of Nishimura [38] and van den Heuvel [57].

Path reconfiguration problems have been studied in this framework. The apparently first problem is the shortest path reconfiguration, introduced by Kaminski et al. [26]. In this problem, we are given an undirected graph with two designated vertices s, t and two s-t shortest paths P and Q. Then, we want to decide whether P can be transformed to Q by a sequence of one-vertex changes in such a way that all the intermediate s-t paths remain the shortest. Bonsma [6] proved that the shortest path reconfiguration is PSPACE-complete, but polynomialtime solvable when the input graph is chordal or claw-free. Bonsma [7] further proved that the problem is polynomial-time solvable for planar graphs. Wrochna [60] proved that the problem is PSPACE-complete even for graphs of bounded bandwidth. Gajjar et al. [22] proved that the problem is polynomial-time solvable for circle graphs, circular-arc graphs, permutation graphs and hypercubes. They also considered a variant where a change can involve k successive vertices; in this variant they proved that the problem is PSPACE-complete even for line graphs. Properties of the adjacency relation in the shortest path reconfiguration have also been studied [4, 5].

Another path reconfiguration problem has been introduced by Amiri et al. [3] who were motivated by a problem in software defined networks. In their setup, we are given a directed graph with edge capacity and two designated vertices s, t. We are also given k pairs of st paths $(P_i, Q_i), i = 1, 2, \ldots, k$, where the number of paths among P_1, P_2, \ldots, P_k (and among Q_1, Q_2, \ldots, Q_k respectively) traversing an edge is at most the capacity of the edge. The problem is to determine whether one set of paths can be transformed to the other set of paths by a sequence of the following type of changes: specify one vertex v and then switch the usable outgoing edges at v from those in the P_i to those in the Q_i . In each of the intermediate situations, there must be a unique path through usable edges in $P_i \cup Q_i$ for each i. See [3] for the precise problem specification. Amiri et al. [3] proved that the problem is NP-hard even when k = 2. For directed acyclic graphs, they also proved that the problem is NP-hard (for unbounded k) but fixed-parameter tractable with respect to k. A subsequent work [2] studied an optimization variant in which the number of steps is to be minimized when a set of "disjoint" changes can be performed simultaneously.

Matching reconfiguration in bipartite graphs can be seen as a certain type of disjoint paths reconfiguration problems. In matching reconfiguration, we are given two matchings (with extra properties) and want to determine whether one matching can be transformed to the other matching by a sequence of local changes. There are several choices for local changes. One of the most studied local change rules is the token jumping rule, where we remove one edge and add one edge at the same time. Ito et al. [24] proved that the matching reconfiguration (under the token jumping rule) can be solved in polynomial time.¹

¹The theorem by Ito et al. [24] only gave a polynomial-time algorithm for a different local change, the so-called

To see a connection of matching reconfiguration with disjoint paths reconfiguration, consider the matching reconfiguration problem in bipartite graphs G under the token jumping rule, where we are given two matchings M, M' of G. Then, we add two extra vertices s, t to G, and for each edge $e \in M$ (and M') we construct a unique s-t path of length three that passes through e. This way, we obtain two s-t linkages \mathcal{P} and \mathcal{P}' from M and M', respectively. It is easy to observe that \mathcal{P} can be reconfigured to \mathcal{P}' in DISJOINT s-t PATHS RECONFIGURATION if and only if M can be reconfigured to M' in the matching reconfiguration problem in G.

There are a lot of studies on the disjoint paths problem and its variants. A natural variant is to maximize the number of vertex-disjoint paths connecting terminal pairs, which is called the maximum disjoint paths problem. Since this problem is NP-hard when k is part of the input, it has been studied from the viewpoint of approximation algorithms. It is known that a simple greedy algorithm achieves an approximation of factor $O(\sqrt{|V|})$ [32], which is the current best approximation ratio for general graphs. Chuzhoy and Kim [9] improved this factor to $\tilde{O}(n^{1/4})$ for grid graphs, and Chuzhoy et al. [10] gave an $\tilde{O}(|V|^{9/19})$ -approximation algorithm for planar graphs. On the negative side, the maximum disjoint paths problem is $2^{\Omega(\sqrt{\log |V|})}$ hard to approximate under some complexity assumption [11], which is the current best hardness result.

Another variant is the shortest non-crossing walks problem. In a graph embedded on a plane, we say that walks are non-crossing if they do not cross each other or themselves, while they may share edges or vertices; see [17] for a formal definition. We can see that this concept is positioned in between disjoint paths and disjoint curves. In the shortest non-crossing walks problem, we are given a plane graph with non-negative edge lengths and k terminal pairs that lie on the boundary of h polygonal obstacles. The objective is to find k non-crossing walks that connect terminal pairs in G of minimum total length, if they exist. For the case of h = 2, Takahashi et al. [55] gave an $O(|V| \log |V|)$ -time algorithm, and Papadopoulou [42] proposed a linear-time algorithm. For general h, Erickson and Nayyeri [17] gave a $2^{O(h^2)}|V| \log k$ -time algorithm, which runs in polynomial time when h is a fixed constant. They also showed that the existence of a feasible solution can be determined in linear time.

1.4 Organization

In Section 2, we introduce some notation and basic concepts in topology. Section 3 deals with rerouting disjoint curves, giving the proof of Theorem 1.2. In Sections 4 and 5, we prove Theorems 1.3, 1.4, and 1.5. Hardness results (Theorems 1.1 and 1.6) are then proven in Section 6.

2 Preliminaries

For a positive integer k, let $[k] = \{1, 2, \dots, k\}$.

Let G = (V, E) be a graph. For a subgraph H of G, the vertex set of H is denoted by V(H). Similarly, for a path P, let V(P) denote the set of vertices in P. For $X \subseteq V$, let N(X) be the set of vertices in $V \setminus X$ that are adjacent to the vertices in X. For a vertex set $U \subseteq V$, let $G \setminus U$ denote the graph obtained from G by removing all the vertices in U and the incident edges. For a path P in G, we denote $G \setminus V(P)$ by $G \setminus P$ to simplify the notation. For disjoint vertex sets $X, Y \subseteq V$, we say that a vertex subset $U \subseteq V \setminus (X \cup Y)$ separates X and Y if $G \setminus U$ contains no path between X and Y. For distinct vertices $s, t \in V, U \subseteq V \setminus \{s, t\}$ is called an *s*-t separator if U separates $\{s\}$ and $\{t\}$.

token addition and removal rule. However, their result can easily be adapted to the token jumping rule, too. See [25].



Figure 2: Local intersection numbers of curves C_1 and C_2 at p.



Figure 3: Algebraic intersection numbers of paths C_1 and C_2 on a graph.

For DISJOINT PATHS RECONFIGURATION (resp. DISJOINT s-t PATHS RECONFIGURATION), an instance is denoted by a triplet $(G, \mathcal{P}, \mathcal{Q})$, where G is a graph and \mathcal{P} and \mathcal{Q} are linkages (resp. s-t linkages). Note that we omit the terminals, because they are determined by \mathcal{P} and \mathcal{Q} . Since any instance has a trivial reconfiguration sequence when k = 1, we may assume that $k \geq 2$. For linkages (resp. s-t linkages) \mathcal{P} and \mathcal{Q} , we denote $\mathcal{P} \leftrightarrow \mathcal{Q}$ if \mathcal{P} and \mathcal{Q} are adjacent. Recall that $\mathcal{P} = (P_1, \ldots, P_k)$ is adjacent to $\mathcal{Q} = (Q_1, \ldots, Q_k)$ if there exists $i \in [k]$ such that $P_j = Q_j$ for $j \in [k] \setminus \{i\}$ and $P_i \neq Q_i$.

For a graph G embedded on a surface Σ , each connected region of $\Sigma \setminus G$ is called a *face* of G. For a face F, its boundary is denoted by ∂F . When a graph G is embedded on a surface Σ , a path in G is sometimes identified with the corresponding curve in Σ . A graph embedded on a plane is called a *plane graph*. A graph is said to be *planar* if it has a planar embedding.

The following notion is well known in topology. See [18, Section 1.2.3] for instance.

Definition 2.1. Let C_1 and C_2 be piecewise smooth oriented curves on an oriented surface and let $p \in C_1 \cap C_2$ be a transverse double point. The *local intersection number* $\varepsilon_p(C_1, C_2)$ of C_1 and C_2 at p is defined by $\varepsilon_p(C_1, C_2) = 1$ if C_1 crosses C_2 from left to right and $\varepsilon_p(C_1, C_2) = -1$ if C_1 crosses C_2 from right to left (see Figure 2). When $\partial C_1 \cap C_2 = C_1 \cap \partial C_2 = \emptyset$, the *algebraic intersection number* $\mu(C_1, C_2) \in \mathbb{Z}$ is defined to be the sum of $\varepsilon_p(C_1, C_2)$ over all $p \in C_1 \cap C_2$ (after a small perturbation if necessary). Note that ∂C_i denotes the set of endpoints of C_i .

When a graph is embedded on an oriented surface, paths in the graph are piecewise smooth curves, and hence we can define the algebraic intersection number for a pair of paths (see Figure 3).

3 Curves on a Plane

In this section, we consider the reconfiguration of curves on a plane and prove Theorem 1.2. Suppose that we are given distinct points $s_1, \ldots, s_k, t_1, \ldots, t_k$ on a plane and linkages \mathcal{P} and \mathcal{Q} that consist of curves on the plane connecting s_i and t_i .

Throughout this section, all intersections of curves are assumed to be transverse double points. Fix $j \in [k]$ and let $\bigcup_{i \in [k]} P_i \cap Q_j = \{s_j, p_1, \dots, p_n, t_j\}$, where the n+2 points are aligned



Figure 4: An example of linkages with $w_1 = x_2 x_1^{-1} x_2^{-1} x_1 x_2^{-1} x_1^{-1} x_2$ and $w_2 = x_1 x_2^{-1} x_1^{-1} x_2 x_1^{-1} x_2^{-1} x_1$.

(I)
$$\begin{array}{c} Q_i \\ \hline P_i \end{array} \iff \begin{array}{c} Q_\ell \\ \hline P_i \end{array} \iff \begin{array}{c} Q_\ell \\ \hline P_i \end{array}$$

Figure 5: Local pictures of isotopies of P_i .

on Q_j in this order. We now define $w_j \in F_k$ by

$$w_j = \prod_{\ell \in [n]} x_{i_\ell}^{\varepsilon_{p_\ell}(P_{i_\ell}, Q_j)}$$

where $i_{\ell} \in [k]$ satisfies $p_{\ell} \in P_{i_{\ell}} \cap Q_j$. Recall that F_k denotes the free group generated by x_1, \ldots, x_k . We give an example in Figure 4.

Remark 3.1. Let ab: $F_k \to \mathbb{Z}^k$ denote the abelianization, that is, the ℓ th entry of ab(w) is the sum of the exponents of x_ℓ 's in w. For distinct $i, j \in [k]$, the *i*th entry of $ab(w_j)$ is equal to the algebraic intersection number $\mu(P_i, Q_j) \in \mathbb{Z}$ of P_i and Q_j . Thus, $w_j \in \langle x_j \rangle$ implies that $\mu(P_i, Q_j) = 0$ for any $i \in [k] \setminus \{j\}$.

In the following two lemmas, we observe the behavior of w_j under certain moves of curves. For $j \in [k]$, let w'_i denote the word defined by a linkage \mathcal{P}' and the curve Q_j .

Lemma 3.2. Let $i \in [k]$ and let $\mathcal{P}' = (P'_1, \ldots, P'_k)$ be a linkage such that $P'_{\ell} = P_{\ell}$ if $\ell \neq i$, and P'_i is isotopic to P_i relative to $\{s_i, t_i\}$ in $\mathbb{R}^2 \setminus \bigcup_{\ell \neq i} P_{\ell}$. Then, $w'_j = w_j$ for $j \in [k] \setminus \{i\}$, and $w'_i = x_i^{e_1} w_i x_i^{e_2}$ for some $e_1, e_2 \in \mathbb{Z}$.

Proof. By the definition of an isotopy (see [18, Section 1.2.5]), P'_i is obtained from P_i by a finite sequence of the moves illustrated in Figure 5. By (I), one intersection of P_i and Q_i is created or eliminated, and thus (I) changes w_i to $w_i x_i^{\pm 1}$ or $x_i^{\pm 1} w_i$. In (II), two intersections of P_i and Q_ℓ are created or eliminated for some $\ell \in [k]$. Since $x_i^{\pm 1} x_i^{\pm 1} = 1$, w_j is unchanged under (II) for any $j \in [k]$.

Recall here that $\langle x_{\ell} \rangle$ denotes the subgroup of F_k generated by x_{ℓ} .

Lemma 3.3. Let γ be a simple curve connecting P_i and s_j $(i \neq j)$ whose interior is disjoint from $\bigcup_{\ell \in [k]} P_\ell$, and define P'_i as illustrated in Figure 6. Let \mathcal{P}' be the linkage obtained from \mathcal{P} by replacing P_i with P'_i . For $\ell \in [k]$, if $w_\ell \in \langle x_\ell \rangle$, then $w'_\ell = w_\ell$.

Proof. Define a group homomorphism $f_{ij}: F_k \to F_k$ by $f_{ij}(x_\ell) = x_\ell$ if $\ell \neq j$, and $f_{ij}(x_j) = x_i x_j x_i^{-1}$. Then, one can check that $w'_\ell = f_{ij}(w_\ell)$ if $\ell \neq j$, and $w'_j = x_i^{-1} f_{ij}(w_j) x_i$ (see Figure 6). Since $w_\ell = x_\ell^{e_\ell}$ for some $e_\ell \in \mathbb{Z}$ by the assumption, we have $w'_\ell = w_\ell$ if $\ell \neq j$. Also, one has

$$w'_{j} = x_{i}^{-1} f_{ij}(w_{j}) x_{i} = x_{i}^{-1} (x_{i} x_{j} x_{i}^{-1})^{e_{j}} x_{i} = w_{j}$$

This completes the proof.



Figure 6: (Left) A move of P_i along γ . (Right) Intersections of P'_i and $\bigcup_{\ell} Q_{\ell}$.



Figure 7: A reconfiguration of P_i to P'_i .

As a consequence of Lemmas 3.2 and 3.3, we obtain the following key lemma.

Lemma 3.4. Suppose that \mathcal{P} is reconfigurable to \mathcal{P}' . For $j \in [k]$, if $w_j \in \langle x_j \rangle$, then $w'_j \in \langle x_j \rangle$. *Proof.* It suffices to consider the case when there is $i \in [k]$ such that $P'_{\ell} = P_{\ell}$ if $\ell \neq i$, and $P'_i \neq P_i$. Since P_i is isotopic to P'_i (relative to $\{s_i, t_i\}$) in \mathbb{R}^2 , the curve P'_i is obtained from P_i by the moves in Lemmas 3.2 and 3.3. Therefore, these lemmas imply that if $w_j \in \langle x_j \rangle$ then $w'_i \in \langle x_j \rangle$.

With this key lemma, we can prove Theorem 1.2 stating that \mathcal{P} is reconfigurable to \mathcal{Q} if and only if $w_j \in \langle x_j \rangle$ for any $j \in [k]$.

Proof of Theorem 1.2. First suppose that \mathcal{P} is reconfigurable to \mathcal{Q} , namely \mathcal{P} is reconfigurable to \mathcal{P}' such that $P'_i \cap Q_i = \{s_i, t_i\}$ and $P'_i \cap Q_j = \emptyset$ for $j \in [k] \setminus \{i\}$. Then, $w'_j = 1$ for any $j \in [k]$. Since \mathcal{P}' is reconfigurable to \mathcal{P} , Lemma 3.4 implies that $w_j \in \langle x_j \rangle$ for any $j \in [k]$.

The converse is shown by induction on the number, say n, of intersections of \mathcal{P} and \mathcal{Q} except their endpoints. The case n = 0 is obvious. Let us consider the case $n \ge 1$. If $P_i \cap Q_j = \emptyset$ for any pair of distinct $i, j \in [k]$, then the reconfiguration is obviously possible. Otherwise, there exists $x_i x_i^{-1}$ or $x_i^{-1} x_i$ in the product of the definition of w_{j^*} for some $i, j^* \in [k]$ (possibly $i = j^*$). This means that P_i can be reconfigured to a curve P'_i as illustrated in Figure 7. This process eliminates at least two intersections and we have $w'_j \in \langle x_j \rangle$ for any $j \in [k]$ by Lemma 3.4. Thus, the induction hypothesis concludes that \mathcal{P}' is reconfigurable to \mathcal{Q} .

By Theorem 1.2 and Remark 3.1, we obtain the following corollary.

Corollary 3.5. Let $\mathcal{P} = (P_1, \ldots, P_k)$ and $\mathcal{Q} = (Q_1, \ldots, Q_k)$ be linkages on a plane (or a sphere). If \mathcal{P} is reconfigurable to \mathcal{Q} , then $\mu(P_i, Q_j) = 0$ for any distinct $i, j \in [k]$.

It is worth mentioning that the converse is not necessarily true as illustrated in Figure 4. This means that a "non-commutative" tool such as the free group F_k is essential to describe the complexity of the reconfiguration of curves on a plane.

4 Algorithms for Planar Graphs

In this section, we consider the reconfiguration in planar graphs and prove Theorems 1.3, 1.4, and 1.5. We deal with one-face instances and two-face instances of DISJOINT PATHS RE-CONFIGURATION in Sections 4.1 and 4.2, respectively. Then, we discuss DISJOINT *s-t* PATHS RECONFIGURATION in Section 4.3. A proof of a key theorem (Theorem 4.2) is postponed to Section 5.

4.1 One-Face Instance

We say that an instance $(G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURATION is a *one-face instance* if G is a plane graph and all the terminals are on the boundary of some face. We show that \mathcal{P} is always reconfigurable to \mathcal{Q} in a one-face instance.

Proposition 4.1. For any one-face instance $(G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURA-TION, \mathcal{P} is reconfigurable to \mathcal{Q} .

Proof. We prove the proposition by induction on the number of vertices in G. Let $I = (G, \mathcal{P}, \mathcal{Q})$ be a one-face instance of DISJOINT PATHS RECONFIGURATION.

If G is not connected, then we can consider each connected component separately. If G is connected but not 2-connected, then there exist subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ for some $v \in V$. For i = 1, 2, define I_i as the restriction of I to G_i if some path in $\mathcal{P} \cup \mathcal{Q}$ uses an edge in G_i incident to v, and define I_i as the restriction of I to $G_i \setminus \{v\}$ otherwise. Since I_1 and I_2 are one-face instances (or trivial instances with at most one terminal pair), the induction hypothesis shows that there exist reconfiguration sequences for I_1 and I_2 . By combining them, we obtain a reconfiguration sequence from \mathcal{P} to \mathcal{Q} .

In what follows, suppose that G is 2-connected. Let F be a face of G whose boundary contains all the terminals. Note that the boundary of F forms a cycle, because G is 2-connected. Since there is a linkage, for some $i \in [k]$, there exists an s_i - t_i path R_i along ∂F that contains no terminals other than s_i and t_i . Since \mathcal{P} and \mathcal{Q} are linkages, P_j and Q_j are disjoint from R_i for $j \in [k] \setminus \{i\}$. Define \mathcal{P}' (resp. \mathcal{Q}') as the linkage that is obtained from \mathcal{P} (resp. \mathcal{Q}) by replacing P_i (resp. Q_i) with R_i . Then, $\mathcal{P} \leftrightarrow \mathcal{P}'$ and $\mathcal{Q} \leftrightarrow \mathcal{Q}'$.

Since $(G \setminus R_i, \mathcal{P}' \setminus \{R_i\}, \mathcal{Q}' \setminus \{R_i\})$ is a one-face instance, $\mathcal{P}' \setminus \{R_i\}$ is reconfigurable to $\mathcal{Q}' \setminus \{R_i\}$ in $G \setminus R_i$ by the induction hypothesis. This implies that \mathcal{P}' is reconfigurable to \mathcal{Q}' in G. Therefore, \mathcal{P} is reconfigurable to \mathcal{Q} in G.

4.2 Two-Face Instance

Let $k \geq 2$. We say that an instance $(G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURATION is a *two-face instance* if G = (V, E) is a plane graph, s_1, \ldots, s_k are on the boundary of some face S, and t_1, \ldots, t_k are on the boundary of another face T. The objective of this subsection is to present a polynomial-time algorithm for two-face instances.

It suffices to consider the case when the graph is 2-connected, since otherwise we can easily reduce to the 2-connected case. Hence, we may assume that the boundary of each face forms a cycle. For ease of explanation, without loss of generality, we assume that G is embedded on \mathbb{R}^2 so that S is an inner face and T is the outer face. Furthermore, we may assume that s_1, \ldots, s_k lie on the boundary of S clockwise in this order and t_1, \ldots, t_k lie on the boundary of T clockwise in this order, because there is a linkage.

A vertex set $U \subseteq V$ is called a *terminal separator* if U separates $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_k\}$. For two curves (or paths) P and Q between ∂S and ∂T that share no endpoints, define $\mu(P,Q)$ as in Definition 2.1. That is, $\mu(P,Q)$ is the number of times P crosses Q from left to right minus the number of times P crosses Q from right to left, where we suppose that P and Q are oriented from ∂S to ∂T . Since $\mu(P_i, Q_j)$ takes the same value for distinct $i, j \in [k]$ (see Appendix B), this value is denoted by $\mu(\mathcal{P}, Q)$. Roughly, $\mu(\mathcal{P}, Q)$ indicates the difference of the numbers of rotations around S of the linkages.

The existence of a linkage shows that the graph has no terminal separator of size less than k. If the graph has no terminal separator of size k, then we can characterize the reconfigurability by using $\mu(\mathcal{P}, \mathcal{Q})$. The following is a key theorem in our algorithm, whose proof is given in Section 5. **Theorem 4.2.** Let $k \geq 2$. Suppose that a two-face instance $(G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURATION has no terminal separator of size k. Then, \mathcal{P} is reconfigurable to \mathcal{Q} if and only if $\mu(\mathcal{P}, \mathcal{Q}) = 0$.

By using this theorem, we can design a polynomial-time algorithm for two-face instances of DISJOINT PATHS RECONFIGURATION and prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that we are given a two-face instance $I = (G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURATION.

We first test whether I has a terminal separator of size k, which can be done in polynomial time by a standard minimum cut algorithm. If there is no terminal separator of size k, then Theorem 4.2 shows that we can easily solve DISJOINT PATHS RECONFIGURATION by checking whether $\mu(\mathcal{P}, \mathcal{Q}) = 0$ or not.

Suppose that we obtain a terminal separator U of size k. Then, we obtain subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = U$, $\{s_1, \ldots, s_k\} \subseteq V(G_1)$, and $\{t_1, \ldots, t_k\} \subseteq V(G_2)$. We test whether $V(P_i) \cap U = V(Q_i) \cap U$ holds for any $i \in [k]$ or not, where we note that each of $V(P_i) \cap U$ and $V(Q_i) \cap U$ consists of a single vertex. If this does not hold, then we can immediately conclude that \mathcal{P} is not reconfigurable to \mathcal{Q} , because $V(P_i) \cap U$ does not change in the reconfiguration. If $V(P_i) \cap U = V(Q_i) \cap U$ for $i \in [k]$, then we consider the instance $I_i = (G_i, \mathcal{P}_i, \mathcal{Q}_i)$ for i = 1, 2, where \mathcal{P}_i and \mathcal{Q}_i are the restrictions of \mathcal{P} and \mathcal{Q} to G_i . That is, I_i is the restriction of I to G_i . Then, we see that \mathcal{P} is reconfigurable to \mathcal{Q}_i for i = 1, 2. Since I_1 and I_2 are one-face or two-face instances, by solving them recursively, we can solve the original instance I in polynomial time. See Algorithm 1 for a pseudocode of the algorithm.

```
Algorithm 1: Algorithm for two-face instances of DPR
   Input: A two-face instance I = (G, \mathcal{P}, \mathcal{Q}) of DPR.
   Output: Is \mathcal{P} reconfigurable to \mathcal{Q}?
 1 Compute a terminal separator U of size k;
 2 if such U does not exist then
 3
       if \mu(\mathcal{P}, \mathcal{Q}) = 0 then
           return YES
 4
       else
 5
           return NO
 6
 7 else
       if V(P_i) \cap U \neq V(Q_i) \cap U for some i \in [k] then
 8
 9
           return NO
       else
10
           Construct G_1 and G_2;
11
           Solve the restriction I_i of I to G_i for i = 1, 2, recursively;
12
           if I_i is a YES-instance for i = 1, 2 then
\mathbf{13}
               return YES
14
           else
15
               return NO
16
```

4.3 Reconfiguration of *s*-*t* Paths

In this subsection, for DISJOINT *s*-t PATHS RECONFIGURATION in planar graphs, we show results that are analogous to Theorems 4.2 and 1.3, which have been already stated in Section 1.2.



Figure 8: (Left) Original graph G. (Right) Modification around s.

Theorem 1.5. Let G = (V, E) be a planar graph with distinct vertices s and t, and let \mathcal{P} and \mathcal{Q} be s-t linkages. If there is no s-t separator of size k, then \mathcal{P} is reconfigurable to \mathcal{Q} .

Proof. Suppose that G, s, t, \mathcal{P} , and \mathcal{Q} are as in the statement, and assume that there is no s-t separator of size k. We fix an embedding of G on the plane. If there is an edge connecting s and t, then s and t are on the boundary of some face, and hence \mathcal{P} is reconfigurable to \mathcal{Q} in the same way as Proposition 4.1. Thus, it suffices to consider the case when there is no edge connecting s and t.

We now construct an instance of DISJOINT PATHS RECONFIGURATION by replacing s and t with large "grids" as follows. Let e_1, e_2, \ldots, e_ℓ be the edges incident to s clockwise in this order. Note that $\ell \ge k + 1$ holds, because G has no s-t separator of size k. For $i \in [\ell]$, we subdivide e_i by introducing p new vertices $v_i^1, v_i^2, \ldots, v_i^p$ such that they are aligned in this order and v_i^1 is closest to s, where p is a sufficiently large integer (e.g., $p \ge |V|^2$). For $i \in [\ell]$ and for $j \in [p]$, we introduce a new edge connecting v_i^j and v_{i+1}^j , where $v_{\ell+1}^j = v_1^j$. Define $s_i = v_i^1$ for $i \in [k]$ and remove s. Then, the graph is embedded on the plane and s_1, \ldots, s_k are on the boundary of some face clockwise in this order; see Figure 8. By applying a similar procedure to t, we modify the graph around t and define t_1, \ldots, t_k that are on the boundary of some face context. Let G' be the obtained graph. Observe that G' contains no terminal separator of size k, because G has no s-t separator of size k.

By rerouting the given s-t linkages \mathcal{P} and \mathcal{Q} around s and t, we obtain linkages \mathcal{P}' and \mathcal{Q}' from $\{s_1, \ldots, s_k\}$ to $\{t_1, \ldots, t_k\}$ in G'. Note that the restrictions of \mathcal{P} and \mathcal{Q} to $G \setminus \{s, t\}$ coincide with those of \mathcal{P}' and \mathcal{Q}' , respectively. Then, we can take \mathcal{P}' and \mathcal{Q}' so that $|\mu(\mathcal{P}', \mathcal{Q}')| \leq |V|$. Furthermore, by using at most |V| concentric cycles around s and t, we can reroute the linkages so that the value $\mu(\mathcal{P}', \mathcal{Q}')$ decreases or increases by one. Therefore, by using $p \geq |V|^2$ concentric cycles, we can reroute \mathcal{P}' and \mathcal{Q}' so that $\mu(\mathcal{P}', \mathcal{Q}')$ becomes zero.

By Theorem 4.2, \mathcal{P}' is reconfigurable to \mathcal{Q}' in G' (in terms of DISJOINT PATHS RECONFIG-URATION). Then, the reconfiguration sequence from \mathcal{P}' to \mathcal{Q}' corresponds to that from \mathcal{P} to \mathcal{Q} in G (in terms of DISJOINT *s*-*t* PATHS RECONFIGURATION). Therefore, \mathcal{P} is reconfigurable to \mathcal{Q} in G.

Theorem 1.4. There is a polynomial-time algorithm for DISJOINT s-t PATHS RECONFIGURA-TION in planar graphs.

Proof. Suppose that we are given a planar graph G = (V, E) with $s, t \in V$ and s-t linkages $\mathcal{P} = \{P_1, \ldots, P_k\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ in G. We first test whether G has an s-t separator of size k. If there is no such a separator, then we can immediately conclude that \mathcal{P} is reconfigurable to \mathcal{Q} by Theorem 1.5.



Figure 9: Construction of G_1 , G_2 , and G_3 .

Suppose that G has an s-t separator of size k. Let X be the inclusionwise minimal vertex set subject to $s \in X$ and N(X) is an s-t separator of size k. Note that such X is uniquely determined by the submodularity of |N(X)| and it can be computed in polynomial time by a standard minimum cut algorithm. Similarly, let Y be the unique inclusionwise minimal vertex set subject to $t \in Y$ and N(Y) is an s-t separator of size k. Let U = N(X), W = N(Y), $G_1 = G[X \cup U], G_2 = G \setminus (X \cup Y)$, and $G_3 = G[Y \cup W]$; see Figure 9. Since $V(P_i) \cap U$ and $V(P_i) \cap W$ do not change in the reconfiguration, we can consider the reconfiguration in G_1, G_2 , and G_3 , separately.

We first consider the reconfiguration in G_1 . Observe that each path in \mathcal{P} contains exactly one vertex in U, and the restriction of \mathcal{P} to G_1 consists of k paths from s to U that are vertexdisjoint except at s. The same for \mathcal{Q} . By the minimality of X, G_1 contains no vertex set of size k that separates $\{s\}$ and U. Therefore, by the same argument as Theorem 1.5, the restriction of \mathcal{P} to G_1 is reconfigurable to that of \mathcal{Q} .

If $U \cap W \neq \emptyset$, then $G \setminus X$ contains no vertex set of size k that separates U and $\{t\}$ by the minimality of Y. In such a case, by shrinking U to a single vertex and by applying the same argument as above, the restriction of \mathcal{P} to $G \setminus X$ is reconfigurable to that of \mathcal{Q} . By combining the reconfiguration in G_1 and that in $G \setminus X$, we obtain a reconfiguration sequence from \mathcal{P} to \mathcal{Q} .

Therefore, it suffices to consider the case when $U \cap W = \emptyset$. In the same way as G_1 , we see that the restriction of \mathcal{P} to G_3 is reconfigurable to that of \mathcal{Q} . This shows that the reconfigurability from \mathcal{P} to \mathcal{Q} in G is equivalent to that in G_2 . By changing the indices if necessary, we may assume that $P_i \cap U = Q_i \cap U$ for $i \in [k]$. If $P_i \cap W \neq Q_i \cap W$ for some $i \in [k]$, then we can conclude that \mathcal{P} is not reconfigurable to \mathcal{Q} . Otherwise, let \mathcal{P}' and \mathcal{Q}' be the restrictions of \mathcal{P} and \mathcal{Q} to G_2 , respectively. Since $(G_2, \mathcal{P}', \mathcal{Q}')$ is a one-face or two-face instance of DISJOINT PATHS RECONFIGURATION, we can solve it in polynomial time by Proposition 4.1 and Theorem 1.3. Therefore, we can test the reconfigurability from \mathcal{P} to \mathcal{Q} in polynomial time; see Algorithm 2.

5 Proof of Theorem 4.2

The necessity ("only if" part) in Theorem 4.2 is immediately derived from Corollary 3.5.

In what follows in this section, we show the sufficiency ("if" part) in Theorem 4.2, which is one of the main technical contributions in this paper. Assume that $\mu(\mathcal{P}, \mathcal{Q}) = 0$ and there is no terminal separator of size k. The objective is to show that \mathcal{P} is reconfigurable to \mathcal{Q} . Our proof is constructive, and based on topological arguments. A similar technique is used in [30, 35, 36, 41].

Algorithm 2: Algorithm for planar s - t paths reconfiguration	
Input: A planar graph G and s-t linkages \mathcal{P} and \mathcal{Q} .	
Output: Is \mathcal{P} reconfigurable to \mathcal{Q} ?	
1 C	Sompute X and Y together with s -t separators U and W of size k ;
2 if	Such separators do not exist or $U \cap W \neq \emptyset$ then
3	return YES
4 else	
5	Construct $G_2 = G \setminus (X \cup Y);$
6	Change the indices so that $P_i \cap U = Q_i \cap U$ for $i \in [k]$;
7	if $P_i \cap W \neq Q_i \cap W$ for some $i \in [k]$ then
8	return NO
9	else
10	Let \mathcal{P}' and \mathcal{Q}' be the restrictions of \mathcal{P} and \mathcal{Q} to G_2 ;
11	if $(G_2, \mathcal{P}', \mathcal{Q}')$ is a YES-instance of DPR then
12	return YES
13	else
14	return NO

5.1 Preliminaries for the Proof

Let C be a simple curve connecting the boundaries of S and T such that C contains no vertex in G, C intersects the boundaries of S and T only at its endpoints, and $\mu(P_i, C) = 0$ for $i \in [k]$. Note that such C always exists, because the last condition is satisfied if C is disjoint from \mathcal{P} . Note also that $\mu(Q_i, C) = 0$ holds for $i \in [k]$, because $\mu(\mathcal{P}, \mathcal{Q}) = 0$.

Since T is the outer face, $\mathbb{R}^2 \setminus (S \cup T)$ forms an annulus (or a cylinder).² Thus, by cutting it along C, we obtain a rectangle whose boundary consists of ∂S , ∂T , and two copies of C. We take infinite copies of this rectangle and glue them together to obtain an infinite long strip R. That is, for $j \in \mathbb{Z}$, let C^j be a copy of C, let R^j be a copy of the rectangle whose boundary contains C^j and C^{j+1} , and define $R = \bigcup_{j \in \mathbb{Z}} R^j$; see Figure 10. By taking C appropriately, we may assume that the copies of s_1, \ldots, s_k lie on the boundary of R^j in this order so that s_1 is closest to C^j and s_k is closest to C^{j+1} . The same for t_1, \ldots, t_k . Note that R is called the universal cover of $\mathbb{R}^2 \setminus (S \cup T)$ in the terminology of topology.

Since G is embedded on $\mathbb{R}^2 \setminus (S \cup T)$, this operation naturally defines an infinite periodic graph $\hat{G} = (\hat{V}, \hat{E})$ on R that consists of copies of G. A path in \hat{G} is identified with the corresponding curve in R. For $v \in V$ and $j \in \mathbb{Z}$, let $v^j \in \hat{V}$ denote the unique vertex in R^j that corresponds to v. Since $\mu(P_i, C) = 0$ for $i \in [k]$, each path in \hat{G} corresponding to P_i is from s_i^j to t_i^j for some $j \in \mathbb{Z}$, and we denote such a path by P_i^j . We define Q_i^j in the same way. Since \mathcal{P} and \mathcal{Q} are linkages in G, $\{P_i^j \mid i \in [k], j \in \mathbb{Z}\}$ and $\{Q_i^j \mid i \in [k], j \in \mathbb{Z}\}$ are sets of vertex-disjoint paths in \hat{G} .

A path in \hat{G} connecting the boundary of R corresponding to ∂S and that corresponding to ∂T is called an $\hat{S}\cdot\hat{T}$ path. For an $\hat{S}\cdot\hat{T}$ path P, let L(P) be the region of $R \setminus P$ that is on the "left-hand side" of P. Formally, let r be a point in R^j for sufficiently small j, and define L(P) as the set of points $x \in R \setminus P$ such that any curve in R between r and x crosses P an even number of times; see Figure 11. For two $\hat{S}\cdot\hat{T}$ paths P and Q, we denote $P \preceq Q$ if $L(P) \subseteq L(Q)$, and denote $P \prec Q$ if $L(P) \subsetneq L(Q)$. For two linkages $\mathcal{P} = (P_1, \ldots, P_k)$ and $\mathcal{Q} = (Q_1, \ldots, Q_k)$ in G with $\mu(P_i, C) = \mu(Q_i, C) = 0$ for $i \in [k]$, we denote $\mathcal{P} \preceq \mathcal{Q}$ if $P_i^j \preceq Q_i^j$ for any $i \in [k]$ and

²More precisely, the annulus is degenerated when $\partial S \cap \partial T \neq \emptyset$, but the same argument works even for this case.



Figure 10: (Left) Curve C in $\mathbb{R}^2 \setminus (S \cup T)$. (Right) Construction of R.



Figure 11: Definition of L(P).

 $j \in \mathbb{Z}$, and denote $\mathcal{P} \prec \mathcal{Q}$ if $\mathcal{P} \preceq \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$.

5.2 Case When $\mathcal{P} \preceq \mathcal{Q}$

In this subsection, we consider the case when $\mathcal{P} \preceq \mathcal{Q}$, and the general case will be dealt with in Section 5.3. In order to show that \mathcal{P} is reconfigurable to \mathcal{Q} , we show the following lemma.

Lemma 5.1. If $\mathcal{P} \prec \mathcal{Q}$, then there exists a linkage \mathcal{P}' such that $\mathcal{P} \leftrightarrow \mathcal{P}'$ and $\mathcal{P} \prec \mathcal{P}' \preceq \mathcal{Q}$.

Proof. To derive a contradiction, assume that such \mathcal{P}' does not exist. Let $\hat{W} := \{\hat{v} \in \hat{V} \mid \hat{v} \in P_i^j \setminus Q_i^j \text{ for some } i \in [k] \text{ and } j \in \mathbb{Z}\}$ and let W be the subset of V corresponding to \hat{W} . If $W = \emptyset$, then take an index $i \in [k]$ such that $P_i \neq Q_i$ and let $\mathcal{P}' = (P'_1, \ldots, P'_k)$ be the set of paths obtained from \mathcal{P} by replacing P_i with Q_i . Since \mathcal{Q} is a linkage and all the vertices in P'_h are contained in Q_h for any $h \in [k], \mathcal{P}'$ is a desired linkage, which contradicts the assumption.

Thus, it suffices to consider the case when $W \neq \emptyset$. Let $u \in W$. Let $\hat{u} \in \hat{W}$ be a vertex corresponding to u and let $i \in [k]$ and $j \in \mathbb{Z}$ be the indices such that $\hat{u} \in P_i^j \setminus Q_i^j$. Since $\hat{u} \in P_i^j \setminus Q_i^j$ implies $\hat{u} \in L(Q_i^j) \setminus L(P_i^j)$, there exists a face \hat{F} of \hat{G} such that $\partial \hat{F}$ contains an edge of P_i^j incident to \hat{u} and $\hat{F} \subseteq L(Q_i^j) \setminus L(P_i^j)$. Define $(P_i^j)'$ as the $s_i^j \cdot t_i^j$ path in \hat{G} with maximal $L((P_i^j)')$ subject to $(P_i^j)' \subseteq P_i^j \cup \partial \hat{F}$; see Figure 12. Note that such a path is uniquely determined and $P_i^j \prec (P_i^j)' \preceq Q_i^j$.

Let P'_i be the $s_i \cdot t_i$ path in G that corresponds to $(P^j_i)'$. If P'_i is disjoint from P_h for any $h \in [k] \setminus \{i\}$, then we can obtain a desired linkage \mathcal{P}' from \mathcal{P} by replacing P_i with P'_i , which contradicts the assumption. Therefore, P'_i intersects P_h for some $h \in [k] \setminus \{i\}$. This together with $P^j_i \prec (P^j_i)'$ shows that $(P^j_i)'$ intersects P^j_{i+1} , where P^j_{k+1} means P^{j+1}_1 . Since P^j_i and P^j_{i+1} are vertex-disjoint, the intersection of $(P^j_i)'$ and P^j_{i+1} is contained in $\partial \hat{F}$, which implies that $\partial \hat{F} \cap P^j_{i+1}$ contains a vertex $\hat{v} \in \hat{V}$; see Figure 12 again. Since $\hat{F} \subseteq L(Q^j_i)$, we obtain $\hat{v} \in L(Q^j_i) \cup Q^j_i \subseteq L(Q^j_{i+1})$, and hence $\hat{v} \notin Q^j_{i+1}$. Let v and F be the vertex and the face of G that correspond to \hat{v} and \hat{F} , respectively. Then, $\hat{v} \in P^j_{i+1} \setminus Q^j_{i+1}$ implies that $\hat{v} \in \hat{W}$ and $v \in W$. Let J be a curve in F from u to v.



Figure 12: The blue thick paths are P_i^j and P_{i+1}^j , and the red dashed path is $(P_i^j)'$. There exists a vertex $\hat{v} \in \partial \hat{F} \cap P_{i+1}^j$.



Figure 13: Each blue path represents P_i . The dotted curve is part of J and the red dotted thick curve is C^* .

By the above argument, for any $u \in W$ on P_i , there exist a vertex $v \in W$ on P_{i+1} and a curve J from u to v contained in some face of G. By repeating this argument and by shifting the indices of P_i if necessary, we obtain v_i and J_i for i = 1, 2, ... such that $v_i \in W$ is on P_i (where the index is modulo k) and J_i is a curve from v_i to v_{i+1} contained in some face. We consider the curve J obtained by concatenating $J_1, J_2, ...$ in this order. Since |W| is finite, this curve visits the same point more than once, and hence it contains a simple closed curve C^* . Since C^* is simple and visits vertices on $P_i, P_{i+1}, ...$ in this order, C^* surrounds S exactly once in the clockwise direction; see Figure 13. In particular, C^* contains exactly one vertex on P_i for each $i \in [k]$. Let U be the set of vertices in V contained in C^* . Then, |U| = k and $G \setminus U$ has no path between $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_k\}$ by the choice of C^* . Furthermore, U contains no terminals, because $U \subseteq W$ and W contains no terminals. Therefore, U is a terminal separator of size k, which contradicts the assumption.

As long as $\mathcal{P} \neq \mathcal{Q}$, we apply this lemma and replace \mathcal{P} with \mathcal{P}' , repeatedly. Then, this procedure terminates when $\mathcal{P} = \mathcal{Q}$, and gives a reconfiguration sequence from \mathcal{P} to \mathcal{Q} . This completes the proof for the case when $\mathcal{P} \preceq \mathcal{Q}$.

5.3 General Case

In this subsection, we consider the case when $\mathcal{P} \preceq \mathcal{Q}$ does not necessarily hold. For $i \in [k]$ and $j \in \mathbb{Z}$, define $P_i^j \lor Q_i^j$ as the $s_i^j \cdot t_i^j$ path in \hat{G} with maximal $L(P_i^j \lor Q_i^j)$ subject to $P_i^j \lor Q_i^j \subseteq P_i^j \cup Q_i^j$;



Figure 14: Construction of $P_i^j \vee Q_i^j$.

see Figure 14. Note that such a path is uniquely determined, $P_i^j \leq P_i^j \lor Q_i^j$, and $Q_i^j \leq P_i^j \lor Q_i^j$. Since \hat{G} is periodic, for any $j \in \mathbb{Z}$, $P_i^j \lor Q_i^j$ corresponds to a common s_i - t_i walk $P_i \lor Q_i$ in G. We now show that $\mathcal{P} \lor \mathcal{Q} := (P_1 \lor Q_1, \ldots, P_k \lor Q_k)$ is a linkage in G.

Lemma 5.2. $\mathcal{P} \lor \mathcal{Q}$ is a linkage in G.

Proof. We first show that $P_i \vee Q_i$ is a path for each $i \in [k]$. Assume to the contrary that $P_i \vee Q_i$ visits a vertex $v \in V$ more than once. Then, for $j \in \mathbb{Z}$, there exist $j_1, j_2 \in \mathbb{Z}$ with $j_1 < j_2$ such that $P_i^j \vee Q_i^j$ contains both v^{j_1} and v^{j_2} . Since the path $P_i^j \vee Q_i^j$ is contained in the subgraph $P_i^j \cup Q_i^j$, without loss of generality, we may assume that P_i^j contains v^{j_2} . This shows that $v^{j_1} \in L(P_i^j) \subseteq L(P_i^j \vee Q_i^j)$, which contradicts that v^{j_1} is contained in $P_i^j \vee Q_i^j$.

We next show that $P_1 \vee Q_1, \ldots, P_k \vee Q_k$ are pairwise vertex-disjoint. Assume to the contrary that $P_i \vee Q_i$ and $P_{i'} \vee Q_{i'}$ contain a common vertex $v \in V$ for distinct $i, i' \in [k]$. Since \hat{G} is periodic, there exist $j, j' \in \mathbb{Z}$ such that $P_i^j \vee Q_i^j$ and $P_{i'}^{j'} \vee Q_{i'}^{j'}$ contain v^0 (i.e., the copy of v in R^0). We may assume that (j, i) is smaller than (j', i') in the lexicographical ordering, that is, either j < j' holds or j = j' and i < i' hold. Since $P_i^j \vee Q_i^j \subseteq P_i^j \cup Q_i^j$, we may also assume that $v^0 \in P_i^j$ by changing the roles of P_i^j and Q_i^j if necessary. Then, we obtain $v^0 \in P_i^j \subseteq L(P_{i'}^{j'}) \subseteq L(P_{i'}^{j'}) \vee Q_{i'}^{j'}$, which contradicts that v^0 is contained in $P_{i'}^{j'} \vee Q_{i'}^{j'}$.

We also see that $\mu(P_i \vee Q_i, C) = 0$ for $i \in [k]$ by definition, and hence $\mu(\mathcal{P}, \mathcal{P} \vee \mathcal{Q}) = 0$. Since $\mathcal{P} \preceq \mathcal{P} \lor \mathcal{Q}$ and $\mu(\mathcal{P}, \mathcal{P} \lor \mathcal{Q}) = 0$, \mathcal{P} is reconfigurable to $\mathcal{P} \lor \mathcal{Q}$ as described in Section 5.2. Similarly, \mathcal{Q} is reconfigurable to $\mathcal{P} \lor \mathcal{Q}$, which implies that $\mathcal{P} \lor \mathcal{Q}$ is reconfigurable to \mathcal{Q} . By combining them, we see that \mathcal{P} is reconfigurable to \mathcal{Q} , which completes the proof of the sufficiency in Theorem 4.2.

6 **PSPACE-Completeness**

We first observe that DISJOINT PATHS RECONFIGURATION and DISJOINT *s-t* PATHS RECON-FIGURATION are in PSPACE by using PSPACE = NPSPACE (a corollary of Savitch's theorem [48]). As a certificate, we receive a reconfiguration sequence. In our polynomial-space algorithm, we read linkages in the certificate one by one, and check whether the linkage is indeed a linkage and whether two consecutive linkages are obtained by a single reconfiguration step. At each step, the working memory only requires to store the graph G and two consecutive linkages. Thus, the algorithm runs in polynomial space. This is a non-deterministic algorithm, but by PSPACE = NPSPACE, we conclude that the problems belong to PSPACE.

For the proof of PSPACE-hardness, we reduce the *nondeterministic constraint logic reconfiguration* (NCL reconfiguration) to our problem. In the NCL reconfiguration, we consider an undirected cubic graph, where each edge is associated with weight one or two and each vertex is incident to three weight-2 edges (called an $OR \ vertex$) or incident to one weight-2 edge and two weight-1 edges (called an $AND \ vertex$). We call such an undirected graph an AND/OR



Figure 15: (Left) A weight-2 edge gadget. (Right) A weight-1 edge gadget.

graph. An NCL configuration is an orientation of the edges of an AND/OR graph in which at each vertex the total weight of the incoming arcs is at least two. A *flip* of an arc is a process of changing the direction of the arc. In the NCL reconfiguration, we are given two NCL configurations of an AND/OR graph, and need to determine whether we can obtain one from the other by a sequence of flips in such a way that all the intermediate orientations are NCL configurations. It is known that the NCL reconfiguration is PSPACE-complete [23], even when the AND/OR graph is planar and of bounded bandwidth [58].

6.1 General Graphs and Two Paths

Theorem 1.1. The DISJOINT s-t PATHS RECONFIGURATION is PSPACE-complete even when k = 2 and the maximum degree of G is four.

Proof. We have already observed that the problem is in PSPACE. To show the PSPACEhardness, we now give a transformation of a given AND/OR graph H = (V(H), E(H)) to an undirected graph G = (V(G), E(G)). Each weight-2 edge $e = \{u, v\} \in E(H)$ of H is mapped to the following weight-2 edge gadget G_e (see Figure 15):

$$V(G_e) = \{w_{e,u}, w_{e,v}, r_{e,1}, r_{e,2}\},\$$

$$E(G_e) = \{\{w_{e,u}, r_{e,1}\}, \{w_{e,u}, r_{e,2}\}, \{w_{e,v}, r_{e,1}\}, \{w_{e,v}, r_{e,2}\}\}.$$

Similarly, each weight-1 edge $e = \{u, v\} \in E(H)$ of H is mapped to the following weight-1 edge gadget G_e (see Figure 15):

$$V(G_e) = \{w_{e,u}, w_{e,v}, b_{e,1}, b_{e,2}\},\$$

$$E(G_e) = \{\{w_{e,u}, b_{e,1}\}, \{w_{e,u}, b_{e,2}\}, \{w_{e,v}, b_{e,1}\}, \{w_{e,v}, b_{e,2}\}\}$$

Let $v \in V(H)$ be an OR vertex, and $e, f, g \in E(H)$ be three weight-2 edges incident to v. Then, v is mapped to the following *OR vertex gadget* G_v (see Figure 16):

$$V(G_v) = \{w_{e,v}, w_{f,v}, w_{g,v}, b_{v,1}, b_{v,2}\},\$$

$$E(G_v) = \{\{w_{e,v}, b_{v,1}\}, \{w_{e,v}, b_{v,2}\}, \{w_{f,v}, b_{v,1}\}, \{w_{f,v}, b_{v,2}\}, \{w_{g,v}, b_{v,1}\}, \{w_{g,v}, b_{v,2}\}\}.$$

Note that the vertices with the same label are identified.

An AND vertex $v \in V(H)$ that is incident to one weight-2 edge e and two weight-1 edges f, g is mapped to the following AND vertex gadget G_v (see Figure 16):

$$V(G_v) = \{w_{e,v}, w_{f,v}, w_{g,v}, r_{v,1}, r_{v,2}, b_{v,1}, b_{v,2}, k_v\},\$$

$$E(G_v) = \{\{w_{e,v}, b_{v,1}\}, \{w_{e,v}, b_{v,2}\}, \{w_{f,v}, r_{v,1}\}, \{w_{g,v}, r_{v,2}\},\$$

$$\{r_{v,1}, k_v\}, \{r_{v,2}, k_v\}, \{b_{v,1}, k_v\}, \{b_{v,2}, k_v\}, \{w_{f,v}, w_{g,v}\}\}.$$



Figure 16: (Left) An OR vertex gadget. (Right) An AND vertex gadget.

Define

$$V(G) = \{s, t\} \cup \bigcup_{v \in V(H)} V(G_v) \cup \bigcup_{e \in E(H)} V(G_e).$$

A vertex of V(G) is red if it is of the form $r_{x,i}$ for some vertex/edge $x \in V(H) \cup E(H)$ and $i \in \{1,2\}$; a vertex of V(G) is blue if it is of the form $b_{x,i}$ for some vertex/edge $x \in V(H) \cup E(H)$ and $i \in \{1,2\}$; a vertex of V(G) is white if it is of the form $w_{e,v}$ for some vertex $v \in V(H)$ and edge $e \in E(H)$; a vertex of V(G) is black if it is s, t or of the form k_v for some vertex $v \in V(H)$. The number of red vertices is equal to $2|V^a(H)| + 2|E_2(H)|$, where $V^a(H)$ is the set of AND vertices and $E_2(H)$ is the set of weight-2 edges; The number of blue vertices is equal to $2|V(H)| + 2|E_1(H)|$, where $E_1(H)$ is the set of weight-1 edges; The number of white vertices is equal to 3|V(H)| = 2|E(H)|; The number of black vertices is equal to $2 + |V^a(H)|$.

Assume that the elements of $V^{\mathbf{a}}(H) \cup E_2(H)$ are arbitrarily ordered as x_1, x_2, \ldots, x_R and the elements of $V(H) \cup E_1(H)$ are arbitrarily ordered as y_1, y_2, \ldots, y_B . Then, define

$$\begin{split} E(G) &= \bigcup_{v \in V(H)} E(G_v) \cup \bigcup_{e \in E(H)} E(G_e) \\ & \cup \{\{r_{x_i,2}, r_{x_{i+1},1}\} \mid i \in \{1, 2, \dots, R-1\}\} \\ & \cup \{\{b_{y_i,2}, b_{y_{i+1},1}\} \mid i \in \{1, 2, \dots, R-1\}\} \\ & \cup \{\{s, r_{x_1,1}\}, \{s, b_{y_1,1}\}\} \cup \{\{r_{x_R,2}, t\}, \{b_{y_B,2}, t\}\}. \end{split}$$

This completes the construction of G. Figure 17 may help the reader understand.

We now describe the way how an NCL configuration of H translates to an *s*-*t* linkage $\{P_1, P_2\}$ of G. We force that P_1 goes through all the red vertices and P_2 goes through all the blue vertices. As a sequence of vertices, P_1 is constructed as follows.

- 1. The path starts at s. Let $r_{x_0,2} = s$.
- 2. Now we iterate over $i \in \{1, \ldots, R\}$. As an invariant at the beginning of the iteration, we assume that we have just visited the vertex $r_{x_{i-1},2}$. Then, the path continues to $r_{x_i,1}$. We consider two separate cases.
 - (a) Consider the case where $x_i = \{u, v\}$ is a weight-2 edge. Let the edge be oriented from u to v in the NCL configuration. Then, the path visits $w_{x_i,u}$, and continues to $r_{x_i,2}$.
 - (b) Consider the case where x_i is an AND vertex that is incident to one weight-2 edge e and two weight-1 edges f and g. If e is oriented toward x_i , then the path visits k_{x_i} and continues to $r_{x_i,2}$. If e is not oriented toward x_i , then the path visits w_{f,x_i}, w_{g,x_i} and continues to $r_{x_i,2}$.
- 3. After the whole iteration, the path now visits $r_{x_R,2}$. Then, the path continues to t and we finish the construction of P_1 .

Similarly, as a sequence of vertices, P_2 is constructed as follows.



Figure 17: Mapping an NCL configuration to an *s*-*t* linkage $\{P_1, P_2\}$. In the NCL configuration, the vertex label A stands for an AND vertex, and the vertex label O stands for an OR vertex. The path P_1 is colored red and the path P_2 is colored blue.

- 1. The path starts at s. Let $b_{y_0,2} = s$.
- 2. Now we iterate over $i \in \{1, \ldots, B\}$. As an invariant at the beginning of the iteration, we assume that we have just visited the vertex $b_{y_{i-1},2}$. Then, the path continues to $b_{y_i,1}$. We consider three separate cases.
 - (a) Consider the case where $y_i = \{u, v\}$ is a weight-1 edge. Let the edge be oriented from u to v in the NCL configuration. Then, the path visits $w_{y_i,u}$, and continues to $b_{y_i,2}$.
 - (b) Consider the case where y_i is an OR vertex that is incident to three weight-2 edges e, f, and g. At least one of those edges is directed toward y_i . Let e be such an edge. Then, the path visits w_{e,y_i} and continues to $b_{y_i,2}$.
 - (c) Consider the case where y_i is an AND vertex that is incident to one weight-2 edge e and two weight-1 edges f and g. If e is oriented toward y_i , then the path visits w_{e,y_i} and continues to $b_{y_i,2}$. If e is not oriented toward y_i , then the path visits k_{y_i} and continues to $b_{y_i,2}$.
- 3. After the whole iteration, the path now visits $b_{y_B,2}$. Then, the path continues to t and we finish the construction of P_2 .

The construction is illustrated in Figure 17. The constructed s-t linkage $\mathcal{P} = \{P_1, P_2\}$ is created from the NCL configuration σ . The following claim ensures that the constructed paths are indeed vertex-disjoint.

Claim 6.1. The paths P_1 and P_2 are internally vertex-disjoint.

Proof. The paths P_1 and P_2 start at s and ends at t. By the construction, the red vertices are visited by P_1 only and the blue vertices are visited by P_2 only. Hence, if they are not vertex-disjoint, they will share a white vertex or a black vertex.

Suppose that they share a white vertex. Let one of the shared white vertices be of the form $w_{e,v}$ for some edge $e \in E(H)$ and an end-vertex v of e.

As the first case, suppose that v is an OR vertex. Note that e is a weight-2 edge in this case. Since $w_{e,v}$ is visited by P_1 , by Step 2(a), e is oriented from v to the other end-vertex of e. Since $w_{e,v}$ is visited by P_2 , by Step 2(b), e is oriented toward v. Thus, the orientation of e is in conflict and we reach a contradiction.

As the second case, suppose that v is an AND vertex. We further distinguish two cases. First, consider the case where e is a weight-2 edge. Then, since $w_{e,v}$ is visited by P_1 , by Step 2(a), e is oriented from v to the other end-vertex of e. On the other hand, since $w_{e,v}$ is visited by P_2 , by Step 2(c), e is oriented toward v. Thus, the orientation of e is in conflict and we reach a contradiction.

Second, consider the case where e is a weight-1 edge. Then, since $w_{e,v}$ is visited by P_2 , by Step 2(a), e is oriented from v. On the other hand, since $w_{e,v}$ is visited by P_1 , by Step 2(b), the weight-2 edge incident to v is not oriented toward v. They together mean that the total weight of incoming arc to v is at most one. This is a contradiction since the orientation does not meet the requirement to be an NCL configuration.

Next, suppose that P_1 and P_2 share a black vertex. Such a black vertex only exists in an AND vertex gadget G_v , where v is an AND vertex of H that is incident to one weight-2 edge e and two weight-1 edges f and g. Namely, the black vertex is of the form k_v . Since k_v is visited by P_1 , by Step 2(b), e is oriented toward v. On the other hand, since k_v is visited by P_2 , by Step 2(c), e is not oriented toward v. This is a contradiction.

In all cases, we derive a contradiction. Thus, P_1 and P_2 are internally vertex-disjoint. \Box

We now study how a flip in an orientation in an NCL configuration corresponds to a constantlength sequence of reconfiguration steps of paths.

Claim 6.2. Let $\mathcal{P} = \{P_1, P_2\}$ be an s-t linkage of G that is created from an NCL configuration σ . If an NCL configuration σ' is obtained from σ by a flip of a single edge, then an s-t linkage $\mathcal{P}' = \{P'_1, P'_2\}$ of G that is created from σ' can be obtained from \mathcal{P} by at most a constant number of reconfiguration steps in G.

Proof. Assume that we are given an NCL configuration σ of H. Let $\mathcal{P} = \{P_1, P_2\}$ be an *s*-t linkage of G that is created from σ . By the construction, the path P_1 passes through all the red vertices; the path P_2 passes through all the blue vertices.

As the first case, consider a flip of a weight-1 edge f that connects two AND vertices u, vfrom (u, v) to (v, u). Then, the blue path P_2 passes through $b_{f,1}, w_{f,u}, b_{f,2}$ of the weight-1 edge gadget in this order. Let P'_2 be a blue path that is created from the NCL configuration obtained after flipping (u, v) to (v, u). To obtain an s-t linkage $\{P_1, P'_2\}$, we replace the part $b_{f,1}, w_{f,u}, b_{f,2}$ of P_2 with $b_{f,1}, w_{f,v}, b_{f,2}$.

This is possible only if P_1 does not pass through $w_{f,v}$; we now observe this is indeed the case. If P_1 passes through $w_{f,v}$, then by the construction of P_1 , the path P_1 passes through $r_{v,1}, w_{f,v}, w_{g,v}, r_{v,2}$ in this order where g is the other weight-1 edge incident to v. This happens only when a unique weight-2 edge incident to v is not oriented toward v. Therefore, after flipping to (v, u) the total sum of the incoming edges to v will be 1. This is a contradiction to the property of an NCL configuration.

As the second case, consider a flip of a weight-2 edge e that connects a vertex u and an AND vertex v from (u, v) to (v, u). Then, the red path P_1 passes through $r_{e,1}, w_{e,u}, r_{e,2}$ of the weight-2 edge gadget in this order. Let P'_1 be a red path that is created from the orientation obtained after flipping to (v, u). To obtain a family of internally vertex-disjoint paths P'_1, P_2 , we want to replace the part $r_{e,1}, w_{e,u}, r_{e,2}$ of P_1 with $r_{e,1}, w_{e,v}, r_{e,2}$.

This is possible only if P_2 does not pass through $w_{e,v}$. However, since e is oriented toward v, P_2 passes through $w_{e,v}$. Thus, in order to make $w_{e,v}$ vacant, we apply the following operations before the above transformation. We first replace the part $r_{v,1}$, k_v , $r_{v,2}$ of P_1 by $r_{v,1}$, $w_{f,v}$, $w_{g,v}$, $r_{v,2}$, where f and g are weight-1 edges incident to v. This is possible since f and g are oriented toward v in σ . Next, we replace the part $b_{v,1}$, $w_{e,v}$, $b_{v,2}$ of P_2 by $b_{v,1}$, k_v , $b_{v,2}$. This way, we can make $w_{e,v}$ vacant with two extra reconfiguration steps.

As the third but final case, consider a flip of a weight-2 edge e that connects a vertex u and an OR vertex v from (u, v) to (v, u). Then, the red path P_1 passes through $r_{e,1}, w_{e,u}, r_{e,2}$ of the weight-2 edge gadget in this order. Let P'_1 be a red path that is created from the orientation obtained after flipping to (v, u). To obtain a family of internally vertex-disjoint paths P'_1, P_2 , we want to replace the part $r_{e,1}, w_{e,u}, r_{e,2}$ of P_1 with $r_{e,1}, w_{e,v}, r_{e,2}$.

This is possible only if P_2 does not pass through $w_{e,v}$. If this is the case, there is no problem. However, in some cases, P_2 indeed passes through $w_{e,v}$. Then, we first replace the part $b_{v,1}, w_{e,v}, b_{v,2}$ with $b_{v,1}, w_{f,v}, b_{v,2}$ or $b_{v,1}, w_{g,v}, b_{v,2}$, where f, g are other weight-2 edges incident to v, to make $w_{e,v}$ vacant. We now observe that this extra step can always be done. Suppose not. Then, all the three vertices $w_{e,v}, w_{f,v}, w_{g,v}$ are occupied by P_2 . This means that the edges e, f, g are oriented as they leave v, and the total sum of edges incoming to v is zero. This is a contradiction to the property of an NCL configuration.

In the second and third cases above, if u is an AND vertex, then we replace the part $b_{u,1}$, k_u , $b_{u,2}$ of P_2 by $b_{u,1}$, $w_{e,u}$, $b_{u,2}$, and replace the part $r_{u,1}$, $w_{f',u}$, $w_{g',u}$, $r_{u,2}$ of P_1 by $r_{u,1}$, k_u , $r_{u,2}$, where f', g' are other weight-2 edges incident to u. Then, we obtain an s-t linkage of G that is created from σ' .

To argue the reverse direction of correspondence, we first observe that a reconfiguration step maintains the property that the paths go through the red and blue vertices.

Claim 6.3. Let $\mathcal{P} = \{P_1, P_2\}$ be an s-t linkage of G such that P_1 goes through all the red vertices and P_2 goes through all the blue vertices. If an s-t linkage $\mathcal{P}' = \{P'_1, P'_2\}$ of G is obtained by a single reconfiguration step in G from \mathcal{P} , then P'_1 goes through all the red vertices and P'_2 goes through all the blue vertices.

Proof. A single reconfiguration step changes one of P_1 and P_2 , but not both. We distinguish two cases.

First, assume that P_1 is reconfigured to another path P'_1 , where in this case $P_2 = P'_2$. Since all the blue vertices are occupied by P_2 , the path P'_1 cannot pass through any blue vertex. The removal of the blue vertices from G results in a graph with the following property: each red vertex separates s and t. Therefore, P'_1 must go through all the red vertices.

Second, assume that P_2 is reconfigured to another path P'_2 , where in this case $P_1 = P'_1$. Since all the red vertices are occupied by P_1 , the path P'_2 cannot pass through any red vertex. The removal of the red vertices from G yields a graph with the following property: each blue vertex separates s and t. Therefore, P'_2 must go through all the blue vertices.

We now describe a canonical way of constructing an NCL configuration σ of H from an *s*-*t* linkage $\mathcal{P} = \{P_1, P_2\}$ of G where P_1 goes through all the red vertices and P_2 goes through all the blue vertices.

Let $e = \{u, v\}$ be a weight-2 edge of H. The weight-2 edge gadget G_e of e has vertices $r_{e,1}$ and $r_{e,2}$ that are visited by P_1 . The path P_1 should visit one of $w_{e,u}$ and $w_{e,v}$, but not both. If P_1 visits $w_{e,u}$, then the edge e is oriented toward v in σ . If P_1 visits $w_{e,v}$, then the edge e is oriented toward u in σ .

Let $e = \{u, v\}$ be a weight-1 edge of H. The weight-1 edge gadget G_e of e has vertices $b_{e,1}$ and $b_{e,2}$ that are visited by P_2 . The path P_2 should visit one of $w_{e,u}$ and $w_{e,v}$, but not both. If P_2 visits $w_{e,u}$, then the edge e is oriented toward v in σ . If P_2 visits $w_{e,v}$, then the edge e is oriented toward u in σ .

This finishes the description of the construction of an NCL configuration. We call the constructed NCL configuration the *canonical NCL configuration obtained from* $\{P_1, P_2\}$. We observe that if $\mathcal{P} = \{P_1, P_2\}$ is an *s*-*t* linkage of *G* that is created from an NCL configuration σ , then σ is the canonical NCL configuration obtained from \mathcal{P} .

Claim 6.4. Let $\mathcal{P} = \{P_1, P_2\}$ be an s-t linkage of G such that P_1 goes through all the red vertices and P_2 goes through all the blue vertices. Then, the canonical NCL configuration σ obtained from \mathcal{P} is indeed an NCL configuration of H.

Proof. We check that σ satisfies the weight requirement at every vertex v of H.

Assume that v is an OR vertex which is incident to three weight-2 edges e, f, and g. We observe that in the OR vertex gadget G_v , one of $w_{e,v}$, $w_{f,v}$ and $w_{g,v}$ is not occupied by the red path P_1 . Suppose not. Then, out of four neighbors of the blue vertex $b_{v,1}$ in G_v , the three vertices $w_{e,v}$, $w_{f,v}$ and $w_{g,v}$ are already occupied. Therefore, the blue path P_2 cannot be vertex-disjoint with P_1 . This is a contradiction.

Therefore, in the canonical NCL configuration σ , one of the edges e, f, and g is oriented toward v. This means the incoming weight to v is at least two, and the weight requirement is satisfied at v.

Assume that v is an AND vertex which is incident to one weight-2 edge e and two weight-1 edges f and g. We observe that in the AND vertex gadget G_v , if $w_{e,v}$ is occupied by P_1 , then $w_{f,v}$ and $w_{g,v}$ are occupied by P_1 , too. To this end, assume that $w_{e,v}$ is occupied by P_1 . Since P_1 and P_2 are vertex-disjoint, and $b_{v,1}$ and $b_{v,2}$ are of degree 3 that share $w_{e,v}$ as one of the common neighbors, the blue path P_2 must go through the black vertex k_v . Since the red vertices $r_{v,1}$ and $r_{v,2}$ are of degree 3 that share k_v as a common neighbor, the red path P_1 must go through $w_{f,v}$ and $w_{g,v}$. This finishes the proof of the observation.

Therefore, if $w_{e,v}$ is not occupied by P_1 , then in the canonical NCL configuration σ the weight-2 edge e is oriented toward v, in which case the weight requirement is satisfied at v. If not, by the observation above, $w_{f,v}$ and $w_{g,v}$ are occupied by P_1 . Since P_1 and P_2 are vertexdisjoint, the blue path P_2 cannot go through any of $w_{f,v}$ and $w_{g,v}$. This means that in the canonical NCL configuration σ the two weight-1 edges f and g are oriented toward v, and in this case the weight requirement is satisfied at v.

Claim 6.5. Let $\mathcal{P} = \{P_1, P_2\}$ be an s-t linkage of G such that P_1 goes through all the red vertices and P_2 goes through all the blue vertices, and let σ be the canonical NCL configuration obtained from \mathcal{P} . Let $\mathcal{P}' = \{P'_1, P'_2\}$ be an s-t linkage of G obtained by a single reconfiguration step from \mathcal{P} and σ' be the canonical NCL configuration obtained from \mathcal{P}' . Then, σ' can be obtained from σ by at most |V(H)| + |E(H)| flips in H.

Proof. Suppose that we now want to transform P_1 to P'_1 in such a way that P'_1 and P_2 are internally vertex-disjoint *s*-*t* paths. By Claim 6.3, P'_1 goes through all the red vertices.

Consider the red vertices $r_{e,1}$ and $r_{e,2}$ for an edge $e = \{u, v\} \in E(H)$. Since P_1 passes through both of them, P_1 must pass through exactly one of $w_{e,u}$ and $w_{e,v}$. The same observation applies to P'_1 , too. If P_1 passes through $w_{e,u}$ and P_2 does not pass through $w_{e,v}$, then we may reroute P_1 to pass through $w_{e,v}$ instead of $w_{e,u}$ so as to obtain P'_1 .

Consider the red vertices $r_{v,1}$ and $r_{v,2}$ for an AND vertex $v \in V(H)$ that is incident to one weight-2 edge e and two weight-1 edges f and g. Then, P_1 must pass through either k_v or $w_{f,v}, w_{g,v}$. The same observation applies to P'_1 , too. If P_1 passes through k_v and P_2 passes through neither $w_{f,v}$ nor $w_{g,v}$, then we may reroute P_1 to pass through $w_{f,v}, w_{g,v}$ so as to obtain P'_1 . If P_1 passes through $w_{f,v}$ and $w_{g,v}$, and P_2 does not pass through k_v , then we may reroute P_1 to pass through k_v so as to obtain P'_1 .

Rerouting within an AND vertex gadget does not correspond to flips in an orientation. On the other hand, rerouting within an edge gadget corresponds to a flip in an orientation. Namely, rerouting P_1 to pass through $w_{e,v}$ instead of $w_{e,u}$ is mapped to a flip of the direction of e. If one rerouting operation involves the reconnection within several edge gadgets, then the operation is mapped to a set of flips of the corresponding edges. We now argue that such a set of flips can be performed sequentially. To this end, imagine that one rerouting operation of P_1 to P'_1 is decomposed into a number of reconfiguration steps each of which involves only one edge gadget. Then, the sequence of those decomposed reconfiguration steps together yields P'_1 from P_1 , and each step corresponds to a single flip in H. Since two or more edge gadgets do not share a vertex in G, such a decomposition is well-defined, and we are done.

Similarly, we may study the case where we transform P_2 to P'_2 in such a way that P_1 and P'_2 are internally vertex-disjoint *s*-*t* paths. Rerouting within an AND vertex gadget and an OR vertex gadget is similar, and does not correspond to a flip in an orientation. On the other hand, rerouting within an edge gadget corresponds to a flip of the corresponding edge.

When we are able to reroute a path within an edge gadget G_e for an edge $e = \{u, v\} \in E(H)$, the other path passes through neither $w_{e,u}$ nor $w_{e,v}$. This means that, even without e, the total weights of incoming arcs to u and v are at least two, respectively. Therefore, the flip described above maintains the property that the resulting orientation is again an NCL configuration.

There can be several gadgets that are involved in one reconfiguration step. However, the number of such gadgets is at most |V(H)| + |E(H)|. Hence, that reconfiguration step corresponds to at most |V(H)| + |E(H)| flips in H.

Let σ' be the canonical NCL configuration in Claim 6.5, and $\mathcal{P}'' = (P_1'', P_2'')$ be the family of two internally vertex-disjoint *s*-*t* paths of *G* that is created from σ' . Note that \mathcal{P}'' does not have to be identical to \mathcal{P}' . However, \mathcal{P}'' can be obtained from \mathcal{P}' by at most $|V(H)| + |V^{a}(H)|$ reconfiguration steps (recall that $V^{a}(H)$ is the set of AND vertices of *H*). This is because the situations in edge gadgets are identical in \mathcal{P}' and \mathcal{P}'' , and reconfiguration steps would be needed only around vertex gadgets.

As a summary, we have proved that the reduction is sound, complete, and polynomially bounded. See Figure 18. We now complete the proof of Theorem 1.1. Let σ and τ be two NCL configurations. Then, define two *s*-*t* linkages $\mathcal{P} = (P_1, P_2)$ and $\mathcal{Q} = (Q_1, Q_2)$ of *G* as they are created from σ and τ , respectively. Suppose that we may transform σ to τ by a sequence of flips in such a way that all the intermediate orientations are NCL configurations. Then, the correspondence above gives a sequence of reconfiguration steps to transform \mathcal{P} to \mathcal{Q} so that all the intermediate sets of two *s*-*t* paths are internally vertex-disjoint. On the other hand, suppose that we may transform \mathcal{P} to \mathcal{Q} by reconfiguration steps. Then, the correspondence above gives a sequence of flips to transform σ to τ so that all the intermediate orientations are NCL configurations.

It is useful to note that the reduction can be modified so that G in the constructed instance of DISJOINT *s-t* PATHS RECONFIGURATION has bounded bandwidth if the AND/OR graph Hin a given instance of the NCL reconfiguration has bounded bandwidth. A brief sketch is given in Appendix C. This shows that Theorem 1.1 holds even for graphs of bounded bandwidth and maximum degree four.

6.2 Planar Graphs of Bounded Bandwidth with Unbounded k

A similar reduction proves the PSPACE-completeness for planar graphs of bounded bandwidth when the number of paths is unbounded.

Theorem 1.6. The DISJOINT PATHS RECONFIGURATION is PSPACE-complete when the graph G is planar and of bounded bandwidth.

Proof. We have already observed that the problem is in PSPACE. To show the PSPACEhardness, we now give a transformation of a given planar AND/OR graph H = (V(H), E(H)) of



Figure 18: The correspondence between flips in NCL configurations and reconfiguration steps for vertex-disjoint paths.



Figure 19: An edge gadget.



Figure 20: An OR vertex gadget.

bounded bandwidth to an undirected planar graph G = (V(G), E(G)) of bounded bandwidth. Each edge $e = \{u, v\} \in E(H)$ of H is mapped to the following *edge gadget* G_e (see Figure 19):

$$V(G_e) = \{w_{e,u}, w_{e,v}, s_e, t_e\}, E(G_e) = \{\{w_{e,u}, s_e\}, \{w_{e,u}, t_e\}, \{w_{e,v}, s_e\}, \{w_{e,v}, t_e\}\}.$$

Let $v \in V(H)$ be an OR vertex, and $e, f, g \in E(H)$ be three weight-2 edges incident to v. Then, v is mapped to the following *OR vertex gadget* G_v (see Figure 20):

$$V(G_v) = \{w_{e,v}, w_{f,v}, w_{g,v}, a_v, b_v, c_v, d_v, s_v, t_v, s_v^{\text{o}}, t_v^{\text{o}}\},\$$

$$E(G_v) = \{\{w_{e,v}, a_v\}, \{w_{e,v}, b_v\}, \{w_{e,v}, t_v\}, \{w_{f,v}, a_v\}, \{w_{f,v}, s_v\},\$$

$$\{w_{g,v}, b_v\}, \{w_{g,v}, s_v\}, \{a_v, c_v\}, \{a_v, d_v\}, \{b_v, c_v\}, \{b_v, d_v\},\$$

$$\{c_v, s_v\}, \{c_v, s_v^{\text{o}}\}, \{c_v, t_v^{\text{o}}\}, \{d_v, t_v\}, \{d_v, s_v^{\text{o}}\}, \{d_v, t_v^{\text{o}}\}\}.$$

Note that the vertices with the same label are identified.

An AND vertex $v \in V(H)$ that is incident to one weight-2 edge e and two weight-1 edges f, g is mapped to the following AND vertex gadget G_v (see Figure 21):

$$V(G_v) = \{w_{e,v}, w_{f,v}, w_{g,v}, s_v, t_v\}, E(G_v) = \{\{w_{e,v}, s_v\}, \{w_{e,v}, t_v\}, \{w_{f,v}, s_v\}, \{w_{g,v}, t_v\}, \{w_{f,v}, w_{g,v}\}\}$$

Define

$$V(G) = \bigcup_{v \in V(H)} V(G_v) \cup \bigcup_{e \in E(H)} V(G_e),$$
$$E(G) = \bigcup_{v \in V(H)} E(G_v) \cup \bigcup_{e \in E(H)} E(G_e).$$

This completes the construction of G. Figure 22 may help the reader understand.



Figure 21: An AND vertex gadget.



Figure 22: Mapping an NCL configuration to a linkage. In the NCL configuration, the vertex label A stands for an AND vertex, and the vertex label O stands for an OR vertex.

We note that if H is planar and of bounded bandwidth, then so is G. To see that G is planar, observe that each gadget can be drawn on the plane without edge crossing in such a way that the vertices shared with other gadgets are placed on its outer face. Then, a plane drawing of G can be obtained from a plane drawing of H by replacing each vertex and edge of H by the corresponding gadget. To see that the bandwidth of G is bounded, observe that each gadget is of constant size. Therefore, by giving a slack of constant width c for each vertex of H, we may obtain an injective mapping of the vertices of G to \mathbb{Z} whose bandwidth is at most c times the bandwidth of H.³

In the constructed graph G, we consider the following pairs of vertices:

$$\{(s_e, t_e) \mid e \in E(H)\} \cup \{(s_v, t_v) \mid v \in V(H)\} \cup \{(s_v^{o}, t_v^{o}) \mid v \in V^{o}(H)\},\$$

where $V^{\circ}(H)$ denotes the set of OR vertices in H. The number of pairs is $|E(H)| + |V(H)| + |V^{\circ}(H)|$.

We now describe the way how an NCL configuration σ of H translates to a linkage

$$\mathcal{P} = \{ P_e \mid e \in E(H) \} \cup \{ P_v \mid v \in V(H) \} \cup \{ P_v^{o} \mid v \in V^{o}(H) \}$$

of G, where P_e joins s_e and t_e for every $e \in E(H)$, P_v joins s_v and t_v for every $v \in V(H)$, and P_v^{o} joins s_v^{o} and t_v^{o} for every $v \in V^{o}(H)$. For convenience, P_e is called a *red* path for every $e \in E(H)$, and P_v^{o} is called a *blue* path for every $v \in V^{o}(H)$. Similarly, for every $v \in V(H)$, P_v is called a *green* path if v is an AND vertex, and a *purple* path if v is an OR vertex.

First, we specify the red paths P_e for all $e = \{u, v\} \in E(H)$. The path P_e connects s_e and t_e and its length is two. Therefore, there will be a unique middle vertex. If the edge e is oriented from u to v in the NCL configuration σ , then the path P_e visits $w_{e,u}$ as its middle vertex. If e is oriented from v to u in σ , then P_e visits $w_{e,v}$ as its middle vertex. This finishes the specification of the red paths P_e .

Next, we specify the green paths P_v that connect s_v and t_v for all AND vertices $v \in V(H)$. Recall that v is incident to a weight-2 edge e and two weight-1 edges f, g. If e is oriented toward

³The constant c can be chosen as 16, but this particular choice is not relevant.

v, then P_v passes through $w_{e,v}$ and its length is two. Otherwise, f and g must be oriented toward v. Then, P_v passes through $w_{f,v}$ and $w_{g,v}$ and its length is three. This finishes the specification of the green paths P_v .

Finally, we specify the purple paths P_v that connect s_v and t_v and the blue paths P_v^{o} that connect s_v^{o} and t_v^{o} for all OR vertices $v \in V^{o}(H)$. Recall that v is incident to three weight-2 edges e, f, g. We have three cases, which are not exclusive. If e is oriented toward v, then the purple path P_v passes through $c_v, a_v, w_{e,v}$ (or $c_v, b_v, w_{e,v}$), and the blue path P_v^{o} passes through d_v . If f is oriented toward v, then P_v passes through $w_{f,v}, a_v, d_v$, and P_v^{o} passes through c_v . If g is oriented toward v, then P_v passes through $w_{g,v}, b_v, d_v$, and P_v^{o} passes through c_v . This finishes the specification of the purple paths P_v and the blue paths P_v^{o} . Note that the lengths of purple paths are always four and the lengths of blue paths are always two.

This finishes the construction of \mathcal{P} . With this construction, we say that the linkage \mathcal{P} is created from the NCL configuration σ .

Claim 6.6. The constructed paths are vertex-disjoint.

Proof. Each path visits vertices in the corresponding gadget only, and within each OR gadget the two constructed paths do not share vertices by the construction rule. Therefore, it suffices to examine the situation at identified vertices between an edge gadget and an AND/OR vertex gadget.

Let v be an OR vertex that are incident to three weight-2 edges e, f, g. Assume that the corresponding OR vertex gadget is constructed as above.

Suppose that the vertex $w_{e,v}$ is shared by two of the constructed paths. One of those paths is the red path P_e that connects s_e and t_e ; the other path is the purple path P_v that connects s_v and t_v . Since P_e passes through $w_{e,v}$, the edge e is not oriented toward v. On the other hand, since P_v passes through $w_{e,v}$, the edge e is oriented toward v. This is a contradiction, and thus $w_{e,v}$ is not shared by two paths.

Suppose that the vertex $w_{f,v}$ is shared by two of the constructed paths. One of those paths is the red path P_f , which means that the edge f is not oriented toward v. Then, the other path is the purple path P_v . By the construction, it holds that f is oriented toward v. This is a contradiction. The same conclusion holds when $w_{g,v}$ is shared by two paths.

Let v be an AND vertex that are incident to one weight-2 edge e and two weight-1 edges f, g. Assume that the corresponding AND vertex gadget is constructed as above.

Suppose that the vertex $w_{e,v}$ is shared by two of the constructed paths. One of those paths is the red path P_e that connects s_e and t_e ; the other path is the green path P_v that connects s_v and t_v . Since P_e passes through $w_{e,v}$, the edge e is not oriented toward v. On the other hand, since P_v passes through $w_{e,v}$, the edge e is oriented toward v. This is a contradiction, and thus $w_{e,v}$ is not shared by two paths.

Suppose that the vertex $w_{f,v}$ is shared by two of the constructed paths. One of those paths is the red path P_f that connects s_f and t_f , which means that the edge f is not oriented toward v. The other path is the green path P_v that connects s_v and t_v , which means that f is oriented toward v. This is a contradiction. The same conclusion holds when $w_{g,v}$ is shared by two paths.

In summary, there is no vertex in the instance G that is shared by two or more constructed paths. $\hfill \Box$

We now study how a flip in an orientation maps to a sequence of reconfiguration steps of paths.

Claim 6.7. Let \mathcal{P} be a linkage of G that is created from an NCL configuration σ . If an NCL configuration σ' is obtained from σ by an flip of a single edge, then a linkage \mathcal{P}' of G that is

created from σ' can be obtained from \mathcal{P} by at most a constant number of reconfiguration steps in G.

Proof. Assume that we are given an NCL configuration σ of H. Let

$$\mathcal{P} = \{ P_e \mid e \in E(H) \} \cup \{ P_v \mid v \in V(H) \} \cup \{ P_v^{o} \mid v \in V^{o}(H) \}$$

be a linkage of G that is created from σ . By the construction, each of the colored paths passes through all the vertices of the corresponding color.

As the first case, consider a flip of a weight-1 edge f that connects two AND vertices u, v from (u, v) to (v, u). Then, the red path P_f passes through $s_f, w_{f,u}, t_f$ of the weight-1 edge gadget in this order. The path P'_f is obtained by replacing it with $s_f, w_{f,v}, t_f$.

This is possible only if $w_{f,v}$ is not passes through by any other paths; we now observe this is indeed the case. Suppose that $w_{f,v}$ is passed through by another path. Then, such a path should be a green path P_v in the AND vertex gadget G_v . When P_v passes through $w_{f,v}$, a unique weight-2 edge e incident to v is not oriented toward v in σ by the construction. Therefore, after flipping to (v, u) the total sum of the incoming edges to v will be 1. This is a contradiction to the property of an NCL configuration.

As the second case, consider a flip of a weight-2 edge e that connects a vertex u and an AND vertex v from (u, v) to (v, u). Then, the red path P_e passes through $s_e, w_{e,u}, t_e$ of the weight-2 edge gadget G_e in this order. The path P'_e is obtained by replacing it with $s_e, w_{e,v}, t_e$.

This is possible only if $w_{e,v}$ is not passes through by an other paths. However, since e is oriented toward v, a green path P_v in the AND vertex gadget G_v passes through $w_{e,v}$. Thus, in order to make $w_{e,v}$ vacant, we apply the following operations before the above transformation. Since e can be flipped, after the flip the total sum of the incoming edges to v should be at least two, which means that the other two weight-1 edges f, g that are incident to v should be directed toward v. Therefore, $w_{v,f}$ and $w_{v,g}$ are not occupied by any other paths in \mathcal{P} . This implies that by rerouting the green path P_v to pass through $w_{v,f}, w_{v,g}$ before rerouting P_e to P'_e we obtain the linkage that corresponds to σ' .

As the third but final case, consider a flip of a weight-2 edge e that connects a vertex u and an OR vertex v from (u, v) to (v, u). Then, the red path P_e passes through $s_e, w_{e,u}, t_e$ of the edge gadget G_e in this order. The P'_e is obtained by replacing it with $s_e, w_{e,v}, t_e$.

This is possible only if $w_{e,v}$ is not passed through by any other paths. If this is the case, there is no problem. However, in some cases, $w_{e,v}$ is indeed passed through by another path. Then, such a path should be a purple path P_v in the OR vertex gadget G_v . By construction, the edge e is oriented toward v in σ . Since e can be flipped, after the flip the total sum of the incoming edges to v should be at least two, which means that one of the other two weight-2 edges f, g that are incident to v should be directed toward v. Therefore, $w_{v,f}$ or $w_{v,g}$ is not occupied by any other paths in \mathcal{P} . This implies that by rerouting the purple path P_v to pass through $w_{v,f}$ or $w_{v,g}$ before rerouting P_e to P'_e , possibly with two more reconfiguration steps involving the blue path P'_v , we obtain the linkage that corresponds to σ' .

In the second and third cases above, if u is an AND vertex, then we reroute a green path P_u in the AND vertex gadget G_u to pass through $w_{e,u}$. Then, we obtain a linkage \mathcal{P}' of G that is created from σ' .

To argue the reverse direction of correspondence, we describe a canonical way of constructing an NCL configuration σ of H from a linkage \mathcal{P} of G.

Let $e = \{u, v\}$ be an edge of H. We note that a path connecting s_e and t_e cannot go through both of $w_{e,u}$ and $w_{e,v}$ due to the vertex-disjointness of the paths in \mathcal{P} . If P_e goes through $w_{e,u}$, then the edge e is oriented toward v in σ . If P_e goes through $w_{e,v}$, then the edge e is oriented toward u in σ . We call the obtained orientation σ the canonical NCL configuration obtained from \mathcal{P} .

Claim 6.8. Let \mathcal{P} be a linkage of G. Then, the canonical NCL configuration σ obtained from \mathcal{P} is indeed an NCL configuration of H.

Proof. We check that σ satisfies the weight requirement at every vertex v of H.

Let v be an OR vertex of H that is incident to three weight-2 edges e, f and g. The corresponding OR vertex gadget is given as in Figure 20. Since the blue path in the gadget G_v contains a vertex c_v or d_v , the purple path in G_v contains at least one of $w_{e,v}$, $w_{f,v}$ and $w_{g,v}$. This shows that one of $w_{e,v}$, $w_{f,v}$ and $w_{g,v}$ is not occupied by any red paths. Therefore, in the canonical NCL configuration σ , one of the edges e, f and g is oriented toward v. This means that the weight requirement is satisfied at v.

Next, let v be an AND vertex of H that is incident to one weight-2 edge e and two weight-1 edges f and g. The green path between s_v to t_v either goes through $w_{e,v}$ or $w_{f,v}, w_{g,v}$. If it goes through $w_{e,v}$, then the red path P_e does not go through $w_{e,v}$ due to vertex-disjointness. In this case, the weight-2 edge e is oriented toward v in σ . If the green path goes through $w_{f,v}$ and $w_{g,v}$, then the red paths P_f and P_g do not go through $w_{f,v}$ and $w_{g,v}$, respectively. In this case, the two weight-1 edges f and g are oriented toward v in σ . Hence, in both cases, the weight requirement is satisfied at v.

Claim 6.9. Let \mathcal{P} be a linkage of G and σ be the canonical NCL configuration obtained from \mathcal{P} . Let a linkage \mathcal{P}' be obtained by a single reconfiguration step from \mathcal{P} and σ' be the canonical NCL configuration obtained from \mathcal{P}' . Then, σ' can be obtained from σ by at most one flip in H.

Proof. We distinguish cases by colors of paths that are reconfigured by one step.

First, consider a case where \mathcal{P} and \mathcal{P}' differ by one red path associated with an edge gadget G_e . Assume that P_e is reconfigured to P'_e . Without loss of generality, assume that P_e passes through $s_e, w_{e,u}, t_e$, and P'_e passes through $s_e, w_{e,v}, t_e$ in this order. Then, to obtain σ' from σ we just need to flip the direction of the edge e.

Next, consider a case where \mathcal{P} and \mathcal{P}' differ by one path that is not a red path. Then, the canonical NCL configuration σ' obtained from \mathcal{P}' is identical to σ . Therefore, we need no flip to construct σ' .

Let σ' be the canonical NCL configuration in Claim 6.9, and \mathcal{P}'' be the linkage created from σ' . Note that \mathcal{P}'' does not have to be identical to \mathcal{P}' . However, \mathcal{P}'' can be obtained from \mathcal{P}' by O(|V(H)|) reconfiguration steps. This is because the situations in edge gadgets are identical in \mathcal{P}' and \mathcal{P}'' , and reconfiguration steps would be needed only around vertex gadgets.

As a summary, we have proved that the reduction is sound, complete, and polynomially bounded. See Figure 23. We now complete the proof of Theorem 1.6. Let σ and τ be two NCL configurations. Then, define two linkages \mathcal{P} and \mathcal{Q} as they correspond to σ and τ , respectively. Suppose that we may transform σ to τ by a sequence of flips in such a way that all the intermediate orientations are NCL configurations. Then, the correspondence above gives a sequence of reconfiguration steps to transform \mathcal{P} to \mathcal{Q} so that all the intermediate families of paths are vertex-disjoint. On the other hand, suppose that we may transform \mathcal{P} to \mathcal{Q} by reconfiguration steps. Then, the correspondence above gives a sequence of flips to transform σ to τ so that all the intermediate orientations are NCL configurations. \Box



Figure 23: The correspondence between flips in NCL configurations and reconfiguration steps for vertex-disjoint paths.

7 Concluding Remarks

We leave several open problems for future research. We proved that DISJOINT PATHS RECON-FIGURATION can be solved in polynomial time when the problem is restricted to the two-face instances. On the other hand, we do not know whether DISJOINT PATHS RECONFIGURATION in planar graphs can be solved in polynomial time for fixed k, and even when k = 2, if we drop the requirement that inputs are two-face instances.

We did not try to minimize the number of reconfiguration steps when a reconfiguration sequence exists. It is an open problem whether a shortest reconfiguration sequence can be found in polynomial time for DISJOINT PATHS RECONFIGURATION restricted to planar twoface instances.

A natural extension of our studies is to consider a higher-genus surface. As a preliminary result, in Appendix A, we give a proof (sketch) to show that when the number k of curves is two, the reconfiguration is *always* possible for any connected orientable closed surface Σ_g of genus $g \ge 1$. Note that this result does not refer to graphs embedded on Σ_g , but only refers to the case when curves can pass through any points on the surface. It is not clear what we can say for DISJOINT PATHS RECONFIGURATION for graphs embedded on Σ_g , $g \ge 1$.

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A Curves on Closed Surfaces

Let Σ_g denote a connected orientable closed surface of genus g. For instance, Σ_0 is a 2-sphere S^2 and Σ_1 is a torus. In contrast to the 2-sphere case, when k = 2, the reconfiguration is always possible on Σ_q for $g \ge 1$.

Theorem A.1. Let $\mathcal{P} = (P_1, P_2)$ and $\mathcal{Q} = (Q_1, Q_2)$ be linkages on Σ_g for $g \ge 1$. Then, \mathcal{P} is always reconfigurable to \mathcal{Q} .

Proof Sketch. A small perturbation allows us to assume that the curves intersect transversally. The proof is by induction on the number of intersection points of \mathcal{P} and \mathcal{Q} , say n. The case n = 4 is obvious since they intersect only at the endpoints. Suppose that the statement holds up to n - 1. We first cut the surface Σ_g along \mathcal{P} and \mathcal{Q} . Let S be a connected component. Then, ∂S is a disjoint union of alternating cycles of edges derived from \mathcal{P} and \mathcal{Q} . If ∂S has a 2-cycle, then one can eliminate this cycle by a single step. On the other hand, if ∂S has at least six edges, then there are two edges derived from P_i for some i = 1, 2, and thus one can find another \mathcal{P}' such that the number of intersections between the curves in \mathcal{P}' and \mathcal{Q} is less than n.

Therefore, we shall consider the case where ∂S is a 4-cycle and show that there is a connected component S such that the genus of the closed surface obtained by capping its boundary with disks is at least 1. Then, one finds a desired \mathcal{P}' after some steps. Assume that there is no such an S, that is, all connected components are disks, and hence we have a cell decomposition of Σ_g . Let m be the number of 2-cells, namely disks. Then, the numbers of 1- and 0-cells are 2m and m+2, respectively. It follows that the Euler characteristic of Σ_g is (m+2) - 2m + m = 2, and thus g = 0 (see [18, Section 1.1.1] for example). This is a contradiction.

B $\mu(P_i, Q_i)$ Takes the Same Value

In this section, we only consider piecewise smooth curves and paths on a plane. For an oriented curve C, let \overline{C} denote the curve with the opposite orientation. We first summarize basic properties of μ that follow from definition.

- (A1) $\mu(C_1, C_2) = -\mu(C_2, C_1) = -\mu(C_1, \overline{C}_2)$ for any oriented curves C_1 and C_2 .
- (A2) Suppose that $C_1 \cup C_2$ is an oriented curve obtained by concatenating two oriented curves C_1 and C_2 , and let C_3 be another oriented curve. Then, $\mu(C_1 \cup C_2, C_3) = \mu(C_1, C_3) + \mu(C_2, C_3)$ and $\mu(C_3, C_1 \cup C_2) = \mu(C_3, C_1) + \mu(C_3, C_2)$.
- (A3) $\mu(C_1, C_2) = 0$ for any oriented closed curves C_1 and C_2 (by the Jordan curve theorem).

By using these properties, we show that $\mu(P_i, Q_i)$ takes the same value in a two-face instance.

Lemma B.1. In a two-face instance $(G, \mathcal{P}, \mathcal{Q})$ of DISJOINT PATHS RECONFIGURATION, $\mu(P_i, Q_j)$ takes the same value for any distinct $i, j \in [k]$.

Proof. Let s_{k+1} and s_{k+2} be points on ∂S such that s_k, s_{k+1}, s_{k+2} , and s_1 lie on ∂S clockwise in this order. Similarly, let t_{k+1} and t_{k+2} be points on ∂T such that t_k, t_{k+1}, t_{k+2} , and t_1 lie on ∂T clockwise in this order. Then, there exist simple curves P_{k+1} from s_{k+1} to t_{k+1} and P_{k+2} from s_{k+2} to t_{k+2} such that $\{P_1, \ldots, P_k, P_{k+1}, P_{k+2}\}$ forms a linkage in $\mathbb{R}^2 \setminus (S \cup T)$. Similarly, let Q_{k+1} and Q_{k+2} be simple curves such that $\{Q_1, \ldots, Q_k, Q_{k+1}, Q_{k+2}\}$ forms a linkage in $\mathbb{R}^2 \setminus (S \cup T)$. In what follows, we show that $\mu(P_i, Q_j) = \mu(P_{k+1}, Q_{k+2})$ for any distinct $i, j \in [k]$.

Let $i, j \in [k]$ with $i \neq j$. Let C_s be a curve in S from s_{k+1} to s_i , and let C_t be a curve in T from t_i to t_{k+1} . Since $P_i \cup C_t \cup \overline{P_{k+1}} \cup C_s$ and $\overline{P_j} \cup Q_j$ are oriented closed curves, by (A3), we obtain $\mu(P_i \cup C_t \cup \overline{P_{k+1}} \cup C_s, \overline{P_j} \cup Q_j) = 0$. Since $\overline{P_j} \cap (P_i \cup C_t \cup \overline{P_{k+1}} \cup C_s) = \emptyset$ and $Q_j \cap (C_s \cup C_t) = \emptyset$, this together with (A2) shows that $\mu(P_i, Q_j) + \mu(\overline{P_{k+1}}, Q_j) = 0$. By (A1), we obtain $\mu(P_i, Q_j) = \mu(P_{k+1}, Q_j)$.

By the same argument, we can show that $\mu(P_{k+1}, Q_j) = \mu(P_{k+1}, Q_{k+2})$. Therefore, we obtain $\mu(P_i, Q_j) = \mu(P_{k+1}, Q_{k+2})$ for any distinct $i, j \in [k]$, which completes the proof. \Box

C Variation of the Reduction for Theorem 1.1.

In this section, we briefly sketch the modification of the proof for Theorem 1.1 to show that the PSPACE-completeness of DISJOINT s-t PATHS RECONFIGURATION also holds for graphs of bounded bandwidth.

The basic line of reduction is identical to the proof for Theorem 1.1. We require the constructed graph G to have bounded bandwidth if the AND/OR graph H in a given instance of the NCL reconfiguration has bounded bandwidth.

To ensure that G has bounded bandwidth, we only change the construction of two paths, the red path P_1 and the blue path P_2 . In the original proof of Theorem 1.1, we have a choice of the ordering along which P_1 goes through all the AND vertex gadgets and weight-2 edge gadgets, and P_2 goes through all the AND vertex gadgets, OR vertex gadgets and weight-1 edge gadgets. We will exploit this freedom.

Since H has bounded bandwidth, there exists an injective map $\pi: V(H) \to \mathbb{Z}$ such that $|\pi(u) - \pi(v)| \leq c$ for some constant c. Let H° be the graph that is obtained from H by subdividing each edge of H, i.e., replacing each edge of H by a path of length two. We construct an injective map $\pi^{\circ}: V(H^{\circ}) \to \mathbb{Z}$ as follows. For each vertex $v \in V(H)$, set $\pi^{\circ}(v) = 4\pi(v)$. For a vertex $w \in V(H^{\circ})$ that is used for subdividing an edge $\{u, v\} \in E(H)$, where $\pi(u) < \pi(v)$, we

set $\pi^{\circ}(w)$ as $\pi^{\circ}(w) \in \{\pi(u) + 1, \pi(u) + 2, \pi(u) + 3\}$. This is possible since the AND/OR graph H is 3-regular. Observe that the bandwidth of π° is at most 4c + 3. For more information on the bandwidth of graphs obtained by graph operations, see [12].

Now, we replace each vertex of H° by the corresponding gadget given in the proof of Theorem 1.1. To ease the presentation, we add one isolated blue vertex to each weight-2 edge gadget, one isolated red vertex to each weight-1 edge gadget, and one isolated red vertex to each OR vertex gadget so that each gadget contains at least one red and at least one blue vertices. Note that each gadget is of constant order (at most eight), even after adding isolated vertices. Therefore, this replacement may increase the bandwidth at most by the factor of eight. Denote by V be the constructed vertex set and by π' the implied injective map of bandwidth at most $8 \cdot (4c+3) + 7$, where π' satisfies that $8\pi^{\circ}(w) \leq \pi'(v) \leq 8\pi^{\circ}(w) + 7$ if $v \in V$ is contained in the gadget corresponding to $w \in V(H^{\circ})$.

We insert s and t to V and set $\pi'(s) = \min\{\pi'(v)\} - 1, \pi'(t) = \max\{\pi'(v)\} + 1$. To wire the red path P_1 , we follow the increasing order of the values of π' . Namely, the path P_1 starts at s, visits the red vertices in the increasing order of π' (with the routing within each gadget as the proof of Theorem 1.1), and terminates at t. Since each gadget contains a red vertex, each edge $\{x, y\}$ of P_1 satisfies $|\pi'(x) - \pi'(y)| \leq 15$. Wiring the blue path P_2 is similarly done. This completes the whole reduction.

The constructed instance has bounded bandwidth, and the correctness is immediate from the proof of Theorem 1.1 since we only makes use of the freedom of ordering for the construction of P_1 and P_2 .

Thus, DISJOINT *s*-*t* PATHS RECONFIGURATION is PSPACE-complete even when k = 2 and *G* has bounded bandwidth and maximum degree four.