VARIATIONAL PRINCIPLE OF RANDOM PRESSURE FUNCTION

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ABSTRACT. This paper aims to develop a convex analysis approach to random pressure functions of random dynamical systems. Using some convex analysis techniques and functional analysis, we establish a variational principle for random pressure function, which extends Bis et al. work (A convex analysis approach to entropy functions, variational principles and equilibrium states, Comm. Math. Phys. (2022) **394** 215-256) and Ruelle's work (Statistical mechanics on a compact set with \mathbb{Z}^{ν} action satisfying expansiveness and specification, Trans. Amer. Math. Soc. (1973) **187** 237-251) to random dynamical systems.

The present paper provides a strategy of obtaining some proper variational principles for entropy-like quantities of dynamical systems to link the topological dynamics and ergodic theory. As applications, we establish variational principles of maximal pattern entropy and polynomial topological entropy of zero entropy systems of Z-actions, mean dimensions of infinite entropy systems acting by amenable groups and preimage entropy-like quantities of non-convertible random dynamical systems.

1. INTRODUCTION

Topological entropy and measure-theoretic entropy are vital topological invariants to understand the complexity of the dynamical systems, which is related by so-called variational principle [Mis75]:

$$h_{top}(X,T) = \sup_{\mu \in M(X,T)} h_{\mu}(T),$$

where $h_{top}(X, T)$ denotes the topological entropy of X, and $h_{\mu}(T)$ denotes the measure-theoretic entropy of the invariant measure μ . Variational principle is a fundamental and important theorem in the theory of topological dynamics, which allows the approaches and tools from the ergodic theory to be invoked to study the dynamical behaviors of topological dynamical systems. The previous work revealed that it is a powerful tool to deal with the problems of local entropy theory, chaotic phenomenons, multi-fractal analysis and other fields of dynamical systems. Topological pressure and its variational principle, equilibrium

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theory constitutes the hardcore part of thermodynamic formalism. Inspired by statistical mechanics, Ruelle [Rue73] introduced the notion of topological pressure for expansive topological dynamical systems, and proved a variational principle for it in terms of an analogues role of measure-theoretic entropy:

$$P(T, f) = \max_{\mu \in \mathcal{M}(X,T)} \{ s(\mu) + \int f d\mu \},$$

where P(T, f) is the topological pressure of the continuous potential f, and $s(\mu) = \inf_{f \in C(X)} \{P(T, f) - \int f d\mu\}$ is given by the idea of topological pressure determining the measure-theoretic entropy. Without the additional assumptions, for any dynamical system Walters [Wal75] established another variational principle in terms of measure-theoretic entropy, which can be stated as follows:

$$P(T, f) = \sup_{\mu \in M(X,T)} \{ h_{\mu}(T) + \int f d\mu \}.$$

Returning to the expansive homeomorphism dynamical systems, the fact that the entropy map $\mu \in M(X,T) \mapsto h_{\mu}(T)$ is upper semicontinuous reclaims $h_{\mu_0}(T) = s(\mu_0)$ for any invariant measure μ_0 (cf. [Wal82, Theorem 9.12]). Thus, the variational principle obtained by Walters gives an extension of Ruelle's result to any dynamical systems. Since then, an important and quite active topic in dynamical systems is how to build some proper bridges to relate the topological dynamics and ergodic theory. A common approach, learned from the classical variational principle, is establishing some proper variational principles between measure-theoretic entropy-like quantities, defined by the measure-preserving structures, and the different types of entropylike quantities. See more results about the variational principles of amenable groups [ST80, OP82], sofic groups [KL16], semi-groups [LMW18, CRV18], weighted dynamical systems [FW16, Tsu23], partially hyperbolic systems [HHW17], random dynamical systems [Bog92, Kif01, DZ15] and references therein. However, not all variants of topological entropy enjoy the so-called variational principles since the system may not exist invariant measures (e.g. dynamical systems acting by semi-groups or non-autonomous dynamical systems), or difficultly formulates an analogous role of measure-theoretic entropy. In this situation, one can either only obtain the partial variational principles or establish Feng-Huang's type of variational principles [FH12].

We address that it is the missing role of measure-theoretic entropy that leads a more difficult task of establishing the proper variational principles for some entropy-like quantities of dynamical systems. Very recently, for every pressure function satisfies monotonicity, translation invariance and convexity, which is summarized from the properties of classical topological pressure, without involving the dynamics, for any metric space Bis et al. [BCMP22] established an abstract variational principle for pressure function in terms of finitely additive probability measures, and characterized the equilibrium measures that attains the supremum. When the dynamics is given, the variational principles of some pressure-like quantities and potential applications are also considered in an elaborate manner [CRV18, BCMP22, CN05]. The goal of this paper is to develop the convex analysis approach to random pressure function in context of random dynamical systems, and establish a variational principle for it involving the invariant measures.

The dynamical systems of Z-actions, which we say deterministic dynamical systems, is given by a single homeomorphism map on a compact metric space. However, in the real world the dynamical system evolves as the random external perturbations by the random transformations and its iterates. The basic framework of random dynamical systems was established by Ulam and von Neumann [UV45], and later by Kakutani [Kak50] when they perused the classical random ergodic theorems. The thermodynamic formalism of random dynamical systems was systematically stated in [Bog92, Arn98, Kif86, Kif01, Liu01]. In the random context, we extend the work of [Rue73, BCMP22] to random pressure function as follows:

Theorem 1.1. Let $G = \mathbb{Z}_{+}^{k}$, $k \geq 1$, or G be a countable infinite discrete amenable group, and let Ω be a locally compact, separable metric space with Borel σ -algebra $\mathcal{B}(\Omega)$. Let T be an amenable random (or \mathbb{Z}_{+}^{k}) dynamical system on compact metric space X over an ergodic measurepreserving system $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, G)$. If Γ is a random pressure function on $L^{1}_{\mathbb{P}}(\Omega, C(X))$, then

$$\Gamma(0) = \max_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} s(\mu),$$

where $s(\mu) = \inf \{ \Gamma(f) - \int f d\mu : f \in L^1_{\mathbb{P}}(\Omega, C(X)) \}$ is a concave upper semi-continuous function on the set of invariant probability measures on $\Omega \times X$.

The above theorem provides a new insight on how to link ergodic theory and topological dynamical systems through the given random pressure function. As applications, we apply it to some certain cases, like zero entropy systems, infinite entropy systems of deterministic dynamical systems, and non-invertible random dynamical systems, see Theorem 4.1, 4.3 and 4.6. However, to obtain the variational principle in Theorem 1.1 more difficulties are encountered in random dynamical systems that do not arise in [BCMP22].

(1) The *Riesz representation theorem* does not work in our current setting since we shall deal with random continuous function rather than the continuous function.

(2) The relativized method (a modification of Misiurewicz's technique [Mis75]), which produces invariant measure of random dynamical systems developed in [Bog92, Kif01], only works for measure-theoretic entropy rather than the integral of probability measures. Besides, unlike Z-actions, constructing an invariant measure having certain marginal on a base system is essential.

We shall use some new ideas and introduce some new tools to overcome the two difficulties. For (1), we will introduce a new tool called *Stone vector lattice* to replace the role of Riesz representation theorem, and show the functional generated by the separation theorem of convex sets is a *pre-integral* on a bounded function space. This allows us to produce some probability measures. For (2), to drop the compactness of Ω , we use *outer measure theory* and lift the probability measures obtained in step (1) to Radon measures. Then the compactness of Radon measures ensures us to get a probability measure by restricting its σ -algebra to Borel σ -algebra. Finally, we employ *Von Neumann's L*¹ *ergodic theorem* to show the marginal of the measure is the probability measure of base system.

The rest of this paper is organized as follows. In section 2, we clarify the basic setting of random dynamical systems and recall some relevant concepts. In section 3, we introduce the notion of random pressure function and prove the Theorem 1.1. In section 4, we exhibit several applications of the Theorem 1.1.

2. Preliminary

In this section, we briefly recall the setting of amenable random dynamical systems and related concepts. A systemic treatment concerning the theories of random dynamical systems are elaborated in several nice monographs [Arn98, Kif86, Cra02, Liu01, DZ15].

Let G be a group and A, B be two subsets of G. By |A| we denote the cardinality of A. The symmetric difference of A and B is defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. A group G is said to be an *amenable group* if there exists a sequence $\{F_n\}_{n\geq 1}$ of non-empty finite subsets of G such that

$$\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0$$

holds for any $g \in G$.

The sequence $\{F_n\}_{n\geq 1}$ is called a Følner sequence of G. Amenable group includes all finite groups, Abelian goups, and all finitely generated groups of sub-exponential growth. An example of non-amenable group is the free group of rank 2. Throughout this paper, we always assume that G is a countable infinite discrete amenable group.

Let $G = \mathbb{Z}_{+}^{k}$, or be an amenable group, and let X be a set. By an *action* of G on the set X we mean a map $\alpha : G \times X \to X$ such that:

(1) $\alpha(e_G, x) = x$ for every $x \in X$;

(2) $\alpha(g_1, \alpha(g_2, x) = \alpha(g_1g_2, x)$ for every $g_1, g_2 \in G$ and $x \in X$.

When g is fixed, we sometimes write $gx := \alpha(g, x)$, or $T_gx := \alpha(g, x)$ for convenience. If G acts continuously on the compact metric space X, we call the pair (X, G) a G-system. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If the G-action on Ω is measure-preserving, that is, $\mathbb{P}(A) = \mathbb{P}(g^{-1}A)$ for all $A \in \mathcal{F}$ and $g \in G$, we call the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, G)$ a measurepreserving system.

We always assume that the σ -algebra of a topological space X, denoted by $\mathcal{B}(X)$, is the usual Borel σ -algebra, which is generated by the set of all open sets of X. Once (X, d) is a compact metric space, we naturally associate a continuous function space C(X) consisting of the set of real-valued continuous functions on X, which becomes a Banach space endowed with the supremum norm $|| \cdot ||_{\infty}$.

An amenable random dynamical system (RDS for short) on compact metric space (X, d) over a measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is generated by a family of continuous self-mappings $\{T_{g,\omega} : g \in G, \omega \in \Omega\}$ on X satisfying

- (i) $(\omega, x) \mapsto T_{g,\omega} x$ is measurable with respect to the product σ -algebra $\mathcal{F} \otimes \mathcal{B}(X)^1$ on $\Omega \times X$;
- (ii) $T_{e_G,\omega}$ is the identity map on X for all ω ;
- (iii) $T_{g_1g_2,\omega} = T_{g_2,g_1\omega} \circ T_{g_1,\omega}$ for all $g_1, g_2 \in G$ and ω .

When $G = \mathbb{Z}_{+}^{k}$ is endowed with the usual additive operation such that conditions (i)-(iii) hold, the system is said to be a *random* \mathbb{Z}_{+}^{k} *dynamical system*, or non-invertible random dynamical system. The measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is a model of noise, and the *G*-system (X, G) is perturbed by the noise. This shall be more clear if we let

$$T: G \times \Omega \times X \to X, (g, \omega, x) \mapsto T(g, \omega, x) := T_{g, \omega}(x),$$

which shows the G-system can be affected by randomly choosing the continuous transformations on X. Specially, when Ω is a single point, no noise is imposed in this case and hence the random dynamical system is exactly reduced to non-random case, the previous G-system (X, G).

To investigate the dynamics of RDS, one can also associate a system $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X), \Theta, G)$ of G-action induced by skew product transformation Θ_g on $\Omega \times X$ given by $\Theta_g(\omega, x) = (g\omega, T_{g,\omega}x)$.

We give two examples of random dynamical systems.

Example 2.1. Let \mathbb{T}^2 be 2-dimensional torus and $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \theta)$ be an ergodic Polish system. Let $f : \Omega \to \mathbb{T}^2$ be a measurable map. Then we have a random dynamical system on \mathbb{T}^2 generated by the random map:

$$T_{\omega}x = Ax + f(\omega), x \in \mathbb{T}^2$$

¹That is the smallest σ -algebra on $\Omega \times X$ such that both the canonical projections $\pi_{\Omega}: \Omega \times X \to \Omega$ and $\pi_X: \Omega \times X \to X$ are measurable.

where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is a hyperbolic matrix.

Example 2.2. Let (Ω, σ) be the full shift over the k-symbols $\{1, ..., k\}$ with the left shift map σ on the product space $\Omega = \{1, ..., k\}^{\mathbb{Z}_+}$. Let \mathbb{P} be a σ -invariant probability measure on Ω . This gives a measurepreserving driving system $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \sigma)$. Let $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m, m \ge 1$, be the m-dimensional torus with homeomorphism maps $\varphi_j, j = 1, ..., k$, on \mathbb{T}^m . Then for every $\omega = (\omega_1, \omega_2, ...)$ the map $T_\omega : \mathbb{T}^m \to \mathbb{T}^m$, given by $T_\omega(x) = \varphi_{\omega_1}(x)$, and the iteration T^m_ω ,

$$T_{\omega}^{n} = \begin{cases} T_{\sigma^{n-1}\omega} \circ T_{\sigma^{n-2}\omega} \circ \cdots \circ T_{\omega}, & \text{for } n \ge 1\\ id, & \text{for } n = 0 \end{cases}$$

give a random \mathbb{Z}_+ -dynamical system.

We continue to collect several concepts related to the invariant measures of amenable random dynamical systems.

For each measurable in (ω, x) and continuous in x real-valued function f on $\Omega \times X$, we set

$$||f|| := \int ||f(\omega)||_{\infty} d\mathbb{P}(\omega),$$

where $||f(\omega)||_{\infty} = \sup_{x \in X} |f(\omega, x)|$. The measurability of $||f(\omega)||_{\infty}$ in ω follows by the separability of X and the measurability of f. Let $L^1_{\mathbb{P}}(\Omega, C(X))$ denote the set of random continuous functions with $||f|| < \infty$. For instance, each element f belonging to L^1 -space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ defines an element of $L^1_{\mathbb{P}}(\Omega, C(X))$ by setting $(\omega, x) \mapsto f(\omega)$; every continuous function $g \in C(X)$ also defines an element of $L^1_{\mathbb{P}}(\Omega, C(X))$ by setting $(\omega, x) \mapsto f(\omega)$; every continuous function $g \in C(X)$ also defines an element of f and g given by $(\omega, x) \mapsto f(\omega)g(x)$. The space $L^1_{\mathbb{P}}(\Omega, C(X))$ becomes a Banach space if we identify f and g by ||f - g|| = 0.

Let π_{Ω} be the canonical projection from $\Omega \times X$ to the first coordinate Ω . A probability measure on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ is said to have marginal \mathbb{P} if the push-forward of μ under π_{Ω} , say $(\pi_{\Omega})_*\mu$, is exactly the probability measure \mathbb{P} of the measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, G)$. The set of such measures is denoted by $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$. It is well-known that [Cra02, Proposition 3.6] any $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$ can disintegrate as $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$, where μ_{ω} is a family of conditional probability measures on X for \mathbb{P} -a.a ω^2 . Equip with the space $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ with the weak*-topology, that is, the smallest topology such that $\mu \mapsto \int f d\mu$ is continuous for all $f \in L^1_{\mathbb{P}}(\Omega, C(X))$, which is compact in sense of weak*-topology [Kif01, Lemma 2.1]. The set of invariant measures, denoted by $\mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$, is the elements of $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$ that is invariant w.r.t. Θ_g for all $g \in G$.

²See [Cra02, Proposition 3.6] for the existence and uniqueness of disintegration.

3. A CONVEX APPROACH TO RANDOM DYNAMICAL SYSTEMS

In this section, we aim to develop a convex analysis approach to random pressure function of random dynamical systems, and establish a variational principle for it.

3.1. Random pressure function. Given a deterministic system (X, T), i.e, a continuous self-map T on the compact metric space X, in Walters' monograph [Wal82, Theorem 9.7, p.214] the topological pressure $P(T, \cdot) : C(X) \to \mathbb{R}$ has many fundamental properties, for instances, monotonicity, translation invariance, convexity, Lipschitz, and cohomology. Especially, it satisfies the following properties:

- (i) Monotonicity: for any $f, g \in C(X)$ with $f \leq g$, one has $P(T, f) \leq P(T, g)$.
- (ii) Translation invariance: for any $f \in C(X)$ and $c \in \mathbb{R}$, one has P(T, f + c) = P(T, f) + c.
- (iii) Convexity: for any $f, g \in C(X)$ and $p \in [0, 1]$, one has $P(T, pf + (1-p)g) \le pP(T, f) + (1-p)P(T, g)$.

In fact, the readers who are familiar with the classical thermodynamic formalism may observe many pressure-like quantities of dynamical systems under group actions still possess the same properties as the classical topological pressure. To treat this phenomenon in a unified manner, Bis et al. [BCMP22, Definition 2.1] axiomatically defined the pressure function³ by requiring a function $\Gamma : C(X) \to \mathbb{R}$ with the properties:

- (1) Monotonicity: for any $f, g \in C(X)$ with $f \leq g$, one has $\Gamma(f) \leq \Gamma(g)$.
- (2) Translation invariance: for any $f \in C(X)$ and $c \in \mathbb{R}$, one has $\Gamma(f+c) = \Gamma(f) + c$.
- (3) Convexity: for any $f, g \in C(X)$ and $p \in [0, 1]$, one has $\Gamma(pf + (1-p)g) \leq p\Gamma(f) + (1-p)\Gamma(g)$.

We borrow this idea to broaden the scope of pressure-like quantities of amenable random dynamical systems. This allows us to formulate more general axiomatic definition for random pressure function on $L^1_{\mathbb{P}}(\Omega, C(X))$.

Definition 3.1. Let $T = (T_{g,\omega})$ be a RDS over the measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, G)$. A function $\Gamma : L^1_{\mathbb{P}}(\Omega, C(X)) \to \mathbb{R}$ is said to be a random pressure function if it satisfies the following conditions:

- (1) Monotonicity: $f \leq g \Rightarrow \Gamma(f) \leq \Gamma(g) \ \forall f, g \in L^1_{\mathbb{P}}(\Omega, C(X)).$
- (2) Translation invariance: $\Gamma(f+c) = \Gamma(f) + c \ \forall f \in L^1_{\mathbb{P}}(\Omega, C(X))$ and $c \in \mathbb{R}$.

³Actually, beyond the continuous function space they define the pressure function on a Banach space **B** over the field \mathbb{R} and X is only assumed to a metric space in a more general framework.

- (3) Convexity: $\Gamma(pf + (1-p)g) \leq p\Gamma(f) + (1-p)\Gamma(g) \ \forall f, g \in L^1_{\mathbb{P}}(\Omega, C(X)) \ and \ p \in [0, 1].$
- (4) Lipschitz: $|\Gamma(f) \Gamma(g)| \le ||f g|| \ \forall f, g \in L^1_{\mathbb{P}}(\Omega, C(X)).$
- (5) Semi-cohomology: $\Gamma(f \circ \Theta_g) \leq \Gamma(f) \ \forall f \in L^1_{\mathbb{P}}(\Omega, C(X))$ and $g \in G$.

Remark 3.2. (i) The random pressure function admits the cohomology property if the equality in (5) of the definition holds.

- (ii) We address that only condition (5) involves the dynamics, while the rest of properties are independent of the given systems.
- (iii) As we have mentioned, if $\Omega = \{\omega\}$ is a single point and \mathbb{P} is a Dirac measure δ_{ω} at ω , then the RDS is reduced to the Gsystem (X, G), and $L^{1}_{\mathbb{P}}(\Omega, C(X))$ is exactly C(X) with supremum norm. Then, in this case, condition (4) in Definition 3.1 can be removed since it can be deduced by using (1) and (2). Indeed, given $f_1, f_2 \in C(X)$, one has

$$\Gamma(f_1) \le \Gamma(f_2 + ||f_1 - f_2||_{\infty}) = \Gamma(f_2) + ||f_1 - f_2||_{\infty}$$

Exchanging the role of f_1 and f_2 gives us the converse inequality.

3.2. The axiomatic definition of $s(\mu)$. Notice that the random pressure function is an abstract definition by summarizing the previous properties from the (random) topological pressure [Bog92, Kif01]. There is no proper measure-theoretic entropy-like quantities for us to link it with random pressure function by variational principle. Corresponding to the role of measure-theoretic entropy in classical variational principles, we introduce an analogous counterpart of measure-theoretic entropy by using the idea of topological pressure determining measuretheoretic entropy.

Our definition is highly inspired by the work of Ruelle [Rue73, Theorem 5.1, p.245] and [Wal82, BCMP22]. Let (X, T) be an expansive dynamical system with specification property. Ruelle formulated a variational principle for topological pressure as follows:

$$P(T, f) = \max_{\mu \in M(X,T)} \{ s(\mu) + \int f d\mu \},\$$

where $s(\mu) = \inf_{f \in C(X)} \{P(T, f) - \int f d\mu\}$. Later, without requiring the additional conditions Walters [Wal82, Theorem 9.12] established variational principle for any dynamical system:

$$P(T, f) = \sup_{\mu \in M(X,T)} \{ h_{\mu}(T) + \int f d\mu \}.$$

Moreover, if the topological entropy of the system is finite, he showed that for any given invariant measure μ_0 , the entropy map $\mu \mapsto h_{\mu}(T)$ defined on M(X,T) is upper semi-continuous (u.s.c.) at μ_0 if and only if $h_{\mu_0}(T) = s(\mu_0)$. In other words, measure-theoretic entropy is completely determined by topological pressure in this case. Now we inject the above idea to the random context.

Definition 3.3. Let $T = (T_{g,\omega})$ be a RDS over the measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, G)$, and let Γ be a random pressure function on $L^1_{\mathbb{P}}(\Omega, C(X))$. For every $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$, we define the measure-theoretic entropy of μ w.r.t. Γ as

$$s(\mu) := \inf\{\Gamma(f) - \int f d\mu : f \in L^1_{\mathbb{P}}(\Omega, C(X))\}.$$

We are in a position to provide some equivalent characterizations for $s(\mu)$ by the certain convex domains of $L^1_{\mathbb{P}}(\Omega, C(X))$.

Proposition 3.4. Under the setting of Definition 3.3, for every $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$, we have

$$s(\mu) = \inf_{f \in \mathcal{A}} \int f d\mu$$
$$= \inf_{f \in \tilde{\mathcal{A}}} \int f d\mu,$$

where

$$\mathcal{A} = \{ f \in L^1_{\mathbb{P}}(\Omega, C(X)) : \Gamma(-f) = 0 \}$$

and

$$\tilde{\mathcal{A}} = \{ f \in L^1_{\mathbb{P}}(\Omega, C(X)) : \Gamma(-f) \le 0 \}.$$

Proof. We first show $s(\mu) = \inf_{f \in \mathcal{A}} \int f d\mu$. This essentially follows from the translation invariance of Γ . Indeed, if $f \in \mathcal{A}$, one has

$$\int f d\mu = \Gamma(-f) - \int -f d\mu \ge s(\mu)$$

Thus, $s(\mu) \leq \inf_{f \in \mathcal{A}} \int f d\mu$. If $f \in L^1_{\mathbb{P}}(\Omega, C(X))$, then $\Gamma(-(\Gamma(f)-f)) = 0$. This shows $\Gamma(f) - f \in \mathcal{A}$ and hence

$$\Gamma(f) - \int f d\mu \ge \inf_{f \in \mathcal{A}} \int f d\mu.$$

We get $s(\mu) \ge \inf_{f \in \mathcal{A}} \int f d\mu$.

The first equality implies that $s(\mu) \geq \inf_{f \in \tilde{\mathcal{A}}} \int f d\mu$. Since $L^1_{\mathbb{P}}(\Omega, C(X))$ is a linear space, we know that if $f \in L^1_{\mathbb{P}}(\Omega, C(X))$ then its inverse element $-f \in L^1_{\mathbb{P}}(\Omega, C(X))$. This shows

$$\begin{split} s(\mu) &= \inf\{\Gamma(f) - \int f d\mu : f \in L^1_{\mathbb{P}}(\Omega, C(X))\} \\ &= \inf\{\Gamma(-f) + \int f d\mu : f \in L^1_{\mathbb{P}}(\Omega, C(X))\} \\ &\leq \inf_{f \in \tilde{\mathcal{A}}} \int f d\mu. \end{split}$$

Let (Ω, d) be a metric space. A function $f : \Omega \to \mathbb{R}$ is called *compactly supported* if the support

$$\operatorname{spt}(f) := \overline{\{\omega \in \Omega : f(\omega) \neq 0\}}$$

is a compact subset of Ω . The set of continuous compactly supported functions on Ω is denoted by $C_c(\Omega)$. Note that a continuous function $f \in C_c(\Omega)$ if and only if there exists compact subset $K \subset \Omega$ such that $f|_{\Omega\setminus K} = 0$. Using this fact, $C_c(\Omega)$ is a real linear space.

Proposition 3.5. We have the following statements:

(1) The set of finite linear combinations of the product of characteristic function of measurable set of Ω and continuous function on X is dense in $L^1_{\mathbb{P}}(\Omega, C(X))$.

(2) The set of bounded function of $L^1_{\mathbb{P}}(\Omega, C(X))$ is dense in $L^1_{\mathbb{P}}(\Omega, C(X))$.

(3) Suppose that Ω is a locally compact and separable metric space with Borel σ -algebra $\mathcal{B}(\Omega)$. Then $C_c(\Omega \times X)$ is dense in $L^1_{\mathbb{P}}(\Omega, C(X))$.

Proof. (1). Let $f \in L^1_{\mathbb{P}}(\Omega, C(X))$ and define the map

$$\hat{f}: (\Omega, \mathcal{F}) \to (C(X), \mathcal{B}(C(X)))$$

given by $\hat{f}(w) = f(\omega, \cdot) \in C(X)$, where $\mathcal{B}(C(X))$ is the Borel σ -algebra on C(X) generated by all open balls of C(X) formed by

$$B(h,r) = \{g \in C(X) : ||g - h||_{\infty} < r\}$$

with $h \in C(X), r > 0$. Let E be a countable dense subset of X. Then

$$||f(\omega, \cdot) - h||_{\infty} = \sup_{x \in X} |f(\omega, x) - h(x)| = \sup_{y \in E} |f(\omega, y) - h(y)|.$$

for all ω . Hence

$$\begin{split} f^{-1}(\overline{B}(h,r)) = &\{\omega \in \Omega : ||f(\omega,\cdot) - h||_{\infty} \le r\} \\ = &\bigcap_{y \in E} \{\omega \in \Omega : |f(\omega,y) - h(y)| \le r\} \in \mathcal{F}, \end{split}$$

which implies that \hat{f} is measurable.

Let $\{g_n : n \ge 1\}$ be a family of countable dense subset of C(X). For $n, k \ge 1$, we define

$$A_{n,k} = \{g \in C(X) : ||g - g_n||_{\infty} < \frac{1}{k}\},\$$

$$B_{1,k} = A_{1,k}, B_{n,k} = A_{n,k} \setminus \bigcup_{j=1}^{n-1} A_{j,k}, n \ge 2.$$

Then $\{\hat{f}^{-1}(B_{n,k}) : n \geq 1\}$ is a measurable partition of Ω . For each $k \geq 1$, there exist positive integer n_k and a \mathbb{P} -measurable set $\Omega_{n_k} = \Omega \setminus \bigcup_{j=1}^{n_k} \hat{f}^{-1}(B_{j,k})$ so that $\mathbb{P}(\Omega_{n_k}) < \frac{1}{k}$. We set

$$h_k(\omega, x) = \sum_{j=1}^{n_k} \chi_{\hat{f}^{-1}(B_{j,k})}(\omega) g_j(x) + \chi_{\Omega_{n_k}} \cdot 0.$$

Therefore,

$$\begin{split} ||f - h_k|| &= \int_{\Omega \setminus \Omega_{n_k}} ||f(\omega) - h_k(\omega)||_{\infty} d\mathbb{P}(\omega) + \int_{\Omega_{n_k}} ||f(\omega)||_{\infty} d\mathbb{P}(\omega) \\ &\leq \int_{\Omega_{n_k}} ||f(\omega)||_{\infty} d\mathbb{P}(\omega) + \frac{1}{k}. \end{split}$$

Since $||f|| < \infty$, one has $\int_{\Omega_k} ||f(\omega)||_{\infty} d\mathbb{P}(\omega) \to 0$ as k goes to ∞ by the absolute continuity of integrals. This shows (1).

(2) is a direct consequence of (1).

(3) Notice that the product σ -algebra $\mathcal{B}(\Omega) \otimes \mathcal{B}(X)$ coincides with the Borel σ -algebra $\mathcal{B}(\Omega \times X)$ generated by the open sets of $\Omega \times X$ since $\Omega \times X$ has a family of countable topology basis. Clearly, $C_c(\Omega \times X)$ is a subset of $L^1_{\mathbb{P}}(\Omega, C(X))$. By (1), it suffices to show for any $f \in$ $L^1_{\mathbb{P}}(\Omega, C(X))$ with the form $\chi_A \cdot g$, $A \in \mathcal{B}(\Omega)$ and $g \in C(X)$, can be approximated by the elements of $C_c(\Omega \times X)$. Ω is σ -compact⁴ since Ω is a locally compact and separable metric space. By [ET13, Theorem B.13, p.425], the probability measure \mathbb{P} on Ω is regular in sense of

$$\mathbb{P}(A) = \sup_{K \subset A, K \text{ compact}} \mathbb{P}(K) = \inf_{A \subset U, U \text{ open}} \mathbb{P}(U).$$

Fix $\epsilon > 0$. Choose a compact set K and an open set U such that $K \subset A \subset U$ and $\mathbb{P}(U \setminus K) < \frac{\epsilon}{2M}$, where $M = \max_{x \in X} |g(x)| + 1$. By the Urysohn's lemma for locally compact space [Rud87], there exists $f \in C_c(\Omega)$ such that $0 \leq f(\omega) \leq 1$, $f|_K \equiv 1$ and the compact support $\operatorname{spt}(f) \subset U$. Since $f(\omega)g(x) = 0$ for any $(\omega, x) \in \Omega \times X \setminus \operatorname{spt}(f) \times \operatorname{spt}(g)$, then $f \cdot g \in C_c(\Omega \times X)$ and

$$||\chi_A g - fg|| = \int_{\Omega} \sup_{x \in X} |\chi_A(\omega)g(x) - f(\omega)g(x)| \mathbb{P}(\omega) < \epsilon.$$

This shows that $C_c(\Omega \times X)$ is a dense subset of $L^1_{\mathbb{P}}(\Omega, C(X))$.

Using Proposition 3.5, we can equivalently write the $s(\mu)$ by shrinking the function space $L^1_{\mathbb{P}}(\Omega, C(X))$ to compactly supported function space $C_c(\Omega \times X)$ and expanding the range of $\Gamma(-\cdot)$ to non-positive values compared with Proposition 3.4.

Lemma 3.6. Suppose that Ω is a locally compact and separable metric space. Let $T = (T_{g,\omega})$ be a RDS over the measure-preserving system $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \theta)$, and let Γ be a random pressure function on $L^1_{\mathbb{P}}(\Omega, C(X))$. Then for any $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$,

$$s(\mu) = \inf_{f \in \tilde{\mathcal{A}}_c} \int f d\mu,$$

where $\tilde{\mathcal{A}}_c = \{ f \in C_c(\Omega \times X) : \Gamma(-f) \le 0 \}.$

⁴A metric space is called σ -compact if it is a countable union of compact subsets.

Proof. It follows from the Proposition 3.4 that we only need to show

$$s(\mu) = \inf_{f \in \tilde{\mathcal{A}}} \int f d\mu \ge \inf_{f \in \tilde{\mathcal{A}}_c} \int f d\mu$$

for any $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$. Fix $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$. The map $f \mapsto \int f d\mu$ is continuous. Indeed, for $f_1, f_2 \in L^1_{\mathbb{P}}(\Omega, C(X))$, one has

$$\begin{split} |\int f_1 d\mu - \int f_2 d\mu| &\leq \int_{\Omega} \int_X |f_1(\omega, x) - f_2(\omega, x)| d\mu_{\omega}(x) d\mathbb{P}(\omega) \\ &\leq \int ||(f_1 - f_2)(\omega)||_{\infty} d\mathbb{P}(\omega) \\ &= ||f_1 - f_2||. \end{split}$$

Let $f \in \mathcal{A}$ and $\gamma > 0$. The Proposition 3.5,(3) allows us to choose a sequence $\{f_n\}$ of the compactly supported continuous functions of $L^1_{\mathbb{P}}(\Omega, C(X))$ such that $f_n \to f + \frac{\gamma}{2}$. Then we get

$$\int f_n d\mu \to \int f + \frac{\gamma}{2} d\mu,$$

and the Lipschitz property and translation property of Γ yields that

$$\Gamma(-f_n) \to \Gamma(-(f+\frac{\gamma}{2})) < 0.$$

Thus, for sufficiently large N one has $f_N \in \tilde{\mathcal{A}}_c$ and $\int f_N d\mu < \int f d\mu + \gamma$. So $\inf_{f \in \tilde{\mathcal{A}}_c} \int f d\mu \leq \int f d\mu$. This implies the desired inequality. \Box

3.3. Variational principle of random pressure function. In this subsection, we give the proof of Theorem 1.1.

3.3.1. Some auxiliary tools. Notice that $s(\mu)$ is pretty different from the (fiber) measure-theoretic entropy. Therefore, the previous method considered in [Rue73, Wal82, Kif01, BCMP22] can not be directly applied to the present case since we are dealing with the integral of invariant measures rather than measure-theoretic entropy. Besides, we also need to give a different construction of invariant measure whose marginal is \mathbb{P} in random situation, which is not involved in the deterministic systems.

The difficult part for us is to get the inequality:

$$\Gamma(0) \le \max_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} s(\mu).$$

The central problem is how to construct a linear functional using the information of random pressure function and relate it with a probability measure on $\Omega \times X$ having margin \mathbb{P} ?

It can be done by the following steps:

(a) We use separation theorem of convex sets to get a linear functional;
(b) We introduce *Stone vector lattice* to deal with random continuous functions, and use *pre-integral* to replace the role of Riesz representation theorem;

(c) We introduce outer measure theory to overcome some problems caused by the convergence of $M(\Omega \times X)$, and obtain a probability measure with margin \mathbb{P} on Ω satisfying the desired inequality.

Before that, we first invoke the aforementioned three powerful tools.

The following *separation theorem of convex sets* presented in [DS88, p.417] allows us to separate two disjoint closed convex sets by a proper linear functional.

Lemma 3.7. Let V be a local convex linear topological space. Suppose that K_1, K_2 are disjoint closed convex subsets and K_1 is compact. Then there exists a continuous real-valued linear functional L on V such that L(x) < L(y) for any $x \in K_1$ and $y \in K_2$.

 $\Omega \times X$ is a locally compact metric space by the assumption of Ω in Theorem 1.1. Although we can apply *Riesz representation theorem* to produce a Radon measure on $\Omega \times X$ whose weak^{*} convergence holds for all $f \in C_c(\Omega \times X)$, the compositions of compactly supported continuous functions on $\Omega \times X$ and the iterations of Θ_g may not belong to $C_c(\Omega \times X)$ since the skew product transformation Θ_g is only measurable. To overcome this difficulty, we shall introduce *pre-integral* to replace the role of Riesz representation theorem.

By \mathcal{L} we denote a family of real-valued functions on the set X. \mathcal{L} is said to be a *vector lattice* if \mathcal{L} is

(i) a linear space;

(ii) $f \lor g := \max\{f, g\} \in \mathcal{L}, \forall f, g \in \mathcal{L}.$

Additionally, \mathcal{L} is said to be a *Stone vector lattice* if \mathcal{L} is a vector lattice and $f \wedge 1 := \min\{f, 1\} \in \mathcal{L}$ for all $f \in \mathcal{L}$.

Given a set X and the vector lattice \mathcal{L} on X, the function L from \mathcal{L} to \mathbb{R} is said to be a *pre-integral* if L is linear, positive, and $L(f_n) \downarrow 0$ whenever $f_n \in \mathcal{L}$ and $f_n(x) \downarrow 0$ for all $x \in X$.

The following lemma comes from [Dud04, Theorem 4.5.2], which is an analogue of the previous Riesz representation theorem in a more general context.

Lemma 3.8. Let X be a set and L be a pre-integral on a Stone vector lattice \mathcal{L} . Then there exists a measure μ on $(X, \sigma(\mathcal{L}))$ such that for all $f \in \mathcal{L}$

$$L(f) = \int f d\mu,$$

where $\sigma(\mathcal{L})$ is the smallest σ -algebra on X such that all functions in \mathcal{L} are measurable.

By means of pre-integral, one can obtain a measure on locally compact metric space $\Omega \times X$. To get an invariant measure, a common trick is considering a standard iterations of the measure under Θ . However, the usual convergence of weak-topology endowed for Borel probability measures on compact sets does not work for the current setting. We place hope on some kind of convergence for locally compact metric space arising in outer measure theory to overcome this obstruction.

We proceed to present some useful concepts and results given in outer measure theory [Fol99, Mat95, Sim18]. Given a set X, a set function $\mu: 2^X \to [0,\infty]$ is said to be an *outer measure* on X if μ satisfies the following conditions:

- (i) $\mu(\emptyset) = 0;$
- (ii) $\mu(A) \le \mu(B), \forall A \subset B;$ (iii) $\forall A_j \subset X, j \ge 1, \ \mu(\cup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \mu(A_j).$

For instance, the usual Lebesgue measure on \mathbb{R}^n ; the counting measure on the set X given by $\mu(A) = \#A$ for any finite subset A of X and $\mu(A) = \infty$ otherwise. If a subset $A \subset X$ satisfies Carathéodory's condition, that is, for any $S \subset X$,

$$\mu(A) = \mu(A \cap S) + \mu(A \setminus S),$$

then A is called a μ -measurable set. Specially, an outer measure μ on a metric space X is called *Borel-regular* if all Borel sets of X are μ measurable sets, and for any $A \subset X$, there exists a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$.

Now let X be a locally compact metric space. An outer measure μ is called a *Radon measure* if the following conditions are satisfied:

- (i) μ is Borel-regular;
- (ii) $\mu(K)$ is finite for each compact set $K \subset X$;
- (ii) $\mu(K)$ is finite for each compact set (iii) for any $A \subset X$, $\mu(A) = \inf_{\substack{U \text{ open}, A \subseteq U}} \mu(U)$; (iv) for any open set $U \subset X$, $\mu(U) = \sup_{\substack{K \text{ compact}, K \subset U}} \mu(K)$.

One says that a sequence of Radon measures $\{\mu_n\}_{n=1}^{\infty}$ on X converges weakly μ if

$$\lim_{n \to \infty} \int f d\mu_n = \int f d\mu$$

for all $\varphi \in C_c(X)$.

Using Urysohn's lemma of locally compact metric space, one can deduce from the definition of Radon measure that the following:

Lemma 3.9. [Mat95, Theorem 1.24] Let X be a locally compact metric space. Suppose that $\{\mu_n\}_{n=1}^{\infty}$ and μ are Radon measures on X such that μ_n weakly converges μ . Then

- (i) for any open set $O \subset X$, $\liminf_{n \to \infty} \mu_n(O) \ge \mu(O)$;
- (ii) for any compact subset $K \subset X$, $\limsup_{n \to \infty} \mu_n(K) \le \mu(K)$.

The following lemma provides a criterion whether a sequence of Radon measures may admit a sub-sequence converging weakly to some Radon measures.

Lemma 3.10. [Sim18, Theorem 5.15] Let X be a locally compact and σ -compact metric space. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Radon measures on X with the property that $\sup_{n\geq 1} \mu_n(K) < \infty$ for each compact set K of X. Then there exists a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ which weakly converges to a Radon measure μ on X.

Carathéodory's theorem [Fol99, Proposition 1.11] says that the set of all μ -measurable sets of X, denoted by $\sigma(\mu)$, is a σ -algebra. Then a Radon measure (with $\mu(X) = 1$) restricts to its Borel σ -algebra gives us a Borel (probability) measure. Conversely, once there is a Borel (probability) measure, we can obtain a Radon measure as follows:

Lemma 3.11. Let X be a locally compact, separable metric space and μ be a Borel probability measure on X. For any $A \subset X$, we define

$$\mu^*(A) = \inf_{B \in \mathcal{B}(X), A \subset B} \mu(B).$$

Then

(i) μ* is a Radon measure on X;
(ii) If f is a μ-integrable function on X, then

(3.1)
$$\int f d\mu = \int f d\mu^*.$$

Proof. (i) Clearly, μ^* is an outer measure on X. By [Sim18, Lemma 5.9], to show that μ^* is a Radon measure on X it suffices to show (a) each open set of X is a countable union of compact sets of X; (b) μ^* takes finite value for compact sets of X;

(c) μ^* is a Borel-regular outer measure on X.

Since each open set of X is a F_{σ} -set and any closed set of σ -compact metric space X is still σ -compact, then each open set can be expressed as a countable union of some compact sets of X. For any $A \subset X$ and $n \geq 1$, one can choose a Borel set $B_n \supset A$ such that $\mu(B_n) < \mu^*(A) + \frac{1}{n}$. We set $B = \bigcap_{n>1} B_n$. Then

$$\mu^*(A) \le \mu^*(B) \le \mu(B) < \mu^*(A) + \frac{1}{n},$$

which implies that $\mu^*(A) = \mu^*(B)$. Now assume that $A \in \mathcal{B}(X)$ and $S \subset X$. Then there exists a Borel set $S_1 \supset S$ such that $\mu^*(S) = \mu^*(S_1)$. Noting that $\mu^*|_{\mathcal{B}(X)} = \mu$, this yields that

$$\mu^*(S) = \mu(S_1) = \mu(S_1 \cap A) + \mu(S_1 \setminus A)$$
$$= \mu^*(S_1 \cap A) + \mu^*(S_1 \setminus A)$$
$$\ge \mu^*(S \cap A) + \mu^*(S \setminus A).$$

On the other hand, we have $\mu^*(S) \leq \mu^*(S \cap A) + \mu^*(S \setminus A)$ by the countable sub-additivity of outer measure μ^* . So μ^* is Borel-regular. This shows μ^* is a Radon measure on X.

(ii) Recall that the integral $\int f d\mu^*$ of f w.r.t. μ^* is defined on the measure space $(X, \sigma(\mu^*), \mu^*|_{\sigma(\mu^*)})$ as the the classical measure theory done. Each μ -measurable function f on X is also μ^* -measurable since

 $\sigma(\mu^*) \supset \mathcal{B}(X)$. Then the equality (3.1) can be obtained by using the fact $\mu^*|_{\mathcal{B}(X)} = \mu$ and a standard approximation technique used in real analysis.

3.3.2. Proof of Theorem 1.1.

Proof. The translation invariance of Γ shows that $\Gamma(-\Gamma(0)) = \Gamma(0) - \Gamma(0) = 0$. By Proposition 3.4, for all $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$ we have $s(\mu) \leq \int \Gamma(0) d\mu = \Gamma(0)$. This implies

$$\sup_{\mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} s(\mu) \le \Gamma(0).$$

To get the converse inequality

(3.2)
$$\Gamma(0) \le \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} s(\mu),$$

 $\mu \in$

we need to show for any $\epsilon > 0$, there exists $\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$ such that $\Gamma(0) - \epsilon < s(\mu)$. By Proposition 3.6, this is equivalent to show

(3.3)
$$\inf_{f \in \tilde{\mathcal{A}}_c} \int f d\mu + \int c d\mu + \epsilon > 0,$$

where $\mathcal{A}_c = \{ f \in C_c(\Omega \times X) : \Gamma(-f) \leq 0 \}$ and $c := -\Gamma(0)$. Equip the bounded function space

$$L_b^1(\Omega, C(X)) = \{ f \in L_{\mathbb{P}}^1(\Omega, C(X)) : f \text{ is bounded on } \Omega \times X \}$$

with subspace topology inherited from $L^1(\Omega, C(X))$, and define

$$\tilde{\mathcal{A}}_b = \{ f \in L^1_b(\Omega, C(X)) : \Gamma(-f) \le 0 \}.$$

Clearly, $\tilde{\mathcal{A}}_c \subset \tilde{\mathcal{A}}_b$. The convexity and Lipschitz property of Γ tells us $\tilde{\mathcal{A}}_b$ is a closed convex subset of $L_b^1(\Omega, C(X))$. Notice that $\Gamma(c) = 0$. Then $-(c + \frac{\epsilon}{2}) \notin \tilde{\mathcal{A}}_b$ and hence $-c \notin \tilde{\mathcal{A}}_b + \frac{\epsilon}{2}$. Consider the disjoint closed convex subsets $K_1 = \{-c\}$ and $K_2 = \tilde{\mathcal{A}}_b + \frac{\epsilon}{2}$. By Lemma 3.7, there exists a continuous real-valued linear functional L on $L_b^1(\Omega, C(X))$ such that for any $f \in \tilde{\mathcal{A}}_b + \frac{\epsilon}{2}$,

$$(3\cdot4) L(f) > L(-c).$$

We first obtain a Borel probability measure μ_0 on $(\Omega \times X, \mathcal{B}(\Omega \times X))$ such that for every $f \in L^1_b(\Omega, C(X))$,

(3.5)
$$\frac{L(f)}{L(1)} = \int f d\mu_0.$$

Let $f \ge 0$ and $\lambda > 0$. By the translation invariance and monotonicity of Γ , one has

$$\Gamma(-(\lambda f + 1 + \Gamma(0))) = \Gamma(-\lambda f) - 1 - \Gamma(0)$$

$$\leq \Gamma(0) - 1 - \Gamma(0) = -1 <$$

Then $L(-c) \leq \lambda L(f) + L(1 + \Gamma(0) + \frac{\epsilon}{2})$ by (3.4) and the linearity of L. If L(f) < 0, by letting $\lambda \to \infty$ we get $L(-c) = -\infty$. This implies

0.

that L(f) must be non-negative. So L is a positive linear functional. Let $\{g_n(\omega, x)\}_n$ be a sequence of $L_b^1(\Omega, C(X))$ such that $g_n(\omega, x)$ is pointwise decreasing to 0 for any fixed $(\omega, x) \in \Omega \times X$. For every fixed ω , one has the function sequences $\{g_n(\omega)\}_n$ on X, given by $g_n(\omega) :=$ $g_n(\omega, \cdot)$, is pointwise decreasing to 0 for all $x \in X$. By Dini's theorem, one has $g_n(\omega)$ converges uniformly 0 for each fixed ω , i.e., $||g_n(\omega)||_{\infty} \downarrow 0$. Then Levi's monotone convergence theorem gives us

$$\lim_{n \to \infty} ||g_n - 0|| = \lim_{n \to \infty} \int ||g_n(\omega)||_{\infty} d\mathbb{P}(\omega) = 0.$$

Since L is positive, continuous and L(0) = 0, this yields that $L(g_n) \downarrow 0$. Since L is positive and can take non-zero value for some f, we choose $g \in L_b^1(\Omega, C(X))$ with $0 \le g \le 1$ such that L(g) > 0. Then

$$L(1) = L(g) + L(1 - g) > 0$$

To sum up, $\frac{L(\cdot)}{L(1)}$ is a pre-integral on Stone vector lattice $L_b^1(\Omega, C(X))$. By Lemma 3.8, there exists a probability measure μ_0 on $(\Omega \times X, \sigma(L_b^1(\Omega, C(X)))$ such that for every $f \in L_b^1(\Omega, C(X))$,

(3.6)
$$\frac{L(f)}{L(1)} = \int f d\mu_0.$$

Since each element in $L_b^1(\Omega, C(X))$ is measurable in (ω, x) , the σ algebra $\sigma(L_b^1(\Omega, C(X))$ generated by the functions of $L_b^1(\Omega, C(X))$ is contained in the product σ -algebra $\mathcal{B}(\Omega) \otimes \mathcal{B}(X)$. On the other hand, let $A \in \mathcal{B}(\Omega)$ and B be a closed subset of X. We set $g_n(\omega, x) :=$ $\chi_A(\omega) \cdot b_n(x) \in L_b^1(\Omega, C(X))$, where $b_n(x) = 1 - \min\{nd(x, B), 1\}$. Then $\lim_{n\to\infty} g_n(\omega, x) = \chi_A(\omega)\chi_B(x)$ for all (ω, x) . Therefore, $A \times B \in$ $\sigma(L_b(\Omega, C(X))$. This shows that $\sigma(L_b(\Omega, C(X)) = \mathcal{B}(\Omega) \otimes \mathcal{B}(X) =$ $\mathcal{B}(\Omega \times X)$, namely μ_0 is a Borel probability measure on $\Omega \times X$.

Next, we construct an invariant probability measure on $\Omega \times X$ with marginal \mathbb{P} . Let $f \in L_b^1(\Omega, C(X))$ and $g \in G$. Then

$$||f \circ \Theta_g|| = \int \sup_{x \in X} |f(g\omega, T_{g,\omega}x)| d\mathbb{P}(\omega) \le \int ||f(g\omega)||_{\infty} d\mathbb{P}(\omega) = ||f||$$

by the fact \mathbb{P} is *G*-invariant. So $f \circ \Theta_g$ belongs to $L_b^1(\Omega, C(X))$. By (3.6) and the linearity of *L*, for all $f \in L_b^1(\Omega, C(X))$, one has

(3.7)
$$\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f\circ\Theta_g)}{L(1)} = \int fd\frac{1}{|F_n|}\sum_{g\in F_n}(\Theta_g)_*\mu_0$$

for any family of Følner sequences $\{F_n\}_{n\geq 1}$ of G. Then

$$\mu_n = \frac{1}{|F_n|} \sum_{\substack{g \in F_n \\ 17}} (\Theta_g)_* \mu_0$$

is a sequence of Borel probability measures on $\Omega \times X$. For any $A \subset \Omega \times X$, we define

$$\mu_n^*(A) = \inf_{B \in \mathcal{B}(\Omega \times X), B \supset A} \mu_n(B).$$

By Lemma 3.11, $\{\mu_n^*\}_{n\geq 1}$ is a sequence of Radon measures on $\Omega \times X$. Since $\mu_n^*(K) = \mu_n(K) \leq 1$ for each compact set K of $\Omega \times X$ and every $n \geq 1$, then by Lemma 3.10, without loss of generality we may assume that μ_n^* converges to a Radon measure μ^* on $\Omega \times X$. Notice that μ^* is Borel-regular. The measure

$$\mu := \mu^*|_{\mathcal{B}(\Omega \times X)}$$

which is a restriction of μ^* on $\mathcal{B}(\Omega \times X)$, is a Borel measure on $\Omega \times X$.

We need to show $\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$. Let $L^{1}(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ denote the usual L^{1} -space with norm $|| \cdot ||_{L^{1}}$. Given $f \in L^{1}(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, by setting $\tilde{f}|_{\{\omega\}\times X} = f(\omega)$ for every ω , this yields a function $\tilde{f} \in L^{1}_{\mathbb{P}}(\Omega, C(X))$. Clearly, $\tilde{\chi}_{A} = \chi_{A\times X}$ for any $A \in \mathcal{B}(\Omega)$. Let $\{F_{n}\}_{n\geq 1}$ be a Følner sequence of G. By L^{1} -mean ergodic theorem [KL16, Theorem 4.23], one has

$$\begin{aligned} ||\frac{1}{|F_n|} \sum_{g \in F_n} \chi_A \circ g - \mathbb{P}(A)|| &= \int |\frac{1}{|F_n|} \sum_{g \in F_n} \chi_A \circ g(\omega) - \mathbb{P}(A)| d\mathbb{P}(\omega) \\ &= ||\frac{1}{|F_n|} \sum_{g \in F_n} \chi_A \circ g - \mathbb{P}(A)||_{L^1} \to 0, n \to \infty. \end{aligned}$$

The continuity of L implies that

(3.8)

$$\lim_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} \tilde{\chi}_A \circ \Theta_g)}{L(1)} = \lim_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} \chi_A \circ g)}{L(1)} = \mathbb{P}(A).$$

Let K be a compact subset of Ω . Since $\mu_n^*|_{\mathcal{B}(\Omega \times X)} = \mu_n$ for each n, one has

 $(3\cdot9)$

$$\mathbb{P}(K) \stackrel{by \ (3\cdot8)}{=} \lim_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} \tilde{\chi}_K \circ \Theta_g)}{L(1)} \stackrel{by \ (3\cdot7)}{=} \lim_{n \to \infty} \int \chi_{K \times X} d\mu_n^*$$
$$= \limsup_{n \to \infty} \mu_n^* (K \times X) \stackrel{by \ \text{Lemma } 3.9}{\leq} \mu^* (K \times X) = \mu (K \times X).$$

By [ET13, Theorem B.13, p.425], \mathbb{P} is regular, that is, for any $A \in \mathcal{B}(\Omega)$

(3.10)
$$\mathbb{P}(A) = \sup_{K \subset A, K \text{ is compact}} \mathbb{P}(K) \le \mu(A \times X), \text{ by } (3.9).$$

The similar arguments give us $\mathbb{P}(A) \ge \mu(A \times X)$. Thus $\mathbb{P}(A) = \mu(A \times X)$ for all $A \in \mathcal{B}(\Omega)$. Specially, we have $\mu(\Omega \times X) = 1$. This shows $(\pi_{\Omega})_*\mu = \mathbb{P}$ and hence $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$.

The remaining is to show μ is Θ_g -invariant for all $g \in G$. For every $f \in C_c(\Omega \times X)$, one has

(3.11)
$$\lim_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} f \circ \Theta_g)}{L(1)} \stackrel{by}{=} \lim_{n \to \infty} \int f d\mu_n$$
$$\stackrel{by \quad \text{Lemma } 3.11, \text{(ii)}}{=} \lim_{n \to \infty} \int f d\mu_n^*$$
$$= \int f d\mu^* = \int f d\mu.$$

Let $f \in L_b^1(\Omega, C(X))$ and choose $f_* \in C_c(\Omega \times X)$ such that $||f - f_*|| < \frac{L(1)\epsilon}{3||L||}$ by Proposition 3.5,(3), where ||L|| is the operator norm of L. By (3.11), for sufficiently large n,

$$\left|\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f_*\circ\Theta_g)}{L(1)}-\int f_*d\mu\right|<\frac{\epsilon}{3}.$$

 So

$$\begin{split} |\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f\circ\Theta_g)}{L(1)} - \int fd\mu| &\leq |\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f\circ\Theta_g)}{L(1)} - \frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f_*\circ\Theta_g)}{L(1)}| + \\ |\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f_*\circ\Theta_g)}{L(1)} - \int f_*d\mu| + |\int f_*d\mu - \int fd\mu| \\ &< \epsilon. \end{split}$$

This shows that

(3.12)
$$\lim_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} f \circ \Theta_g)}{L(1)} = \int f d\mu$$

holds for all $f \in L_b^1(\Omega, C(X))$. Let $h \in G$ and $f \in L_b^1(\Omega, C(X))$. Replacing f by $f \circ \Theta_h$ in (3.12), we have

$$\begin{split} |\int f \circ \Theta_h d\mu - \int f d\mu| &= \lim_{n \to \infty} \frac{1}{L(1)|F_n|} |L(\sum_{g \in hF_n \Delta F_n} f \circ \Theta_g)| \\ &\leq \lim_{n \to \infty} \frac{||L|| \cdot ||f||}{L(1)} \frac{|hF_n \Delta F_n|}{|F_n|} = 0. \end{split}$$

Note that $C_c(\Omega \times X)$ is dense in $L^1_{\mathbb{P}}(\Omega, C(X))$. So is $L^1_b(\Omega, C(X))$. A standard approximation approach shows that $\int f \circ \Theta_h d\mu = \int f d\mu$ for all $f \in L^1_{\mathbb{P}}(\Omega, C(X))$ and $h \in G$. This shows $\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$.

Finally, we show μ exactly satisfies inequality (3.3):

$$\inf_{f \in \tilde{\mathcal{A}}_c} \int f d\mu + \int_{19} c d\mu + \epsilon > 0.$$

For each $f \in \mathcal{A}_c$ and $g \in G$, one has $f \circ \Theta_g \in \mathcal{A}_b$ by the semi-cohomology of Γ . It follows from (3.11) that

(3.13)
$$\int f d\mu = \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} \frac{L(f \circ \Theta_g)}{L(1)} \ge \frac{1}{L(1)} \inf_{f \in \tilde{\mathcal{A}}_b} L(f),$$

which implies that $s(\mu) \geq \frac{1}{L(1)} \inf_{g \in \tilde{\mathcal{A}}_b} L(g)$ by Proposition 3.6. Therefore, one has

$$s(\mu) + \int cd\mu + \epsilon \ge \frac{1}{L(1)} \inf_{f \in \tilde{\mathcal{A}}_b + \frac{\epsilon}{2}} L(f) + \frac{L(c)}{L(1)} + \frac{\epsilon}{2}$$
$$= \frac{1}{L(1)} (\inf_{f \in \tilde{\mathcal{A}}_b + \frac{\epsilon}{2}} L(f) + L(c)) + \frac{\epsilon}{2} > 0, \text{ by } (3\cdot 4).$$

Recall that $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ is endowed with the weak*-topology, which is compact. It is easy to see that $\mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$ is a non-empty closed subset of $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$. Since $s(\mu)$ is upper semi-continuous concave function on $\mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$, then the set

$$M_{max}(T,\Gamma) := \{ \mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G) : \Gamma(0) = s(\mu) \}$$

is a non-empty compact convex subset of $\mathcal{M}_{\mathbb{P}}(\Omega \times X, G)$. Thus, the supremum can be attained for some invariant measures.

Suppose that $\Omega = \{\omega\}$ is a single point, and $G = \mathbb{Z}_+^k$, or G is an amenable group continuously acting on the compact metric space X with continuous self-maps $\{T_g : g \in G\}$ on X. If the pressure function $\Gamma : C(X) \to \mathbb{R}$ satisfies the following properties:

- (1) Monotonicity: $f \leq g \Rightarrow \Gamma(f) \leq \Gamma(g) \ \forall f, g \in C(X).$
- (2) Translation invariance: $\Gamma(f+c) = \Gamma(f) + c \ \forall f \in C(X)$ and $c \in \mathbb{R}$.
- (3) Convexity: $\Gamma(pf + (1-p)g) \le p\Gamma(f) + (1-p)\Gamma(g) \ \forall f, g \in C(X)$ and $p \in [0, 1]$.
- (4) Semi-cohomology: $\Gamma(f \circ T_g) \leq \Gamma(f) \ \forall f \in C(X) \text{ and } g \in G$,

then, by Theorem 1.1, we can obtain the following variational principle of pressure function for dynamical systems of G-actions.

Theorem 3.12. Let $G = \mathbb{Z}_{+}^{k}$, $k \geq 1$, or G be an amenable group continuously acting on the compact metric space (X, d). If Γ is a pressure function on C(X), then

$$\Gamma(0) = \max_{\mu \in \mathcal{M}(X,G)} s(\mu),$$

where $s(\mu) = \inf \{ \Gamma(f) - \int f d\mu : f \in C(X) \}$ is a concave upper semicontinuous function on the set of G-invariant Borel probability measures $\mathcal{M}(X, G)$.

4. Some applications of main result

In this section, we exhibit several applications of the Theorem 1.1, and formulate the new variational principles of the entropy-like quantities of dynamical systems, including some entropy-like quantities of the zero and infinite entropy systems, and preimage entropy-like quantities of non-invertible random dynamical systems. The main results are Theorem 4.1, 4.3 and 4.6.

4.1. Zero entropy systems. According to the possible value of topological entropy, dynamical systems can divided into zero entropy systems, finite entropy systems and infinite entropy systems. Although the zero entropy systems are viewed as a class of "simple" system with lower topological complexity, it may possess the complicated dynamical behaviors detecting by some zero entropy-like quantities, for instance, topological sequence entropy, maximal pattern entropy, and polynomial topological entropy are such candidates.

In this subsection, we are interested in the connections between the topological and ergodic aspects of zero entropy-systems, and we establish variational principles for these zero entropy-like quantities.

We first review the precise definitions of the aforementioned zero entropy-like quantities.

4.1.1. Topological sequence entropy. Let (X, T) be a topological dynamical system (TDS for short), where X is a compact metric space with a metric d and $T : X \to X$ is a homeomorphism map. Given a sequence $S = \{i_0, i_1, \dots\} \subset \mathbb{Z}_{\geq 0}$ of non-negative integers with $i_0 < i_1 < \dots \nearrow \infty$, we define the Bowen metric d_n^S w.r.t. the sequence S as

$$d_n^S(x,y) = \max_{0 \le j \le n-1} d(T^{i_j}(x), T^{i_j}(y))$$

for any $x, y \in X$.

A set F is an (n, ϵ, S) -separated set of X if $d_n^S(x, y) > \epsilon$ for any distinct $x, y \in F$. Let $f \in C(X)$. The topological sequence pressure w.r.t. S is defined by

$$P_S(T,f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{F \subset X} \{ \sum_{x \in F} e^{\sum_{j=0}^{n-1} f(T^{i_j}x)} \},$$

where the supremum is taken over the set of (n, ϵ, S) -separated sets of X.

By letting f = 0 and $h_{top}^{S}(T) := P_{S}(T, 0)$, we call $h_{top}^{S}(T)$ the topological sequence entropy of X w.r.t. S.

For any such sequence S, Goodman [Goo74, Theorem 3.1] established the partial variational principle of topological sequence entropy:

$$\sup_{\mu \in M(X,T)} h^S_{\mu}(T) \le h^S_{top}(T),$$

where $h^{S}_{\mu}(T)$ is the measure-theoretic sequence entropy of μ w.r.t. *S* introduced by Kushnirenko [Kus67], and for some TDSs the strict inequality can occur.(cf.[Goo74, Section 5])

Taking the supremum S over all sub-sequences of non-negative integers, the supremum of the topological sequence pressure of f is given by

$$P^*(T, f) = \sup_S P_S(T, f).$$

Letting f be zero potential and $h_{top}^*(T,X) := P^*(T,0)$, we call $h_{top}^*(T,X)$ maximal pattern entropy(or supremum of topological sequence entropy)⁵ of X.

We proceed to recall another topological invariant introduced by Katok and Thouvenot [KT97], which is useful for classifying the zero entropy-systems.

4.1.2. Polynomial topological entropy. Let (X, T) be a TDS and $f \in C(X)$ be a continuous potential. Let $\lceil u \rceil$ denote the smallest integer greater than u. We define polynomial topological pressure of f as

$$P_{pol}(T,f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{\log n} \log \sup_{F \subset X} \{ \sum_{x \in F} e^{\sum_{j=0}^{\lceil \log n \rceil} f(T^j x)} \},$$

where the supremum ranges over the set of (n, ϵ) -separated sets of X^6 .

In definition of polynomial topological pressure, we only consider the Birkhoff sum of x whose time begins with 0 and ends with $\lceil \log n \rceil$ rather than n as we have done for the classical topological pressure. Such a formulation is to guarantee that the polynomial topological pressure $P_{pol}(T, \cdot) : C(X) \to \mathbb{R}$ satisfies the translation invariance $P_{pol}(T, f + c) = P_{pol}(T, f) + c$.

When f = 0 is zero potential, we call $h_{top}^{pol}(T, X) := P_{pol}(T, 0)$ the polynomial topological entropy of X^7 . It seems that a proper variational principle for polynomial topological entropy is still missing for lasting a long time due to the lack of the role of measure-theoretic entropy. Therefore, we have the following question:

Question 1: how to define proper measure-theoretic entropy-like quantities such that the variational principles hold for maximal pattern entropy and polynomial topological entropy?

⁵In [HY09], for any TDS (X, T), Huang and Ye defined the maximal pattern entropy of X, and proved that it is equal to the supremum of topological sequence entropy over all subsequences of non-negative integers, which explains the reason why we call $h_{top}^*(T, X)$ the maximal pattern entropy.

⁶Namely, the (n, ϵ, S) -separated sets of X by considering $S = \mathbb{Z}_{\geq 0}$.

⁷It is also called *slow entropy* in the original literature [KT97].

4.1.3. Variational principle of zero entropy-like quantities. It is immediate to check from the definitions that both $P^*(T, \cdot)$ and $P_{pol}(T, \cdot)$ are pressure functions on C(X). Thus, by Theorem 3.12 we have the following variational principles:

Theorem 4.1. Let (X, T) be a TDS.

(i) If
$$h_{top}^{*}(T, X) < \infty$$
, then
 $h_{top}^{*}(T, X) = \max_{\mu \in M(X,T)} h_{\mu}^{*}(T)$,
where $h_{\mu}^{*}(T) = \inf_{f \in C(X)} \{P^{*}(T, f) - \int f d\mu\}$.
(ii) If $h_{top}^{pol}(T, X) < \infty$, then
 $h_{top}^{pol}(T, X) = \max_{\mu \in M(X,T)} h_{\mu}^{pol}(T)$,
where $h_{\mu}^{pol}(T) = \inf_{f \in C(X)} \{P_{pol}(T, f) - \int f d\mu\}$.

4.2. Infinite entropy systems.

4.2.1. Mean dimensions of \mathbb{Z} -actions. Contrary to the zero entropy systems, infinite entropy systems admit the quiet complicated dynamics, while we get no extra information about the systems except the entropy is infinite. To capture the topological complexity of infinite entropy systems, mean dimension, introduced by Gromov [Gro99], and metric mean dimension, introduced by Lindenstrauss and Weiss [LW00], are two vital quantities for us to understand the geometric structures and topological structures of infinite entropy systems. Recall that the classical variational principle states that the topological entropy is the supremum of the measure-theoretic entropy over the set of invariant measures, that is,

$$h_{top}(X,T) = \sup_{\mu \in M(X,T)} h_{\mu}(T).$$

It is the missing role of measure-theoretic counterpart in infinite entropy systems that prevents us to obtain some proper variational principles for mean dimension and metric mean dimension. Very recently, Lindenstrauss and Tsukamoto's pioneering work [LT18] gave the first analogue of classical variational principle. More precisely, by injecting the rate-distortion theory into ergodic theory, for any TDS (X, T)with a metric *d* they established the following variational principle for (upper) metric mean dimension:

$$\overline{\mathrm{mdim}}_{M}(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X, T)} R_{\mu, L^{\infty}}(\epsilon)$$

where $\overline{\mathrm{mdim}}_M(T, X, d)$ denotes upper metric mean dimension of X, and $R_{\mu,L^{\infty}}(\epsilon)$ denotes L^{∞} -rate distortion dimension of μ . See [CDZ22] for an extension of this result. We remark that a counter-example is given

in [LT18, Section VIII] to show the order of $\sup_{\mu \in M(X,T)}$ and $\limsup_{\epsilon \to 0}$ can not be exchanged. Thus, the variational principle

$$\overline{\mathrm{mdim}}_M(T, X, d) = \sup_{\mu \in M(X, T)} \{ \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} R_{\mu, L^{\infty}}(\epsilon) \}$$

does not hold for any TDS.

One says that a TDS (X, T) admits marker property if for any N > 0 there exists an open set $U \subset X$ with the property that

$$U \cap T^n U = \emptyset, 1 \le n \le N$$
, and $X = \bigcup_{n \in \mathbb{Z}} T^n U$.

For instance, aperiodic minimal systems and their extensions have marker property. This property was used to obtain the Voronoi tiling of \mathbb{Z}^k [GLT16, Section 4], which was extensively used to deal with the embedding problems of dynamical systems. For systems with marker property, Lindenstruass and Weiss [LT19] also proved the variational principle for mean dimension:

$$\operatorname{mdim}(X,T) = \min_{d \in \mathscr{D}(X)} \sup_{\mu \in M(X,T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} R_{\mu,L^{1}}(\epsilon)$$

where $\operatorname{mdim}(X, T)$ denotes the mean dimension of X, $R_{\mu,L^1}(\epsilon)$ denotes L^1 -rate distortion dimension of μ^8 , and $\mathscr{D}(X)$ is the set of all compatible metrics on the topology of X. We remark that in [LT19] it holds that for any TDS

$$\operatorname{mdim}(X,T) \leq \inf_{d \in \mathscr{D}(X)} \sup_{\mu \in M(X,T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} R_{\mu,L^{1}}(\epsilon)$$
$$\leq \inf_{d \in \mathscr{D}(X)} \overline{\operatorname{mdim}}_{M}(T,X,d),$$

then to get the desired equality the marker property is imposed to find the metric such that $\operatorname{mdim}(X,T) = \operatorname{mdim}_M(T,X,d)^9$. Voronoi tiling is still not developed yet for amenable groups since the structure of the group does not supply an easy generalization of the variational principle to amenable groups. Besides, since the marker property implies nonaperiodicity, while a "simple" example: the full shift on $[0,1]^{\mathbb{Z}}$ with a metric given by $d(x,y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n|$, has lots of periodic points such that Lindenstrauss-Tsukamoto's variational principle holds for mean dimension:

$$\overline{\mathrm{mdim}}_{M}(\sigma, [0, 1]^{\mathbb{Z}}, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} R_{\mu, L^{1}}(\epsilon) = 1,$$

where $\mu = \mathcal{L}^{\otimes \mathbb{Z}}$, and \mathcal{L} is the Lebesgue measure on [0, 1]. Based on these facts, we pose the following question:

⁸Due to the pretty lengthy definitions, we refer the readers to [LT18] for the precise definitions of L^1 and L^{∞} - rate distortion functions.

⁹It is an open problem posed in [GLT16, LT19] whether for any TDS (X, T), there exists a metric on X such that $\operatorname{mdim}(X, T) = \overline{\operatorname{mdim}}_M(T, X, d)$.

Question 2: For any G-system (X, G), without assuming the marker property, how to define proper measure-theoretic entropy-like quantities such that the variational principles hold for mean dimension and metric mean dimension?

4.2.2. Mean dimensions of G-actions. We first introduce the mean dimension with potential of G-actions. Let (X, G) be a G-system. Denote by $\mathcal{F}(G)$ the set of non-empty finite subsets of G. Given an open open cover \mathcal{U} of X, we define $\operatorname{ord}_x(\mathcal{U}) = \sum_{U \in \mathcal{U}} \chi_U(x) - 1$ for every $x \in X$ and

$$\mathcal{D}(\mathcal{U}) = \min_{\mathcal{V}\succ\mathcal{U}} \operatorname{ord}(\mathcal{V}),$$

where the infimum is taken over all finite open cover of X that refines \mathcal{U} , and $\operatorname{ord}(\mathcal{V}) = \max_{x \in X} \operatorname{ord}_x(\mathcal{V})$ is the order of \mathcal{V} .

Let $F \in \mathcal{F}(G)$ and define $\mathcal{U}_F = \bigvee_{g \in F} g^{-1} \mathcal{U}$. Then the function $F \in \mathcal{F}(G) \mapsto \mathcal{D}(\mathcal{U}_F)$ satisfies the condition of Ornstein-Weiss theorem [OW87] and hence there exists a negative number(only depending on G and the map), which we denote it by $\mathrm{mdim}(G,\mathcal{U})$, such that the limit

$$\lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U}_{F_n})}{|F_n|}$$

exists and does not depend on the choice of Følner sequences $\{F_n\}_{n\geq 1}$ of G. Then the mean dimension of X is defined by

$$\operatorname{mdim}(X,G) = \sup_{\mathcal{U}} \operatorname{mdim}(G,\mathcal{U}),$$

where the supremum is taken over all finite open covers of X.

To apply our main result, we need to define a notion called mean dimension with potential, however we do not follow Tsukamoto's approach [Tsu20, p.4] since it does not satisfy the desired convexity when the mean dimension with potential is regarded as a function on C(X).

Fix a Følner sequence $\{F_n\}_{n\geq 1}$ of G. The set of finite open cover \mathcal{V} that refines \mathcal{U} and attains the minimal $\mathcal{D}(\mathcal{U}_{F_n})$ is denoted by \mathcal{P}_n . Let f be a continuous potential on X. Let $F \in \mathcal{F}(G)$ and define $S_{F_n}(x) = \sum_{g \in F_n} f(gx)$ as the Birkhoff sum of x w.r.t. F. We set

$$\mathrm{mdim}(G, f, \mathcal{U}; \{F_n\}) := \limsup_{n \to \infty} \frac{1}{|F_n|} \sup_{\mathcal{V} \in \mathcal{P}_n} \mathrm{ord}(f, \mathcal{V}),$$

where $\operatorname{ord}(f, \mathcal{V}) = \max_{x \in X} \{ \operatorname{ord}_x(\mathcal{V}) + S_{F_n} f(x) \}$. We define the mean dimension of X with potential f as

$$\operatorname{mdim}(X, G, f; \{F_n\}) = \sup_{\mathcal{U}} \operatorname{mdim}(G, f, \mathcal{U}; \{F_n\}).$$

It is easy to see that $\operatorname{mdim}(X, G) = \operatorname{mdim}(X, G, 0; \{F_n\})$ if f is zero potential.

Next, we follow Tsukamoto's approach [Tsu20] to define the metric mean with potential of G-actions. Let $F \in \mathcal{F}(G)$ and d be a metric on X. The Bowen metric w.r.t. F is given by

$$d_F(x,y) = \max_{g \in F} d(gx,gy).$$

A set $E \subset X$ is a (d_F, ϵ) -separated set if $d_F(x, y) > \epsilon$ for any $x, y \in F$ with $x \neq y$. Let $\epsilon > 0$ and $f \in C(X)$. We put

$$P_F(X, f, d, \epsilon) = \sup\{\sum_{x \in E} e^{S_F f(x)} : E \text{ is a } (d_F, \epsilon) \text{-separated set of } X\}.$$

Then the metric mean dimension of X with potential f^{10} is given by

$$\overline{\mathrm{mdim}}_M(G, X, f, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \limsup_{n \to \infty} \frac{1}{|F_n|} \log P_{F_n}(X, f \cdot \log \frac{1}{\epsilon}, d, \epsilon).$$

Letting f = 0 and $\operatorname{mdim}_M(G, X, d) := \operatorname{mdim}_M(G, X, 0, d)$, we call $\operatorname{mdim}_M(G, X, d)$ the metric mean dimension of X.

4.2.3. Variational principles of infinite entropy-like quantities.

Proposition 4.2. Let (X, G) be a *G*-system. Then $mdim(G, \cdot, \mathcal{U}; \{F_n\})$ and $\overline{mdim}_M(G, X, \cdot, d)$ are two pressure functions on C(X).

Proof. For every Følner sequence $\{F_n\}$ of G, we show the quantity $\operatorname{mdim}(G, \cdot, \mathcal{U}; \{F_n\})$ is a pressure function. The remaining statements can be similarly verified. By the definition, we only need to check the cohomology.

Let $\{F_n\}$ be a Følner sequence of G. Fix $g \in G$ and $\mathcal{V} \in \mathcal{P}_n$. Then

$$\operatorname{ord}(f \circ g, \mathcal{V}) = \max_{x \in X} \{ \operatorname{ord}_x(\mathcal{V}) + S_{gF_n} f(x) \}$$
$$\leq \max_{x \in X} \{ \operatorname{ord}_x(\mathcal{V}) + S_{F_n} f(x) \} + |gF_n \triangle F_n| \cdot ||f||_{\infty}.$$

This implies that $\operatorname{mdim}(G, f \circ g, \mathcal{U}; \{F_n\}) \leq \operatorname{mdim}(G, f, \mathcal{U}; \{F_n\})$ for all $g \in G$.

Therefore, we can apply the Theorem 3.12 to the two types of pressure functions and obtain the following variational principles for mean dimensions whose forms are more close to the classical ones, which do not assume that the marker property.

Theorem 4.3. Let (X, G) be a G-system.

(i) If $\operatorname{mdim}(X,G) < \infty$, then for every Følner sequence $\{F_n\}$ of G

$$\operatorname{mdim}(X,G) = \max_{\mu \in \mathcal{M}(X,G)} \operatorname{mdim}_{\mu}(X,G;\{F_n\}),$$

where $\operatorname{mdim}_{\mu}(X, G; \{F_n\}) = \inf_{f \in C(X)} \{\operatorname{mdim}(X, G, f; \{F_n\}) - \int f d\mu \}.$

¹⁰Actually, by some more effort one can show the quantity $\overline{\mathrm{mdim}}_M(G, X, f, d)$ is independent of the choice of Følner sequence $\{F_n\}$ of G.

(ii) If
$$\overline{\mathrm{mdim}}_{M}(G, X, d) < \infty$$
, then
 $\overline{\mathrm{mdim}}_{M}(G, X, d) = \max_{\mu \in \mathcal{M}(X, G)} \overline{\mathrm{mdim}}_{M}(G, \mu, X, d),$
where $\overline{\mathrm{mdim}}_{M}(G, \mu, X, d) = \inf_{f \in C(X)} \{\overline{\mathrm{mdim}}_{M}(G, X, f, d) - \int f d\mu \}$

4.3. Non-invertible RDSs.

4.3.1. Preimage entropies and variational principles of \mathbb{Z}^+ -action. We review some progress of variational principles for preimage entropy-like quantities. A general fact is if T is a homeomorphism map, then the topological entropies of the forward orbits and backward orbits coincide, that is, $h_{top}(T, X) = h_{top}(T^{-1}, X)$. For non-invertible map T on X, the cardinality of the iterated preimage set $T^{-n}x$ of x is in general not a single point and even uncountable, so the backward orbits may possess pretty complicated preimage structures. To describe how "noninvertible" a system is and how "non-invertibility" contributes to the entropy, many types of preimage entropy-like quantities were introduced and investigated from the topological and measure-theoretical viewpoints through the growth rate of preimages and the condition entropy [LP92, Hur95, NP99, CN05, WZ21, WZ22].

Cheng and Newhouse [CN05] defined topological preimage entropy of X as

$$h_{pre}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X, k \ge n} \log s_n(T, T^{-k}x, \epsilon),$$

where $s_n(T, Z, \epsilon)$ denotes the maximal cardinality of the (n, ϵ) -separated sets of a non-empty subset Z of X, and established a variational principle for it in terms of measure-theoretic preimage entropy:

$$h_{pre}(T) = \sup_{\mu \in M(X,T)} h_{pre,\mu}(T).$$

where $h_{pre,\mu}(T) = \sup_{\alpha \in \mathcal{P}_X} h_{pre,\mu}(T,\alpha) = \sup_{\alpha \in \mathcal{P}_X} \lim_{n \to \infty} \frac{1}{n} h_{\mu}(\alpha^n | \mathcal{B}^-)$, and $\mathcal{B}^- = \bigcap_{j=0}^{\infty} T^{-j} \mathcal{B}$ is the infinite past σ -algebra related to the Borel σ -algebra \mathcal{B} . Unfortunately, the recent work in [W23, SYZ23] posed some examples showing that the Cheng-Newhouse's variational principle fails, that is, there exists TDSs such that $\sup_{\mu \in M(X,T)} h_{pre,\mu}(T) < h_{pre}(T)$. Now, turning eyes to two kind of pointwise preimage entropy:

$$h_m(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log s_n(T, T^{-n}x, \epsilon),$$

$$h_p(T) = \sup_{x \in X} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, T^{-n}x, \epsilon).$$

For systems with uniform separation of preimages¹¹, Wu and Zhu [WZ21, Theorem B, Corollary A.1] established variational principles for it in

¹¹If for some $\epsilon_0 > 0$, $d(x, y) < \epsilon_0$ and T(x) = T(y) implies that x = y.

terms of pointwise measure-theoretic preimage entropy:

$$h_p(T) = h_m(T) = \sup_{\mu \in M(X,T)} h_{m,\mu}(T) < \infty.$$

where $h_{m,\mu}(T) = \sup_{\alpha \in \mathcal{P}_X} h_{m,\mu}(T,\alpha) = \sup_{\alpha \in \mathcal{P}_X} \limsup_{n \to \infty} \frac{1}{n} h_{\mu}(\alpha^n | T^{-n} \mathcal{B}).$

In the context of non-invertible RDS, Wang et al. [WWZ23, Theorem B] extended the partial result in [WZ21] to random pointwise preimage pressure $P_{pre,m}(T, f)$, yet the corresponding variational principle is still vacant for another random pointwise preimage pressure $P_{pre,p}(T, f)$.

4.3.2. Preimage entropies-like quantities of non-invertible RDSs. We assume that $G = \mathbb{Z}_+$. A random \mathbb{Z}_+ dynamical system(non-invertible RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is generated by mappings $T_{\omega} : X \to X$ with iterates

$$T_{\omega}^{n} = \begin{cases} T_{\theta^{n-1}\omega} \circ T_{\theta^{n-2}\omega} \circ \cdots \circ T_{\omega}, & \text{for } n \ge 1\\ id, & \text{for } n = 0 \end{cases}$$

such that $(\omega, x) \mapsto T_{\omega} x$ is measurable and $x \mapsto T_{\omega} x$ is continuous for all ω .

We define a family of Bowen metrics $\{d_n^{\omega} : n \in \mathbb{N}, \omega \in \Omega\}$ on X, where

$$d_n^{\omega}(x,y) = \max_{0 \le j \le n-1} d(T_{\omega}^j x, T_{\omega}^j y),$$

for any $x, y \in X$. Given $f \in L^1_{\mathbb{P}}(\Omega, C(X))$, we set

$$S_n f(\omega, x) = \sum_{j=0}^{n-1} f \circ \Theta^j(\omega, x) = \sum_{j=0}^{n-1} f(\theta^j \omega, T_\omega^j x).$$

Let *E* be a non-empty set of *X*. A set $F \subset E$ is an (ω, n, ϵ) -separated set if $d_n^{\omega}(x, y) > \epsilon$ for any $x, y \in F$ with $x \neq y$. Let $s_n(T, E, \omega, \epsilon)$ denote the maximal cardinality of (ω, n, ϵ) -separated sets of *E*.

Topological preimage entropy was firstly introduced by Cheng and Newhouse [CN05] and was extended to topological preimage pressure by Zeng et al. [ZYZ07], which were further investigated by the several authors for non-invertible RDSs [Z07, MC09, ZLL09]. We define

$$P_{pre,n}(T, f, \omega, \epsilon, k)$$

= $\sup_{x \in X} \sup \{ \sum_{y \in F} e^{S_n f(\omega, y)} : F \text{ is an } (\omega, n, \epsilon) \text{-separated set of } T_{\omega}^{-k} x \}$

The quantity $P_{pre,n}(T, f, \omega, \epsilon, k)$ is measurable in ω [ZLL09, Lemma 2.1]. Then we set

$$P_{pre}(T, f, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \sup_{k \ge n} \int \log P_{pre,n}(T, f, \omega, \epsilon, k) d\mathbb{P}(\omega).$$

The random topological preimage pressure of f is defined by

$$P_{pre}^{*}(T, f) = \lim_{\epsilon \to 0} P_{pre}(T, f, \epsilon).$$

Letting f = 0 and $h_{pre}^*(T) := P_{pre}^*(T,0)$, we call $h_{pre}^*(T)$ random topological preimage entropy of X.

A pointwise approach is considered to define pointwise preimage entropy by Hurley [Hur95]. See also [WZ21, LWZ20, WWZ23] for the extension of pointwise preimage entropies in both \mathbb{Z}_+ -actions and noninvertible RDSs. We put

$$P_{pre,n}(T, f, \omega, x, \epsilon) = \sup\{\sum_{y \in E} e^{S_n f(\omega, y)} : E \text{ is an } (\omega, n, \epsilon) \text{-separated set of } T_{\omega}^{-n} x\}.$$

In [WWZ23], Wang, Wu and Zhu showed that both the quantities $P_{pre,n}(T, f, \omega, x, \epsilon)$ and $\sup_{x \in X} P_{pre,n}(T, f, \omega, x, \epsilon)$ are measurable in ω . This allows us to define

$$P_{pre,m}(T, f, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \int \log \sup_{x \in X} P_{pre,n}(T, f, \omega, x, \epsilon) d\mathbb{P}(\omega),$$
$$P_{pre,p}(T, f, \epsilon) = \sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} \int \log P_{pre,n}(T, f, \omega, x, \epsilon) d\mathbb{P}(\omega).$$

The random pointwise preimage pressures of f are defined by

$$P_{pre,m}^{*}(T,f) = \lim_{\epsilon \to 0} P_{pre,m}(T,f,\epsilon),$$

$$P_{pre,p}^{*}(T,f) = \lim_{\epsilon \to 0} P_{pre,p}(T,f,\epsilon).$$

Letting $f = 0, h_{pre,m}^*(T) := P_{pre,m}^*(T,0)$ and $h_{pre,p}^*(T) := P_{pre,p}^*(T,0),$ we call $h_{pre,m}^*(T), h_{pre,p}^*(T)$ random topological preimage entropies of X. For any $f \in L^1_{\mathbb{P}}(\Omega, C(X))$, it is clear that

$$P_{pre,p}^{*}(T,f) \le P_{pre,m}^{*}(T,f) \le P_{pre}^{*}(T,f),$$

and the three types of preimage topological pressures do not depend on the choice of the compatible metrics on X.

4.3.3. Variational principles of preimage entropy-like quantities. We show the preimage pressures-like quantities are random pressure functions on $L^1_{\mathbb{P}}(\Omega, C(X))$.

Proposition 4.4. Let $T = (T_{\omega})$ be a RDS over the measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Let $f, g \in L^1_{\mathbb{P}}(\Omega, C(X))$. Then $P^*_{pre}(T, \cdot)$ satisfies the following statements:

- (1) $h_{pre}^*(T) ||f|| \leq P_{pre}^*(T, f) \leq h_{pre}^*(T) + ||f||.$ (2) $P_{pre}^*(T, \cdot) : L_{\mathbb{P}}^1(\Omega, C(X)) \longrightarrow \mathbb{R} \cup \{\infty\}$ is either finite value or constantly ∞ .
- (3) (monotonicity) If $f \leq g$, then $P_{pre}^*(T, f) \leq P_{pre}^*(T, g)$.
- (4) (translation invariance) $P_{mre}^{*}(T, f + c) = P_{mre}^{*}(T, f) + c$ for any $c \in \mathbb{R}$.

(5) (Lipschitz and convexity) If $h_{pre}^*(T) < \infty$, then

$$|P_{pre}^{*}(T, f) - P_{pre}^{*}(T, g)| \le ||f - g||$$

and $P_{pre}^{*}(T, \cdot)$ is convex on $L^{1}_{\mathbb{P}}(\Omega, C(X))$. (6) (cohomology) $P_{pre}^{*}(T, f + g \circ \Theta - g) = P_{pre}^{*}(T, f)$.

Furthermore, the above properties are also valid for $P^*_{pre,p}(T, \cdot)$, $P^*_{pre,m}(T, \cdot)$.

Proof. These properties can be essentially directly verified by definitions. Due to the completeness, we only give a sketch for $P_{pre}^*(T, \cdot)$.

(1-2). It follows from the inequality

$$e^{\sum_{j=0}^{n-1} - ||f(\theta^{j}\omega)||_{\infty}} P_{pre,n}(T,0,\omega,\epsilon,k) \le P_{pre,n}(T,f,\omega,\epsilon,k)$$
$$\le e^{\sum_{j=0}^{n-1} ||f(\theta^{j}\omega)||_{\infty}} P_{pre,n}(T,0,\omega,\epsilon,k).$$

for any $k \geq n$ and $\omega \in \Omega$.

(5). Let $0 < \epsilon < 1, k \ge n$ and $\omega \in \Omega$. Then

$$P_{pre,n}(T, f, \omega, \epsilon, k) \le e^{\sum_{j=0}^{n-1} ||(f-g)(\theta^j \omega)||_{\infty}} P_{pre,n}(T, g, \omega, \epsilon, k)$$

which implies that $P_{pre}^*(T, f) \leq P_{pre}^*(T, g) + ||f - g||$. Exchanging the role of f and g one has

$$P_{pre}^{*}(T,g) \le P_{pre}^{*}(T,f) + ||f - g||$$

Let $p \in [0,1]$, and let E be an (ω, n, ϵ) -separated set of $T_{\omega}^{-k}x$. Applying Hölder's inequality,

$$\sum_{y \in E} e^{pS_n f(\omega, y) + (1-p)S_n g(\omega, y)} \le (\sum_{y \in E} e^{S_n f(\omega, y)})^p (\sum_{y \in E} e^{S_n g(\omega, y)})^{(1-p)},$$

which yields that $P_{pre}^*(T, pf + (1-p)g) \le pP_{pre}^*(T, f) + (1-p)P_{pre}^*(T, g)$. (6) Let F be an (ω, n, ϵ) -separated set of $T_{\omega}^{-k}x$. Notice that for every

 (ω, x) and $0 < \epsilon < 1$,

$$\sum_{y \in F} (1/\epsilon)^{S_n(f+g \circ \Theta - g)(\omega, y)} = \sum_{y \in F} (1/\epsilon)^{S_n f(\omega, y) + g(\theta^n \omega, T^n_\omega y) - g(\omega, y)}.$$

Then the result holds by considering the inequality

$$\sum_{y \in F} (1/\epsilon)^{S_n f(\omega, y) - ||(g(\theta^n \omega))|_{\infty} - ||(g(\omega))|_{\infty}}$$

$$\leq \sum_{y \in F} (1/\epsilon)^{S_n (f + g \circ \Theta - g)(\omega, y)}$$

$$\leq \sum_{x \in F} (1/\epsilon)^{S_n f(\omega, y) + ||(g(\theta^n \omega))|_{\infty} + ||(g(\omega))|_{\infty}}.$$

Definition 4.5. Let $T = (T_{\omega})$ be a RDS over the measure-preserving system with $h_{pre}^*(T) < \infty$. For any $\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X)$, we respectively define the measure-theoretic preimage entropy, measure-theoretic pointwise preimage entropies of μ

$$h_{pre,\mu}^{*}(T) = \inf\{P_{pre}^{*}(T,f) - \int f d\mu : f \in L^{1}_{\mathbb{P}}(\Omega, C(X))\},\$$

$$h_{p,\mu}^{*}(T) = \inf\{P_{pre,p}^{*}(T,f) - \int f d\mu : f \in L^{1}_{\mathbb{P}}(\Omega, C(X))\},\$$

$$h_{m,\mu}^{*}(T) = \inf\{P_{pre,m}^{*}(T,f) - \int f d\mu : f \in L^{1}_{\mathbb{P}}(\Omega, C(X))\}.$$

Based on the Proposition 4.4 and Theorem 1.1, one can formulate the following variational principles of random preimage entropy-like quantities without requiring the uniform separation of preimages.

Theorem 4.6. Let Ω be a locally compact and separable metric space with Borel σ -algebra $\mathcal{B}(\Omega)$. Let $T = (T_{\omega})$ be a random \mathbb{Z}_+ dynamical system over an ergodic measure-preserving system $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \theta)$. If $h_{pre}^*(T) < \infty$, then

$$h_{pre}^{*}(T) = \max_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} h_{pre,\mu}^{*}(T),$$

$$h_{pre,p}^{*}(T) = \max_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} h_{p,\mu}^{*}(T),$$

$$h_{pre,m}^{*}(T) = \max_{\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times X, G)} h_{m,\mu}^{*}(T).$$

Remark 4.7. (i) When Ω is a single point, for systems with the property of uniform separation of preimages, Li, Wu and Zhu [LWZ20, Theorem A, Proposition 4.2] showed that

$$h_{m,\mu}(T) = \inf\{P_{pre,m}(T,f) - \int f d\mu : f \in C(X)\},\$$

= $\inf\{P_{pre,p}(T,f) - \int f d\mu : f \in C(X)\},\$

where $P_{pre,m}(T, f)$, $P_{pre,p}(T, f)$ are point-wise preimage pressure of f respectively.

Hence, the Definition 4.3 is compatible with $h_{m,\mu}(T)$ under \mathbb{Z}^+ -actions, which also provides a strategy to extend the variational principles of pointwise preimage entropies given in [WZ21] to any finite preimage entropy systems without assuming the condition of the uniform separation of preimages.

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