

# APPROXIMATE CONTROL OF THE MARKED LENGTH SPECTRUM BY SHORT GEODESICS

KAREN BUTT

ABSTRACT. The marked length spectrum (MLS) of a closed negatively curved manifold  $(M, g)$  is known to determine the metric  $g$  under various circumstances. We show that in these cases, (approximate) values of the MLS on a sufficiently large finite set approximately determine the metric. Our approach is to recover the hypotheses of our main theorems in [But22], namely multiplicative closeness of the MLS functions on the entire set of closed geodesics of  $M$ . We use mainly dynamical tools and arguments, but take great care to show the constants involved depend only on concrete geometric information about the given Riemannian metrics, such as the dimension, sectional curvature bounds, and injectivity radii.

## 1. INTRODUCTION

The marked length spectrum of a closed negatively curved Riemannian manifold  $(M, g)$  of negative curvature is a function on the free homotopy classes of closed curves in  $M$  which assigns to each class the length of its unique geodesic representative. It is conjectured that this function completely determines the metric  $g$  up to isometry [BKB<sup>+</sup>85], and this is known under various conditions, namely in dimension 2 [Ota90, Cro90], in dimensions 3 or more when one of the metrics is locally symmetric [Ham99, BCG95], and in general when the metrics are sufficiently close in a suitable  $C^k$  topology [GL19, GKL22]. (See the introductions to [But22, GL19] for more detailed discussions.)

In the case where  $(M, g)$  has constant negative curvature and dimension at least 3, the fundamental group of  $M$  already determines  $g$  by Mostow Rigidity [Mos73]. When  $\dim M = 2$ , the Teichmüller space of all such metrics has finite dimension  $6 \operatorname{genus}(M) - 6$ . Here it is known that the marked length spectrum on a sufficiently large finite subset determines the metric up to isometry. (See [FM11, Theorem 10.7] and the introduction to [Ham03].)

In the case of variable curvature, on the other hand, the space of all negatively curved metrics on  $M$  is infinite-dimensional, so no finite set can suffice. It is nevertheless natural to ask if finitely many closed geodesics can approximately determine the metric. As far as we know, this question has not been previously considered in the literature. In this paper, we do this two cases: in dimension 2, and in dimension at least 3 when one of the metrics is locally symmetric. As mentioned above, these are two of the main cases where it is already known that the full marked length spectrum determines the metric up to isometry.

In [But22], we showed that in each of the above situations, two metrics are bi-Lipschitz-equivalent with constant close to 1 when the marked length spectra are multiplicatively close. In light of this, we answer our question by first proving that for arbitrary closed negatively curved manifolds, finitely many closed geodesics determine the full marked length spectrum approximately (Theorem 1.2). In fact, we do not require the lengths of the finitely many free homotopy classes to coincide exactly, but only approximately.

**1.1. Statement of the main result.** To state our main result precisely, we first introduce some notation. Let  $\mathcal{L}_g$  denote the marked length spectrum of  $(M, g)$ . Since the set of free homotopy classes of  $M$  can be identified with conjugacy classes in the fundamental group  $\Gamma$  of  $M$ , we will write  $\mathcal{L}_g(\gamma)$  for the length of the geodesic representative of the conjugacy class of  $\gamma \in \Gamma$  with respect to the metric  $g$ . If  $(N, g_0)$  is another negatively curved Riemannian manifold with fundamental group isomorphic to  $\Gamma$ , there is a homotopy equivalence  $f : M \rightarrow N$  inducing this isomorphism. If  $f_*$  denotes the induced map on fundamental groups, then  $\mathcal{L}_{g_0} \circ f_*$  makes sense as a function on  $\Gamma$  as well. Our work investigates what can be said about  $g$  and  $g_0$  satisfying the hypothesis below:

**Hypothesis 1.1.** For  $L > 0$ , let  $\Gamma_L := \{\gamma \in \Gamma \mid \mathcal{L}_g(\gamma) \leq L\}$ . Now let  $\varepsilon > 0$  small and suppose

$$1 - \varepsilon \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(f_*\gamma)} \leq 1 + \varepsilon$$

for all  $\gamma \in \Gamma_L$ .

If  $L$  is sufficiently large, we obtain estimates for the ratio  $\mathcal{L}_g/\mathcal{L}_{g_0}$  on all of  $\Gamma$  in terms of  $\varepsilon$  and  $L$  in Theorem 1.2 below. Moreover, our estimates do not depend on the particular pair of metrics under consideration; they are uniform for all  $(M, g)$  and  $(N, g_0)$  with pinched sectional curvatures and injectivity radii bounded away from zero.

**Theorem 1.2.** *Let  $(M, g)$  and  $(N, g_0)$  be closed Riemannian manifolds of dimension  $n$  with sectional curvatures contained in the interval  $[-\Lambda^2, -\lambda^2]$ . Let  $\mathcal{L}_g$  and  $\mathcal{L}_{g_0}$  denote their marked length spectra. Let  $\Gamma$  denote the fundamental group of  $N$  and let  $i_N$  denote its injectivity radius. Suppose there is a homotopy equivalence  $f : M \rightarrow N$  and let  $f_*$  denote the induced map on fundamental groups. Then there is  $L_0 = L_0(n, \Gamma, \lambda, \Lambda, i_N)$  so that the following holds: Suppose the marked length spectra  $\mathcal{L}_g$  and  $\mathcal{L}_{g_0}$  satisfy Hypothesis 1.1 for some  $\varepsilon > 0$  and  $L \geq L_0$ . Then there exist constants  $C > 0$  and  $0 < \alpha < 1$ , depending only on  $n, \Gamma, \lambda, \Lambda, i_N$ , so that*

$$1 - (\varepsilon + CL^{-\alpha}) \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(f_*\gamma)} \leq 1 + (\varepsilon + CL^{-\alpha})$$

for all  $\gamma \in \Gamma$ .

*Remark 1.3.* Note that  $M$  and  $N$  need not be diffeomorphic *a priori*, as shown by Farrell–Jones [FJ89, FJ94] and Aravinda–Farrell [AF04, AF03]. All of our results cover this case as well.

*Remark 1.4.* Throughout the proof of Theorem 1.2, we use several constants which depend on  $i_M$ , the injectivity radius of  $M$ . This dependence on  $i_M$  can be replaced with a dependence on  $i_N$  in light of Hypothesis 1.1. Indeed, since the injectivity radius of a negatively curved manifold is half the length of the shortest closed geodesic [Pet06, p.178], we can write  $2i_M = l_g(\gamma)$  for some homotopy class  $\gamma$ . So long as  $L$  is sufficiently large for  $\Gamma_L$  to be non-empty, we have  $l_g(\gamma) \geq (1 - \varepsilon)l_{g_0}(\gamma)$ . Since  $l_{g_0}(\gamma) \geq 2i_N$  for any  $\gamma \in \Gamma$ , we obtain  $i_M \geq (1 - \varepsilon)i_N$ .

*Remark 1.5.* One can obtain similar conclusions to those in Theorem 1.2 by combining Proposition 2.4, the proof of which occupies the vast majority of this paper, with finite Livsic theorems such as [GL21, Theorem 1.2] and [Kat90]. However, our direct method in Section 2 yields estimates which depend only on concrete geometric information (dimension, sectional curvature, injectivity radius) and not on the given flows; see Remark 2.10.

**1.2. Applications to rigidity.** Since the conclusion of Theorem 1.2 is the main hypothesis in [But22], we recover versions of all our quantitative marked length spectrum rigidity results from only finite data, i.e., only assuming Hypothesis 1.1. Let  $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, L, n, \Gamma, \lambda, \Lambda, i_M) = \varepsilon + CL^{-\alpha}$  as in the conclusion of Theorem 1.2. This theorem states that Hypothesis 1.1 results in

$$1 - \tilde{\varepsilon} \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(f_*\gamma)} \leq 1 + \tilde{\varepsilon} \quad (1.1)$$

for all  $\gamma \in \Gamma$ . Applying Theorems 1.2 and 1.4 in [But22] yields the following two corollaries.

**Corollary 1.6.** *Let  $(M, g)$  be a closed Riemannian manifold of dimension at least 3 with fundamental group  $\Gamma$  and sectional curvatures contained in the interval  $[-\Lambda^2, 0)$ . Let  $(N, g_0)$  be a locally symmetric space. Assume there is a homotopy equivalence  $f : M \rightarrow N$  and let  $f_*$  denote the induced map on fundamental groups. Then there exists small enough  $\varepsilon_0$  (depending on  $\Gamma$ ) so that whenever  $\tilde{\varepsilon} \leq \varepsilon_0$  and (1.1) holds, there is a  $C^2$  map  $F : M \rightarrow N$  homotopic to  $f$  and constants  $c_1(\tilde{\varepsilon}, n, \Gamma, \Lambda) < 1$ ,  $C_2(\tilde{\varepsilon}, n, \Gamma, \Lambda) > 1$  such that for all  $v \in TM$  we have*

$$c_1 \|v\|_g \leq \|dF(v)\|_{g_0} \leq C_2 \|v\|_g. \quad (1.2)$$

More precisely, there is a constant  $C = C(n, \Gamma, \Lambda)$  so that  $c_1 = 1 - C\tilde{\varepsilon}^{1/8(n+1)} + O(\tilde{\varepsilon}^{1/4(n+1)})$  and  $C_2 = 1 + C\tilde{\varepsilon}^{1/8(n+1)} + O(\tilde{\varepsilon}^{1/4(n+1)})$ .

*Remark 1.7.* If  $\tilde{N}$  is a real, complex or quaternionic hyperbolic space, we can take  $c_1 = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$  and  $C_2 = 1 + C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$ . See [But22, Theorem 1.2].

**Corollary 1.8.** *Let  $(M, g)$  be a closed negatively curved Riemannian manifold with fundamental group  $\Gamma$ . Let  $(N, g_0)$  be another closed negatively curved manifold with fundamental group  $\Gamma$  and assume the geodesic flow on  $T^1N$  has  $C^{1+\beta}$  Anosov splitting for some  $0 < \beta < 1$ . Suppose the marked length spectra of  $M$  and  $N$  are  $\tilde{\varepsilon}$ -close as in (1.1). Then there is a constant  $C$  depending only on  $\tilde{N}$  such that*

$$(1 - C\tilde{\varepsilon}^\beta)(1 - \tilde{\varepsilon})^n \text{Vol}(M) \leq \text{Vol}(N) \leq (1 + C\tilde{\varepsilon}^\beta)(1 + \tilde{\varepsilon})^n \text{Vol}(M).$$

If, in addition,  $(N, g_0)$  is locally symmetric and  $\varepsilon$  is sufficiently small (depending on  $n = \dim N$ ), then  $\beta$  can be replaced with 2 in the above estimates and the constant  $C$  depends only on  $n$ .

Applying [But22, Theorem 1.1] yields the following.

**Corollary 1.9.** *Let  $M$  be a compact surface of higher genus with fundamental group  $\Gamma$ . Fix  $\lambda, \Lambda, v_0, D_0 > 0$ . Given any  $A > 1$ , there exists a large enough  $L = L(A, \lambda, \Lambda, v_0, D_0, \Gamma)$  so that for any pair of Riemannian metrics  $g$  and  $h$  on  $M$  with sectional curvatures in the interval  $[-\Lambda^2, -\lambda^2]$ , volume bounded below by  $v_0$ , diameter bounded above by  $D_0$  and marked length spectra satisfying  $\mathcal{L}_g(\gamma) = \mathcal{L}_h(\gamma)$  for all  $\{\gamma \in \Gamma \mid \mathcal{L}_g(\gamma) \leq L\}$ , there is an  $A$ -Lipschitz map  $f : (M, g) \rightarrow (M, h)$ .*

**1.3. Structure of the paper.** In Section 2, we start by stating the key dynamical facts used in our proof of Theorem 1.2. Specifically, we use an estimate for the size of a covering of the unit tangent bundle  $T^1M$  by certain small “flow boxes” in addition to a Hölder estimate for a certain orbit equivalence between the geodesic flows of  $M$  and  $N$ . We then prove the theorem assuming these two facts. See the introduction to Section 2 below for a rough sketch of the argument.

The rest, and vast majority, of this paper is devoted to proving the above-mentioned covering lemma and Hölder estimate. The proofs rely on a few well-established consequences of the hyperbolicity of the geodesic flow. However, the standard results from the theory of Anosov flows (uniformly hyperbolic flows) are stated very generally and thus contain a multitude of constants which depend on the given flow in arguably mysterious ways. As a result, considerable technical difficulties arise in ensuring the constants depend only on select geometric and topological properties of  $(M, g)$  and  $(N, g_0)$ .

The main components of this analysis are as follows. In Section 3, we use geometric arguments involving horospheres to investigate the local product structure of the geodesic flow, a key mechanism responsible for many of the salient features of hyperbolic dynamical systems. Indeed, the results of this section are used to prove both the covering lemma and the Hölder estimate. The covering lemma is then quickly proved in Section 4. Before proving the desired Hölder estimate, we show that the homotopy equivalence  $f : M \rightarrow N$  (via which we are able to compare the marked length spectrum functions  $\mathcal{L}_g$  and  $\mathcal{L}_{g_0}$ ) can be taken to be a quasi-isometry with controlled quasi-isometry constants, i.e., depending only on  $n$ ,  $\Gamma$ ,  $\lambda$ ,  $\Lambda$ ,  $i_M$ ,  $i_N$ . This is done in Section 5. Finally, in Section 6, we prove the orbit equivalence of geodesic flows in [Gro00] is Hölder continuous, also with controlled constants.

**Acknowledgements.** I am very grateful to my advisor Ralf Spatzier for many helpful discussions and for help reviewing this paper. I thank Vaughn Climenhaga, Dmitry Dolgopyat, Andrey Gogolev, and Thang Nguyen for helpful conversations. I also thank Nicholas Wawrykow for help with making the figures. This research was supported in part by NSF grant DMS-2003712.

## 2. PROOF OF MAIN THEOREM

In this section, we will prove Theorem 1.2 assuming two key statements: a covering lemma (Lemma 2.1 below) and a Hölder estimate (Proposition 2.4 below). These statements are proved in Sections 4 and 6, respectively.

The basic idea is to start by covering the unit tangent bundle  $T^1M$  with finitely many sufficiently small “flow boxes”, that is, sets obtained by flowing local transversals for some small fixed time interval  $(0, \delta)$ . On the one hand, any periodic orbit of the flow that visits each of these boxes at most once is short, i.e., has period at most  $\delta$  times the total number of boxes. On the other hand, any periodic orbit that is long, i.e., of length more than  $\delta$  times the number of boxes, must return to at least one of the boxes more than once before it closes up. In other words, long periodic orbits contain shorter almost-periodic segments. By the Anosov closing lemma, these are in turn shadowed by periodic orbits. This allows us to approximate the lengths of long closed geodesics with sums of lengths of short ones. We then use a Hölder continuous orbit equivalence  $\mathcal{F} : T^1M \rightarrow T^1N$  to argue that similar approximations hold for the corresponding closed geodesics in  $N$ . From this, we are able to estimate the ratio of  $\mathcal{L}_g(\gamma)/\mathcal{L}_{g_0}(\gamma)$  for all long geodesics  $\gamma$  given our assumed estimate holds for short ones (Hypothesis 1.1).

We now introduce the precise statements of the aforementioned covering lemma and Hölder estimate. Let  $W^{si}$  for  $i = s, u$  denote the strong stable and strong unstable foliations for the geodesic flow  $\phi^t$  on the unit tangent bundle  $T^1M$ . For  $\delta > 0$ , let  $W_\delta^{si}(v) = W^{si}(v) \cap B(v, \delta)$ , where  $B(v, \delta)$  denotes a ball of radius  $\delta$  in  $T^1M$  with respect to the Sasaki metric. (See Section 3 for some background on the stable/unstable foliations and the Sasaki metric.)

Let  $P(v, \delta) = \cup_{v' \in W_\delta^{ss}(v)} W_\delta^{su}(v')$  and let  $R(v, \delta) = \cup_{t \in (-\delta/2, \delta/2)} \phi^t P(v, \delta)$ . We will call  $R(v, \delta)$  a  $\delta$ -rectangle. For our proof of Theorem 1.2, we use the following estimate for the number of  $\delta$ -rectangles needed to cover  $T^1M$ .

**Lemma 2.1.** *Let  $i_M$  denote the injectivity radius of  $M$ . There is small enough  $\delta_0 = \delta_0(n, \lambda, \Lambda, i_M)$  and a constant  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$  so that for any  $\delta < \delta_0$ , there is a covering of  $T^1M$  by at most  $C/\delta^{2n+1}$   $\delta$ -rectangles.*

*Remark 2.2.* The main difficulty is showing that the constant  $C$  does not depend on the metric  $g$ , but only on  $n, \Gamma, \lambda, \Lambda, \text{diam}(M)$ .

*Remark 2.3.* Rectangles of the form  $R(v, \delta)$  are often used to construct Markov partitions, e.g. in [Rat73]. However, in Lemma 2.1, we are not constructing a partition, meaning we do not require the rectangles to be measurably disjoint.

Now consider the geodesic flows  $\phi^t$  and  $\psi^t$  on  $T^1M$  and  $T^1N$ , respectively. Recall that a homeomorphism  $\mathcal{F} : T^1M \rightarrow T^1N$  is an *orbit equivalence* if there is some function (cocycle)  $a(t, v)$  so that

$$\mathcal{F}(\phi^t v) = \psi^{a(t, v)} \mathcal{F}(v)$$

for all  $v \in T^1M$  and for all  $t \in \mathbb{R}$ . Since  $M$  and  $N$  are homotopy-equivalent compact negatively curved manifolds, such an  $\mathcal{F}$  exists by [Gro00]. Our proof of Theorem 1.2 relies on the following estimates for the regularity of  $\mathcal{F}$ .

**Proposition 2.4.** *Suppose  $(M, g)$  and  $(N, g_0)$  are a pair of homotopy-equivalent compact Riemannian manifolds with sectional curvatures contained in the interval  $[-\Lambda^2, -\lambda^2]$ . Let  $i_M$  and  $i_N$  denote their respective injectivity radii. Then there exists an orbit equivalence of geodesic flows  $\mathcal{F} : T^1M \rightarrow T^1N$  which is  $C^1$  along orbits and transversally Hölder continuous. More precisely, there is small enough  $\delta_0 = \delta_0(\lambda, \Lambda, i_M)$  together with constants  $C$  and  $A$ , depending only on  $n, \Gamma, \lambda, \Lambda, i_M, i_N$ , so that the following hold:*

- (1)  $d(\mathcal{F}(v), \mathcal{F}(\phi^t v)) \leq At$  for all  $v \in T^1\tilde{M}$  and  $t \in \mathbb{R}$ ,
- (2)  $d(\mathcal{F}(v), \mathcal{F}(w)) \leq Cd(v, w)^{A^{-1}\lambda/\Lambda}$  for all  $v, w \in T^1\tilde{M}$  with  $d(v, w) < \delta_0$ .

*Remark 2.5.* It is a standard fact that any orbit equivalence of Anosov flows is  $C^0$ -close to a Hölder continuous one; in other words, there are constants  $C$  and  $\alpha$ , depending on the given flows, i.e., on the metrics  $g$  and  $g_0$ , so that  $d(\mathcal{F}(v), \mathcal{F}(w)) \leq Cd(v, w)^\alpha$  [FH19, Theorem 6.4.3]. However, we are claiming the stronger statement that for the orbit equivalence in [Gro00], there is a uniform choice of  $C$  and  $\alpha$  for all  $(M, g)$  and  $(N, g_0)$  with pinched sectional curvatures and injectivity radii bounded away from 0.

To prove Theorem 1.2, we start with a covering of  $T^1M$  by  $\delta$ -rectangles (see Lemma 2.1). Let  $\delta_0$  be as in Proposition 2.4, then make  $\delta_0$  smaller if necessary so that Lemma 2.1 holds as well. This choice of  $\delta_0$  depends only on  $n, \lambda, \Lambda, i_M$ . Now fix  $\delta \leq \delta_0$ , together with a covering  $T^1M = \cup_{i=1}^m R(v_i, \delta)$ . By Lemma 2.1, we can take  $m \leq C\delta^{2n+1}$ . Since  $\delta$  is now fixed, we use the notation  $R_i$  for the rectangle  $R(v_i, \delta)$  and  $P_i$  for the transversal  $P(v_i, \delta)$ .

Let  $v \in T^1M$ . Then  $v \in P_i$  if and only if  $\phi^t v \in R_i$  for all  $t \in (-\delta/2, \delta/2)$ . Moreover, if  $v$  is tangent to a closed geodesic of length  $\tau$ , then for any rectangle  $R_i$ , the set

$$\{t \in (-\delta/2, \tau - \delta/2) \mid \phi^t v \cap R_i \neq \emptyset\}$$

is a (possibly empty) disjoint union of intervals of length  $\delta$ .

**Definition 2.6.** Fix a covering of  $T^1M$  by  $\delta$ -rectangles  $R_1, \dots, R_m$  as above. Suppose  $\eta$  is a closed geodesic of length  $\tau$  with  $\eta'(0) = v$ . Suppose that for each  $i$ , the set

$$\{t \in (-\delta/2, \tau - \delta/2) \mid \phi^t v \cap R_i \neq \emptyset\}$$

consists of at most a single interval. Then we say  $\eta$  is a *short geodesic* (with respect to the covering  $R_1, \dots, R_m$ ).

*Remark 2.7.* Let  $L = L(\delta) = C\delta^{-2n}$ , where  $C$  is the constant in the statement of Lemma 2.1. If  $\eta$  is a short geodesic, then  $l_g(\eta) \leq m\delta \leq C\delta^{-2n} = L$ .

**Proposition 2.8.** *Let  $\gamma$  be any closed geodesic in  $M$ . Then there is  $k \in \mathbb{N}$  (depending on  $\gamma$ ) and short geodesics  $\eta_1, \dots, \eta_{k+1}$  so that*

$$\left| l_g(\gamma) - \sum_{i=1}^{k+1} l_g(\eta_i) \right| < 2kC\delta$$

for some constant  $C = C(\lambda, \Lambda, i_M)$ .

*Proof.* If  $\gamma$  is already a short geodesic, then  $k = 0$  and  $\eta_1 = \gamma$ . If not, then let  $i$  be the smallest index so that  $\gamma$  crosses through  $R_i$  in at least two time intervals. Let  $v \in P_i$  tangent to  $\gamma$  and let  $t_1 > 0$  be the first time so that  $\phi^{t_1} v \in P_i$ . By the Anosov closing lemma, there is  $w_1$  tangent to a closed geodesic  $\gamma_1$  of length  $t'_1$  with  $|t_1 - t'_1| < C\delta$ , where  $C$  depends only on the sectional curvature bounds  $\lambda$  and  $\Lambda$  and the injectivity radius  $i_M$  (see Lemma 3.12). Similarly, applying the Anosov closing lemma to the orbit segment  $\{\phi^t v \mid t \in [t_1, \tau]\}$  gives  $w_2$  tangent to a closed geodesic  $\gamma_2$  of length  $t'_2$  with  $|(\tau - t_1) - t'_2| < C\delta$ . This means  $|l_g(\gamma) - l_g(\gamma_1) - l_g(\gamma_2)| < 2C\delta$ .

Iterating the above process, we can “decompose”  $\gamma$  into short geodesics. More precisely, if  $\gamma_1$  is not a short geodesic, then there is some other rectangle  $R_j$  through which  $\gamma_1$  crosses twice. By the same argument as above, we get  $|l_g(\gamma_1) - l_g(\gamma_{1,1}) - l_g(\gamma_{1,2})| < 2C\delta$  for some  $\gamma_{1,1}, \gamma_{1,2} \in \Gamma$ . Continuing in this manner, we get the desired conclusion.  $\square$

Next, we show that  $l_{g_0}(\gamma)$  is still well-approximated by the sum of the  $g_0$ -lengths of the *same* free homotopy classes  $\eta_1, \dots, \eta_{k+1}$  that were used to do the approximation with respect to  $g$ . For this, we use the estimates for the regularity of the orbit equivalence  $\mathcal{F} : T^1M \rightarrow T^1N$  in Proposition 2.4. Recall that  $a(t, v)$  denotes the time-change cocycle, i.e.  $\mathcal{F}(\phi^t v) = \psi^{a(t, v)} \mathcal{F}(v)$ .

**Lemma 2.9.** *Let  $\gamma$  and  $\eta_1, \dots, \eta_{k+1}$  as in Proposition 2.8. Then an analogous estimate holds in  $(N, g_0)$ , namely,*

$$\left| l_{g_0}(\gamma) - \sum_{i=1}^{k+1} l_{g_0}(\eta_i) \right| < 2kC\delta^\alpha,$$

where  $C$  depends only on  $\Gamma, \lambda, \Lambda, i_M, i_N$ , and  $\alpha$  is the Hölder exponent in the statement of Proposition 2.4.

*Proof.* As in the proof of Proposition 2.8, let  $v \in T^1M$  tangent to  $\gamma$ . By the Anosov closing lemma, there is  $w_1 \in T^1M$  tangent to a closed geodesic  $\gamma_1$  of length  $t'_1$  such that  $d(v, w_1) < C\delta$ , for some  $C = C(\lambda, \Lambda, i_M)$  (Lemma 3.12). Additionally,  $d(\phi^{t_1} v, \phi^{t'_1} w_1) < C\delta$ .

By Proposition 2.4, we know  $d(\mathcal{F}(v), \mathcal{F}(w_1)) < C\delta^\alpha$ . Moreover, since  $\mathcal{F}(v)$  and  $\mathcal{F}(w_1)$  remain  $C\delta^\alpha$ -close after being flowed by times  $a(t_1, v)$  and  $a(t'_1, w)$ , respectively, it follows

that  $|a(t_1, v) - a(t'_1, w_1)| < 2C\delta^\alpha$ . (We defer the short proof of this fact to Section 3; see Lemma 3.2.)

Similarly, the Anosov closing lemma applied to the orbit segment  $\{\phi^t v \mid t \in [t_1, l(\gamma)]\}$  gives  $w_2$  tangent to a closed geodesic  $\gamma_2$  of length  $t'_2$ . By an analogous argument,  $|a(l_g(\gamma) - t_1, v) - a(t'_2, w_2)| < 2C\delta^\alpha$ . Since  $a(t, v)$  is a cocycle we get  $|a(l_g(\gamma), v) - a(t'_1, w_1) - a(t'_2, w_2)| < 4C\delta^\alpha$ .

Using that  $\mathcal{F}$  is a  $\Gamma$ -equivariant orbit-equivalence, it follows that  $a(l_g(\gamma), v) = l_{g_0}(\gamma)$  whenever  $v \in T^1M$  is tangent to the closed geodesic  $\gamma$ . So the estimate in the previous paragraph can be rewritten as  $|l_{g_0}(\gamma) - l_{g_0}(\gamma_2) - l_{g_0}(\gamma_2)| < 4C\delta^\alpha$ . As such, we can iterate the process in Proposition 2.8 and get an additive error of  $4C\delta^\alpha$  at each stage.  $\square$

*Proof of Theorem 1.2.* Recall from Remark 2.7 that  $L = L(\delta) = C\delta^{-2n}$  for some  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$ . Since we fixed  $\delta \leq \delta_0 = \delta_0(n, \lambda, \Lambda, i_M)$ , we see that  $L \geq L_0 = L(\delta_0)$ . By Lemma 2.1, this choice of  $L_0$  depends only on  $n, \Gamma, \lambda, \Lambda, i_M$ .

Recall as well that we are assuming

$$1 - \varepsilon \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + \varepsilon$$

for all  $\gamma \in \Gamma_L := \{\gamma \in \Gamma \mid l_g(\gamma) \leq L\}$  (see Hypothesis 1.1). We then have

$$l_g(\gamma) \leq \sum_{i=1}^{k+1} l_g(\gamma_i) + 2kC\delta \quad (\text{Proposition 2.8})$$

$$\leq (1 + \varepsilon) \sum_{i=1}^{k+1} l_{g_0}(\gamma_i) + 2kC\delta \quad (\text{Hypothesis 1.1})$$

$$\leq (1 + \varepsilon)l_{g_0}(\gamma) + (1 + \varepsilon)2k(2C'\delta^\alpha + C\delta) \quad (\text{Proposition 2.9})$$

$$\leq (1 + \varepsilon)l_{g_0}(\gamma) + kC''\delta^\alpha.$$

Using this, we consider the ratio

$$\begin{aligned} \frac{l_g(\gamma)}{l_{g_0}(\gamma)} &\leq (1 + \varepsilon) + \frac{kC''\delta^\alpha}{l_{g_0}(\gamma)} \\ &\leq 1 + \varepsilon + \frac{kC''\delta^\alpha}{\sum_{i=1}^{k+1} l_{g_0}(\gamma_i) - 2k\delta} \quad (\text{Proposition 2.9}) \\ &\leq 1 + \varepsilon + \frac{kC''\delta^\alpha}{2ki_N - 2k\delta} \\ &= 1 + \varepsilon + \frac{C''\delta^\alpha}{2i_N - 2\delta}. \end{aligned}$$

In the last inequality, we used the fact that  $l_{g_0}(\gamma) \geq 2i_N$  for all  $\gamma$ .

Finally, by the definition of  $L$  in Remark 2.7, we have  $\delta = CL^{-1/2n}$ , where  $C$  is a constant depending only on  $n, \Gamma, a, b, i_M$ . So we can write that the ratio  $l_g(\gamma)/l_{g_0}(\gamma)$  is between  $1 \pm (\varepsilon + C'L^{-\alpha/2n})$ , where  $\alpha$  is the Hölder exponent in the statement of Proposition 2.4.  $\square$

*Remark 2.10.* There is a way to obtain approximate control of the marked length spectrum from finitely many geodesics by combining Proposition 2.4 with the finite Livsic theorem in [GL21], but our direct method above yields better estimates.

Let  $a(t, v)$  denote the time change function for the orbit equivalence  $\mathcal{F}$  in Proposition 2.4. By the definition of  $a(t, v)$  in (6.3) (see also [But22, Lemma 2.24]), this cocycle is

differentiable in the  $t$  direction. Let  $a(v) = \frac{d}{dt}|_{t=0}a(t, v)$ . It follows from (6.3) and Lemma 6.8 that  $a(v)$  is of  $C^\alpha$  regularity, where  $\alpha$  is the same Hölder exponent as in the statement of Proposition 2.4. It follows from Lemma 6.2 and the proof of [But22, Lemma 2.24] that  $\|a(v)\|_{C^0} \leq A$ , where  $A$  is the constant in Lemma 6.2. Hence,  $\|a\|_{C^\alpha} \leq A + C$ , where  $C$  is the constant in Proposition 2.4.

Now let  $\{\phi^t v\}_{0 \leq t \leq l_g(\gamma)}$  be the  $g$ -geodesic representative of the free homotopy class  $\gamma$ . Then  $l_{g_0}(\gamma) = \int_0^{l_g(\gamma)} a(\phi^t v) dt$ . Let  $f(v) = (a(v) - 1)/\|a - 1\|_{C^\alpha}$ . Then  $\|f\|_{C^\alpha} \leq 1$  and Hypothesis 1.1 implies

$$\frac{1}{l_g(\gamma)} \left| \int_0^{l_g(\gamma)} f(\phi^t v) dt \right| \leq \frac{\varepsilon}{A + C}$$

for all  $\gamma \in \Gamma_L$ . Setting  $L = \left(\frac{\varepsilon}{C+A}\right)^{-1/2}$  means that  $f$  satisfies the hypotheses of Theorem 1.2 in [GL21]. This theorem implies that for all  $\gamma \in \Gamma$ , the ratio  $\mathcal{L}_g/\mathcal{L}_{g_0}$  is between  $1 \pm C' \left(\frac{\varepsilon}{C+A}\right)^\tau$ , where  $C'$  and  $\tau$  are constants depending on the given flow. Our direct method above yields an exponent of  $\alpha/4n$  in place of  $\tau$ .

### 3. LOCAL PRODUCT STRUCTURE

We consider the distance  $d$  on  $T^1M$  induced by the *Sasaki metric*  $g^S$  on  $T^1M$ , which is in turn defined in terms of the Riemannian inner product  $g$  on  $M$  (see [dC92, Exercise 3.2] for the definition). Throughout the rest of this paper, we will make use of the following standard facts relating the Sasaki distance  $d$  to the distance  $d_M$  on  $M$  coming from the Riemannian metric  $g$  and the distance  $d_{T_q^1M}$  on  $S^{n-1} \cong T_q^1M$ . Let  $v, w \in T^1M$  be unit tangent vectors with footpoints  $p$  and  $q$  respectively. Let  $v' \in T_q^1M$  be the vector obtained by parallel transporting  $v$  along the geodesic joining  $p$  and  $q$ . Then we have

$$d_M(p, q), d_{T_q^1M}(v', w) \leq d(v, w) \leq d_M(p, q) + d_{T_q^1M}(v', w). \quad (3.1)$$

For convenience, we will often write  $d$  in place of  $d_M$  when it is clear from context that we are considering the distance between points as opposed to between unit tangent vectors.

Recall the geodesic flow on the unit tangent bundle of a negatively curved manifold is Anosov, and thus has *local product structure*. This means every point  $v$  has a neighborhood  $V$  which satisfies: for all  $\varepsilon > 0$ , there is  $\delta > 0$  so that whenever  $x, y \in V$  with  $d(x, y) \leq \delta$  there is a point  $[x, y] \in V$  and a time  $|\sigma(x, y)| < \varepsilon$  such that

$$[x, y] = W^{ss}(x) \cap W^{su}(\phi^{\sigma(x,y)}y)$$

[FH19, Proposition 6.2.2]. Moreover, there is a constant  $C_0 = C_0(\delta)$  so that  $d(x, y) < \delta$  implies  $d_{ss}(x, [x, y]), d_{su}(\phi^{\sigma(x,y)}[x, y], y) \leq C_0 d(x, y)$ , where  $d_{ss}$  and  $d_{su}$  denote the distances along the strong stable and strong unstable manifolds, respectively.

To describe the stable and unstable distances  $d_{ss}$  and  $d_{su}$ , we first recall the stable and unstable manifolds  $W^{ss}$  and  $W^{su}$  for the geodesic flow have the following geometric description (see, for instance, [Bal95, p. 72]). Let  $v \in T^1\tilde{M}$ . Let  $p \in \tilde{M}$  be the footpoint of  $v$  and let  $\xi \in \partial\tilde{M}$  be the forward projection of  $v \in T^1\tilde{M}$  to the boundary. Let  $B_{\xi,p}$  denote the Busemann function on  $\tilde{M}$  and let  $H_{\xi,p}$  denote its zero set. Then the lift of  $W^{ss}(v)$  to  $T^1\tilde{M}$  is given by  $\{-\text{grad}B_{\xi,p}(q) \mid q \in H_{\xi,p}\}$ . If  $\eta$  denotes the projection of  $-v$  to the boundary  $\partial\tilde{M}$ , then the lift of  $W^{su}(v)$  to  $T^1\tilde{M}$  is analogously given by  $\{\text{grad}B_{\eta,p}(q) \mid q \in H_{\eta,p}\}$ .

Now let  $v \in T^1M$  and  $w \in W^{ss}(v)$ . Let  $p$  and  $q$  denote the footpoints of  $v$  and  $w$  respectively. Define the stable distance  $d_{ss}(v, w)$  to be the horospherical distance  $h(p, q)$ , i.e.,



the distance obtained from restricting the Riemannian metric  $g$  on  $\tilde{M}$  to a given horosphere. The unstable distance is defined analogously.

From the above description of  $W^{ss}$  and  $W^{su}$  in terms of normal fields to horospheres, it follows that the local product structure for the geodesic flow enjoys stronger properties than those for a general Anosov flow given in the first paragraph. First, the product structure is globally defined, meaning the neighborhood  $V$  in the first paragraph can be taken to be all of  $T^1M$  (see, for instance, [Cou04]). Second, the bound on the temporal function  $\sigma$  can be strengthened:

**Lemma 3.1.** *If  $d(v, w) < \delta$ , then  $|\sigma(v, w)| < \delta$ . for all  $\delta$*

*Proof.* Let  $p$  and  $q$  denote the footpoints of  $v$  and  $w$  respectively. Then by (3.1), we know  $d(p, q) < \delta$ . Let  $\xi$  denote the forward boundary point of  $v$  and let  $\eta$  denote the backward boundary point of  $w$ . Let  $p' \in H_{\xi, p}$  and  $q' \in H_{\eta, q}$  be points on the geodesic through  $\eta$  and  $\xi$ . Then  $d(p', q') = |\sigma(v, w)|$ . Moreover, since the geodesic segment through  $p'$  and  $q'$  is orthogonal to both  $H_{\xi, p}$  and  $H_{\eta, q}$ , it minimizes the distance between these horospheres. In other words,  $|\sigma(v, w)| = d(p', q') \leq d(p, q) < \delta$ .  $\square$

This allows us to deduce the following key lemma, which was used in the proof of Proposition 2.9.

**Lemma 3.2.** *Consider the geodesic flow  $\phi^t$  on the universal cover  $T^1\tilde{M}$ . Suppose  $d(v, w) < \delta_1$  and  $d(\phi^s v, \phi^t w) < \delta_2$ . Then  $|s - t| < \delta_1 + \delta_2$ .*

*Proof.* Since  $[\phi^s v, \phi^s w] = [\phi^s v, \phi^t w]$ , we have

$$\phi^{\sigma(\phi^s v, \phi^s w)} \phi^s w = \phi^{\sigma(\phi^s v, \phi^t w)} \phi^t w.$$

Thus  $\sigma(\phi^s v, \phi^s w) + s = \sigma(\phi^s v, \phi^t w) + t$ . Rearranging gives

$$s - t = \sigma(\phi^s v, \phi^t w) - \sigma(\phi^s v, \phi^s w) = \sigma(\phi^s v, \phi^t w) - \sigma(v, w).$$

By Lemma 3.1, the absolute value of the right hand side is bounded above by  $\delta_1 + \delta_2$ , which completes the proof.  $\square$

Now assume  $(M, g)$  has sectional curvatures between  $-\Lambda^2$  and  $-\lambda^2$ . We will show the constant  $C_0$  in the definition of local product structure can be taken to depend only on  $\lambda$ ,  $\Lambda$  and  $\text{diam}(M)$ , whereas *a priori* it depends on the metric  $g$ . For our purposes, it will suffice to show the following proposition, which is formulated using the Sasaki distance  $d$  between vectors in  $T^1M$  instead of the stable/unstable distances  $d_{ss}$  and  $d_{su}$  between vectors on the same horosphere. In fact, we will show later (Lemma 6.5) that the Sasaki distance  $d$  between vectors on the same stable/unstable manifold is comparable to  $d_{ss}$  and  $d_{su}$ , respectively.

**Proposition 3.3.** *Suppose  $(M, g)$  has sectional curvatures between  $-\Lambda^2$  and  $-\lambda^2$ . Then there is small enough  $\delta_0 = \delta_0(\lambda, \Lambda, \text{diam}(M))$  so that the following holds. Let  $u \in T_p^1\tilde{M}$ . Let  $u_1 \in W^{ss}(u)$  and  $u_2 \in W^{ss}(x)$  so that  $d(u_1, u_2) \leq \text{diam}(T^1M)$ , where  $d$  denotes the distance in the Sasaki metric. Then there exists a constant  $C_0 = C_0(\lambda, \Lambda, \text{diam}(M))$  so that whenever  $d(u_1, u_2) < \delta_0$ , we have  $d(u, u_i) \leq C_0 d(u_1, u_2)$  for  $i = 1, 2$ .*

*Remark 3.4.* In our context, the dependence of the constant  $C_0$  on the diameter of  $M$  can be replaced with a dependence on the injectivity radius  $i_M$ . Indeed, by [Gro82, Section 0.3], the volume of  $M$  is bounded above by a constant  $V_0$  depending only on  $n$ ,  $\Gamma$ , and  $\lambda$ . A standard argument (see, for instance, the proof of Lemma 3.9 in [But22]) then shows the diameter is bounded above by  $D_0 = D_0(i_M, V_0, \Lambda)$ .

Our proof of Proposition 3.3 relies on the geometry of horospheres, and we use many of the methods and results from the paper [HIH77] of the same title. However, we additionally consider the Sasaki distances between unit tangent vectors in  $T^1\tilde{M}$  instead of just distances between points in  $\tilde{M}$ .

Let  $\xi \in \partial\tilde{M}$  and let  $B = B_\xi$  be the associated Busemann function. Suppose  $p \in \tilde{M}$  is such that  $B(p) = 0$ . Let  $v \in T_p^1\tilde{M}$  perpendicular to  $\text{grad}B(p)$  and consider the geodesic  $\gamma(s) = \exp_p(sv)$ . Define  $f(s) = B(\gamma(s))$ . This is the distance from  $\gamma(s)$  to the zero set of  $B$ . Moreover,  $f'(s) = \langle \text{grad}B, \gamma' \rangle = \cos\theta$ , where  $\theta$  is the angle between  $\gamma'(s)$  and  $\text{grad}B(\gamma(s))$ . In particular,  $f'(0) = 0$ .

**Lemma 3.5.** *For all  $s \in \mathbb{R}$  we have  $f(s) \leq \frac{\Lambda}{2}s^2$  and  $\cos\theta(s) = f'(s) \leq \Lambda s$ .*

*Proof.* We have  $f''(s) = \langle \nabla_{\gamma'} \text{grad}B, \gamma' \rangle = \langle \nabla_{\gamma'_T} \text{grad}B, \gamma'_T \rangle$ , where  $\gamma'_T$  denotes the component of  $\gamma'$  which is tangent to the horosphere through  $\xi$  and  $\gamma(s)$ . Note  $\|\gamma'_T\| = \sin(\theta)$ , where as before,  $\theta$  is the angle between  $\gamma'(s)$  and  $\text{grad}B(\gamma(s))$ .

Thus  $f''(s) = \langle J'(0), J(0) \rangle$ , where  $J$  is the stable Jacobi field along the geodesic through  $\gamma(s)$  and  $\xi$  with  $J(0) = \gamma'_T(s)$ . (See, for instance, [BCG95, p.750–751].) By [Bal95, Proposition IV.2.9 ii)], we have  $\|J'(0)\| \leq \Lambda\|J(0)\|$ , which shows  $f''(s) \leq \Lambda\|J(0)\|^2 \leq \Lambda$ . Since  $f(0)$  and  $f'(0)$  are both 0, Taylor's theorem implies that for any  $s$ , there is  $\tilde{s} \in [0, s]$  so that  $f(s) = \frac{f''(\tilde{s})}{2}s^2$ . Thus,  $f(s) \leq \frac{\Lambda}{2}s^2$  for all  $s \geq 0$ . Moreover, since  $f'(0) = 0$ , integrating  $f''(s)$  shows  $\cos\theta = f'(s) \leq \Lambda s$ .  $\square$

**Lemma 3.6.** *Fix  $S > 0$ . Then there is a constant  $c = c(\lambda, S)$  such that for all  $s \in [0, S]$  we have  $f(s) \geq \frac{c}{2}s^2$  and  $\cos\theta = f'(s) \geq cs$ .*

*Proof.* As in [HIH77, Section 4], we use  $f_\lambda(s)$  to denote the analogue of the function  $f(s)$ , but defined in the space of constant curvature  $-\lambda^2$ . By considering the appropriate comparison triangles, it follows that  $f(s) \geq f_\lambda(s)$  and  $f'(s) \geq f'_\lambda(s)$  [HIH77, Lemma 4.2]. As in the proof of the previous lemma, we know  $f''_\lambda(s) = \langle J(0), J'(0) \rangle$ , where  $\|J(0)\| = \sin\theta$ . Solving the Jacobi equation explicitly in constant curvature gives  $f''_\lambda(s) = \lambda \sin^2\theta$ . For all  $s \in [0, S]$ , this is bounded below by  $\lambda \sin^2\theta(S)$ , which is a constant depending only on the value of  $S$  and the space of constant curvature  $-\lambda^2$ . In other words, there is a constant  $c = c(\lambda, S)$  so that  $f''_\lambda(s) \geq c$  for all  $s \in [0, S]$ . As in the proof of the previous lemma, Taylor's theorem then implies  $f_\lambda(s) \geq \frac{c}{2}s^2$ , and integrating  $f''_\lambda$  on the interval  $[0, s]$  gives  $f'_\lambda(s) \geq cs$ .  $\square$

*Remark 3.7.* From the above proof it is evident that  $f'_\lambda(s)/s \rightarrow 0$  as  $s \rightarrow \infty$ , and as such the only way to get a positive lower bound for  $\cos\theta/s$  is to restrict to a compact interval  $[0, S]$ . This is reasonable for our purposes, since in the end, we will be applying the results of this section to the compact manifold  $M$  as opposed to its universal cover  $\tilde{M}$ . In Hypothesis 3.8 below, we explain how we choose  $S$  based on  $\text{diam}(M)$ .

For the proofs of the next several lemmas, we will consider the following setup (see Figure 1 below). Let  $u$  be a unit tangent vector with footpoint  $p$ . Let  $v \in T_p^1M$  perpendicular to  $u$  and let  $\gamma(t) = \exp_p(tv)$ . Fix  $s > 0$  and let  $u_1 \in W^{ss}(u)$  be such that the geodesic determined by  $u_1$  passes through  $\gamma(s)$ . Let  $p_1$  denote the footpoint of  $u_1$ . Let  $\eta$  denote the geodesic segment joining  $p$  and  $p_1$  and let  $\alpha$  denote the angle this segment makes with the vector  $u$ . Let  $q$  be the orthogonal projection of  $p_1$  onto the geodesic  $\gamma$ . Consider the geodesic right triangle with vertices  $p_1, q, \gamma(s)$ . Let  $\theta$  denote the angle at  $\gamma(s)$  and let  $\theta_1$  denote the angle at  $p_1$ .

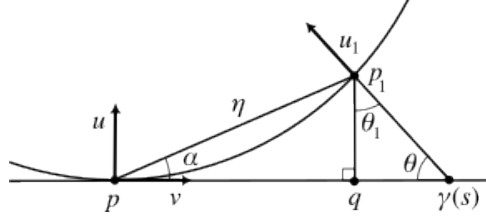


FIGURE 1.

**Hypothesis 3.8.** For our purposes, it is reasonable to assume  $d(p, p_1) \leq \text{diam}(M)$ , where  $p, p_1 \in \tilde{M}$ . By [HIH77, Theorem 4.6, Proposition 4.7], this forces  $s \leq S$ , where  $S$  is a constant depending only on  $\text{diam}(M)$  and the lower sectional curvature bound  $-\Lambda^2$ . So we assume  $s \leq S$  from now on.

**Lemma 3.9.** *Let  $u_1 \in W^{ss}(u)$  as in Figure 1, and assume  $s \leq S$  (see Hypothesis 3.8). Then there is a constant  $C = C(\Lambda, \text{diam}(M))$  so that  $d(u, u_1) \leq Cs$ . If  $u_2 \in W^{ss}(u)$ , then  $d(u, u_2) \leq Cs$  as well.*

*Proof.* Consider the setup in Figure 1. Let  $\eta$  denote the geodesic joining  $p$  and  $p_1$  and let  $P_\eta : T_p M \rightarrow T_{p_1} M$  denote parallel transport along this geodesic. Recall

$$d(u, u_1) \leq d_M(p, p_1) + d_{T_{p_1} M}(Pu, u_1).$$

To bound  $d_M(p, p_1)$ , we use the triangle inequality together, Lemma 3.5, and Hypothesis 3.8:

$$d(p, p_1) \leq d(p, q) + d(p_1, q) \leq s + d(p_1, \gamma(s)) \leq s + \Lambda s^2/2 \leq (1 + \Lambda S/2)s.$$

To bound  $d_{T_{p_1} M}(Pu, u_1)$ , we first find bounds for the angles  $\theta$  and  $\theta_1$ . We know from Lemma 3.5 that  $\sin(\pi/2 - \theta) = \cos \theta \leq \Lambda s$ . Moreover,  $\sin(\pi/2 - \theta) \geq (2/\pi)(\pi/2 - \theta)$  for  $0 \leq \pi/2 - \theta \leq \pi/2$ . Since the interior angles of geodesic triangles in  $M$  sum to less than  $\pi$ , we know  $\theta + \theta_1 < \pi/2$ . Thus,  $\theta_1 < \pi/2 - \theta \leq (\pi/2)\Lambda s$ .

Now let  $\alpha$  denote the angle between  $u$  and  $\eta'$  at the point  $p$ . Then  $\alpha$  is also the angle between  $Pu$  and  $\eta'$  at the point  $p_1$ , since parallel transport is an isometry and  $\eta'$  is a geodesic. Since the angle sum of the geodesic triangle with vertices  $p, p_1$  and  $q$  is less than  $\pi$ , the angle in  $T_{p_1} M$  between  $\eta'$  and  $[p_1, q]$  is strictly less than  $\alpha$ . Thus if we rotate  $\eta'$  towards  $Pu$ , we must pass through the tangent vector to  $[p_1, q]$  along the way. Hence  $d_{T_{p_1} M}(Pu, u_1) < \theta_1 \leq (\pi/2)\Lambda s$ , which completes the proof of the upper bound for  $d(u, u_1)$ . The estimate for  $d(u, u_2)$  follows by an analogous argument.  $\square$

**Lemma 3.10.** *Again, consider the setup in Figure 1 and Hypothesis 3.8. For all  $s \in [0, S]$  we have  $\theta_1 \geq cs$  for some  $c = c(\lambda, \Lambda, \text{diam}(M))$ .*

*Proof.* Consider the following comparison triangle with vertices  $p'_1, q', x$  in the space of constant curvature  $-\Lambda^2$ : suppose there is a right angle at the vertex  $q'$  and the lengths of the two legs are equal to  $d_M(q, p_1)$  and  $d_M(q, \gamma(s))$ . Let  $\theta'$  denote the angle at  $x$  and let  $\theta'_1$  denote the angle at  $p'_1$ . Since triangles in  $M$  are thicker than in the space of constant curvature  $-\Lambda^2$ , we have  $\theta_1 \geq \theta'_1$  and  $\cos(\theta') \geq \cos(\theta)$ . Now by [Bea12, Theorem 7.11.3] we have

$$\theta'_1 \geq \sin(\theta'_1) = \frac{\cos(\theta')}{\cosh(\Lambda d(q, p_1))} \geq \frac{\cos(\theta)}{\cosh(\Lambda d(\gamma(s), p_1))}.$$

By Lemma 3.6, we can bound the numerator below by  $cs$  for some  $c = c(\lambda, \Lambda, \text{diam}(M))$ . Using Lemma 3.5 and Hypothesis 3.8, we get  $d(\gamma(s), p_1) = f(s) \leq \Lambda s^2/s \leq \Lambda S^2/2$ . So the denominator is bounded above by some constant depending only  $\lambda, \Lambda, \text{diam}(M)$ , which completes the proof.  $\square$

**Lemma 3.11.** *Let  $u \in T_p^1 M$ . Let  $u_1 \in W^{ss}(u)$  be such that the footpoints  $p$  and  $p_1$  of  $u$  and  $u_1$  are distance  $t$  apart. Then  $d(u, u_1) \leq (1 + \Lambda)t$ .*

*Proof.* Let  $\eta$  denote the geodesic joining  $p$  and  $p_1$ . Let  $P_\eta : T_p M \rightarrow T_{p_1} M$  denote parallel transport along  $\eta$ . Let  $v_0 \in T_p^1 M$  be the vector contained in the plane spanned by  $u$  and  $\eta'(0)$  so that  $\langle u, v_0 \rangle = 0$  and  $\langle \eta'(0), v_0 \rangle > 0$ . Let  $V(s)$  denote the parallel vector field along  $\eta(s)$  with initial value  $V(0) = v_0$ . Let  $\theta(s)$  be the angle between  $V(s)$  and  $-\text{grad}B(\eta(s))$ . Then  $\theta_1 = \pi/2 - \theta(t)$  is the angle between  $u_1$  and  $P_\eta u$ . We have

$$\sin(\pi/2 - \theta) = \cos(\theta) = \langle V(t), -\text{grad}B(\eta(t)) \rangle = \int_0^t \langle V(s), \nabla_V \text{grad}B(\eta(s)) \rangle ds.$$

By the same argument as in the proof of Lemma 3.5, this integral is bounded above by  $\Lambda t$ . Hence,  $d(u, u_1) \leq d_M(p, p_1) + d_{T_{p_1} M}(P_\eta u, u_1) \leq t + \Lambda t$ .  $\square$

*Proof of Proposition 3.3.* Consider the hypersurface formed by taking the exponential image of  $\text{grad}B(p)^\perp$ . For  $i = 1, 2$ , let  $x_i$  denote the point on this hypersurface which is on the geodesic determined by  $u_i$ . Let  $v_i \in T_p^1 M$  perpendicular to  $u$  for  $i = 1, 2$  such that  $\exp_p(s_i v_i) = x_i$ , where  $s_i = d(p, x_i)$ . Let  $\gamma_i(s) = \exp_p(s v_i)$ .

Now suppose without loss of generality that  $s_1 \geq s_2$  and let  $s = s_1$ . By Lemma 3.9, we have  $d(u, u_1) \leq Cs$  for some  $C = C(\Lambda, \text{diam}(M))$ . So it suffices to bound  $d(u_1, u_2)/s$  from below by some constant depending only on the desired parameters. Now let  $c = c(\lambda, \Lambda, \text{diam}(M))$  be the constant from the statement of Lemma 3.10, and let  $\beta$  such that  $c - 2(1 + \Lambda)\beta = c/2$ . Let  $s' = \beta s$ . Let  $q_1$  and  $q_2$  denote the orthogonal projections of  $p_1$  and  $p_2$  onto the tangent plane  $\text{grad}B(p)^\perp$ . See Figure 2 below.

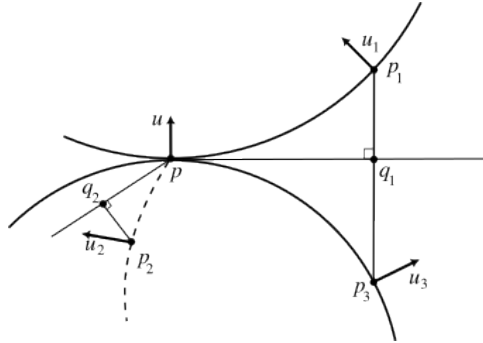


FIGURE 2.

We consider the cases  $d(q_1, q_2) \geq s'$  and  $d(q_1, q_2) \leq s'$  separately. In the first case, we obtain  $d(u_1, u_2) \geq d_M(p_1, p_2) \geq d(q_1, q_2) \geq \beta s_1$ , which shows  $d(u, u_1) \leq \frac{c}{\beta} d(u_1, u_2)$ .

We now consider the case  $d(\gamma_1(s), \gamma_2(s)) \leq s'$ . The geodesic determined by  $p_1$  and  $q_1$  intersects the unstable horosphere  $W^{su}(u)$  at some point we will call  $p_3$ . Let  $u_3 = \text{grad}B_{\eta, p}(p_3)$ . As before, let  $\theta_1$  denote the angle between  $u_1$  and the geodesic determined by the points  $p_1$  and  $q_1$ . Then  $d(u_1, u_3) \geq \theta_1 \geq cs$  by Lemma 3.10.

We now bound  $d(u_2, u_3)$  from above. Lemma 3.11 gives  $d(u_2, u_3) \leq (1 + \Lambda)d(p_2, p_3)$ . By the triangle inequality and Lemma 3.5,  $d(p_2, p_3) \leq s' + \Lambda s^2$ . So if  $s' \leq \beta s$ , we obtain  $d(u_2, u_3) \leq (1 + \Lambda)(\beta + \Lambda s)s$ . We now claim that there is  $\delta_0$  sufficiently small so whenever  $d(u_1, u_2) < \delta_0$ , we also have  $s$  is small enough to guarantee  $\Lambda s \leq \beta$ . To see this, first note that  $d(u_1, u_2) \geq d(p_1, p_2) \geq d(p_1, q_1)$ . Now consider a comparison right triangle in the space of constant curvature  $-\lambda^2$  with hypotenuse equal to  $d(\gamma(s_0), p_1) = f(s)$  and an angle  $\theta$  equal to the angle between  $\text{grad}B$  and  $\gamma'$  at the point  $\gamma(s_0)$ . Let  $x$  denote the length of the side opposite to the angle  $\theta$ . Then, using the fact that triangles in  $M$  are thinner than this comparison triangle, together with [Bea12, Theorem 7.11.2 ii)], Lemma 3.6 and Hypothesis 3.8, gives

$$\sinh(d(p_1, q_1)) \geq \sinh(x) = \sin(\theta(s)) \sinh(f(s)) \geq \sin(\theta(S)) \sinh(cs^2).$$

So if  $d(u_1, u_2) \leq \delta$  we see that  $\sinh(cs^2) \leq C\delta$ , where  $C$  depends only on  $\lambda$  and  $S$ . In other words, there is small enough  $\delta_0 = \delta_0(\lambda, \Lambda, \text{diam}(M))$  to ensure  $s$  is as small as desired, which in this case means small enough for  $\Lambda s \leq \beta = \beta(\lambda, \Lambda, \text{diam}(M))$ . Thus, we now have  $d(u_2, u_3) \leq 2(1 + \Lambda)\beta s$ .

Finally,  $d(u_1, u_2) \geq d(u_1, u_3) - d(u_2, u_3) \geq s_1(c - 2(1 + \Lambda)\beta)$ . By the choice of  $\beta$ , this is bounded below by  $\frac{c}{2}s$ . Hence  $d(u, u_1) \leq Cs_1 \leq \frac{2C}{c}d(u_1, u_2)$ . Reversing the roles of  $u_1$  and  $u_2$  and repeating the same argument gives the analogous upper bound for  $d(u, u_2)$ .  $\square$

Proposition 3.3 allows us to deduce the following refinement of the Anosov Closing Lemma, where we can say the constants involved depend only on concrete geometric information about  $(M, g)$ , namely the diameter and the sectional curvature bounds. Note that now the setting is  $T^1M$  as opposed to the universal cover  $T^1\tilde{M}$ .

**Lemma 3.12.** *There is  $\delta_0 = \delta_0(\lambda, \Lambda, \text{diam}(M))$  sufficiently small so that the following holds. Suppose  $v, \phi^t v \in T^1M$  so that  $d(v, \phi^t v) < \delta \leq \delta_0$ . Then either  $v$  and  $\phi^t v$  are on the same local flow line or there is  $w$  with  $d(v, w) < C\delta$  so that  $w$  is tangent to a closed geodesic of length  $t' \in [t - C\delta, t + C\delta]$ , where  $C$  is a constant depending only on the diameter of  $M$  and the sectional curvature bounds  $\lambda$  and  $\Lambda$ .*

*Proof.* Let  $\delta_0$  be the constant in Proposition 3.3. The proof of the usual Anosov Closing Lemma in [Fra18, Figure 2] (see also [Bow75, 3.6, 3.8]) shows the constant  $C$  depends only on the local product structure constant  $C_0$ . By Proposition 3.3, we know this depends only on  $\lambda, \Lambda, \text{diam}(M)$ .  $\square$

#### 4. COVERING LEMMA

In this section, we prove the following covering lemma, which was one of the key statements we used in the proof of the main theorem.

**Lemma 2.1.** *There is small enough  $\delta_0 = \delta_0(n, \lambda, \Lambda, \text{diam}(M))$  together with a constant  $C = C(n, \Gamma, \lambda, \Lambda, \text{diam}(M))$  so that for any  $\delta < \delta_0$ , there is a covering of  $T^1M$  by at most  $C/\delta^{2n+1}$   $\delta$ -rectangles.*

We start with a preliminary lemma.

**Lemma 4.1.** *Let  $B(v, \delta)$  be a ball of radius  $\delta$  in  $T^1M$  with respect to the Sasaki metric. There is small enough  $\delta_0 = \delta_0(n)$ , depending only on the dimension  $n$ , so that for all  $\delta < \delta_0$  we have  $\text{vol}(B(v, \delta)) \geq c\delta^{2n+1}$  for some constant  $c = c(n)$ .*

*Proof.* First we claim  $B(v, \delta) \supset B_M(p, \delta/2) \times B_{S^{n-1}}(v, \delta/2)$ , where  $B_M(p, \delta/2)$  is a ball of radius  $\delta/2$  in  $M$  and  $B_{S^{n-1}}(v, \delta/2)$  is a ball of radius  $\delta/2$  in the unit tangent sphere  $T_p^1 M$ . This follows immediately from (3.1). Since  $M$  is negatively curved, Theorem 3.101 ii) in [GHL90] implies  $\text{vol} B_M(p, \delta/2) \geq \beta_n \delta^n / 2^n$ , where  $\beta_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . By Theorem 3.98 in [GHL90], we have  $\text{vol} B_{S^{n-1}}(v, \delta/2) = \frac{\beta_{n-1} \delta^{n-1}}{2^{n-1}} (1 - \frac{n-1}{6(n+1)} \delta^2 + o(\delta^4))$ . Then for  $\delta$  less than some small enough  $\delta_0$ , we can write

$$B_{S^{n-1}}(v, \delta/2) \geq \frac{\beta_{n-1} \delta^{n-1}}{2^{n-1}} \left( 1 - 2 \frac{n-1}{6(n+1)} \delta^2 \right) \geq c \delta^{n+1},$$

for some  $c = c(n)$ . The quantity  $\delta_0$  depends only on the coefficients of the Taylor expansion of  $\text{vol} B_{S^{n-1}}(v, \delta/2)$ , which depend only on the geometry of  $S^{n-1}$ . So we can say  $\delta_0$  depends only on  $n$ . Therefore, the volume of the Sasaki ball  $B(v, \delta)$  is bounded below by  $c \delta^{2n+1}$  for some other constant  $c = c(n)$  depending only on  $n$ .  $\square$

*Proof of Lemma 2.1.* Let  $\delta_0$  and  $C$  as in Proposition 3.3. Let  $c = 1/C$  and let  $\delta < \delta_0/2c$ . Let  $v_1, \dots, v_m$  be a maximal  $c\delta$ -separated set in  $T^1 M$  with respect to the Sasaki metric. We claim that the balls  $B(v_1, c\delta), \dots, B(v_m, c\delta)$  cover  $T^1 M$ . If not, there is some  $v$  such that  $d(v, v_i) \geq c\delta$  for all  $i$ . This contradicts the fact that  $v_1, \dots, v_m$  was chosen to be a *maximal*  $c\delta$ -separated set.

This implies that the rectangles  $R(v_1, \delta) \dots R(v_m, \delta)$  cover  $T^1 M$  as well. Indeed, let  $w \in B(v, c\delta)$ . Then by Lemma 3.1 there is a time  $\sigma = \sigma(v, w) < c\delta$  and a point  $[v, w] \in T^1 M$  so that  $[v, w] = W^{ss}(v) \cap W^{su}(\phi^\sigma w)$ . Thus  $d(v, \phi^\sigma w) \leq \delta_0$  and Proposition 3.3 implies  $d_{ss}(v, [v, w]), d_{su}([v, w], \phi^\sigma w) < Cc\delta = \delta$  as desired.

Now we estimate  $m$ . Since  $v_1, \dots, v_m$  is  $c\delta$ -separated, it follows that for  $i \neq j$  we have  $B(v_i, c\delta/2) \cap B(v_j, c\delta/2) = \emptyset$ . Hence

$$m \inf_i \text{vol}(B(v_i, c\delta/2)) \leq \text{vol}(T^1 M) = \text{vol}(S^{n-1}) \text{vol}(M).$$

By [Gro82, 0.3 Thurston's Theorem], we have  $\text{vol}(M)$  is bounded above by a constant depending only on  $n$ ,  $\Gamma$  and the upper sectional curvature bound  $-\lambda^2$ . This, together with Lemma 4.1, gives  $m \leq C/\delta^{2n+1}$  for some constant  $C = C(n, \Gamma, \lambda, \Lambda, \text{diam}(M))$ .  $\square$

## 5. PSEUDO-ISOMETRY ESTIMATES

Recall  $(M, g)$  and  $(N, g_0)$  are compact negatively curved manifolds with a given isomorphism between their fundamental groups. Since  $M$  and  $N$  are  $K(\pi, 1)$  spaces, there is a homotopy equivalence  $M \rightarrow N$  inducing this isomorphism; moreover, we can assume it is of  $C^1$  regularity, since every continuous map is homotopic to a differentiable one. Now lift this  $C^1$  homotopy equivalence to a map  $f : \tilde{M} \rightarrow \tilde{N}$ , which is equivariant with respect to the actions of  $\Gamma \cong \pi_1(M) \cong \pi_1(N)$  on  $\tilde{M}$  and  $\tilde{N}$ . It is well-known that  $f$  is a *pseudo-isometry* (see, for instance, [BP92, Proposition C.1.2]), meaning there exist constants  $A$  and  $B$  so that for all  $x_1, x_2 \in \tilde{M}$  we have

$$A^{-1} d_g(x_1, x_2) - B \leq d_{g_0}(f(x_1), f(x_2)) \leq A d_g(x_1, x_2). \quad (5.1)$$

This is a special case of the perhaps more widely used concept of a *quasi-isometry*, which is when there is also an additive constant on the righthand side of (5.1). In our case, however, the absence of this additive constant is crucial for our remaining arguments.

In this section, we show the constants  $A$  and  $B$  depend only on the fundamental group  $\Gamma$ , the injectivity radius of  $(M, g)$  and the sectional curvature bounds for  $(M, g)$  and  $(N, g_0)$  (Proposition 5.1 below).

**Proposition 5.1.** *Suppose  $f : \tilde{M} \rightarrow \tilde{N}$  is a  $\Gamma$ -equivariant  $C^1$  map as above. Now let  $g$  and  $g_0$  be Riemannian metrics on  $M$  and  $N$  with sectional curvatures contained in the interval  $[-\Lambda^2, -\lambda^2]$ , and suppose the injectivity radii of  $(M, g)$  and  $(N, g_0)$  are bounded below by  $i_M$  and  $i_N$ , respectively. Then there are constants  $A$  and  $B$  depending only  $n, \Gamma, a, b, i_M$  so that (5.1) holds for all  $x_1, x_2 \in \tilde{M}$ .*

We start by finding a uniform Lipschitz bound for  $f$ , in other words, proving the second inequality in the above proposition (Corollary 5.4 below). A key tool we use is *Gromov compactness*. Let  $\mathcal{M}(D_0, v_0, \Lambda)$  be the space of all Riemannian metrics on  $M$  with diameter at most  $D_0$ , volume at least  $v_0$  and absolute sectional curvatures at most  $\Lambda^2$ . This space satisfies certain pre-compactness properties. We will use a refinement of Gromov's theorem due to Greene–Wu [GW88], namely that any sequence  $(M, g_n) \in \mathcal{M}(D_0, v_0, \Lambda)$  has a subsequence  $(M, g_{n_k})$  converging in the following sense: there is a Riemannian metric  $g_\infty$  on  $M$  such that in local coordinates we have  $g_{n_k}^{ij} \rightarrow g_\infty^{ij}$  in the  $C^{1,\alpha}$  norm and the limiting  $g_\infty^{ij}$  have regularity  $C^{1,\alpha}$ , for some  $0 < \alpha < 1$ .

**Lemma 5.2.** *Suppose  $f : M \rightarrow N$  is a  $C^1$  map. Suppose  $g$  and  $g_0$  are Riemannian metrics on  $M$  and  $N$  with  $(M, g) \in \mathcal{M}(D_0, v_0, \Lambda)$  and  $(N, g_0) \in \mathcal{M}(D'_0, v'_0, \Lambda')$ . Then there exists a constant  $A = A(f, D_0, D'_0, v_0, v'_0, \Lambda, \Lambda')$  so that*

$$\|df\|_{g, g_0} := \sup_{v \in TM} \frac{\|df_p(v)\|_{g_0}}{\|v\|_g} \leq A.$$

*Proof.* If  $g^n \rightarrow g$  in the  $C^{1,\alpha}$  topology, then, in particular,  $\|v\|_{g^n} \rightarrow \|v\|_{g_\infty}$  uniformly on compact sets. This means if  $g_n \rightarrow g$  and  $g_0^n \rightarrow g_0$ , then  $df_{g^n, g_0^n} \rightarrow df_{g, g_0}$ .

Now suppose for contradiction that the statement of the lemma is false. This means there are sequences  $g^n \in \mathcal{M}(D_0, v_0, \Lambda)$  and  $g_0^n \in \mathcal{M}(D'_0, v'_0, \Lambda')$  so that  $\|df\|_{g^n, g_0^n} \rightarrow \infty$ . After passing to convergent subsequences, we have  $\|df\|_{g^n, g_0^n} \rightarrow \|df\|_{g^\infty, g_0^\infty}$  for some  $C^{1,\alpha}$  Riemannian metrics  $g^\infty$  and  $g_0^\infty$ . Since  $f$  is  $C^1$  and the unit tangent bundle of  $M$  is compact, the derivative  $df(v)$  is uniformly bounded in  $v$ . In other words,  $\|df\|_{g^\infty, g_0^\infty} < \infty$ , which is a contradiction. So the statement of the lemma must be true.  $\square$

**Lemma 5.3.** *Suppose that  $(M, g)$  has sectional curvatures in the interval  $[-\Lambda^2, -\lambda^2]$  and injectivity radius at least  $i_M$ . Then there are constants  $v_0 = v_0(i_M, n)$  and  $D_0 = D_0(n, \lambda, \Gamma, i_M)$  so that  $(M, g) \in \mathcal{M}(D_0, v_0, \Lambda)$ .*

*Proof.* First, the desired absolute sectional curvature bound holds by assumption. Second, by Gromov's systolic inequality, we know  $\text{vol}(M, g) \geq v_0$ , where  $v_0$  is a constant depending only on  $n$  and  $i_M$  [Gro83, 0.1.A]. It now remains to bound the diameter from above. By [Gro82, Section 0.3], the volume is bounded above by a constant  $V_0$  depending only on  $n, \Gamma$ , and  $\lambda$ . A standard argument (see, for instance, the proof of Lemma 3.9 in [But22]) shows the diameter is bounded above by  $D_0 = D_0(i_M, V_0, \Lambda)$ .  $\square$

**Corollary 5.4.** *Let  $(M, g)$  and  $(N, g_0)$  be closed Riemannian manifolds of dimension  $n$  with sectional curvatures in the interval  $[-\Lambda^2, -\lambda^2]$ , and assume there is an isomorphism between their fundamental groups. Then there is an  $A$ -Lipschitz map  $f : M \rightarrow N$  inducing this isomorphism, where  $A$  depends only on  $n, \Gamma, \lambda, \Lambda$ , and the injectivity radii  $i_M$  and  $i_N$ .*

Given that  $f$  is  $A$ -Lipschitz, we now show the second estimate in the definition of pseudo-isometry. We follow the approach of [BP92, Proposition C.12], but we need to check the constants depend only on the desired parameters  $n, \Gamma, \lambda, \Lambda$  and  $i_N$ .

First, let  $h : \tilde{N} \rightarrow \tilde{M}$  be a  $C^1$  homotopy inverse of  $f$ . By Corollary 5.4,  $h$  is also  $A$ -Lipschitz. Now consider the following fundamental domain  $D_M$  for the action of  $\Gamma$  on  $\tilde{M}$  (see [BP92, Proposition C.1.3]). Fix  $p \in \tilde{M}$ . Let

$$D_M = \{x \in \tilde{M} \mid d(x, p) \leq d(x, \gamma.p) \forall \gamma \in \Gamma\}. \quad (5.2)$$

**Claim 5.5.** *The diameter of  $D_M$  satisfies  $\text{diam}(D_M) \leq 2 \text{diam}(M)$ .*

*Proof.* Let  $x \in \tilde{M}$  so that  $d(p, x) > \text{diam}(M)$ . This means there is some  $\gamma \in \Gamma$  so that  $d(x, \gamma.p) < d(x, p)$ . In other words, any geodesic in  $\tilde{M}$  starting at  $p$  stays in  $D_M$  for a time of at most  $\text{diam}(M)$ . So if  $x_1, x_2 \in D_M$  so that  $d(x_1, x_2) = \text{diam}(D_M)$ , then  $d(x_1, x_2) \leq d(x_1, p) + d(x_2, p) \leq 2 \text{diam}(M)$ , which proves the claim.  $\square$

**Claim 5.6.** *For all  $x \in \tilde{M}$ , we have  $d(h \circ f(x), x) \leq 2(1 + A^2)\text{diam}(M)$ .*

*Proof.* Since  $h$  and  $f$  are both continuous and  $\Gamma$ -equivariant, so is  $h \circ f$ , and thus it suffices to check the statement for  $x$  in a compact fundamental domain  $D_M$ . Since  $f$  and  $h$  are  $A$ -Lipschitz, it follows that the function  $x \mapsto d(h \circ f(x), x)$  is  $(1 + A^2)$ -Lipschitz:

$$|d(h \circ f(x), x) - d(h \circ f(y), y)| \leq d(h \circ f(x), h \circ f(y)) + d(x, y) \leq (1 + A^2)d(x, y).$$

Noting  $d(x, y) \leq D_M \leq 2 \text{diam}(M)$  completes the proof.  $\square$

*Proof of Proposition 5.1.* We can now use the argument in [BP92] verbatim. By the previous claim, we obtain

$$d(h(f(x_1)), h(f(x_2))) \geq d(x_1, x_2) - 4(1 + A^2)\text{diam}(M).$$

Then, the Lipschitz bounds for  $f$  and  $h$  give

$$d(f(x_1), f(x_2)) \geq A^{-1}d(h \circ f(x_1), h \circ f(x_2)) \geq A^{-1}(d(x_1, x_2) - 4(1 + A^2)\text{diam}(M)),$$

which completes the proof.  $\square$

## 6. HÖLDER ESTIMATE

In this section, we show the following, which was one of the main ingredients in the proof of Theorem 1.2.

**Proposition 2.4.** *Suppose  $(M, g)$  and  $(N, g_0)$  are a pair of homotopy-equivalent compact Riemannian manifolds with sectional curvatures contained in the interval  $[-\Lambda^2, -\lambda^2]$ . Let  $i_M$  and  $i_N$  denote their respective injectivity radii. Then there exists an orbit equivalence of geodesic flows  $\mathcal{F} : T^1M \rightarrow T^1N$  which is  $C^1$  along orbits and transversally Hölder continuous. More precisely, there is small enough  $\delta_0 = \delta_0(\lambda, \Lambda, i_M)$  together with constants  $C$  and  $A$ , depending only on  $n, \Gamma, \lambda, \Lambda, i_M, i_N$ , so that the following hold:*

- (1)  $d(\mathcal{F}(v), \mathcal{F}(\phi^t v)) \leq At$  for all  $v \in T^1\tilde{M}$  and  $t \in \mathbb{R}$ ,
- (2)  $d(\mathcal{F}(v), \mathcal{F}(w)) \leq Cd(v, w)^{A^{-1}\lambda/\Lambda}$  for all  $v, w \in T^1\tilde{M}$  with  $d(v, w) < \delta_0$ .

We will take  $\mathcal{F}$  to be the map in [Gro00], whose construction we now recall. As in Section 5, consider a  $C^1$  homotopy equivalence  $M \rightarrow N$ , which we lift to a  $\Gamma$ -equivariant map



$f : \tilde{M} \rightarrow \tilde{N}$ . By Proposition 5.1, there are constants  $A$  and  $B$ , depending only on  $n, \lambda, \Lambda, \Gamma, i_M$ , so that

$$A^{-1}d(p, q) - B \leq d(f(p), f(q)) \leq Ad(p, q). \quad (6.1)$$

Let  $\eta$  be a bi-infinite geodesic in  $\tilde{M}$  and let  $\zeta = \bar{f}(\eta)$  be the corresponding geodesic in  $\tilde{N}$ , where  $\bar{f} : \partial^2 \tilde{M} \rightarrow \partial^2 \tilde{N}$  is obtained from extending the quasi-isometry  $f$  to a map  $\partial \tilde{M} \rightarrow \partial \tilde{N}$ .

Let  $P_\zeta : \tilde{N} \rightarrow \zeta$  denote the orthogonal projection. Note this projection is  $\Gamma$ -equivariant, i.e.,  $\gamma P_\zeta(x) = P_\zeta(\gamma x)$ . If  $(p, v) \in T^1 \tilde{M}$  is tangent to  $\eta$ , define  $\mathcal{F}_0(p, v)$  to be the tangent vector to  $\zeta$  at the point  $P_\zeta \circ \bar{f}(p)$ . Thus  $\mathcal{F}_0 : T^1 \tilde{M} \rightarrow T^1 \tilde{N}$  is a  $\Gamma$ -equivariant map which sends geodesics to geodesics. As such, we can define a cocycle  $b(t, v)$  to be the time which satisfies

$$\mathcal{F}_0(\phi^t v) = \psi^{b(t, v)} \mathcal{F}_0(v). \quad (6.2)$$

It is possible for a fiber of the orthogonal projection map to intersect the quasi-geodesic  $\bar{f}(\eta)$  in more than one point; thus,  $\mathcal{F}_0$  is not necessarily injective. In order to obtain an injective orbit equivalence, we follow the method in [Gro00] and average the function  $b(t, v)$  along geodesics. Let

$$a_l(t, v) = \frac{1}{l} \int_t^{t+l} b(s, v) ds. \quad (6.3)$$

There is a large enough  $l$ , depending only on the quasi-isometry constants  $A$  and  $B$  and the upper sectional curvature bound  $-\lambda^2$ , so that  $t \mapsto a_l(t, v)$  is injective for all  $v$ . (The idea of the proof is to compute the derivative  $\frac{d}{dt} a_l(t, v)$  and choose large enough  $l$  so that it is always nonzero. For details, see [But22, Lemma 2.24].) An injective orbit equivalence  $\mathcal{F}$  is then given by

$$\mathcal{F}(v) = \psi^{a_l(0, v)} \mathcal{F}_0(v)$$

for  $a_l$  injective as above. (See [But22, Proposition 2.25] for a proof that  $\mathcal{F}$  is injective.)

We now proceed to find a Hölder estimate for  $\mathcal{F}$ . Most of the work is finding estimates for the map  $\mathcal{F}_0$  from (6.2) (Proposition 6.6).

We will repeatedly use the following fact about quasi-geodesics remaining bounded distance away from their corresponding geodesics:

**Lemma 6.1** (Theorem III.H.1.7 of [BH13]). *Let  $f$  be the quasi-isometry from Section 5. Let  $c(t)$  be any geodesic in  $\tilde{M}$  and let  $\eta$  be its corresponding geodesic in  $\tilde{N}$  obtained from the boundary map  $\bar{f} : \partial \tilde{M} \rightarrow \partial \tilde{N}$ . Then there is a constant  $R$ , depending only on the pseudo-isometry constants  $A$  and  $B$  of  $f$  and the upper sectional curvature bound  $-\lambda^2$  for  $N$ , so that  $d(f(c(t)), P_\eta(f(c(t)))) \leq R$  for any  $t \in \mathbb{R}$ .*

**Lemma 6.2.** *Let  $b(t, v)$  as in (6.2). Let  $A, B$  as in (6.1). Then  $b(t, v)$  satisfies*

$$At - B' \leq b(t, v) \leq At.$$

for all  $t$ , where  $B'$  is a constant depending only on  $\lambda, A, B$ .

*Proof.* Recall  $b(t, v) = d(P_\eta f(p), P_\eta f(q))$ , which is bounded above by  $d(f(p), f(q))$  because orthogonal projection is a contraction in negative curvature. This quantity is in turn bounded above by  $At$ , using the Lipschitz bound for  $f$  in (6.1).

Next, let  $R$  be the constant in Lemma 6.1. Then  $d(f(p), P_\eta(f(p))) \leq R$ , which implies  $b(t, v) \geq d(f(p), f(q)) - 2R$ . The desired estimate then follows from the lower bound for  $d(f(p), f(q))$  in (6.1).  $\square$

**Lemma 6.3.** *There is small enough  $\delta_0 = \delta_0(\Lambda)$  so that for any  $\delta \leq \delta_0$  the following holds. Fix  $v \in T^1\tilde{M}$  and let  $x \in \tilde{M}$  be a point such that the orthogonal projection  $P_v(x)$  of  $x$  onto the bi-infinite geodesic determined by  $v$  is the footpoint of  $v$ . Let  $w \in W^{su}(v)$  and suppose further that  $d_{su}(v, w) < \delta$ . Then there is a constant  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$  so that  $d(P_v(x), P_w(x)) < C\delta$ .*

*Proof.* Let  $p$  and  $q$  denote the footpoints of  $v$  and  $w$ , respectively. Let  $u \in T_p^1M$  be the vector tangent to the curve in the horosphere connecting  $p$  and  $q$ . Let  $\gamma(s) = \exp_p(su)$ . Let  $s_0$  be such that  $\gamma(s_0)$  intersects the geodesic determined by  $w$ . We claim there are positive constants  $\delta_0 = \delta_0(\Lambda)$  and  $C = C(\Lambda)$  so that if  $d(v, w) \leq \delta \leq \delta_0$  then  $s_0 \leq C\delta$ . By [HIH77, Proposition 4.7] we know  $\tanh(\Lambda s_0) \leq Cd_{su}(v, w) \leq C\delta$ , where  $C$  is some constant depending only on  $\Lambda$ , which proves the claim.

Now let  $\theta$  denote the angle between the geodesic segment  $[x, \gamma(s_0)]$  and the geodesic determined by  $w$ . We start by showing  $\theta$  is close to  $\pi/2$ . In the case where  $x$  and  $p$  coincide, the above angle  $\theta$  is the same as the angle  $\theta$  in Lemma 3.5. Thus,  $\cos \theta \leq \Lambda s_0$ .

Otherwise, let  $t_0 = d(x, p) \neq 0$ . We consider two further cases:  $d(x, \gamma(s_0)) \leq \delta$  and  $d(x, \gamma(s_0)) \geq \delta$ . For the proof in the first case, we start by noting that

$$\begin{aligned} d(p, P_w(x)) &\leq d(p, \gamma(s_0)) + d(P_w(x), \gamma(s_0)) \\ &\leq d(p, q) + d(q, \gamma(s_0)) + d(P_w(x), \gamma(s_0)) + d(x, \gamma(s_0)). \end{aligned}$$

Since  $d(v, w) < \delta$  (by assumption), so is  $d(p, q)$ . By Lemma 3.5,  $d(q, \gamma(s_0)) \leq \Lambda s_0^2 \leq C\delta^2$ . Finally, note  $d(x, P_w(x)) \leq d(x, \gamma(s_0))$ . So applying the hypothesis  $d(x, \gamma(s_0)) \leq \delta$  completes the proof in this case.

Now we consider the case  $d(x, \gamma(s_0)) \geq \delta$ . Let  $v_0 \in T_x^1M$  such that  $\exp_x(t_0 v_0) = p$ . For  $0 < s \leq s_0$ , let  $v(s) \in T_{x_0}M$  such that  $\exp_x(t_0 v(s)) = \gamma(s)$ . Then  $X(s) := \frac{d}{dt}|_{t=t_0} \exp_x(tv(s))$  is a vector field along  $\gamma(s)$ . The hypothesis  $d(x, \gamma(s_0)) \geq \delta$  allows us to bound

$$\frac{\|X(s)\|}{\|X(s_0)\|} = \frac{d(x, \gamma(s))}{d(x, \gamma(s_0))} \leq 1 + \frac{s_0}{d(x, \gamma(s_0))} \leq 1 + \frac{s_0}{\delta} \leq 1 + C, \quad (6.4)$$

where  $C$  is a constant depending only on  $\Lambda$ .

We now claim there is a constant  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$  so that

$$\cos \theta = \frac{\langle \text{grad} B_\xi(\gamma(s_0)), X(s_0) \rangle}{\|X(s_0)\|} \leq C s_0. \quad (6.5)$$

Since  $\langle \text{grad} B_\xi(\gamma(0)), X(0) \rangle = 0$ , the fundamental theorem of calculus gives

$$\langle \text{grad} B_\xi(\gamma(s_0)), X(s_0) \rangle = \int_0^{s_0} \frac{d}{ds} \langle \text{grad} B_\xi(\gamma(s)), X(s) \rangle ds.$$

So the desired bound for  $\cos(\theta)$  follows from bounding the integrand from above by  $C\|X(s_0)\|$  for all  $s \in [0, s_0]$ . In light of (6.4), it suffices to find an upper bound of the form  $C\|X(s)\|$ . To this end, we rewrite integrand using the product rule:

$$\frac{d}{ds} \langle \text{grad} B(\gamma(s)), X(s) \rangle = \langle \nabla_{\gamma'} \text{grad} B(\gamma(s)), X(s) \rangle + \langle \text{grad} B(\gamma(s)), \nabla_{\gamma'} X(s) \rangle. \quad (6.6)$$

The first term on the righthand side is bounded above by

$$\|X(s)\| |\langle \nabla \text{grad} B_{\gamma'}(\gamma(s)), u \rangle| = \|X(s)\| \text{Hess} B_\xi(\gamma'(s), u)$$

for some unit vector  $u$ . Next, using that the Hessian is symmetric bilinear form, together with Lemma 3.5, we have

$$\text{Hess}B_\xi(\gamma'(s), u) \leq \frac{1}{4}\text{Hess}B_\xi(\gamma'(s) + u, \gamma'(s) + u) \leq \frac{\Lambda}{4}\|\gamma'(s) + u\| \leq \frac{\Lambda}{2}.$$

Now we consider the second term in (6.6). First note that  $\nabla_{\gamma'}X(s) = J'_s(t_0)$ , where  $J_s(t)$  is the Jacobi field along the geodesic  $\eta_s(t) = \exp_x(tv(s))$  with initial conditions  $J_s(0) = 0$  and  $J'_s(0) = v(s)$ . In order to bound  $\|J'_s(t_0)\|$ , we let  $e_1(t) = \eta'(t), e_2(t), \dots, e_n(t)$  be a parallel orthonormal frame along  $\eta(t)$ . Let  $f_1(t), \dots, f_n(t)$  such that  $J_s(t) = \sum_{i=1}^n f_i(t)e_i(t)$ . The fact that  $J_s$  satisfies the Jacobi equation means  $f''_1(t) = 0$ , so  $f_1(t) = \langle v(s), \eta'(0) \rangle t$  and  $|f'_1(t)| \leq \|v(s)\| = \|X(s)\|$ . Now let  $J_s^\perp$  denote the component of  $J_s$  which is perpendicular to  $\eta_s$ . By [Bal95, Proposition IV.2.5], we have

$$\|(J_s^\perp)'(t_0)\| \leq \cosh(\Lambda t_0)\|(J_s^\perp)'(0)\| \leq \cosh(\Lambda R)\|X(s)\|,$$

where  $R$  is the constant in Lemma 6.1. This completes the verification of (6.5).

Now let  $q'$  be the orthogonal projection of  $x$  onto the geodesic determined by  $w$ . We use our bound for  $\cos\theta$  to show  $d(p, q')$  is small. Consider the geodesic triangle with vertices  $x, q'$  and  $\gamma(s_0)$ . The angle at  $q'$  is  $\pi/2$  by definition of orthogonal projection, and we have just shown the angle  $\theta$  at  $\gamma(s_0)$  satisfies  $\cos\theta \leq Cs_0$ , where  $s_0 < C\delta$ . Then by [Bea12, Theorem 7.11.2 iii)]  $\tanh(d(q', \gamma(s_0))) \leq C\delta \tanh(d(x, q')) \leq C\delta$ , where  $C$  is the constant in (6.5). Thus, for  $\delta_0$  sufficiently small in terms of  $C$ , we see that  $d(q', \gamma(s_0)) \leq 2C\delta$  whenever  $\delta < \delta_0$ . Now recall from the first paragraph that  $d(p, \gamma(s_0)) = s_0 \leq C\delta$ . Noting that  $d(p, q') \leq d(p, \gamma(s_0)) + d(q', \gamma(s_0))$  completes the proof.  $\square$

**Proposition 6.4.** *Let  $\delta_0 = \delta_0(\Lambda)$  be as small as in the previous lemma. Suppose  $w \in W^{su}(v)$  and  $d_{su}(v, w) < \delta_0$ . Then there is a constant  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$  so that  $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd_{su}(v, w)^{A\lambda/\Lambda}$ , where  $A$  is the constant in Proposition 5.1. The analogous statement holds if  $w \in W^{ss}(v)$  instead.*

*Proof.* Let  $p$  and  $q$  denote the footpoints of  $v$  and  $w$ , respectively. By definition,  $\mathcal{F}_0(v) = P_{\eta_1}(f(p))$  and  $\mathcal{F}_0(w) = P_{\eta_2}(f(q))$  for the appropriate bi-infinite geodesics  $\eta_1$  and  $\eta_2$  in  $\tilde{N}$ . By the triangle inequality,

$$d(P_{\eta_1}(f(p)), P_{\eta_2}(f(q))) \leq d(P_{\eta_1}(f(p)), P_{\eta_1}(f(q))) + d(P_{\eta_1}(f(q)), P_{\eta_2}(f(q))). \quad (6.7)$$

We start by estimating the first term. Let  $d_{su}(v, w) = \delta$ . Then  $d(p, q) < \delta$ . By (6.1), we have  $d(f(p), f(q)) < A\delta$ . Since orthogonal projection is a contraction in negative curvature, the second term is bounded above by  $d(f(p), f(q)) \leq Ad(p, q)$ .

Thus it remains to bound  $d(P_{\eta_1}(f(q)), P_{\eta_2}(f(q)))$ , which we do by applying Lemma 6.3. Since  $\mathcal{F}(v)$  and  $\mathcal{F}(w)$  are on the same weak unstable leaf, there is  $w'$  on the orbit of  $w$  so that  $\mathcal{F}(v)$  and  $\mathcal{F}(w')$  are on the same strong unstable leaf. In light of Lemma 6.3, it suffices to find a Hölder estimate for  $d_{su}(\mathcal{F}(v), \mathcal{F}(w'))$ .

Again, let  $\delta = d_{su}(v, w)$  for simplicity. Since the unstable distance exponentially expands under the geodesic flow, there is some positive time  $t$  so that  $d_{su}(\phi^t v, \phi^t w) = 1$ . More precisely, [HIH77, Proposition 4.1] implies  $\Lambda t \geq \log(1/\delta)$ .

Next, note that  $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w')) \leq d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq 2R + A$ , where  $R$  is as in Lemma 6.1 and  $A$  is the Lipschitz constant for  $f$ . Indeed, since  $f$  is  $A$ -Lipschitz, we have  $d(f(\phi^t v), f(\phi^t w)) \leq A$ , and  $d(f(\phi^t v), P_{\eta_1}f(\phi^t v)) \leq R$ . By [HIH77, Theorem 4.6], we also have the bound  $d_{su}(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w')) \leq \frac{2}{\Lambda} \sinh(\Lambda(2R + A)/2)$ .

By [HIH77, Proposition 4.1], we have the following estimate for how the unstable distance gets contracted under the geodesic flow:

$$d_{su}(\mathcal{F}_0(v), \mathcal{F}_0(w')) \leq e^{-\lambda b(t,v)} d_{su}(\psi^{b(t,v)} \mathcal{F}_0(v), \psi^{b(t,v)} \mathcal{F}_0(w')).$$

Now recall  $b(t, v) \geq A^{-1}t - B'$  from Lemma 6.2. This, together with the previous paragraph, gives

$$d_{su}(\mathcal{F}(v), \mathcal{F}(w')) \leq e^{-\lambda(A^{-1}t - B')} \frac{2}{\Lambda} \sinh(\Lambda(2R + A)/2) = Ce^{-\lambda A^{-1}t}$$

for some constant  $C = C(\lambda, \Lambda, A, B)$ . Finally, we use  $t \geq \frac{\log(1/\delta)}{\Lambda}$  to obtain  $d_{su}(\mathcal{F}_0(v), \mathcal{F}_0(w')) \leq C\delta^{A^{-1}\lambda/\Lambda}$  for some other constant  $C = C(\lambda, A, B)$ . By Lemma 6.3, the second term in (6.7) is thus bounded above by  $C\delta^{A^{-1}\lambda/\Lambda}$  for some other constant  $C = C(\lambda, \Lambda, A, B)$ , which completes the proof.  $\square$

**Lemma 6.5.** *There is small enough  $\delta_0$ , depending only on the curvature bounds  $\lambda$  and  $\Lambda$ , so that if  $w \in W^{ss}(v)$  and  $d(v, w) < \delta_0$ , then*

$$c_1 d(v, w) \leq d_{ss}(v, w) \leq c_2 d(v, w),$$

where  $c_1$  and  $c_2$  are constants depending only on  $\lambda$  and  $\Lambda$ . The analogous statement holds for  $d_{su}$ .

*Proof.* By [HIH77, Theorem 4.6], we have  $d_{ss}(v, w) \leq \frac{\Lambda}{2} \sinh(2/\Lambda d(p, q))$ . Thus, if  $d(p, q)$  is small enough (depending on  $\Lambda$ ), we have  $h(p, q) \leq \frac{4}{\Lambda} d(p, q) \leq \frac{4}{\Lambda} d(v, w)$ .

By Lemma 3.11,  $d(v, w) \leq (1 + \Lambda)d(p, q)$ . By the other estimate in [HIH77, Theorem 4.6], there is a constant  $C$ , depending only on  $\lambda$ , so that  $d(p, q) \leq Ch(p, q)$  for all  $p, q$  with  $d(p, q)$  sufficiently small in terms of  $\lambda$ .  $\square$

**Proposition 6.6.** *There exists small enough  $\delta_0$ , depending only on  $\lambda$ ,  $\Lambda$ ,  $\text{diam}(M)$ , so that for any  $v, w \in T^1\tilde{M}$  satisfying  $d(v, w) < \delta_0$  we have  $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v, w)^{A^{-1}\lambda/\Lambda}$  for some constant  $C = C(n, \Gamma, \lambda, \Lambda, i_M)$ .*

*Proof.* By Lemma 3.1, we know that for any  $v, w \in T^1M$  with  $d(v, w) = \delta$ , there is a time  $\sigma = \sigma(v, w) \in [-\delta, \delta]$  and a point  $[v, w] \in T^1\tilde{M}$  so that

$$[v, w] = W^{ss}(v) \cap W^{su}(\phi^\sigma w).$$

Let  $\alpha = A^{-1}\lambda/\Lambda$  be the exponent from Proposition 6.4. Applying Proposition 6.4, followed by Lemma 6.5 and Proposition 3.3, and finally Lemma 3.1, we have

$$d(\mathcal{F}_0(v), \mathcal{F}_0([v, w])) \leq C' d_{ss}(v, [v, w])^\alpha \leq Cd(v, \phi^\sigma w)^\alpha \leq C(2d(v, w))^\alpha$$

for some constants  $C$  and  $C'$  depending only on  $n, \Gamma, \lambda, \Lambda, i_M, \text{diam}(M)$ . By a similar argument,

$$d(\mathcal{F}_0([v, w], \phi^\sigma w) \leq Cd(v, w)^\alpha.$$

Finally, as in the beginning of the proof of Proposition 6.4, we have

$$d(\mathcal{F}(w), \mathcal{F}(\phi^\sigma w)) \leq A\delta.$$

Now,  $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v, w)^\alpha$  follows from the triangle inequality.  $\square$

**Lemma 6.7.** *There is a constant  $C = C(\lambda, \Lambda, t)$  so that  $d(\phi^t v, \phi^t w) < Cd(v, w)$  for all  $d(v, w) \leq \delta_0$ , where  $\delta_0$  depends only on  $\lambda, \Lambda, \text{diam}(M)$ .*

*Proof.* As before, consider  $[v, w] = W^{ss}(v) \cap W^{su}(\phi^{\sigma(v,w)}w)$ . The distance between  $w$  and  $\phi^{\sigma(v,w)}$  remains constant under application of  $\phi^t$ , and since  $v$  and  $[v, w]$  are on the same stable leaf, their distance contracts under application of  $\phi^t$ . Finally, since  $[v, w]$  and  $\phi^{\sigma(v,w)}w$  are on the same strong unstable leaf, [HIH77, Proposition 4.1], Lemma 6.5 and Proposition 3.3 imply

$$d(\phi^t[v, w], \phi^t \phi^{\sigma(v,w)}w) \leq e^{\Lambda t} d_{su}([v, w], \phi^{\sigma(v,w)}w) \leq e^{\Lambda t} C d(v, w)$$

for some constant  $C$  depending only on  $\lambda, \Lambda, \text{diam}(M)$ .  $\square$

**Lemma 6.8.** *Let  $C$  denote the constant in Proposition 6.6, and let  $\alpha = A^{-1}\lambda/\Lambda$  denote the Hölder exponent. Then there is a constant  $C_1 = C_1(C, t)$  so that*

$$|b(t, v) - b(t, w)| \leq C_1 d(v, w)^\alpha.$$

*Proof.* By Proposition 6.6, we have  $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq C d(v, w)^\alpha$  and  $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq C d(\phi^t v, \phi^t w)^\alpha$ . Applying Lemma 6.7 shows  $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq C_1 d(v, w)^\alpha$ , where  $C_1$  depends on  $C$  and  $t$ . The desired result now follows from Lemma 3.2.  $\square$

*Proof of Proposition 2.4.* We want to find a Hölder estimate for  $\mathcal{F}_l(v) = \psi^{a_l(0,v)}\mathcal{F}_0(v)$ , where  $a_l(0, v) = \frac{1}{l} \int_0^l b(t, v) dt$ . By the triangle inequality,

$$d(\mathcal{F}_l(v), \mathcal{F}_l(w)) \leq d(\psi^{a_l(0,v)}\mathcal{F}_0(v), \psi^{a_l(0,v)}\mathcal{F}_0(w)) + d(\psi^{a_l(0,v)}\mathcal{F}_0(w), \psi^{a_l(0,w)}\mathcal{F}_0(w)).$$

To bound the first term, note that for all  $t \in [0, l]$  we have  $b(t, v) \leq At \leq Al$  by Lemma 6.2. Hence the average  $a_l(0, v)$  is bounded above by  $A$ . By Lemma 6.7 and Proposition 6.6, we have

$$d(\psi^{a_l(0,v)}\mathcal{F}_0(v), \psi^{a_l(0,v)}\mathcal{F}_0(w)) \leq C d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq C d(v, w)^\alpha,$$

where  $C$  depends only on  $l$  and the constant from Proposition 6.6. As such,  $C$  depends only on  $n, \Gamma, \lambda, \Lambda, i_M$ . By Lemma 6.8, the second term is bounded above by

$$|a_l(0, v) - a_l(0, w)| \leq \frac{1}{l} \int_0^l |b(t, v) - b(t, w)| dt \leq C d(v, w)^\alpha,$$

where  $C$  again depends only on  $n, \Gamma, \lambda, \Lambda, i_M$ .  $\square$

## REFERENCES

- [AF03] CS Aravinda and FT Farrell, *Exotic negatively curved structures on Cayley hyperbolic manifolds*, Journal of Differential Geometry **63** (2003), no. 1, 41–62.
- [AF04] ———, *Exotic structures and quaternionic hyperbolic manifolds*, Algebraic Groups and Arithmetic (2004), 507–524.
- [Bal95] Werner Ballmann, *Lectures on spaces of nonpositive curvature*, vol. 25, Springer Science & Business Media, 1995.
- [BCG95] Gérard Besson, Gilles Courtois, and Sylvestre Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geometric & Functional Analysis GAFA **5** (1995), no. 5, 731–799.
- [Bea12] Alan F Beardon, *The geometry of discrete groups*, vol. 91, Springer Science & Business Media, 2012.
- [BH13] Martin R Bridson and André Haefliger, *Metric spaces of non-positive curvature*, vol. 319, Springer, 2013.
- [BKB<sup>+</sup>85] Keith Burns, Anatole Katok, W Ballman, M Brin, P Eberlein, and R Osserman, *Manifolds with non-positive curvature*, Ergodic Theory and Dynamical Systems **5** (1985), no. 2, 307–317.
- [Bow75] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Math **470** (1975), 78–104.
- [BP92] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Springer, 1992.

- [But22] Karen Butt, *Quantitative marked length spectrum rigidity*, arXiv preprint math/2203.12128 (2022).
- [Cou04] Yves Coudene, *Topological dynamics and local product structure*, Journal of the London Mathematical Society **69** (2004), no. 2, 441–456.
- [Cro90] Christopher B Croke, *Rigidity for surfaces of non-positive curvature*, Commentarii Mathematici Helvetici **65** (1990), no. 1, 150–169.
- [dC92] Manfredo Perdigao do Carmo, *Riemannian geometry*, Birkhäuser, 1992.
- [FH19] Todd Fisher and Boris Hasselblatt, *Hyperbolic flows*, 2019.
- [FJ89] F Thomas Farrell and Lowell E Jones, *Negatively curved manifolds with exotic smooth structures*, Journal of the American Mathematical Society **2** (1989), no. 4, 899–908.
- [FJ94] FT Farrell and LE Jones, *Complex hyperbolic manifolds and exotic smooth structures*, Inventiones mathematicae **117** (1994), no. 1, 57–74.
- [FM11] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton university press, 2011.
- [Fra18] Steven Frankel, *Coarse hyperbolicity and closed orbits for quasigeodesic flows*, Annals of Mathematics **188** (2018), no. 1, 1–48.
- [GHL90] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, vol. 2, Springer, 1990.
- [GKL22] Colin Guillarmou, Gerhard Knieper, and Thibault Lefeuvre, *Geodesic stretch, pressure metric and marked length spectrum rigidity*, Ergodic Theory and Dynamical Systems **42** (2022), no. 3, 974–1022.
- [GL19] Colin Guillarmou and Thibault Lefeuvre, *The marked length spectrum of Anosov manifolds*, Annals of Mathematics **190** (2019), no. 1, 321–344.
- [GL21] Sébastien Gouëzel and Thibault Lefeuvre, *Classical and microlocal analysis of the x-ray transform on Anosov manifolds*, Analysis & PDE **14** (2021), no. 1, 301–322.
- [Gro82] Michael Gromov, *Volume and bounded cohomology*, Publications Mathématiques de l’IHÉS **56** (1982), 5–99.
- [Gro83] Mikhael Gromov, *Filling Riemannian manifolds*, Journal of Differential Geometry **18** (1983), no. 1, 1–147.
- [Gro00] Mikhail Gromov, *Three remarks on geodesic dynamics and fundamental group*, Enseign. Math.(2) **46** (2000), 391–402.
- [GW88] Robert Greene and Hung-Hsi Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific Journal of Mathematics **131** (1988), no. 1, 119–141.
- [Ham99] Ursula Hamenstädt, *Cocycles, symplectic structures and intersection*, Geometric & Functional Analysis GAFA **9** (1999), no. 1, 90–140.
- [Ham03] Ursula Hamenstadt, *Length functions and parameterizations of Teichmüller space for surfaces with cusps*, Annales-academiae scientiarum fennicae mathematica, vol. 28, Academia Scientiarum fennica, 2003, pp. 75–88.
- [HIH77] Ernst Heintze and Hans-Christoph Im Hof, *Geometry of horospheres*, Journal of Differential geometry **12** (1977), no. 4, 481–491.
- [Kat90] Svetlana Katok, *Approximate solutions of cohomological equations associated with some Anosov flows*, Ergodic Theory and Dynamical Systems **10** (1990), no. 2, 367–379.
- [Mos73] G Daniel Mostow, *Strong rigidity of locally symmetric spaces*, no. 78, University of Tokyo Press, 1973.
- [Ota90] Jean-Pierre Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Annals of Mathematics **131** (1990), no. 1, 151–162.
- [Pet06] Peter Petersen, *Riemannian geometry. second edition*, vol. 171, Springer, 2006.
- [Rat73] Marina Ratner, *Markov partitions for Anosov flows on  $n$ -dimensional manifolds*, Israel Journal of Mathematics **15** (1973), no. 1, 92–114.