
ONE GRADIENT FRANK-WOLFE FOR DECENTRALIZED ONLINE CONVEX AND SUBMODULAR OPTIMIZATION *

Tuan-Anh Nguyen
LIG, Inria
University Grenoble-Alpes
Saint-Martin-d'Hères, France
tuan-anh.nguyen@inria.fr

Nguyen Kim Thang
LIG, Inria, CNRS, Grenoble INP
University Grenoble-Alpes
Saint-Martin-d'Hères, France
kim-thang.nguyen@univ-grenoble-alpes.fr

Denis Trystram
LIG, Inria, CNRS, Grenoble INP
University Grenoble-Alpes
Saint-Martin-d'Hères, France
denis.trystram@imag.fr

ABSTRACT

Decentralized learning has been studied intensively in recent years motivated by its wide applications in the context of federated learning. The majority of previous research focuses on the offline setting in which the objective function is static. However, the offline setting becomes unrealistic in numerous machine learning applications that witness the change of massive data. In this paper, we propose *decentralized online* algorithm for convex and continuous DR-submodular optimization, two classes of functions that are present in a variety of machine learning problems. Our algorithms achieve performance guarantees comparable to those in the centralized offline setting. Moreover, on average, each participant performs only a *single* gradient computation per time step. Subsequently, we extend our algorithms to the bandit setting. Finally, we illustrate the competitive performance of our algorithms in real-world experiments.

Keywords Decentralized Learning · Online Optimization · Frank-Wolfe Algorithm · Submodular Function

1 Introduction

Learning over data generated by sensors and mobile devices has gained a high interest in recent years due to the continual interaction with users and the environment on a timely basis. The patterns related to user's behavior, preference, and the surrounding stochastic events become a promising source for machine learning applications to be more and more reliable. However, collecting such data in a centralized location has become problematic due to privacy concerns and the high cost of data transfer over the network. Consequently, the learning methods that can leave the data locally while efficiently exploiting data patterns, such as decentralized learning, are emerging as an alternative to traditional centralized learning.

Under the optimization scheme, learning in a decentralized manner consists of multiple interconnected agents cooperating to optimize a global objective function where each agent detains partial information of the interested function. Several works [1, 2, 3, 4, 5] have considered this setting for convex and strongly convex functions. [6] also study the problem when the objective function is generally non-convex whereas [7, 8] proposes a decentralized algorithm to maximize monotone submodular functions for both continuous and discrete domains. However, these works only consider the offline setting which is not realistic since data constantly evolve in many real-world applications. In this paper, we study decentralized online algorithms for optimizing both convex and submodular functions.

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Problem definition. Formally, we are given a convex set $\mathcal{K} \subseteq \mathbb{R}^d$ (w.l.o.g one can assume that $\mathcal{K} \subseteq [0, 1]^d$) and a set of agents connected over a network represented by a graph $G = (V, E)$ where $n = |V|$ is the number of agents. At every time $1 \leq t \leq T$, each agent $i \in V$ can communicate with (and only with) its immediate neighbors, i.e., adjacent agents in G and takes a decision $\mathbf{x}_t^i \in \mathcal{K}$. Subsequently, a cost/reward function $f_t^i : \mathcal{K} \rightarrow \mathbb{R}$ is revealed adversarially and locally to agent i . Note that in the *bandit* setting, agent i observes only the value $f_t^i(\mathbf{x}_t^i)$ instead of the whole function f_t^i . Although each agent i observes only function f_t^i (or the value $f_t^i(\mathbf{x}_t^i)$ in the bandit setting), agent i is interested in the cumulating cost/reward $F_t(\cdot)$ where $F_t(\cdot) := \frac{1}{n} \sum_{j=1}^n f_t^j(\cdot)$. In particular, at time t , the cost/reward of agent i with the its chosen \mathbf{x}_t^i is $F_t(\mathbf{x}_t^i)$.

In the context of convex minimization, the functions f_t^i 's are convex and the goal of each agent i is to minimize the total cumulating cost $\sum_{t=1}^T F_t(\mathbf{x}_t^i)$ via local communication with its immediate neighbors. Our objective is to design an algorithm with small regret. An online algorithm is \mathcal{R}_T -*regret* if for every agent $1 \leq i \leq n$,

$$\sum_{t=1}^T F_t(\mathbf{x}_t^i) - \min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T F_t(\mathbf{x}) \leq \mathcal{R}_T$$

In the context of monotone DR-submodular maximization, the functions f_t^i 's are monotone DR-submodular. Roughly speaking, a bounded differentiable and non-negative function $F : [0, 1]^d \rightarrow \mathbb{R}_+$ is *DR-submodular* if for every $\mathbf{x}, \mathbf{y} \in [0, 1]^d$ satisfying $x_i \leq y_i, \forall i \in [d]$, we have $\nabla F(\mathbf{x}) \geq \nabla F(\mathbf{y})$. The goal of each agent i is to maximize the total cumulating reward $\sum_{t=1}^T F_t(\mathbf{x}_t^i)$, again via local communication with its immediate neighbors. Our objective is to design an algorithm with an approximation ratio as close to 1 as possible and together with a small regret. An online algorithm has a ρ -*regret* of \mathcal{R}_T if for every agent $1 \leq i \leq n$,

$$\rho \cdot \max_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T F_t(\mathbf{x}) - \sum_{t=1}^T F_t(\mathbf{x}_t^i) \leq \mathcal{R}_T$$

1.1 Our contribution

The challenge in designing robust and efficient algorithms for these problems is to simultaneously address the following issues:

- Uncertainty (online setting, agents observe their loss functions only after selecting their decisions).
- Partial information (decentralized setting, agents know only its local loss functions while attempting to minimize the cumulated cost).
- Low computation/communication resources of agents (so it is desirable that each agent performs a small number of gradient computations and communications).
- Additionally, in the bandit setting, one has only limited feedback (agents can only observe the function value of their decisions).

We present performance-guaranteed algorithms for solving the constraint convex and continuous DR-submodular optimization problem in the decentralized and online setting with *only* one gradient evaluation and low communications per agent per time step on average. Specifically, our algorithms achieve the regret and the $(1 - \frac{1}{e})$ -regret bounds of $O(T^{4/5})$ for both convex and monotone continuous DR-submodular functions. Using a one-point gradient estimator [9], we extend the algorithms to the bandit setting in which the gradient is unavailable to the agents. We obtain the $(1 - \frac{1}{e})$ -regret bound of $O(T^{8/9})$ for the bandit setting. It should be noted that the $(1 - \frac{1}{e})$ -regret of $O(T^{4/5})$ and $O(T^{8/9})$ matches the regret guarantees in the centralized online settings. Besides, one can convert the algorithm to be projection-free (by selecting suitable oracles). This property allows the algorithm to be implemented in various contexts based on the computing capacity of local devices. We demonstrate the practical application of our algorithm on a Movie Recommendation problem and present a thorough analysis of different aspects of the performance guarantee, the effects of network topology, and decentralization, which are predictably explained by our theoretical results.

1.2 Related Works

Decentralized Online Optimization. [11] introduced decentralized online projected subgradient descent and showed vanishing regret for convex and strongly convex functions. In contrast, [12] extended distributed dual averaging technique to the online setting, using a general regularized projection for both unconstrained and constrained optimization. A distributed variant of online conditional gradient [13] was designed and analyzed in [14] that requires

| Algorithm | Stochastic Gradient | $(1 - 1/e)$ -Regret | Communications | Gradient Evaluations |
|-------------------------|---------------------|------------------------|------------------------------|----------------------|
| DMFW | Yes | $\mathcal{O}(T^{1/2})$ | $2 \cdot T^{3/2}$ | $T^{3/2}$ |
| Monode-FW | Yes | $\mathcal{O}(T^{4/5})$ | $2 \cdot \text{\#neighbors}$ | 1 |
| Bandit Monode-FW | - | $\mathcal{O}(T^{8/9})$ | $2 \cdot \text{\#neighbors}$ | - |

Table 1: Comparison of previous work on *adversarial* decentralized online monotone DR-submodular maximization (DMFW [10]) and our proposed algorithms (in bold). The communications and gradient evaluations are measured per agent per time step.

linear minimizers and uses exact gradients. However, computing exact gradients may be prohibitively expensive for moderately sized data and intractable when a closed-form does not exist. [10] proposes a decentralized online algorithm for maximizing monotone submodular function on a time-varying network using stochastic gradient estimate and multiple optimization oracles. This work achieves the optimal regret bound of $\mathcal{O}(T^{1/2})$ but requires $T^{3/2}$ -gradient evaluation and communication per function. In this work, we advance further in designing a distributed algorithm that uses stochastic gradient estimates and requires only one gradient evaluation.

Monotone DR-submodular Maximization. The maximization of monotone DR-submodular functions has been investigated in both offline and online settings. For the offline case, [15] examined the problem where the constraint set is a down-closed convex set and demonstrated that the greedy method [16], a variation of the Frank-Wolfe algorithm, ensures a $(1 - 1/e)$ -approximation. [17] demonstrated the restriction of the greedy method in a stochastic environment where only unbiased gradient estimates are available. Later, [18] introduced an algorithm for maximizing monotone DR-submodular function over the general convex set using new variance reduction techniques to accomplish $(1 - 1/e)$ -approximation in a stochastic setting. [19] suggested a method that achieves $(1 - 1/e, \mathcal{O}(\sqrt{T}))$ -regret for maximizing monotone DR-submodular over a general convex set in an online setting. Subsequently, [20] introduced an approach that reduces the number of per-function gradient evaluations from $T^{3/2}$ to 1, while maintaining the same approximation ratio of $(1 - 1/e)$. They also presented a bandit approach that achieves an expected $(1 - 1/e)$ -approximation ratio with regret $T^{8/9}$ to tackle the same problem.

2 Preliminaries and Notations

We begin by explaining the notations and concepts that will be used throughout the paper. Given an undirected graph $G = (V, E)$, the set of neighbors of an agent $i \in V$ is denoted $N(i) := \{j \in V : (i, j) \in E\}$. Consider the following symmetric matrix $\mathbf{W} \in \mathbb{R}_+^{n \times n}$ with entry W_{ij} defined as follows.

$$W_{ij} = \begin{cases} \frac{1}{1 + \max\{d_i, d_j\}} & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E, i \neq j \\ 1 - \sum_{j \in N(i)} W_{ij} & \text{if } i = j \end{cases}$$

where $d_i = |N(i)|$, the degree of vertex i . In fact, the matrix \mathbf{W} is doubly stochastic, i.e $\mathbf{W}\mathbf{1} = \mathbf{W}^T\mathbf{1} = \mathbf{1}$ and so it inherits several useful properties of doubly stochastic matrices. We use boldface letter e.g \mathbf{x} to represent vectors. We denote by $\mathbf{x}_{q,k}^i$ the decision vector of agent i at time step k of phase q . If not specified otherwise, we suppose that the constraint set \mathcal{K} is a bounded convex set with diameters $D = \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{K}} \|\mathbf{x} - \mathbf{y}\|$ and radius $R = \sup_{\mathbf{x} \in \mathcal{K}} \|\mathbf{x}\|$. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we note $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i \forall i$. We note \mathbb{B}_d and \mathbb{S}_d the d -dimensional unit ball and the unit sphere, respectively.

A continuous function $F : [0, 1]^d \rightarrow \mathbb{R}_+$ is *DR-submodular* if for any vectors $\mathbf{x}, \mathbf{y} \in [0, 1]^d$ such that $\mathbf{x} \leq \mathbf{y}$, for a constant $\alpha > 0$ and any basis vectors $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ such that $\mathbf{x} + \alpha\mathbf{e}_i \in [0, 1]^d$ and $\mathbf{y} + \alpha\mathbf{e}_i \in [0, 1]^d$.

$$F(\mathbf{x} + \alpha\mathbf{e}_i) - F(\mathbf{x}) \geq F(\mathbf{y} + \alpha\mathbf{e}_i) - F(\mathbf{y}) \quad (1)$$

For a differentiable function, the DR-property is equivalent to $\nabla F(\mathbf{x}) \geq \nabla F(\mathbf{y}), \forall \mathbf{x} \leq \mathbf{y} \in [0, 1]^d$. More over, if F is twice-differentiable, the DR-property is equivalent to all entries of the Hessian matrix being non-positive i.e $\forall 1 \leq i, j \leq d, \frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0$. A function F is *monotone* if $\forall \mathbf{x} \leq \mathbf{y} \in [0, 1]^d$, we have $F(\mathbf{x}) \leq F(\mathbf{y})$.

A function F is β -smooth if for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$:

$$F(\mathbf{y}) \leq F(\mathbf{x}) + \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\beta}{2} \|\mathbf{y} - \mathbf{x}\|^2$$

or equivalently $\|\nabla F(\mathbf{x}) - \nabla F(\mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|$. Also, we say a function F is G -Lipschitz if for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq G \|\mathbf{x} - \mathbf{y}\|$$

In this paper, we employ linear optimization oracles in our algorithm to solve an online linear optimization problem given a feedback function and a constraint set. In particular, in the online linear optimization problem, one must choose $\mathbf{u}^t \in \mathcal{K}$ at every time $1 \leq t \leq T$. The adversary then discloses a vector \mathbf{d}^t and feedbacks the cost function $\langle \cdot, \mathbf{d}^t \rangle$ where the goal is to minimize the regret of the linear objective. Several algorithms [13], including the projection-free follow-the-perturbed-leader algorithm offer an optimum regret bound of $\mathcal{R}_T = O(\sqrt{T})$ for the online linear optimization problem. One of these methods can be used as an oracle to solve the online linear optimization problem.

In practice, it may not be possible to use a full gradient due to the vast quantity of data and processing restrictions. To address this issue, our approach utilizes an unbiased stochastic gradient in place of the gradient and proposes a variance reduction technique for distributed optimization based on a rigorous analysis that may be applied to problems of independent interest. We make the following assumptions for the next two sections.

Assumption 1. We let k_0 to be the smallest integer such that $\lambda_2(\mathbf{W}) \leq \left(\frac{k_0}{k_0+1}\right)^2$.

Assumption 2. The function f_t is G -Lipschitz and β -smooth. Its stochastic gradient $\tilde{\nabla} f_t(\mathbf{x})$ is unbiased, uniformly upper-bounded and has a bounded variance, i.e., $\mathbb{E} \left[\tilde{\nabla} f_t(\mathbf{x}) \right] = \nabla f_t(\mathbf{x})$, $\|\tilde{\nabla} f_t(\mathbf{x})\| \leq G_0$, and $\mathbb{E} \left[\left\| \tilde{\nabla} f_t(\mathbf{x}) - \nabla f_t(\mathbf{x}) \right\|^2 \right] \leq \sigma_0^2$.

Assumption 3. For all $t \in [T]$ and $i \in [n]$, $\sup_{\mathbf{x} \in \mathcal{K}} |f_t^i(\mathbf{x})| \leq B$

Assumption 4. There exist a number $r \geq 0$ such that $r\mathbb{B}_d \subseteq \mathcal{K}$

3 Full Information Setting

This section thoroughly describes the algorithm for both convex and DR-submodular optimization. Recall that each agent receives a function f_t^i at every time $t \in [T]$. We partition time steps into Q blocks, each of size K so that $T = QK$. For each block $q \in [Q]$, we define f_q^i as the average of the K functions within the block. Additionally, each agent $1 \leq i \leq n$ maintains K online linear optimization oracles $\mathcal{O}_{i,1}, \dots, \mathcal{O}_{i,K}$. Let $\sigma_q \in \mathfrak{S}_K$ be a random permutation of function indexes for all agents.

At a high level, at each block q , the agent i performs K -steps of Frank-Wolfe algorithm, where the update vector is a combination of the oracles' outputs and the aggregate of its neighbors' current decisions. The final decision \mathbf{x}_q^i for the block q is disclosed at the end of K steps, such that at each time step in the block, agent i plays the same decision \mathbf{x}_q^i .

More specifically, following the Frank-Wolfe steps, agent i performs K gradient updates using the estimators $f_{\sigma_q(k)}^i$. It calculates the stochastic gradient of the permuted function $f_{\sigma_q(k)}^i$ evaluated at the corresponding decision vector $\mathbf{x}_{q,k}^i$ and thereafter exchanges information with its neighbors. It then computes a variance reduction version $\tilde{\mathbf{a}}_{q,k}^i$ of the vector $\tilde{\mathbf{d}}_{q,k}^i$ and returns $\langle \tilde{\mathbf{a}}_{q,k}^i, \cdot \rangle$ as the cost function at time $\sigma_q^{-1}(k)$ to the oracle \mathcal{O}_k^i . The vectors $\tilde{\mathbf{d}}_{q,k}^i$ are subtly constructed to capture progressively more information on the accumulating cost functions.

Note that the use of random permutation σ_q is crucial here. By that, all the permuted functions $f_{\sigma_q(k)}^i$ become an estimation of f_q^i , i.e., $\mathbb{E}[f_{\sigma_q(k)}^i] = f_q^i$. Therefore the gradient of $f_{\sigma_q(k)}^i$ is likewise an estimation of the gradient of f_q^i . One can think of f_q^i as an artificial objective function for which we have access to its gradient estimates, where each estimation is one gradient evaluation per function within the block. As a result, conducting K gradient updates of f_q^i turns out to be executing one gradient update for each of the K functions. Using this approach, initiated in [20], we can effectively reduce the gradient evaluation number to 1 for each arriving function f_t^i .

Since we deal with both convex and submodular, there are modifications to adapt for both kinds of optimization problem. The online optimization oracle's objective function should be minimized for convex optimization and maximized for submodular optimization. The decision update for convex problems is a convex combination of the aggregated

neighbors' decisions $\mathbf{y}_{q,k}^i$ and the oracle's output $\mathbf{v}_{q,k}^i$, i.e.,

$$\mathbf{x}_{q,k+1}^i = (1 - \eta_k) \mathbf{y}_{q,k}^i + \eta_k \mathbf{v}_{q,k}^i, \quad \eta_k \in [0, 1] \quad (2)$$

whereas the update for the submodular optimization problem is achieved by shifting the aggregated decisions towards the direction of the oracle's output by a step-size η_k , i.e.,

$$\mathbf{x}_{q,k+1}^i = \mathbf{y}_{q,k}^i + \eta_k \mathbf{v}_{q,k}^i, \quad \eta_k \in [0, 1] \quad (3)$$

For convex functions, the initialization can be any random point inside the constraint set, however for submodular functions, this value should be set to 0.

The formal description is given in Algorithm 1. The proof of the following lemmas and theorems can be found in Appendix A.

Algorithm 1 Monode Frank-Wolfe

Input: A convex set \mathcal{K} , a time horizon T , a block size K , online linear optimization oracles $\mathcal{O}_{i,1}, \dots, \mathcal{O}_{i,K}$ for each agent $1 \leq i \leq n$, step sizes $\eta_k \in (0, 1)$ for all $1 \leq k \leq K$, number of blocks $Q = T/K$

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1: Initialize linear optimizing oracle  $\mathcal{O}_k^i$  for all  $1 \leq k \leq K$ 
2: for  $q = 1$  to  $Q$  do
3:   for every agent  $1 \leq i \leq n$  do
4:     Initialize  $\mathbf{x}_{q,1}^i$  and set  $\tilde{\mathbf{a}}_{i,0}^t \leftarrow 0$ 
5:     for  $1 \leq k \leq K$  do
6:       Let  $\mathbf{v}_{q,k}^i$  be the output of oracle  $\mathcal{O}_k^i$  at phase  $q$ .
7:       Send  $\mathbf{x}_{q,k}^i$  to all neighbours  $N(i)$ 
8:       Once receiving  $\mathbf{x}_{q,k}^j$  from all neighbours  $j \in N(i)$ , set  $\mathbf{y}_{q,k}^i \leftarrow \sum_j W_{ij} \mathbf{x}_{q,k}^j$ .
9:       Update  $\mathbf{x}_{q,k+1}^i$  as (2) or (3).
10:    end for
11:    Choose  $\mathbf{x}_q^i \leftarrow \mathbf{x}_{q,K+1}^i$  and agent  $i$  plays the same  $\mathbf{x}_q^i$  for every time  $t$  in phase  $q$ .
12:    Let  $\sigma_q$  be a random permutation of  $1, \dots, K$  — times in phase  $q$ .
13:    for  $1 \leq k \leq K$  do
14:      Let  $s = \sigma_q^{-1}(k)$ 
15:      Query the values of  $\tilde{\nabla} f_k^i(\mathbf{x}_{q,s}^i)$ 
16:    end for
17:    Set  $\tilde{\mathbf{g}}_{q,1}^i \leftarrow \tilde{\nabla} f_{\sigma_q(1)}^i(\mathbf{x}_{q,1}^i)$ 
18:    for  $1 \leq k \leq K$  do
19:      Send  $\tilde{\mathbf{g}}_{q,k}^i$  to all neighbours  $N(i)$ .
20:      After receiving  $\tilde{\mathbf{g}}_{q,k}^j$  from all neighbours  $j \in N(i)$ , compute  $\tilde{\mathbf{d}}_{q,k}^i \leftarrow \sum_{j \in N(i)} W_{ij} \tilde{\mathbf{g}}_{q,k}^j$  and  $\tilde{\mathbf{g}}_{q,k+1}^i \leftarrow$ 
       $(\tilde{\nabla} f_{\sigma_q(k+1)}^i(\mathbf{x}_{q,k+1}^i) - \tilde{\nabla} f_{\sigma_q(k)}^i(\mathbf{x}_{q,k}^i)) + \tilde{\mathbf{d}}_{q,k}^i$ 
21:       $\tilde{\mathbf{a}}_{q,k}^i \leftarrow (1 - \rho_k) \cdot \tilde{\mathbf{a}}_{q,k-1}^i + \rho_k \cdot \tilde{\mathbf{d}}_{q,k}^i$ .
22:      Feedback function  $\langle \tilde{\mathbf{a}}_{q,k}^i, \cdot \rangle$  to oracles  $\mathcal{O}_k^i$ . (The cost of the oracle  $\mathcal{O}_k^i$  at block  $q$  is  $\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i \rangle$ .)
23:    end for
24:  end for
25: end for
    
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Lemma 1. Let $V_d = 2nG \left(\frac{\lambda_2}{1-\lambda_2} + 1 \right)$, the local gradient is uniformly upper-bounded, i.e., $\forall i \in [n], \forall k \in [K]$,

$$\|\mathbf{d}_{q,k}^i\| \leq V_d.$$

Lemma 2. Under Assumption 2 and let $\sigma_1^2 = 4n \left[\left(\frac{G+G_0}{\lambda_2-1} \right)^2 + 2\sigma_0^2 \right]$. For $i \in [n], k \in [K]$, the variance of the local stochastic gradient is uniformly bounded i.e $\mathbb{E} \left[\|\mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i\|^2 \right] \leq \sigma_1^2$

Theorem 1. Given a convex set \mathcal{K} and assume that F_t is a convex function. Setting $Q = T^{2/5}, K = T^{3/5}, T = QK$ and step-size $\eta_k = \frac{1}{k}$. Let $\rho_k = \frac{2}{(k+3)^{2/3}}$ and $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$ when $k \in [1, \frac{K}{2}]$ and $k \in [\frac{K}{2} + 1, K]$ respectively.

Then, the expected regret of Algorithm 1 is at most

$$\mathbb{E}[\mathcal{R}_T] \leq (GD + 2\beta D^2) T^{2/5} + \left(C + 6D(N + \sqrt{M})\right) T^{4/5} + \frac{3}{5}\beta D^2 T^{2/5} \log(T) \quad (4)$$

where $N = k_0 \cdot nG \max\{\lambda_2 \left(1 + \frac{2}{1-\lambda_2}\right), 2\}$ and $M = \max\{M_1, M_2\}$ where $M_0 = 4(V_d^2 + \sigma_1^2) + 128V_d^2$, $M_1 = \max\left\{5^{2/3} \left(V_d + \frac{2}{4^{2/3}}G_0\right)^2, M_0\right\}$ and $M_2 = 2.55(V_d^2 + \sigma_1^2) + \frac{28V_d^2}{3}$

Theorem 2. Given a convex set \mathcal{K} and assume that the function F_t is monotone continuous DR-submodular. Setting $Q = T^{2/5}$, $K = T^{3/5}$, $T = QK$ and step-size $\eta_k = \frac{1}{K}$. Let $\rho_k = \frac{2}{(k+3)^{2/3}}$ and $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$ when $1 \leq k \leq \frac{K}{2} + 1$ and $\frac{K}{2} + 1 \leq k \leq K$ respectively. Then, the expected $(1 - \frac{1}{e})$ -regret is at most

$$\mathbb{E}[\mathcal{R}_T] \leq \frac{3}{2}\beta D^2 T^{2/5} + \left(C + 3D(N + \sqrt{M})\right) T^{4/5} \quad (5)$$

where the constants are defined in Theorem 1

As stated in the preceding paragraph, the distinction between convex and submodular optimization can be found in line 9 of Algorithm 1 and in the oracle optimization subroutine. To achieve the regret bound mentioned in Theorems 1 and 2, we use follow-the-perturbed-leader as the oracle with regret $\mathcal{R}_T = C\sqrt{T}$. In the case of convex optimization, one may use online gradient descent to obtain the same outcome, but this method is more computationally intensive because it involves a projection step onto the constraint set.

4 Bandit Setting

This section describes a bandit algorithm for a decentralized submodular maximization. We let \mathcal{K} be a down-closed convex set. A major difference between this algorithm and the previous one is the function's value $f_t^i(\mathbf{x}_t^i)$ being the only information provided to the agent. It does not know of the value incurred if it had chosen another decision in the constraint set. As a consequence, this setting makes access to the gradient impossible for the agent. To circumvent this limitation, we use the one-point gradient estimate [9] and adapt the biphasic bandit setting [20] to our decentralized algorithm.

We recall that for a function f_t defined on $\mathcal{K} \subset \mathbb{R}^d$, it admits a δ -smoothed version for any $\delta > 0$, given as

$$\hat{f}_{t,\delta}(\mathbf{x}_t) = \mathbb{E}_{\mathbf{v} \sim \mathbb{B}_d} [f_t(\mathbf{x}_t + \delta \mathbf{v})]$$

where \mathbf{v} is drawn uniformly random from the d -dimensional unit ball. The value of $\hat{f}_{t,\delta}$ at a point \mathbf{x} is the average of f_t evaluated across the d -dimensional ball of radius δ centered at \mathbf{x} . This function inherits various functional properties from f_t , therefore becomes a suitable approximation for f_t , as shown in the following lemma.

Lemma 3 (Lemma 2 [21], Lemma 6.6 [13]). *Let f be a monotone continuous DR-submodular function. If f is β -smooth, G -Lipschitz, then so is \hat{f}_δ and we have $\|\hat{f}_\delta(\mathbf{x}) - f(\mathbf{x})\| \leq \delta G$. More over, if we choose \mathbf{u} uniformly from the unit sphere \mathbb{S}^{d-1} , the following equation holds*

$$\nabla \hat{f}_{t,\delta}(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{S}^{d-1}} \left[\frac{d}{\delta} f_t(\mathbf{x} + \delta \mathbf{u}) \mathbf{u} \right] \quad (6)$$

Lemma 3 shows that a decision that maximizes $\hat{f}_{t,\delta}$ can also maximize f_t approximately. The δ -smooth version additionally provides a one-point gradient estimate that can be used to estimate the gradient of f_t by evaluating the function at a random point on the $(d-1)$ -dimensional sphere of radius δ . It is important to note that the point $\mathbf{x} + \delta \mathbf{u}$ may be outside the set when \mathbf{x} is near to the constraint set's boundary. For this reason, we let $\mathcal{K}' \subset \mathcal{K}$ be the δ -interior of \mathcal{K} that verifies: $\forall \mathbf{x} \in \mathcal{K}', \mathbb{B}(\mathbf{x}, \delta) \subset \mathcal{K}$, and solve the optimization problem on the new set \mathcal{K}' . By shrinking the constraint set down to \mathcal{K}' , we assure that the point $\mathbf{x} + \delta \mathbf{u}$ is in \mathcal{K} for any point \mathbf{x} in \mathcal{K}' . More over, if the distance $d(\mathcal{K}', \mathcal{K})$ between \mathcal{K}' and \mathcal{K} is small enough, we can approximately get the optimal regret bound on the original constraint set \mathcal{K} by running the bandit algorithm on \mathcal{K}' . The detail on the construction of \mathcal{K}' is given in Lemma 4

The biphasic setting consist of partitioning T into Q blocks of size L , with each block consisting of two phases: exploration and exploitation. Each agent i performs $K < L$ steps of exploration by updating the decision vector $\mathbf{x}_{q,k}^i$ using equation (3). During the exploration phase, rather than playing the final decision as in Algorithm 1, the agent draws uniformly a random vector $\mathbf{u}_{q,k}^i$ from \mathbb{S}_{d-1} and plays $\mathbf{x}_{q,k}^i + \delta \mathbf{u}_{q,k}^i$ for the function $f_{\sigma_q(k)}^i$, as it can only estimate

the gradient at the point it plays. The gradient estimate $\tilde{\mathbf{h}}_{q,k}^i$ is then computed using equation (6), followed by a local aggregation and variance reductions steps, the final step consisting of feeding the variance reduction vector $\tilde{\mathbf{a}}_{q,k}^i$ back to the oracle \mathcal{O}_k^i . The remaining $L - K$ iterations are used for exploitation, where each agent plays the final decision \mathbf{x}_q^i to obtain a high reward. We give the detail of every step in Algorithm 2. Appendix B contains the analysis of Theorem 3

Algorithm 2 Bandit Monode Frank-Wolfe

Input: Smoothing radius δ , δ -interior \mathcal{K}' with lower bound \underline{u} , a time horizon T , a block size L , number of exploration step K . Online linear optimization oracles $\mathcal{O}_{i,1}, \dots, \mathcal{O}_{i,K}$ for each player $1 \leq i \leq n$, step sizes $\eta_k, \rho_k \in (0, 1)$ for all $1 \leq k \leq K$, number of blocks $Q = T/L$

```

1: Initialize linear optimizing oracle  $\mathcal{O}_k^i$  for all  $1 \leq k \leq K$ 
2: for  $q = 1$  to  $Q$  do
3:   for every agent  $1 \leq i \leq n$  do
4:     Initialize  $\mathbf{x}_{q,1}^i \leftarrow \underline{u}$  and set  $\tilde{\mathbf{a}}_{i,0}^i \leftarrow 0$ 
5:     Update  $\mathbf{x}_{q,k}^i$  using line 5 to 10 of Algorithm 1. Choose  $\mathbf{x}_q^i \leftarrow \mathbf{x}_{q,K+1}^i$ 
6:     Let  $\sigma_q$  be a random permutation of  $1, \dots, L$  — times in phase  $q$ .
7:     for  $1 \leq \ell \leq L$  do
8:       Let  $s = \sigma_q^{-1}(\ell)$ 
9:       if  $\ell \leq K$  then
10:        play  $f_{q,\ell}^i(\mathbf{x}_{q,s}^i + \delta \mathbf{u}_{q,s}^i)$  where  $\mathbf{u}_{q,s}^i \in \mathbb{S}^{d-1}$ . - Exploration
11:       else
12:        play  $f_{q,\ell}^i(\mathbf{x}_q^i)$ . - Exploitation
13:       end if
14:     end for
15:     Set  $\tilde{\mathbf{g}}_{q,1}^i \leftarrow \frac{d}{\delta} f_{\sigma_q(1)}^i(\mathbf{x}_{q,1}^i + \delta \mathbf{u}_{q,1}^i) \mathbf{u}_{q,1}^i$ 
16:     for  $1 \leq k \leq K$  do
17:       Let  $\tilde{\mathbf{h}}_{q,k}^i = \frac{d}{\delta} f_{\sigma_q(k)}^i(\mathbf{x}_{q,k}^i + \delta \mathbf{u}_{q,k}^i) \mathbf{u}_{q,k}^i$ 
18:       Send  $\tilde{\mathbf{g}}_{q,k}^i$  to all neighbours  $N(i)$ .
19:       After receiving  $\tilde{\mathbf{g}}_{q,k}^j$  from all neighbours  $j \in N(i)$ , compute  $\tilde{\mathbf{d}}_{q,k}^i \leftarrow \sum_{j \in N(i)} W_{ij} \tilde{\mathbf{g}}_{q,k}^j$ 
20:        $\tilde{\mathbf{g}}_{q,k+1}^i \leftarrow \tilde{\mathbf{h}}_{q,k+1}^i - \tilde{\mathbf{h}}_{q,k}^i + \tilde{\mathbf{d}}_{q,k}^i$ 
21:        $\tilde{\mathbf{a}}_{q,k}^i \leftarrow (1 - \rho_k) \cdot \tilde{\mathbf{a}}_{q,k-1}^i + \rho_k \cdot \tilde{\mathbf{d}}_{q,k}^i$ .
22:       Feedback function  $\langle \tilde{\mathbf{a}}_{q,k}^i, \cdot \rangle$  to oracles  $\mathcal{O}_k^i$ . (The cost of the oracle  $\mathcal{O}_k^i$  at block  $q$  is  $\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i \rangle$ .)
23:     end for
24:   end for
25: end for

```

Lemma 4 (Lemma 1, [20]). *Let \mathcal{K} is down-closed convex set and δ is sufficiently small such that $\alpha = \frac{\sqrt{d+1}}{r} \delta < 1$. The set $\mathcal{K}' = (1 - \alpha)\mathcal{K} + \delta \mathbf{1}$ is convex, compact and down-closed δ -interior of \mathcal{K} satisfies $d(\mathcal{K}, \mathcal{K}') \leq \left(\sqrt{d} \left(\frac{R}{e} + 1 \right) + \frac{R}{r} \right) \delta$*

Theorem 3. *Let \mathcal{K} be a down-closed convex and compact set. We suppose the δ -interior \mathcal{K}' verify Lemma 4. Let $Q = T^{2/9}$, $L = T^{7/9}$, $K = T^{2/3}$, $\delta = \frac{r}{\sqrt{d+2}} T^{-1/9}$ and $\rho_k = \frac{2}{(k+2)^{2/3}}$, $\eta_k = \frac{1}{K}$. Then the expected $(1 - \frac{1}{e})$ -regret is upper bounded*

$$\mathbb{E}[\mathcal{R}_T] \leq Z T^{8/9} + \frac{\beta D^2}{2} T^{1/9} + \frac{3}{2} D \frac{d(\sqrt{d} + 2)}{r} P_{n,\lambda_2} T^{2/9} + \beta D^2 T^{3/9} \quad (7)$$

where we note $Z = (1 - \frac{1}{e}) \left(\sqrt{d} \left(\frac{R}{e} + 1 \right) + \frac{R}{r} \right) G \frac{r}{\sqrt{d+2}} + (2 - \frac{1}{e}) G \frac{r}{\sqrt{d+2}} + 2\beta + C$ and $P_{n,\lambda_2} = k_0 \cdot nB \max \left\{ \lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2 \right\} + 4^{1/3} \left(24n^2 \left(\frac{1}{\lambda_2-1} + 1 \right)^2 + 8n \left(\frac{1}{(\lambda_2-1)^2} + 2 \right) \right)^{1/2}$

5 Experiments

We run the algorithm on a movie recommendation problem, with the goal of identifying a set of k movies that satisfy all users. Our setting is closely related to the one in [7] and [8]. We use the MovieLens dataset, which contains one million

ratings ranging from 1 to 5 from 6000 users on 3883 movies. We divided the data set into T batches B_1, \dots, B_T , with each batch B_t containing ratings from 50 users. We chose Complete, Line, Grid, and Erdos-Renyi graphs with linked probability 0.2. We set the number of nodes/agents equals to 10, 25, and 50. At each iteration t , the agent i receives a subset of ratings $B_t^i \subset B_t$. Let \mathcal{M} be the set of movies and \mathcal{U} the set of users; we note $r(u, m)$ the rating of user $u \in \mathcal{U}$ for movies $m \in \mathcal{M}$. Let $S \subset \mathcal{M}$ a collection of movies such that $|S| = k$, the facility location function associated to each agent i denoted,

$$f(B_t^i, S) = f_t^i(S) = \frac{1}{|B_t^i|} \sum_{u \in B_t^i} \max_{m \in S} r(u, m) \quad (8)$$

We denote by $\mathcal{K} = \{ \mathbf{x} \in [0, 1]^d \mid \sum_{j=1}^d x_j = k \}$. The multilinear extension of f_t^i is defined as,

$$F_t^i(\mathbf{x}) = \sum_{S \subset \mathcal{M}} f_t^i(S) \prod_{j \in S} x_j \prod_{\ell \notin S} (1 - x_\ell), \quad \forall \mathbf{x} \in \mathcal{K} \quad (9)$$

The goal is to maximize the global objective function $F_t(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n F_t^i(\mathbf{x})$, subject to $\mathbf{x} \in \mathcal{K}$ while using only local communication and partial information for each local functions.

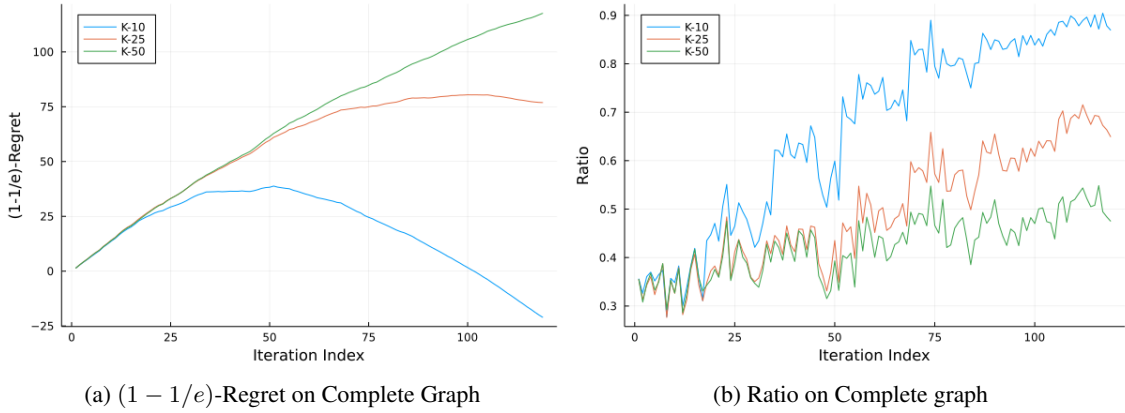


Figure 1: Algorithm performance on complete graph. The number of nodes is 10, 25 and 50.

Figure 1a shows the $(1 - \frac{1}{e})$ -regret of the algorithm for $k = 20$ on a complete graph with different node's configuration. We observe that increasing network size leads to a decrease in regret value, which is expected in a decentralized setting because information distributed across a larger set of nodes makes reaching consensus more difficult. Recall that the algorithm uses the same value for each function f_t in block q . If we set $K = 17$ and $Q = 6$, we can expect a stepwise-like curve since the objective function's value changes significantly at each round $t \bmod 17$. In a small graph configuration, this value change is more pronounced, bringing the cumulative sum of the objective function closer to the $(1 - \frac{1}{e})$ -optimal value. Figure 1b depicts the ratio of our algorithm's objective value on a complete graph to an offline centralized Frank-Wolfe. As t increases, the ratio approaches one, demonstrating that our algorithm's performance is comparable to that of an offline setting if we run the algorithm for many rounds, particularly in a 10-nodes configuration. Thus, the results validate our theoretical analysis in the previous section.

Figure 2 shows the average value of the objective function over T rounds for all graph types when the number of movie suggestions k is varied in a 50-node configuration. The average degree for Erdos-Renyi, Complete, Grid, and Line is 5.8, 51, 5.4, and 4, respectively. As a result, we observe lower performance on less connected graphs when compared to other graph settings. We also notice that increasing the value of k is equivalent to relaxing the cardinality constraint, which results in better performance on the objective function.

6 Concluding remarks

In this paper, we propose a decentralized online algorithm for optimizing convex and monotone continuous DR-submodular functions with regret and $(1 - \frac{1}{e})$ -regret bound of $O(T^{4/5})$. The extension of the algorithm to the bandit setting ensures a $(1 - \frac{1}{e})$ -regret bound of $O(T^{8/9})$. A detailed analysis is given when the constraint set is either a general convex set or a downward-closed convex set under full information and bandit settings, respectively. In addition, the experiment results on a real-life movie recommendation problem assess the interest of the proposed algorithm for learning in decentralized settings.

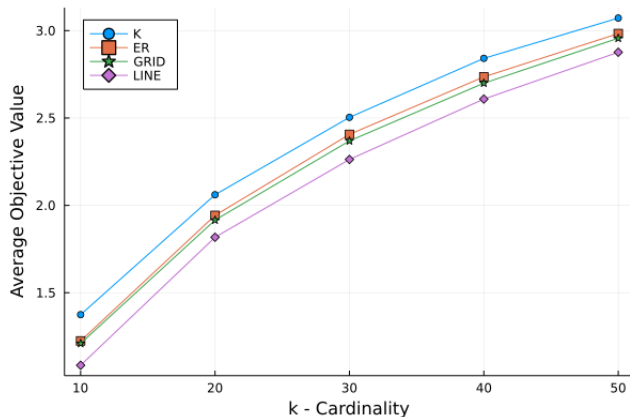


Figure 2: Average objective value over T rounds in function of the cardinality constraint.

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Supplementary File for One Gradient Frank-Wolfe for Decentralized Online Convex and Submodular Optimization

A Theoretical Analysis for Section 3

In the analysis, we note $\sigma_q(k)$ to be the permutation of k at phase q . We define the average function of the remaining $(K - k)$ functions as

$$\bar{F}_{q,k}(\mathbf{x}) = \frac{1}{K-k} \sum_{\ell=k+1}^K F_{\sigma_q(\ell)}(\mathbf{x}) = \frac{1}{K-k} \sum_{\ell=k+1}^K \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^i(\mathbf{x}) \quad (10)$$

where $F_{\sigma_q(\ell)}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^i(\mathbf{x})$. We also define

$$\hat{f}_{q,k}^i = \frac{1}{K-k} \sum_{\ell=k+1}^K f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i), \quad \nabla \hat{f}_{q,k}^i = \frac{1}{K-k} \sum_{\ell=k+1}^K \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) \quad (11)$$

as the average of the remaining $(K - k)$ functions and stochastic gradients of $f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i)$ respectively. Then we note,

$$\hat{F}_{q,k} = \frac{1}{n} \sum_{i=1}^n \hat{f}_{q,k}^i, \quad \nabla \hat{F}_{q,k} = \frac{1}{n} \sum_{i=1}^n \nabla \hat{f}_{q,k}^i, \quad (12)$$

In the same spirit of $\hat{f}_{q,k}^i$, we define

$$\hat{\mathbf{g}}_{q,k}^i = \frac{1}{K-k} \sum_{\ell=k+1}^K \mathbf{g}_{q,\ell}^i, \quad \hat{\mathbf{d}}_{q,k}^i = \frac{1}{K-k} \sum_{\ell=k+1}^K \mathbf{d}_{q,\ell}^i \quad (13)$$

In the rest of the analysis, we let $\mathcal{F}_{q,1} \subset \dots \subset \mathcal{F}_{q,k}$ to be the σ -field generated by the permutation up to time k and $\mathcal{H}_{q,1} \subset \dots \subset \mathcal{H}_{q,k}$ another σ -field generated by the randomness of the stochastic gradient estimate up to time k .

Assumption 5. Let $\{\tilde{\mathbf{d}}_t\}_1^T$ be a sequence such that $\mathbb{E}[\tilde{\mathbf{d}}_t | \mathcal{H}_{t-1}] = \mathbf{d}_t$ where \mathcal{H}_{t-1} is the filtration of the stochastic estimate up to $t - 1$.

Lemma 5 (Fact 1, [6]). Let $\mathbf{x}^1, \dots, \mathbf{x}^n \in \mathbb{R}^d$ be a set of vector and $\mathbf{x}_{avg} := \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$ their average. Let \mathbf{W} be non-negative doubly stochastic matrix. The output of one round of the average consensus update $\bar{\mathbf{x}} = \sum_{j=1}^n W_{ij} \mathbf{x}^j$ satisfy :

$$\sqrt{\sum_{i=1}^n \|\bar{\mathbf{x}}^i - \mathbf{x}_{avg}\|^2} \leq |\lambda_2(\mathbf{W})| \cdot \sqrt{\sum_{i=1}^n \|\mathbf{x}^i - \mathbf{x}_{avg}\|^2}$$

where $\lambda_2(\mathbf{W})$ is the second largest eigenvalue of \mathbf{W} .

Lemma 6. For all $1 \leq q \leq Q$ and $1 \leq k \leq K$, we have

$$\bar{\mathbf{x}}_{q,k+1} = \bar{\mathbf{x}}_{q,k} + \eta_k \left(\frac{1}{n} \sum_{i=1}^n \mathbf{v}_{q,k}^i - \bar{\mathbf{x}}_{q,k} \right) \quad (14)$$

for convex case, and

$$\bar{\mathbf{x}}_{q,k+1} = \bar{\mathbf{x}}_{q,k} + \frac{1}{K} \left(\frac{1}{n} \sum_{i=1}^n \mathbf{v}_{q,k}^i \right) \quad (15)$$

for submodular case.

Proof.

$$\begin{aligned}
 \bar{\mathbf{x}}_{q,k+1} &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_{q,k+1}^i && \text{(Definition of } \bar{\mathbf{x}}_{q,k+1}\text{)} \\
 &= \frac{1}{n} \sum_{i=1}^n ((1 - \eta_k) \mathbf{y}_{q,k}^i + \eta_k \mathbf{v}_{q,k}^i) && \text{(Definition of } \mathbf{x}_{q,k}^i\text{)} \\
 &= \frac{1}{n} \sum_{i=1}^n \left[(1 - \eta_k) \left(\sum_{j=1}^n \mathbf{W}_{ij} \mathbf{x}_{q,k}^j \right) + \eta_k \mathbf{v}_{q,k}^i \right] && \text{(Definition of } \mathbf{y}_{q,k}^i\text{)} \\
 &= (1 - \eta_k) \frac{1}{n} \sum_{i=1}^n \left[\sum_{j=1}^n \mathbf{W}_{ij} \mathbf{x}_{q,k}^j \right] + \frac{1}{n} \eta_k \sum_{i=1}^n \mathbf{v}_{q,k}^i \\
 &= (1 - \eta_k) \frac{1}{n} \sum_{j=1}^n \left[\mathbf{x}_{q,k}^j \sum_{i=1}^n \mathbf{W}_{ij} \right] + \frac{1}{n} \eta_k \sum_{i=1}^n \mathbf{v}_{q,k}^i \\
 &= (1 - \eta_k) \frac{1}{n} \sum_{j=1}^n \mathbf{x}_{q,k}^j + \frac{1}{n} \eta_k \sum_{i=1}^n \mathbf{v}_{q,k}^i && (\sum_{i=1}^n W_{ij} = 1 \text{ for every } j) \\
 &= (1 - \eta_k) \bar{\mathbf{x}}_{q,k} + \frac{1}{n} \eta_k \sum_{i=1}^n \mathbf{v}_{q,k}^i \\
 &= \bar{\mathbf{x}}_{q,k} + \eta_k \left(\frac{1}{n} \sum_{i=1}^n \mathbf{v}_{q,k}^i - \bar{\mathbf{x}}_{q,k} \right)
 \end{aligned}$$

where we use a property of W which is $\sum_{i=1}^n W_{ij} = 1$ for every j . The proof of the second equation is similar. \square

Lemma 7. For all $k \in \{1, \dots, K\}$, it holds that

$$\nabla \hat{F}_{q,k} = \frac{1}{n} \sum_{i=1}^n \nabla \hat{f}_{q,k}^i = \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{g}}_{q,k}^i = \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{d}}_{q,k}^i \quad (16)$$

Proof. First, we verify that $\forall \ell \in \{1, \dots, K\}$

$$\frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) = \frac{1}{n} \sum_{i=1}^n \mathbf{g}_{q,\ell}^i = \frac{1}{n} \sum_{i=1}^n \mathbf{d}_{q,\ell}^i \quad (17)$$

For the base step $\ell = 1$, we have $\mathbf{g}_{q,\ell}^i = \nabla f_{\sigma_q(1)}^i(\mathbf{x}_{q,1}^i)$. Averaging over n yields

$$\frac{1}{n} \sum_{i=1}^n \mathbf{g}_{q,1}^i = \frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(1)}^i(\mathbf{x}_{q,1}^i)$$

Since \mathbf{W} is doubly stochastic, we have

$$\frac{1}{n} \sum_{i=1}^n \mathbf{d}_{q,1}^i = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n W_{ij} \mathbf{g}_{q,1}^j = \frac{1}{n} \sum_{j=1}^n \mathbf{g}_{q,1}^j \sum_{i=1}^n W_{ij} = \frac{1}{n} \sum_{j=1}^n \mathbf{g}_{q,1}^j \quad (18)$$

For the recurrence step, recall the definition of $\mathbf{g}_{q,\ell}^i$,

$$\mathbf{g}_{q,\ell}^i = \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) - \nabla f_{\sigma_q(\ell-1)}^i(\mathbf{x}_{q,\ell-1}^i) + \mathbf{d}_{q,\ell-1}^i$$

Averaging over n and using the recurrence hypothesis $\frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(\ell-1)}^i(\mathbf{x}_{q,\ell-1}^i) = \frac{1}{n} \sum_{i=1}^n \mathbf{d}_{q,\ell-1}^i$, we deduce that

$$\frac{1}{n} \sum_{i=1}^n \mathbf{g}_{q,\ell}^i = \frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) \quad (19)$$

Also, using the same techniques in equation (18) for $\mathbf{d}_{q,\ell}^i$, we complete the verification for equation (17). The proof of Lemma 7 can be deduced from equation (17) by averaging over $\ell \in \{k+1, \dots, K\}$ \square

Lemma 8. Suppose that each of $f_{\sigma_q(k)}^i$ is β -smooth. Using the Frank-Wolfe update of $\mathbf{x}_{q,k}^i$, the average of the remaining $(K - k)$ gradient approximation $\hat{\mathbf{d}}_{q,k}^i$ satisfies

$$\max_{i \in [1, n]} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\| \right] \leq \begin{cases} \frac{N}{k} & k \in \left[1, \frac{K}{2} \right] \\ \frac{N}{K - k + 1} & k \in \left[\frac{K}{2} + 1, K \right] \end{cases}$$

where $N = nk_0 \max\{\lambda_2 \left(1 + \frac{2}{1 - \lambda_2}\right), 2\}$.

Proof. We will prove the lemma by induction following the idea from Lemma 2 of [6]. Let's define following variables

$$\hat{\mathbf{d}}_{q,k}^{cat} = \left[\hat{\mathbf{d}}_{q,k}^{1\top}, \dots, \hat{\mathbf{d}}_{q,k}^{n\top} \right]^\top, \quad \hat{\mathbf{g}}_{q,k}^{cat} = \left[\hat{\mathbf{g}}_{q,k}^{1\top}, \dots, \hat{\mathbf{g}}_{q,k}^{n\top} \right]^\top, \quad \nabla \hat{F}_{q,k}^{cat} = \left[\nabla \hat{F}_{q,k}^\top, \dots, \nabla \hat{F}_{q,k}^\top \right]^\top \quad (20)$$

and let the slack variables as

$$\delta_{q,k}^i := \nabla f_{q,k}^i - \nabla f_{q,k-1}^i, \quad \bar{\delta}_{q,k} := \frac{1}{n} \sum_{i=1}^n \left(\nabla f_{q,k}^i - \nabla f_{q,k-1}^i \right) = \nabla \hat{F}_{q,k} - \nabla \hat{F}_{q,k-1} \quad (21)$$

then, following the definition in 20, we note

$$\delta_{q,k}^{cat} = \left[\delta_{q,k}^{1\top}, \dots, \delta_{q,k}^{n\top} \right]^\top, \quad \bar{\delta}_{q,k}^{cat} = \left[\bar{\delta}_{q,k}^\top, \dots, \bar{\delta}_{q,k}^\top \right]^\top$$

By Lemma 5, we have

$$\begin{aligned} \left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\|^2 &= \sum_{i=1}^n \left\| \hat{\mathbf{d}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\|^2 \\ &\leq \lambda_2^2 \sum_{i=1}^n \left\| \hat{\mathbf{g}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\|^2 \\ &= \lambda_2^2 \left\| \hat{\mathbf{g}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\|^2 \end{aligned} \quad (22)$$

We can deduce that

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] &\leq \lambda_2 \mathbb{E} \left[\left\| \hat{\mathbf{g}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] \\ &= \lambda_2 \mathbb{E} \left[\left\| \delta_{q,k}^{cat} + \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k}^{cat} + \nabla \hat{F}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] \\ &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] + \mathbb{E} \left[\left\| \delta_{q,k}^{cat} - \bar{\delta}_{q,k}^{cat} \right\| \right] \right) \\ &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] + \mathbb{E} \left[\left\| \delta_{q,k}^{cat} \right\| \right] \right) \end{aligned} \quad (23)$$

since

$$\left\| \delta_{q,k}^{cat} - \bar{\delta}_{q,k}^{cat} \right\|^2 = \sum_{i=1}^n \left\| \delta_{q,k}^i - \bar{\delta}_{q,k} \right\|^2 \leq \sum_{i=1}^n \left\| \delta_{q,k}^i \right\|^2 - n \left\| \bar{\delta}_{q,k} \right\|^2 \leq \sum_{i=1}^n \left\| \delta_{q,k}^i \right\|^2 = \left\| \delta_{q,k}^{cat} \right\|^2 \quad (24)$$

Notice that we can bound the expected value of δ^{cat} by

$$\begin{aligned}
 \mathbb{E} \left[\|\delta_{q,k}^{cat}\|^2 \right] &= \mathbb{E} \left[\sum_{i=1}^n \|\delta_{q,k}^i\|^2 \right] = \sum_{i=1}^n \mathbb{E} \left[\left\| \nabla \hat{f}_{q,k}^i - \nabla \hat{f}_{q,k-1}^i \right\|^2 \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \nabla \hat{f}_{q,k}^i - \nabla \hat{f}_{q,k-1}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \frac{\sum_{\ell=k+1}^K \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i)}{K-k} - \frac{\sum_{\ell=k}^K \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i)}{K-k+1} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \frac{\sum_{\ell=k+1}^K \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i)}{(K-k)(K-k+1)} - \frac{\nabla f_{\sigma_q(k)}^i(\mathbf{x}_{q,k}^i)}{K-k+1} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &\leq n \left(\frac{2G}{K-k+1} \right)^2
 \end{aligned} \tag{25}$$

using Jensen's inequality, we can deduce that

$$\mathbb{E} [\|\delta_{q,k}^{cat}\|] \leq \sqrt{\mathbb{E} \left[\|\delta_{q,k}^{cat}\|^2 \right]} \leq \frac{2\sqrt{n}G}{K-k+1} \tag{26}$$

We are now proving the lemma by induction, when $k = 1$, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,1}^{cat} - \nabla \hat{F}_{q,1}^{cat} \right\|^2 \right] &= \mathbb{E} \left[\sum_{i=1}^n \left\| \hat{\mathbf{d}}_{q,1}^i - \nabla \hat{F}_{q,1} \right\|^2 \right] \leq \lambda_2^2 \mathbb{E} \left[\sum_{i=1}^n \left\| \hat{\mathbf{g}}_{q,1}^i - \nabla \hat{F}_{q,1} \right\|^2 \right] \\
 &\leq \lambda_2^2 \mathbb{E} \left[\sum_{i=1}^n \left\| \nabla \hat{f}_{q,1}^i - \nabla \hat{F}_{q,1} \right\|^2 \right] \leq \lambda_2^2 \mathbb{E} \left[\sum_{i=1}^n \left\| \nabla \hat{f}_{q,1}^i \right\|^2 \right] \leq n\lambda_2^2 G^2
 \end{aligned}$$

where we have used Lipschitzness of f in the last inequality. We now suppose that $1 \leq k \leq k_0$, from equations (23) and (26)

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] + \mathbb{E} [\|\delta_{q,k}^{cat}\|] \right) \\
 &\leq \lambda_2^{k-1} \sqrt{n}G + 2 \sum_{\tau=1}^k \lambda_2^\tau \sqrt{n}G \\
 &\leq \lambda_2 \sqrt{n}G + 2 \frac{\lambda_2}{1-\lambda_2} \sqrt{n}G \\
 &= \lambda_2 \sqrt{n}G \left(1 + \frac{2}{1-\lambda_2} \right)
 \end{aligned} \tag{27}$$

We set $N_0 = k_0 \sqrt{n}G \max\{\lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2\}$, we claim that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] \leq \frac{N_0}{k}$ for $k \in [k_0, \frac{K}{2} + 1]$. Recall that $K - k + 1 \geq k - 1$, by equations (23) and (26) and induction hypothesis, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] + \mathbb{E} [\|\delta_{q,k}^{cat}\|] \right) \\
 &\leq \lambda_2 \left(\frac{N_0}{k-1} + \frac{2\sqrt{n}G}{K-k+1} \right) \\
 &\leq \lambda_2 \left(\frac{N_0}{k-1} + \frac{2\sqrt{n}G}{k-1} \right) \\
 &\leq \lambda_2 \left(\frac{N_0 + 2\sqrt{n}G}{k-1} \right) \\
 &\leq \lambda_2 \left(N_0 \frac{k_0 + 1}{k_0(k-1)} \right) \\
 &\leq \frac{N_0}{k}
 \end{aligned} \tag{28}$$

where we have used the fact that $\lambda_2(\mathbf{W}) \frac{k_0 + 1}{k_0(k-1)} \leq \frac{1}{k}$ in the last inequality. When $k \in [\frac{K}{2} + 1, K]$, we claim that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] \leq \frac{N_0}{K - k + 1}$. The base case $k = \frac{K}{2} + 1$ is verified by equation (28),

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] \leq \frac{N_0}{\frac{K}{2} + 1} \leq \frac{N_0}{\frac{K}{2}} \leq \frac{N_0}{K - (\frac{K}{2} + 1) + 1} \quad (29)$$

For $k \geq \frac{K}{2} + 2$, using equations (23) and (26) and the induction hypothesis, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat} - \nabla \hat{F}_{q,k-1}^{cat} \right\| \right] + \mathbb{E} \left[\left\| \delta_{q,k}^{cat} \right\| \right] \right) \\ &\leq \lambda_2 \left(\frac{N_0}{K - k + 2} + \frac{2\sqrt{n}G}{K - k + 1} \right) \\ &\leq \lambda_2 \left(\frac{N_0 + 2G}{K - k + 1} \right) \\ &\leq \lambda_2 \left(N_0 \frac{k_0 + 1}{k_0(K - k + 1)} \right) \\ &\leq \frac{N_0}{K - k + 1} \end{aligned} \quad (30)$$

Recall that

$$\frac{1}{\sqrt{n}} \mathbb{E} \left[\sum_{i=1}^n \left\| \hat{\mathbf{d}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\| \right] \leq \mathbb{E} \left[\left(\sum_{i=1}^n \left\| \hat{\mathbf{d}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\|^2 \right)^{1/2} \right] = \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat} - \nabla \hat{F}_{q,k}^{cat} \right\| \right] \quad (31)$$

The desired result followed from equations (28), (30) and (31) where $N = \sqrt{n}N_0$

$$\max_{i \in [1, n]} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^i - \nabla \hat{F}_{q,k} \right\| \right] \leq \begin{cases} \frac{N}{k} & k \in \left[1, \frac{K}{2} \right] \\ \frac{N}{K - k + 1} & k \in \left[\frac{K}{2} + 1, K \right] \end{cases} \quad (32)$$

□

Lemma 9. For $i \in [n], k \in [K]$. Let $V_d = 2nG \left(\frac{\lambda_2}{1 - \lambda_2} + 1 \right)$, the local gradient is upper-bounded, i.e $\left\| \hat{\mathbf{d}}_{q,k}^i \right\| \leq V_d$

Proof. We use the same notation introduced in equation (20). Let's define

$$\mathbf{d}_{q,k}^{cat} = [\mathbf{d}_{q,k}^{1\top}, \dots, \mathbf{d}_{q,k}^{n\top}]^\top \in \mathbb{R}^{nd}, \quad \nabla f_{\sigma_q(k)}^{cat} = [\nabla f_{\sigma_q(k)}^1(\mathbf{x}_{q,k}^1)^\top, \dots, \nabla f_{\sigma_q(k)}^n(\mathbf{x}_{q,k}^n)^\top]^\top \in \mathbb{R}^{nd} \quad (33)$$

and

$$\nabla F_{\sigma_q(k)}^{cat} = [\nabla F_{\sigma_q(k)}^\top, \dots, \nabla F_{\sigma_q(k)}^\top]^\top = \left[\frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(k)}^i(\mathbf{x}_{q,k}^i)^\top, \dots, \frac{1}{n} \sum_{i=1}^n \nabla f_{\sigma_q(k)}^i(\mathbf{x}_{q,k}^i)^\top \right]^\top \quad (34)$$

Using the local gradient update, we have

$$\begin{aligned}
 \mathbf{d}_{q,k}^{cat} &= (\mathbf{W} \otimes I_d) \left(\nabla f_{\sigma_q(k)}^{cat} - \nabla f_{\sigma_q(k-1)}^{cat} + \mathbf{d}_{q,k-1}^{cat} \right) \\
 &= (\mathbf{W} \otimes I_d) \left(\nabla f_{\sigma_q(k)}^{cat} - \nabla f_{\sigma_q(k-1)}^{cat} \right) + (\mathbf{W} \otimes I_d)^2 \left(\nabla f_{\sigma_q(k-1)}^{cat} - \nabla f_{\sigma_q(k-2)}^{cat} + \mathbf{d}_{q,k-2}^{cat} \right) \\
 &= \sum_{\tau=1}^{k-1} (\mathbf{W} \otimes I_d)^{k-\tau} \left(\nabla f_{\sigma_q(\tau+1)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right) + (\mathbf{W} \otimes I_d)^k \nabla f_{\sigma_q(1)}^{cat} \\
 &= \sum_{\tau=1}^{k-1} (\mathbf{W} \otimes I_d)^{k-\tau} \left(\nabla f_{\sigma_q(\tau+1)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right) + (\mathbf{W} \otimes I_d)^k \nabla f_{\sigma_q(1)}^{cat} \\
 &\quad - \sum_{\tau=1}^{k-1} \left(\nabla F_{\sigma_q(\tau+1)}^{cat} - \nabla F_{\sigma_q(\tau)}^{cat} \right) - \nabla F_{\sigma_q(1)}^{cat} + \nabla F_{\sigma_q(k)}^{cat} \\
 &= \sum_{\tau=1}^{k-1} \left[\left(\mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \left(\nabla f_{\sigma_q(\tau+1)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right) \\
 &\quad + \left[\left(\mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \nabla f_{\sigma_q(1)}^{cat} + \left(\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \nabla f_{\sigma_q(k)}^{cat}
 \end{aligned} \tag{35}$$

where the fourth equality holds since $\nabla F_{\sigma_q(k)}^{cat} - \sum_{\tau=1}^{k-1} \left(\nabla F_{\sigma_q(\tau+1)}^{cat} - \nabla F_{\sigma_q(\tau)}^{cat} \right) - \nabla F_{\sigma_q(1)}^{cat} = 0$. The fifth equality can be deduced using $\nabla F_{\sigma_q(k)}^{cat} = \left(\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \nabla f_{\sigma_q(k)}^{cat}$ and $(\mathbf{W} \otimes I_d)^k = (\mathbf{W}^k \otimes I_d)$. Recall that $\|\mathbf{W} \otimes I_d\| = \|\mathbf{W}\|$. Taking the norm on equation (35), we have

$$\|\mathbf{d}_{q,k}^{cat}\| \leq 2\sqrt{n}G \sum_{\tau=1}^{k-1} \lambda_2^{k-\tau} + \sqrt{n}G (\lambda_2^k + 1) \leq 2\sqrt{n}G \left(\frac{\lambda_2}{1-\lambda_2} + 1 \right) \tag{36}$$

where we have used $\left\| \nabla f_{\sigma_q(\tau+1)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right\| \leq 2\sqrt{n}G$, $\|\mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T\| \leq \lambda_2^k$ and $\|\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T\| \leq 1$ in the first inequality. We have $\forall i \in [n]$

$$\|\mathbf{d}_{q,k}^i\| \leq \sum_{i=1}^n \|\mathbf{d}_{q,k}^i\| \leq \sqrt{n} \left(\sum_{i=1}^n \|\mathbf{d}_{q,k}^i\|^2 \right)^{1/2} = \sqrt{n} \|\mathbf{d}_{q,k}^{cat}\| \tag{37}$$

one can obtain the desired result. \square

Lemma 10. Under Assumption 2 and let $\sigma_1^2 = 4n \left[\left(\frac{G+G_0}{\lambda_2-1} \right)^2 + 2\sigma_0^2 \right]$. For $i \in [n]$, $k \in [K]$, the variance of the local stochastic gradient is uniformly bounded.

$$\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] \leq \sigma_1^2$$

Proof. We denote $\tilde{\mathbf{d}}^{cat}$ the stochastique version of \mathbf{d}^{cat} , following equation (35), we have

$$\begin{aligned}
 \tilde{\mathbf{d}}_{q,k}^{cat} &= \sum_{\tau=1}^{k-1} \left[\left(\mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \left(\tilde{\nabla} f_{\sigma_q(\tau+1)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} \right) \\
 &\quad + \left[\left(\mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \tilde{\nabla} f_{\sigma_q(1)}^{cat} + \left(\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \tilde{\nabla} f_{\sigma_q(k)}^{cat}
 \end{aligned} \tag{38}$$

Then, we have

$$\begin{aligned}
 \mathbf{d}_{q,k}^{cat} - \tilde{\mathbf{d}}_{q,k}^{cat} &= \sum_{\tau=1}^{k-1} \left[\left(\mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \left(\nabla f_{\sigma_q(\tau+1)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau+1)}^{cat} + \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right) \\
 &\quad + \left[\left(\mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_d \right] \left(\nabla f_{\sigma_q(1)}^{cat} - \tilde{\nabla} f_{\sigma_q(1)}^{cat} \right) \\
 &\quad + \left(\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \left(\nabla f_{\sigma_q(k)}^{cat} - \tilde{\nabla} f_{\sigma_q(k)}^{cat} \right)
 \end{aligned} \tag{39}$$

By Assumption 2 and Jensen's inequality, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \nabla f_{\sigma_q(\tau)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} \right\|^2 \right] &= \mathbb{E} \left[\sum_{i=1}^n \left\| \nabla f_{\sigma_q(\tau)}^i(\mathbf{x}_{q,\tau}^i) - \tilde{\nabla} f_{\sigma_q(\tau)}^i(\mathbf{x}_{q,\tau}^i) \right\|^2 \right] \\ &\leq \sqrt{\sum_{i=1}^n \mathbb{E} \left[\left\| \nabla f_{\sigma_q(\tau)}^i(\mathbf{x}_{q,\tau}^i) - \tilde{\nabla} f_{\sigma_q(\tau)}^i(\mathbf{x}_{q,\tau}^i) \right\|^2 \right]} \leq \sqrt{n} \sigma_0 \end{aligned} \quad (40)$$

The second moment of equation (39) is written as

$$\begin{aligned} &\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{cat} - \tilde{\mathbf{d}}_{q,k}^{cat} \right\|^2 \right] \\ &\leq \mathbb{E} \left[\left(\sum_{\tau=1}^{k-1} \left\| \mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\| \left\| \nabla f_{\sigma_q(\tau+1)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau+1)}^{cat} + \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right\| \right)^2 \right] \\ &\quad + \mathbb{E} \left[\left\| \left(\mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \left(\nabla f_{\sigma_q(1)}^{cat} - \tilde{\nabla} f_{\sigma_q(1)}^{cat} \right) + \left(\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \right) \left(\nabla f_{\sigma_q(k)}^{cat} - \tilde{\nabla} f_{\sigma_q(k)}^{cat} \right) \right\|^2 \right] \\ &\leq \mathbb{E} \left[\left(\sum_{\tau=1}^{k-1} \left\| \mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\| \left\| \nabla f_{\sigma_q(\tau+1)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau+1)}^{cat} + \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right\| \right)^2 \right] \\ &\quad + 4 \left(\mathbb{E} \left[\left\| \mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\|^2 \left\| \nabla f_{\sigma_q(1)}^{cat} - \tilde{\nabla} f_{\sigma_q(1)}^{cat} \right\|^2 \right] + \mathbb{E} \left[\left\| \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\|^2 \left\| \nabla f_{\sigma_q(k)}^{cat} - \tilde{\nabla} f_{\sigma_q(k)}^{cat} \right\|^2 \right] \right) \\ &\leq 4n(G+G_0)^2 \left(\sum_{\tau=1}^{k-1} \lambda_2^{k-\tau} \right)^2 + 4n\sigma_0^2(\lambda_2^{2k} + 1) \\ &\leq 4n(G+G_0)^2 \left(\frac{\lambda_2}{1-\lambda_2} \right)^2 + 4n\sigma_0^2(\lambda_2 + 1) \leq 4n \left[\left(\frac{G+G_0}{\frac{1}{\lambda_2} - 1} \right)^2 + 2\sigma_0^2 \right] \end{aligned} \quad (41)$$

where the first inequality holds since $\mathbb{E} \left[\nabla f_{\sigma_q(\tau+1)}^{cat} - \tilde{\nabla} f_{\sigma_q(\tau+1)}^{cat} + \tilde{\nabla} f_{\sigma_q(\tau)}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right] = 0$. The second inequality follows the fact that $\|a+b\|^2 \leq 4(\|a\|^2 + \|b\|^2)$. The third inequality comes from Assumption 2 and the analysis in Lemma 1. Finally, one can obtain the desired result by noticing $\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] \leq \sum_{i=1}^n \mathbb{E} \left[\left\| \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] = \mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{cat} - \tilde{\mathbf{d}}_{q,k}^{cat} \right\|^2 \right]$ \square

Lemma 11 (Lemma 6, [20]). *Under Assumption 5, Lemma 1, Lemma 2 and setting $\rho_k = \frac{2}{(k+3)^{2/3}}$ and $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$ for $k \in [\frac{K}{2}]$ and $k \in [\frac{K}{2} + 1, K]$ respectively, we have*

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \right\| \right] \leq \begin{cases} \frac{\sqrt{M}}{(k+4)^{1/3}} & k \in \left[\frac{K}{2} \right] \\ \frac{\sqrt{M}}{(K-k+1)^{1/3}} & i \in \left[\frac{K}{2} + 1, K \right] \end{cases} \quad (42)$$

where $M = \max\{M_1, M_2\}$ where $M_1 = \max\{5^{2/3}(V_d + L_0)^2, M_0\}$, $M_0 = 4(V_d^2 + \sigma^2) + 32\sqrt{2}V_d$ and $M_2 = 2.55(V_d^2 + \sigma^2) + \frac{7\sqrt{2}V_d}{3}$ and $L_0 = \frac{2}{4^{2/3}} \left\| \tilde{\mathbf{d}}_{q,1}^i \right\|$

Proof. In order to prove the lemma, we only need to bound $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right]$, following the decomposition in [20], we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &= \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - (1 - \rho_k) \tilde{\mathbf{a}}_{q,k-1}^i - \rho_k \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] \\
 &= \rho_k^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] + (1 - \rho_k)^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\|^2 \right] \\
 &\quad + (1 - \rho_k)^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\quad + 2\rho_k(1 - \rho_k) \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\rangle \right] \\
 &\quad + 2\rho_k(1 - \rho_k) \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\
 &\quad + 2(1 - \rho_k)^2 \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i, \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right]
 \end{aligned} \tag{43}$$

The first part of the above equation is written as

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i + \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &\leq \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i \right\|^2 + \left\| \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 + 2 \langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \rangle \mid \mathcal{F}_{q,k-1} \right] \right]
 \end{aligned} \tag{45}$$

Using the definition of $\hat{\mathbf{d}}_{q,k-1}^i$, Lemma 1 and Lemma 2 and law of total expectation, we have

$$\mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] = \mathbb{E} \left[\text{Var}_\sigma \left(\mathbf{d}_{q,k}^i \mid \mathcal{F}_{q,k-1} \right) \right] \leq \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \mathbf{d}_{q,k}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \leq V_{\mathbf{d}}^2 \tag{46}$$

$$\mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \leq \sigma_1^2 \tag{47}$$

Recall that $\mathcal{H}_{q,k}$ is the filtration related to the randomness of $\tilde{\mathbf{d}}_{q,k}^i$ and $\hat{\mathbf{d}}_{q,k-1}^i$ and $\mathbf{d}_{q,k}^i$ is $\mathcal{F}_{q,k}$ -measurable, then one can write

$$\begin{aligned}
 &\mathbb{E} \left[\mathbb{E}_\sigma \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\rangle \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\rangle \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \right\rangle \mid \mathcal{F}_{q,k} \right] \right] \\
 &= \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbb{E}_\sigma \left[\mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k} \right] \right\rangle \right] \quad (\text{by } \mathcal{F}_{q,k}\text{-measurability}) \\
 &= \mathbb{E} \left[\mathbb{E}_{\tilde{\mathbf{d}}} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbb{E}_\sigma \left[\mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k} \right] \right\rangle \mid \mathcal{H}_{q,k-1} \right] \right] \\
 &= \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbb{E}_{\tilde{\mathbf{d}}} \left[\mathbb{E}_\sigma \left[\mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k} \right] \mid \mathcal{H}_{q,k-1} \right] \right\rangle \right] \\
 &= \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \mathbf{d}_{q,k}^i, \mathbb{E}_\sigma \left[\mathbb{E}_{\tilde{\mathbf{d}}} \left[\mathbf{d}_{q,k}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{H}_{q,k-1} \right] \mid \mathcal{F}_{q,k} \right] \right\rangle \right] \quad (\text{by Fubini's theorem}) \\
 &= 0
 \end{aligned} \tag{48}$$

where the last equation holds since $\mathbb{E}_{\tilde{\mathbf{d}}} \left[\tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{H}_{q,k-1} \right] = \mathbf{d}_{q,k}^i$. Combining equations (46) to (48), equation (45) is upper bounded by

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \right\|^2 \right] \leq V_{\mathbf{d}}^2 + \sigma_1^2 \triangleq V \tag{49}$$

We are now bounding $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\|^2 \right]$, using the definition of $\hat{\mathbf{d}}_{q,k}^i$ and Lemma 1. We have

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\|^2 \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\|^2 \mid \mathcal{F}_{q,k-2} \right] \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \frac{\sum_{\ell=k}^K \mathbf{d}_{q,\ell}^i}{K-k+1} - \frac{\sum_{\ell=k-1}^K \mathbf{d}_{q,\ell}^i}{K-k+2} \right\|^2 \mid \mathcal{F}_{q,k-2} \right] \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \frac{\sum_{\ell=k}^K \mathbf{d}_{q,\ell}^i}{K-k+1} - \frac{\sum_{\ell=k}^K \mathbf{d}_{q,\ell}^i}{K-k+2} - \frac{\mathbf{d}_{q,k-1}^i}{K-k+2} \right\|^2 \mid \mathcal{F}_{q,k-2} \right] \right] \\
 &= \mathbb{E} \left[\mathbb{E}_\sigma \left[\left\| \frac{\sum_{\ell=k}^K \mathbf{d}_{q,\ell}^i}{(K-k+1)(K-k+2)} - \frac{\mathbf{d}_{q,k-1}^i}{K-k+2} \right\|^2 \mid \mathcal{F}_{q,k-2} \right] \right] \\
 &\leq \mathbb{E} \left[\mathbb{E}_\sigma \left[\left(\frac{\sum_{\ell=k}^K \|\mathbf{d}_{q,\ell}^i\|}{(K-k+1)(K-k+2)} + \frac{\|\mathbf{d}_{q,k-1}^i\|}{K-k+2} \right)^2 \mid \mathcal{F}_{q,k-2} \right] \right] \\
 &\leq \frac{4V_{\mathbf{d}}^2}{(K-k+2)^2} \triangleq \frac{L}{(K-k+2)^2} \tag{50}
 \end{aligned}$$

More over, we have

$$\begin{aligned}
 & \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\rangle \right] \\
 &= \mathbb{E} \left[\mathbb{E}_{\sigma, \tilde{\mathbf{d}}} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\rangle \mid \mathcal{F}_{q,k-1}, \mathcal{H}_{q,k-1} \right] \right] \\
 &= \mathbb{E} \left[\left\langle \mathbb{E}_{\sigma, \tilde{\mathbf{d}}} \left[\hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k-1}, \mathcal{H}_{q,k-1} \right], \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\rangle \right] \\
 &= 0 \tag{51}
 \end{aligned}$$

since $\mathbb{E}_{\tilde{\mathbf{d}}} \left[\tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{H}_{q,k-1} \right] = \hat{\mathbf{d}}_{q,k}^i$ and $\mathbb{E}_\sigma \left[\hat{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k-1} \right] = \hat{\mathbf{d}}_{q,k}^i$. Using the same argument, we can deduce

$$\begin{aligned}
 & \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\
 &= \mathbb{E} \left[\mathbb{E}_{\sigma, \tilde{\mathbf{d}}} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i, \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \mid \mathcal{F}_{q,k-1}, \mathcal{H}_{q,k-1} \right] \right] \\
 &= \mathbb{E} \left[\left\langle \mathbb{E}_{\sigma, \tilde{\mathbf{d}}} \left[\hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{d}}_{q,k}^i \mid \mathcal{F}_{q,k-1}, \mathcal{H}_{q,k-1} \right], \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\
 &= 0 \tag{52}
 \end{aligned}$$

where we have use law of total expectation and conditional unbiasedness of $\tilde{\mathbf{d}}_{q,k}^i$. Using Young's inequality and equation (50), one can write

$$\begin{aligned}
 & \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i, \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\
 &\leq \mathbb{E} \left[\frac{1}{2\alpha_k} \left\| \hat{\mathbf{d}}_{q,k-1}^i - \hat{\mathbf{d}}_{q,k-2}^i \right\|^2 + \frac{\alpha_k}{2} \left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\leq \frac{L}{2\alpha_k(K-k+2)^2} + \frac{\alpha_k}{2} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \tag{53}
 \end{aligned}$$

With the above analysis, we can deduce that

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \rho_k^2 V + (1-\rho_k)^2 \frac{L}{(K-k+2)^2} + (1-\rho_k)^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\quad + (1-\rho_k)^2 \left(\frac{L}{\alpha_k(K-k+2)^2} + \alpha_k \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \right) \tag{54}
 \end{aligned}$$

Setting $\alpha_k = \frac{\rho_k}{2}$, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \rho_k^2 V + (1 - \rho_k)^2 \left(1 + \frac{2}{\rho_k} \right) \frac{L}{(K - k + 2)^2} \\
 &\quad + (1 - \rho_k)^2 \left(1 + \frac{\rho_k}{2} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\leq \rho_k^2 V + \left(1 + \frac{2}{\rho_k} \right) \frac{L}{(K - k + 2)^2} + (1 - \rho_k) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \tag{55}
 \end{aligned}$$

For $k \leq \frac{K}{2} + 1$, we set $\rho_k = \frac{2}{(k+3)^{2/3}}$ and recall that $K - k + 2 \geq k$, equation (55) is written as:

$$\begin{aligned}
 &\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \\
 &\leq \frac{4}{(k+3)^{4/3}} V + \left(1 + (k+3)^{2/3} \right) \frac{L}{k^2} + \left(1 - \frac{2}{(k+3)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\leq \frac{4}{(k+3)^{4/3}} V + \left(1 + (k+3)^{2/3} \right) \frac{16L}{(k+3)^2} + \left(1 - \frac{2}{(k+3)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\leq \frac{4}{(k+3)^{4/3}} V + \frac{16L}{(k+3)^{4/3}} + \frac{16L}{(k+3)^{4/3}} + \left(1 - \frac{2}{(k+3)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\leq \frac{4V + 32L}{(k+3)^{4/3}} + \left(1 - \frac{2}{(k+3)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\
 &\triangleq \frac{M_0}{(k+3)^{4/3}} + \left(1 - \frac{2}{(k+3)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \tag{56}
 \end{aligned}$$

We consider the base step where $k = 1$,

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,0}^i - \tilde{\mathbf{a}}_{q,1}^i \right\|^2 \right] &= \mathbb{E} \left[\left\| \frac{1}{K} \sum_{\ell=1}^K \mathbf{d}_{q,\ell}^i - \frac{2}{4^{2/3}} \tilde{\mathbf{d}}_{q,1}^i \right\|^2 \right] \\
 &\leq \left(V_{\mathbf{d}} + \frac{2}{4^{2/3}} \left\| \tilde{\mathbf{d}}_{q,1}^i \right\| \right)^2 \\
 &\leq \left(V_{\mathbf{d}} + \frac{2}{4^{2/3}} G_0 \right)^2 \\
 &\triangleq (V_{\mathbf{d}} + L_0)^2 \tag{57}
 \end{aligned}$$

Set $M_1 = \max \left\{ 5^{2/3} (V_{\mathbf{d}} + L_0)^2, M_0 \right\}$. For $k \in [\frac{K}{2} + 1]$, we claim that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \leq \frac{M_1}{(k+4)^{2/3}}$.

Suppose the claim holds for $k - 1$, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \frac{M_1}{(k+3)^{4/3}} + \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \left(1 - \frac{2}{(k+3)^{2/3}} \right) \\
 &\leq \frac{M_1}{(k+3)^{4/3}} + \frac{M_1}{(k+3)^{2/3}} \cdot \frac{(k+3)^{2/3} - 2}{(k+3)^{2/3}} \\
 &\leq \frac{M_1 ((k+3)^{2/3} - 1)}{(k+3)^{4/3}} \\
 &\leq \frac{M_1}{(k+4)^{2/3}} \tag{58}
 \end{aligned}$$

since $\frac{(k+3)^{2/3}-1}{(k+3)^{4/3}} \leq \frac{1}{(k+4)^{2/3}}$. For $k \in [\frac{K}{2}+1, K]$, we set $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$, thus

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \frac{2.55V}{(K-k+2)^{4/3}} + \left(1 + \frac{4}{3} (K-k+2)^{2/3} \right) \frac{L}{(K-k+2)^2} \\
 &\quad + \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \left(1 - \frac{1.5}{(K-k+2)^{2/3}} \right) \\
 &\leq \frac{2.55V}{(K-k+2)^{4/3}} + \frac{L}{(K-k+2)^{4/3}} + \frac{4}{3} \frac{L}{(K-k+2)^{4/3}} \\
 &\quad + \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \left(1 - \frac{1.5}{(K-k+2)^{2/3}} \right) \\
 &\leq \frac{2.55V + 7L/3}{(K-k+2)^{4/3}} + \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \left(1 - \frac{1.5}{(K-k+2)^{2/3}} \right) \\
 &\triangleq \frac{M_2}{(K-k+2)^{4/3}} + \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^i - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \left(1 - \frac{1.5}{(K-k+2)^{2/3}} \right) \tag{59}
 \end{aligned}$$

Let $M = \max \{M_1, M_2\}$ and $k \in [\frac{K}{2}+1, K]$, we claim that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \leq \frac{M}{(K-k+1)^{2/3}}$. The base step is verified by equation (58). We now suppose the claim holds for $k-1$, let's prove for k .

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \frac{M}{(K-k+2)^{4/3}} + \frac{M}{(K-k+2)^{2/3}} \cdot \frac{(K-k+2)^{2/3} - 1.5}{(K-k+2)^{2/3}} \\
 &= \frac{M \left((K-k+2)^{2/3} - 0.5 \right)}{(K-k+2)^{4/3}} \\
 &\leq \frac{M}{(K-k+1)^{2/3}} \tag{60}
 \end{aligned}$$

Thus, from equation (58) and equation (60), we have

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \leq \begin{cases} \frac{M}{(k+4)^{2/3}} & k \in \left[1, \frac{K}{2} \right] \\ \frac{M}{(K-k+1)^{2/3}} & k \in \left[\frac{K}{2} + 1, K \right] \end{cases} \tag{61}$$

Thus, using Jensen inequality, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] &\leq \sqrt{\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right]} \\
 &= \sqrt{\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right]} \\
 &\leq \begin{cases} \frac{\sqrt{M}}{(k+4)^{1/3}} & k \in \left[1, \frac{K}{2} \right] \\ \frac{\sqrt{M}}{(K-k+1)^{1/3}} & k \in \left[\frac{K}{2} + 1, K \right] \end{cases} \tag{62}
 \end{aligned}$$

□

Claim 1.

$$\mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \nabla \hat{F}_{q,k-1} \right\| \right] \leq \beta D \tag{63}$$

Proof of claim. Recall the definition of $\bar{F}_{q,k-1}$ and $\hat{F}_{q,k-1}$,

$$\begin{aligned}\bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) &= \frac{1}{K-k+1} \sum_{\ell=k}^K \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^i(\bar{\mathbf{x}}_{q,k}) \\ \hat{F}_{q,k-1} &= \frac{1}{K-k+1} \sum_{\ell=k}^K \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i)\end{aligned}$$

we have,

$$\begin{aligned}& \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_q^k) - \nabla \hat{F}_{q,k-1} \right\| \right] \\ &= \mathbb{E} \left[\left\| \frac{1}{K-k+1} \cdot \frac{1}{n} \sum_{\ell=k}^K \sum_{i=1}^n \left(\nabla f_{\sigma_q(\ell)}^i(\bar{\mathbf{x}}_{q,k}) - \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) \right) \right\| \right] \\ &\leq \mathbb{E} \left[\frac{1}{K-k+1} \cdot \frac{1}{n} \sum_{\ell=k}^K \sum_{i=1}^n \left\| \nabla f_{\sigma_q(\ell)}^i(\bar{\mathbf{x}}_{q,k}) - \nabla f_{\sigma_q(\ell)}^i(\mathbf{x}_{q,\ell}^i) \right\| \right] \\ &\leq \mathbb{E} \left[\frac{1}{K-k+1} \cdot \frac{1}{n} \sum_{\ell=k}^K \sum_{i=1}^n \beta \|\bar{\mathbf{x}}_{q,k} - \mathbf{x}_{q,\ell}^i\| \right] \quad (\text{by } \beta\text{-smoothness}) \\ &\leq \beta D \quad (64)\end{aligned}$$

□

Claim 2.

$$\sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \leq \beta D + \left(N + \sqrt{M} \right) 3K^{2/3} \quad (65)$$

Proof of claim.

$$\begin{aligned}& \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\ &\leq \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \nabla \hat{F}_{q,k-1} \right\| \right] + \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1} - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\ &\leq \beta D + \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1} - \hat{\mathbf{d}}_{q,k-1}^i \right\| \right] + \sum_{k=1}^K \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \quad (66)\end{aligned}$$

where we have used Claim 1 and triangle inequality in the last inequality. Using Lemma 8, we have

$$\begin{aligned}& \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1} - \hat{\mathbf{d}}_{q,k-1}^i \right\| \right] \\ &= \sum_{k=1}^{K/2} \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1} - \hat{\mathbf{d}}_{q,k-1}^i \right\| \right] + \sum_{k=K/2+1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1} - \hat{\mathbf{d}}_{q,k-1}^i \right\| \right] \\ &\leq \sum_{k=1}^{K/2} \frac{N}{k} + \sum_{k=K/2+1}^K \frac{N}{K-k+1} \quad (67)\end{aligned}$$

By Lemma 11, we also have

$$\begin{aligned}
 & \sum_{k=1}^K \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\
 &= \sum_{k=1}^{K/2} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] + \sum_{k=K/2+1}^K \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^i - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\
 &\leq \sum_{k=1}^{K/2} \frac{\sqrt{M}}{(k+4)^{1/3}} + \sum_{k=K/2+1}^K \frac{\sqrt{M}}{(K-k+1)^{1/3}}
 \end{aligned} \tag{68}$$

Combining equation (67) and equation (68), equation (66) is written as

$$\begin{aligned}
 & \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_q^k) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\
 &\leq \beta D + \sum_{k=1}^{K/2} \left(\frac{N}{k} + \frac{\sqrt{M}}{(k+4)^{1/3}} \right) + \sum_{k=K/2+1}^K \left(\frac{N}{K-k+1} + \frac{\sqrt{M}}{(K-k+1)^{1/3}} \right) \\
 &\leq \beta D + (N + \sqrt{M}) \sum_{k=1}^{K/2} \frac{1}{(k+4)^{1/3}} + (N + \sqrt{M}) \sum_{k=K/2+1}^K \frac{1}{(K-k+1)^{1/3}} \\
 &\leq \beta D + (N + \sqrt{M}) \sum_{k=1}^{K/2} \frac{1}{k^{1/3}} + (N + \sqrt{M}) \sum_{l=1}^{K/2} \frac{1}{l^{1/3}} \\
 &\leq \beta D + 2(N + \sqrt{M}) \int_0^{K/2} \frac{1}{s^{1/3}} ds \\
 &\leq \beta D + 2(N + \sqrt{M}) \frac{3}{2} \left(\frac{K}{2} \right)^{2/3} \\
 &\leq \beta D + (N + \sqrt{M}) 3K^{2/3}
 \end{aligned} \tag{69}$$

□

A.1 Proof of Theorem 1

Theorem 3. *Given a convex set \mathcal{K} with diameters D . Assume that function F_t are convex, β -smooth and $\|\nabla F_t\| \leq G$ for $t \in [T]$. Setting $Q = T^{2/5}$, $K = T^{3/5}$, $T = QK$ and step-size $\eta_k = \frac{1}{k}$. Let $\rho_k = \frac{2}{(k+3)^{2/3}}$ and $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$ when $k \in [1, \frac{K}{2}]$ and $k \in [\frac{K}{2} + 1, K]$ respectively. Then, the expected regret of Algorithm 1 is at most*

$$\mathbb{E}[\mathcal{R}_T] \leq (GD + 2\beta D^2) T^{2/5} + \left(C + 6D(N + \sqrt{M}) \right) T^{4/5} + \frac{3}{5} \beta D^2 T^{2/5} \log(T) \tag{70}$$

where $N = k_0 \cdot nG \max\{\lambda_2 \left(1 + \frac{2}{1-\lambda_2}\right), 2\}$ and $M = \max\{M_1, M_2\}$ where $M_0 = 4(V_d^2 + \sigma_1^2) + 128V_d^2$, $M_1 = \max\left\{5^{2/3} \left(V_d + \frac{2}{4^{2/3}} G_0\right)^2, M_0\right\}$ and $M_2 = 2.55(V_d^2 + \sigma_1^2) + \frac{28V_d^2}{3}$. All the constant are defined in Lemma 1, Lemma 2, Lemma 8 and Lemma 11.

Proof.

$$\begin{aligned}
 & \mathbb{E} \left[\bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k+1}) - \bar{F}_{q,k-1}(\mathbf{x}^*) \right] \\
 &\leq (1 - \eta_k) \mathbb{E} \left[\bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \bar{F}_{q,k-1}(\mathbf{x}^*) \right] + \frac{\eta_k}{n} \sum_{i=1}^n \mathbb{E} \left[\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle \right] \\
 &\quad + \frac{\eta_k}{n} D \sum_{i=1}^n \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] + \frac{\beta}{2} \eta_k^2 D^2
 \end{aligned} \tag{71}$$

As $\mathbb{E} [\bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \bar{F}_{q,k-1}(\mathbf{x}^*)] = \mathbb{E} [\bar{F}_{q,k-2}(\bar{\mathbf{x}}_{q,k}) - \bar{F}_{q,k-2}(\mathbf{x}^*)]$, we can apply equation (71) recursively for $k \in \{1, \dots, K\}$, thus

$$\begin{aligned} & \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_q) - \bar{F}_{q,0}(\mathbf{x}^*)] \\ & \leq \prod_{k=1}^K (1 - \eta_k) \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_{q,1}) - \bar{F}_{q,0}(\mathbf{x}^*)] + \sum_{k=1}^K \prod_{k'=k+1}^K (1 - \eta_{k'}) \frac{\eta_k}{n} \sum_{i=1}^n \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle] \\ & \quad + \sum_{k=1}^K \prod_{k'=k+1}^K (1 - \eta_{k'}) \frac{\eta_k}{n} D \sum_{i=1}^n \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\|] + \frac{\beta}{2} D^2 \sum_{k=1}^K \prod_{k'=k+1}^K (1 - \eta_{k'}) \eta_k^2 \end{aligned} \quad (72)$$

Choosing $\eta_k = \frac{1}{k}$, we have

$$\prod_{k=r}^K (1 - \eta_k) \leq \exp \left(- \sum_{k=r}^K \frac{1}{k} \right) \leq \frac{r}{K}$$

We have then,

$$\begin{aligned} & \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_q) - \bar{F}_{q,0}(\mathbf{x}^*)] \\ & \leq \frac{1}{K} \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_{q,1}) - \bar{F}_{q,0}(\mathbf{x}^*)] + \sum_{k=1}^K \frac{k+1}{K} \cdot \frac{1}{k} \cdot \frac{1}{n} \sum_{i=1}^n \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle] \\ & \quad + \sum_{k=1}^K \frac{k+1}{K} \cdot \frac{1}{k} \cdot \frac{1}{n} D \sum_{i=1}^n \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\|] + \frac{\beta}{2} D^2 \sum_{k=1}^K \frac{k+1}{K} \cdot \frac{1}{k^2} \end{aligned} \quad (73)$$

Which maybe simplified by using $\frac{k+1}{K} \cdot \frac{1}{k} \leq \frac{2}{K}$.

$$\begin{aligned} & \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_q) - \bar{F}_{q,0}(\mathbf{x}^*)] \\ & \leq \frac{1}{K} \mathbb{E} [\bar{F}_{q,0}(\bar{\mathbf{x}}_{q,1}) - \bar{F}_{q,0}(\mathbf{x}^*)] + \frac{2}{K} \cdot \frac{1}{n} \sum_{k=1}^K \sum_{i=1}^n \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle] \\ & \quad + \frac{2}{K} \cdot \frac{1}{n} D \sum_{k=1}^K \sum_{i=1}^n \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\|] + \frac{\beta D^2}{2} \frac{2}{K} \sum_{k=1}^K \frac{1}{k} \\ & \leq \frac{GD}{K} + \frac{2}{K} \cdot \frac{1}{n} \sum_{k=1}^K \sum_{i=1}^n \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle] \\ & \quad + \frac{2}{K} \cdot D \left(\beta D + (N + \sqrt{M}) 3K^{2/3} \right) + \frac{\beta D^2}{K} \log K \end{aligned} \quad (74)$$

where we have used Claim 2, G -Lipschitz property of $\bar{F}_{q,0}$ and boundedness of \mathcal{K} . Since $T = QK$ and assume that the oracle at round k has a regret of order $\mathcal{O}(\sqrt{Q})$, i.e

$$\mathbb{E} \left[\sum_{q=1}^Q \langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{v}_{q,k}^i - \mathbf{x}^* \rangle \right] \leq C\sqrt{Q}$$

then, the expected regret of the algorithm upper bounded by

$$\begin{aligned} \mathbb{E} [\mathcal{R}_T] & = \mathbb{E} \left[\sum_{q=1}^Q K (\bar{F}_{q,0}(\bar{\mathbf{x}}_q) - \bar{F}_{q,0}(\mathbf{x}^*)) \right] \\ & \leq QGD + CKQ^{1/2} + 2QD \left(\beta D + (N + \sqrt{M}) 3K^{2/3} \right) + Q\beta D^2 \log K \\ & \leq QGD + CKQ^{1/2} + 2Q\beta D^2 + 6D (N + \sqrt{M}) QK^{2/3} + Q\beta D^2 \log K \\ & \leq (GD + 2\beta D^2) Q + CKQ^{1/2} + 6D (N + \sqrt{M}) QK^{2/3} + Q\beta D^2 \log K \end{aligned} \quad (75)$$

Setting $Q = T^{2/5}$ and $K = T^{3/5}$, we have

$$\mathbb{E}[\mathcal{R}_T] \leq (GD + 2\beta D^2) T^{2/5} + \left(C + 6D(N + \sqrt{M})\right) T^{4/5} + \frac{3}{5}\beta D^2 T^{2/5} \log(T) \quad (76)$$

□

A.2 Proof of Theorem 2

Lemma 12. *If F_t is monotone continuous DR-submodular and β -smoothness, $\mathbf{x}_{t,k+1} = \mathbf{x}_{t,k} + \frac{1}{K}\mathbf{v}_{t,k}$ for $k \in [1, \dots, K]$, then*

$$\begin{aligned} F_t(\mathbf{x}^*) - F_t(\mathbf{x}_{t,k+1}) &\leq (1 - 1/K) [F_t(\mathbf{x}^*) - F_t(\mathbf{x}_{t,k})] \\ &\quad - \frac{1}{K} [-\|\nabla F_t(\mathbf{x}_{t,k}) - \mathbf{d}_{t,k}\| D + \langle \mathbf{d}_{t,k}, \mathbf{v}_{t,k} - \mathbf{x}^* \rangle] + \frac{\beta D^2}{2K^2} \end{aligned} \quad (77)$$

Proof. The proof is essentially based on the analysis of [19]. By β -smoothness of F_t ,

$$\begin{aligned} F_t(\mathbf{x}_{t,k+1}) &\geq F_t(\mathbf{x}_{t,k}) + \langle F_t(\mathbf{x}_{t,k}), \mathbf{x}_{t,k+1} - \mathbf{x}_{t,k} \rangle - \frac{\beta}{2} \|\mathbf{x}_{t,k+1} - \mathbf{x}_{t,k}\|^2 \\ &\geq F_t(\mathbf{x}_{t,k}) + \frac{1}{K} \langle F_t(\mathbf{x}_{t,k}), \mathbf{v}_{t,k} \rangle - \frac{\beta D^2}{2K^2} \quad (\text{since } \|\mathbf{v}_{t,k}\| \leq D) \\ &\geq F_t(\mathbf{x}_{t,k}) + \frac{1}{K} [\langle \nabla F_t(\mathbf{x}_{t,k}) - \mathbf{d}_{t,k}, \mathbf{v}_{t,k} - \mathbf{x}^* \rangle + \langle \nabla F_t(\mathbf{x}_{t,k}), \mathbf{x}^* \rangle + \langle \mathbf{d}_{t,k}, \mathbf{v}_{t,k} - \mathbf{x}^* \rangle] - \frac{\beta D^2}{2K^2} \end{aligned} \quad (78)$$

By Cauchy-Schwarz's inequality, note that,

$$\langle \nabla F_t(\mathbf{x}_{t,k}) - \mathbf{d}_{t,k}, \mathbf{v}_{t,k} - \mathbf{x}^* \rangle \geq -\|\nabla F_t(\mathbf{x}_{t,k}) - \mathbf{d}_{t,k}\| D$$

Using concavity along non-negative direction and monotonicity of F_t , we have,

$$\begin{aligned} F_t(\mathbf{x}^*) - F_t(\mathbf{x}_{t,k}) &\leq F_t(\mathbf{x}^* \vee \mathbf{x}_{t,k}) - F_t(\mathbf{x}_{t,k}) \\ &\leq \langle \nabla F_t(\mathbf{x}_{t,k}), (\mathbf{x}^* \vee \mathbf{x}_{t,k}) - \mathbf{x}_{t,k} \rangle \\ &= \langle \nabla F_t(\mathbf{x}_{t,k}), (\mathbf{x}^* - \mathbf{x}_{t,k}) \vee 0 \rangle \\ &\leq \langle \nabla F_t(\mathbf{x}_{t,k}), \mathbf{x}^* \rangle \end{aligned} \quad (79)$$

then, equation (78) becomes

$$\begin{aligned} F_t(\mathbf{x}_{t,k+1}) &\geq F_t(\mathbf{x}_{t,k}) + \langle F_t(\mathbf{x}_{t,k}), \mathbf{x}_{t,k+1} - \mathbf{x}_{t,k} \rangle - \frac{\beta}{2} \|\mathbf{x}_{t,k+1} - \mathbf{x}_{t,k}\|^2 \\ &\geq F_t(\mathbf{x}_{t,k}) + \frac{1}{K} [-\|\nabla F_t(\mathbf{x}_{t,k}) - \mathbf{d}_{t,k}\| D + F_t(\mathbf{x}^*) - F_t(\mathbf{x}_{t,k}) + \langle \mathbf{d}_{t,k}, \mathbf{v}_{t,k} - \mathbf{x}^* \rangle] - \frac{\beta D^2}{2K^2} \end{aligned} \quad (80)$$

Adding and subtracting $F_t(\mathbf{x}^*)$ and multiply both side by -1 yields lemma 12. □

Theorem 4. *Given a convex set \mathcal{K} with diameters D . Assume that functions F_t are monotone continuous DR-Submodular, β -smooth and G -Lipschitz. Setting $Q = T^{2/5}$, $K = T^{3/5}$, $T = QK$ and step-size $\eta_k = \frac{1}{K}$. Let $\rho_k = \frac{2}{(k+3)^{2/3}}$ and $\rho_k = \frac{1.5}{(K-k+2)^{2/3}}$ when $1 \leq k \leq \frac{K}{2} + 1$ and $\frac{K}{2} + 1 \leq k \leq K$ respectively. Then, the expected $(1 - \frac{1}{e})$ -regret is at most*

$$\mathbb{E}[\mathcal{R}_T] \leq \frac{3}{2}\beta D^2 T^{2/5} + \left(C + 3D(N + \sqrt{M})\right) T^{4/5} \quad (81)$$

where the constant are defined in Theorem 1

Proof. We apply Lemma 12 with $F_t = \bar{F}_{q,k-1}$, $\mathbf{x}_{t,k} = \bar{\mathbf{x}}_{q,k}$ and $\mathbf{d}_{t,k} = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{a}}_{q,k}^i$, we have

$$\begin{aligned} \bar{F}_{q,k-1}(\mathbf{x}^*) - \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k+1}) &\leq \left(1 - \frac{1}{K}\right) [\bar{F}_{q,k-1}(\mathbf{x}^*) - \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k})] \\ &\quad + \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\| D + \langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}^* - \mathbf{v}_{q,k}^i \rangle] + \frac{\beta D^2}{2K^2} \end{aligned} \quad (82)$$

As $\mathbb{E} [\bar{F}_{q,k-1}(\mathbf{x}^*) - \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k})] = \mathbb{E} [\bar{F}_{q,k-2}(\mathbf{x}^*) - \bar{F}_{q,k-2}(\bar{\mathbf{x}}_{q,k})]$, we can apply equation (82) recursively for $k \in \{1, \dots, K\}$, thus

$$\begin{aligned} \mathbb{E} [\bar{F}_{q,0}(\mathbf{x}^*) - \bar{F}_{q,0}(\bar{\mathbf{x}}_q)] &\leq \left(1 - \frac{1}{K}\right)^K \mathbb{E} [\bar{F}_{q,0}(\mathbf{x}^*) - \bar{F}_{q,0}(\bar{\mathbf{x}}_{q,1})] \\ &+ \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\| D] + \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}^* - \mathbf{v}_{q,k}^i \rangle] + \frac{\beta D^2}{2K} \end{aligned} \quad (83)$$

Note that $\left(1 - \frac{1}{K}\right)^K \leq \frac{1}{e}$ and $\bar{F}_{q,0}(\bar{\mathbf{x}}_{q,1}) \geq 0$, we have

$$\begin{aligned} \mathbb{E} \left[\left(1 - \frac{1}{e}\right) \bar{F}_{q,0}(\mathbf{x}^*) - \bar{F}_{q,0}(\bar{\mathbf{x}}_q) \right] &\leq \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\| D] \\ &+ \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}^* - \mathbf{v}_{q,k}^i \rangle] + \frac{\beta D^2}{2K} \end{aligned} \quad (84)$$

Let $T = QK$, using Claim 2 and note that the oracle has a regret $\mathcal{R}_Q \leq C\sqrt{Q}$. We have

$$\begin{aligned} \mathbb{E} [\mathcal{R}_T] &= \mathbb{E} \left[\sum_{q=1}^Q K \left[\left(1 - \frac{1}{e}\right) \bar{F}_{q,0}(\mathbf{x}^*) - \bar{F}_{q,0}(\bar{\mathbf{x}}_q) \right] \right] \\ &\leq \frac{D}{n} \sum_{q=1}^Q \sum_{k=1}^K \mathbb{E} [\|\nabla \bar{F}_{q,k-1}(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i\|] + \frac{1}{n} \sum_{q=1}^Q \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} [\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}^* - \mathbf{v}_{q,k}^i \rangle] + \frac{\beta}{2} QD^2 \\ &\leq QD \left(\beta D + (N + \sqrt{M}) 3K^{2/3} \right) + KC\sqrt{Q} + \frac{\beta QD^2}{2} \end{aligned} \quad (85)$$

Setting $Q = T^{2/5}$ and $K = T^{3/5}$, the expected regret of the algorithm is upper bounded by

$$\begin{aligned} \mathbb{E} [\mathcal{R}_T] &\leq T^{2/5} \left(\beta D^2 + (N + \sqrt{M}) 3T^{2/5} \right) + CT^{4/5} + \frac{\beta D^2 T^{2/5}}{2} \\ &\leq \frac{3}{2} \beta D^2 T^{2/5} + (C + 3D(N + \sqrt{M})) T^{4/5} \end{aligned} \quad (86)$$

□

B Theoretical analysis for Section 4

Let $f_t^\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{v} \in \mathbb{B}^d} [f_t(\mathbf{x} + \delta \mathbf{v})]$ and recall its gradient $\nabla f_t^\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \in \mathbb{S}^{d-1}} \left[\frac{d}{\delta} f_t(\mathbf{x} + \delta \mathbf{u}) \mathbf{u} \right]$. We define the average function

$$\bar{F}_{q,k}^\delta(\mathbf{x}) = \frac{1}{L-k} \sum_{\ell=k+1}^L F_{\sigma_q(\ell)}^\delta(\mathbf{x}) = \frac{1}{L-k} \sum_{\ell=k+1}^L \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}) \quad (87)$$

and the average of the remaining $(L-k)$ functions of $f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i)$ over n agents as

$$\hat{F}_{q,k}^\delta = \frac{1}{n} \sum_{i=1}^n \hat{f}_{q,k}^{i,\delta} = \frac{1}{L-k} \sum_{\ell=k+1}^L \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i) \quad (88)$$

where $F_{\sigma_q(\ell)}^\delta(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x})$ and $\hat{f}_{q,k}^{i,\delta} = \frac{1}{L-k} \sum_{\ell=k+1}^L f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i)$. Then, the one-point gradient $\nabla \bar{F}_{q,k}^\delta$ and $\nabla \hat{F}_{q,k}^\delta$ come naturally with the above definitions. Let $\mathcal{H}_{q,1} \subset \dots \subset \mathcal{H}_{q,k}$ be the σ -fields generated by the randomness of the stochastic gradient estimate up to time k .

$$\mathbf{g}_{q,k}^{i,\delta} = \mathbb{E} [\tilde{\mathbf{g}}_{q,k}^i | \mathcal{H}_{q,k-1}], \quad \mathbf{d}_{q,k}^{i,\delta} = \mathbb{E} [\tilde{\mathbf{d}}_{q,k}^i | \mathcal{H}_{q,k-1}], \quad \nabla f_{\sigma_q(k)}^{i,\delta}(\mathbf{x}_{q,k}^i) = \mathbb{E} [\tilde{\mathbf{h}}_{q,k}^i] \quad (89)$$

and

$$\hat{\mathbf{g}}_{q,k}^{i,\delta} = \frac{1}{L-k} \sum_{\ell=k+1}^L \mathbf{g}_{q,\ell}^{i,\delta}, \quad \hat{\mathbf{d}}_{q,k}^{i,\delta} = \frac{1}{L-k} \sum_{\ell=k+1}^L \mathbf{d}_{q,\ell}^{i,\delta} \quad (90)$$

Lemma 13. For $i \in [n], k \in [K]$. Let $V_d^\delta = 2n \frac{d}{\delta} B \left(\frac{\lambda_2}{1-\lambda_2} + 1 \right)$, the local gradient is upper-bounded, i.e $\|\mathbf{d}_{q,k}^{i,\delta}\| \leq V_d^\delta$

Lemma 14. Under Assumption 3, the variance of the local gradient estimate is uniformly bounded, i.e

$$\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{d}}_{q,k}^{i,\delta} \right\|^2 \right] \leq 4n \left(\frac{d}{\delta} B \right)^2 \left[\frac{1}{\left(\frac{1}{\lambda_2} - 1 \right)^2} + 2 \right] \quad (91)$$

Proof. By Assumption 3, we have

$$\mathbb{E} \left[\left\| \nabla f_{\sigma_q(\tau)}^{cat} - \tilde{\mathbf{h}}_{q,\tau}^{cat} \right\|^2 \right] = \mathbb{E} \left[\sum_{i=1}^n \left\| \nabla f_{\sigma_q(\tau)}^{i,\delta}(\mathbf{x}_{q,\tau}^i) - \tilde{\mathbf{h}}_{q,\tau}^i \right\|^2 \right] \leq n \left(\frac{d}{\delta} B \right)^2 \quad (92)$$

Following the same analysis in equation (41), we have

$$\begin{aligned} & \mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{cat} - \tilde{\mathbf{d}}_{q,k}^{cat} \right\|^2 \right] \\ & \leq \mathbb{E} \left[\left(\sum_{\tau=1}^{k-1} \left\| \mathbf{W}^{k-\tau} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\| \left\| \nabla f_{\sigma_q(\tau+1)}^{cat} - \tilde{\mathbf{h}}_{q,\tau+1}^{cat} + \tilde{\mathbf{h}}_{q,\tau}^{cat} - \nabla f_{\sigma_q(\tau)}^{cat} \right\| \right)^2 \right] \\ & \quad + 4 \left(\mathbb{E} \left[\left\| \mathbf{W}^k - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\|^2 \left\| \nabla f_{\sigma_q(1)}^{cat} - \tilde{\mathbf{h}}_{q,1}^{cat} \right\|^2 \right] + \mathbb{E} \left[\left\| \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right\|^2 \left\| \nabla f_{\sigma_q(k)}^{cat} - \tilde{\mathbf{h}}_{q,k}^{cat} \right\|^2 \right] \right) \\ & \leq 4n \left(\frac{d}{\delta} B \right)^2 \left(\sum_{\tau=1}^{k-1} \lambda_2^{k-\tau} \right)^2 + 4n \left(\frac{d}{\delta} B \right)^2 (\lambda_2^{2k} + 1) \\ & \leq 4n \left(\frac{d}{\delta} B \right)^2 \left(\frac{\lambda_2}{1-\lambda_2} \right)^2 + 4n \left(\frac{d}{\delta} B \right)^2 (\lambda_2 + 1) \leq 4n \left(\frac{d}{\delta} B \right)^2 \left[\frac{1}{\left(\frac{1}{\lambda_2} - 1 \right)^2} + 2 \right] \end{aligned} \quad (93)$$

The lemma follows by remarking that $\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{d}}_{q,k}^{i,\delta} \right\|^2 \right] \leq \mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{cat} - \tilde{\mathbf{d}}_{q,k}^{cat} \right\|^2 \right]$ □

Let $\mathbf{x}^* = \arg \max_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$, $\mathbf{x}_\delta^* = \arg \max_{\mathbf{x} \in \mathcal{K}'} \sum_{t=1}^T f_t(\mathbf{x})$ Let $\mathbf{z}_{q,k}^i = \mathbf{x}_{q,k}^i + \delta \mathbf{u}_{q,k}^i$, we define $\bar{\mathbf{z}}_{q,k} = \frac{1}{n} \sum_{i=1}^n \mathbf{z}_{q,k}^i$ for $1 \leq k \leq K$.

Lemma 15. Under Assumption 1 and Assumption 3. Let $N = k_0 \cdot n B \frac{d}{\delta} \max \left\{ \lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2 \right\}$. Then, for $k \in [K]$, we have

$$\max_{i \in [1,n]} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{i,\delta} - \nabla \hat{F}_{q,k}^\delta \right\|^2 \right] \leq \frac{N}{k} \quad (94)$$

Proof. The proof is essentially based on the one of Lemma 8. Note that we keep the same notation with a superscript δ to indicate the smooth version of f and related variables. By definition of the one-point gradient estimator and

Assumption 3, equation (25) becomes

$$\begin{aligned}
 \mathbb{E} \left[\left\| \delta_{q,k}^{cat,\delta} \right\|^2 \right] &= \mathbb{E} \left[\sum_{i=1}^n \left\| \delta_{q,k}^{i,\delta} \right\|^2 \right] = \sum_{i=1}^n \mathbb{E} \left[\left\| \nabla \hat{f}_{q,k}^{i,\delta} - \nabla \hat{f}_{q,k-1}^{i,\delta} \right\|^2 \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \nabla \hat{f}_{q,k}^{i,\delta} - \nabla \hat{f}_{q,k-1}^{i,\delta} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \frac{\sum_{\ell=k+1}^L \nabla f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i)}{L-k} - \frac{\sum_{\ell=k}^L \nabla f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i)}{L-k+1} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{E} \left[\left\| \frac{\sum_{\ell=k+1}^L \nabla f_{\sigma_q(\ell)}^{i,\delta}(\mathbf{x}_{q,\ell}^i)}{(L-k)(L-k+1)} - \frac{\nabla f_{\sigma_q(k)}^{i,\delta}(\mathbf{x}_{q,k}^i)}{L-k+1} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\
 &\leq n \left(\frac{2B \frac{d}{\delta}}{L-k+1} \right)^2
 \end{aligned} \tag{95}$$

By Jensen's inequality, we deduce that

$$\mathbb{E} \left[\left\| \delta_{q,k}^{cat,\delta} \right\| \right] \leq \sqrt{\mathbb{E} \left[\left\| \delta_{q,k}^{cat,\delta} \right\|^2 \right]} \leq \frac{2\sqrt{n}B \frac{d}{\delta}}{L-k+1} \tag{96}$$

When $k = 1$, following the same derivation in equation (27), we have

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,1}^{cat,\delta} - \nabla \hat{F}_{q,1}^{cat,\delta} \right\|^2 \right] \leq \lambda_2^2 \mathbb{E} \left[\sum_{i=1}^n \left\| \hat{\mathbf{g}}_{q,1}^{i,\delta} - \nabla \hat{F}_{q,1}^{i,\delta} \right\|^2 \right] \leq n \lambda_2^2 \frac{d^2}{\delta^2} B^2$$

Let $k \in [2, k_0]$, from equation (23) and equation (96)

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat,\delta} - \nabla \hat{F}_{q,k}^{cat,\delta} \right\| \right] &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat,\delta} - \nabla \hat{F}_{q,k-1}^{cat,\delta} \right\| \right] + \mathbb{E} \left[\left\| \delta_{q,k}^{cat,\delta} \right\| \right] \right) \\
 &\leq \lambda_2^{k-1} \sqrt{n} \frac{d}{\delta} B + 2 \sum_{\tau=1}^k \lambda_2^\tau \sqrt{n} \frac{d}{\delta} B \\
 &\leq \lambda_2 \sqrt{n} \frac{d}{\delta} B + 2 \frac{\lambda_2}{1-\lambda_2} \sqrt{n} \frac{d}{\delta} B \\
 &= \lambda_2 \sqrt{n} \frac{d}{\delta} B \left(1 + \frac{2}{1-\lambda_2} \right)
 \end{aligned} \tag{97}$$

Let $N_0 = k_0 \cdot \sqrt{n} \max \left\{ \lambda_2 B \frac{d}{\delta} \left(1 + \frac{2}{1-\lambda_2} \right), 2B \frac{d}{\delta} \right\}$. We claim that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat,\delta} - \nabla \hat{F}_{q,k}^{cat,\delta} \right\| \right] \leq \frac{N_0}{k}$ when $k \in [k_0, K]$.

Let $L \geq 2K$, we have then $\frac{1}{L-k+1} \leq \frac{1}{2K-k+1} \leq \frac{1}{K+1} \leq \frac{1}{k+1}$. Thus, using the induction hypothesis, we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k}^{cat,\delta} - \nabla \hat{F}_{q,k}^{cat,\delta} \right\| \right] &\leq \lambda_2 \left(\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{cat,\delta} - \nabla \hat{F}_{q,k-1}^{cat,\delta} \right\| \right] + \mathbb{E} \left[\left\| \delta_{q,k}^{cat,\delta} \right\| \right] \right) \\
 &\leq \lambda_2 \left(\frac{N_0}{k-1} + \frac{2\sqrt{n}B \frac{d}{\delta}}{L-k+1} \right) \\
 &\leq \lambda_2 \left(\frac{N_0}{k-1} + \frac{2\sqrt{n}B \frac{d}{\delta}}{k+1} \right) \\
 &\leq \lambda_2 \left(N_0 \frac{k_0+1}{k_0(k-1)} \right) \\
 &\leq \frac{N_0}{k}
 \end{aligned} \tag{98}$$

Using the inequality in equation (31) and the above result, the lemma is then proven. \square

Lemma 16 (Lemma 10, Lemma 11 [20]). *Under Lemma 13 and lemma 14 and setting $\rho_k = \frac{2}{(k+3)^{2/3}}$, we have*

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \leq \frac{\sqrt{M_0}}{(k+3)^{1/3}}, \quad k \in [K] \quad (99)$$

$$\text{where } M_0 = 4^{2/3} \frac{d^2}{\delta^2} B^2 \left[24n^2 \left(\frac{1}{\frac{1}{\lambda_2} - 1} + 1 \right)^2 + 8n \left(\frac{1}{\left(\frac{1}{\lambda_2} - 1 \right)^2} + 2 \right) \right]$$

Proof. The proof follows the same idea in Lemma 10 and Lemma 11 of [20] with some changes in the constant values. We will evoke in details in the following section. Following the same decomposition in the proof of Lemma 11, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &= \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - (1 - \rho_k) \tilde{\mathbf{a}}_{q,k-1}^i - \rho_k \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \right] \\ &= \rho_k^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \right] + (1 - \rho_k)^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta} \right\|^2 \right] \\ &\quad + (1 - \rho_k)^2 \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\ &\quad + 2\rho_k(1 - \rho_k) \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta}, \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta} \right\rangle \right] \\ &\quad + 2\rho_k(1 - \rho_k) \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta}, \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\ &\quad + 2(1 - \rho_k)^2 \mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta}, \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \\ \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \right] &= \mathbb{E} \left[\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \\ &\leq \mathbb{E} \left[\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \mathbf{d}_{q,k}^{i,\delta} \right\|^2 + \left\| \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 + 2 \left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \mathbf{d}_{q,k}^{i,\delta}, \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\rangle \mid \mathcal{F}_{q,k-1} \right] \right] \end{aligned} \quad (100)$$

By the definition in equation (90), we have $\mathbb{E} \left[\mathbf{d}_{q,k}^{i,\delta} \mid \mathcal{F}_{q,k-1} \right] = \hat{\mathbf{d}}_{q,k-1}^{i,\delta}$, using Lemma 13, we have

$$\mathbb{E} \left[\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \mathbf{d}_{q,k}^{i,\delta} \right\|^2 \mid \mathcal{F}_{q,k-1} \right] \right] \leq (V_{\mathbf{d}}^\delta)^2 \quad (101)$$

Invoking Lemma 14, we have

$$\mathbb{E} \left[\left\| \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \right] \leq \sigma_2^2 \quad (102)$$

and

$$\mathbb{E} \left[\mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \mathbf{d}_{q,k}^{i,\delta}, \mathbf{d}_{q,k}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\rangle \mid \mathcal{F}_{q,k-1} \right] \right] = 0 \quad (103)$$

by following the same analysis in equation (48). We now claim that equation (100) is bounded above by

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta} \right\|^2 \right] \leq (V_{\mathbf{d}}^\delta)^2 + \sigma_2^2 \triangleq V^\delta \quad (104)$$

More over, taking the idea from equations (50) to (53), we have

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta} \right\|^2 \right] \leq \frac{4(V_{\mathbf{d}}^\delta)^2}{(L - k + 2)^2} \triangleq \frac{L^\delta}{(L - k + 2)^2} \quad (105)$$

$$\mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^{i,\delta}, \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta} \right\rangle \right] = 0 \quad (106)$$

$$\mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{d}}_{q,k}^{i,\delta}, \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] = 0 \quad (107)$$

and

$$\mathbb{E} \left[\left\langle \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \hat{\mathbf{d}}_{q,k-2}^{i,\delta}, \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\rangle \right] \leq \frac{L^\delta}{2\alpha_k(L-k+2)^2} + \frac{\alpha_k}{2} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \quad (108)$$

by using Young's inequality. Setting $\alpha_k = \frac{\rho_k}{2}$ similarly to Lemma 11, we have

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \leq \rho_k^2 V^\delta + \left(1 + \frac{2}{\rho_k} \right) \frac{L^\delta}{(L-k+2)^2} + (1-\rho_k) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \quad (109)$$

Setting $L \geq 2K$ and $\rho_k = \frac{2}{(k+2)^{2/3}}$, we have then $\frac{1}{L-k+2} \leq \frac{1}{2K-k+2} \leq \frac{1}{K+2} \leq \frac{1}{k+2}$. Following the derivation from Lemma 11 of [20], Equation (109) can be bounded above by

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \rho_k^2 V^\delta + \left(1 + \frac{2}{\rho_k} \right) \frac{L^\delta}{(k+2)^2} + (1-\rho_k) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\ &\leq \frac{4^{2/3} (2V^\delta + L^\delta)}{(k+2)^{4/3}} + \left(1 - \frac{2}{(k+2)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\ &\triangleq \frac{M_0}{(k+2)^{4/3}} + \left(1 - \frac{2}{(k+2)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \end{aligned} \quad (110)$$

Assume that $\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] \leq \frac{M_0}{(k+3)^{2/3}}$ for $k \in [K]$. When $k = 1$, by definition of $\tilde{\mathbf{a}}_{q,1}^i$ and $\hat{\mathbf{d}}_{q,0}^{i,\delta}$, we have

$$\mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,0}^{i,\delta} - \tilde{\mathbf{a}}_{q,1}^i \right\|^2 \right] \leq \left(V_d^\delta + \frac{2}{3^{2/3}} \frac{d}{\delta} B \right)^2 \quad (111)$$

Thus, since $\sigma_2 \geq \frac{2}{3^{2/3}} \frac{d}{\delta} B$, one can observe that

$$\frac{M_0}{(1+2)^{2/3}} = 2V^\delta + L^\delta \geq 2V^\delta = 2 \left((V_d^\delta)^2 + \sigma_2^2 \right) \geq (V_d^\delta + \sigma_2)^2 \geq \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,0}^{i,\delta} - \tilde{\mathbf{a}}_{q,1}^i \right\|^2 \right] \quad (112)$$

Suppose that the induction hypothesis holds for $k-1$, one can easily verify for k since

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\|^2 \right] &\leq \frac{M_0}{(k+2)^{4/3}} + \left(1 - \frac{2}{(k+2)^{2/3}} \right) \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-2}^{i,\delta} - \tilde{\mathbf{a}}_{q,k-1}^i \right\|^2 \right] \\ &\leq \frac{M_0}{(k+2)^{4/3}} + \left(1 - \frac{2}{(k+2)^{2/3}} \right) \frac{M_0}{(k+2)^{2/3}} \\ &\leq M_0 \frac{(k+2)^{2/3} - 1}{(k+3)^{4/3}} \\ &\leq \frac{M_0}{(k+3)^{2/3}} \end{aligned} \quad (113)$$

□

Claim 3. Under Claim 1, Lemma 15 and Lemma 16.

$$\sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \leq \beta D + \frac{3}{2} \left(N + \sqrt{M_0} \right) K^{2/3} \quad (114)$$

Proof of claim.

$$\begin{aligned}
 \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] &\leq \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k}) - \nabla \hat{F}_{q,k-1}^\delta \right\| \right] + \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1}^\delta - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\
 &\leq \beta D + \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \hat{F}_{q,k-1}^\delta - \hat{\mathbf{d}}_{q,k-1}^{i,\delta} \right\| \right] + \sum_{k=1}^K \mathbb{E} \left[\left\| \hat{\mathbf{d}}_{q,k-1}^{i,\delta} - \tilde{\mathbf{a}}_{q,k}^i \right\| \right] \\
 &\leq \beta D + \sum_{k=1}^K \frac{N}{k} + \sum_{k=1}^K \frac{\sqrt{M_0}}{(k+3)^{1/3}} \\
 &\leq \beta D + (N + \sqrt{M_0}) \sum_{k=1}^K \frac{1}{(k+3)^{1/3}} \\
 &\leq \beta D + \frac{3}{2} (N + \sqrt{M_0}) K^{2/3} \tag{115}
 \end{aligned}$$

where Claim 1 is still verified in the second inequality since $f_{\sigma_q(\ell)}^{i,\delta}$ is β -smooth and the third inequality is the result of Lemma 15 and Lemma 16 \square

Claim 4.

$$\mathbb{E} \left[\sum_{q=1}^Q \sum_{\ell=1}^L \left(1 - \frac{1}{e} \right) F_{\sigma_q(\ell)}^\delta(\mathbf{x}_\delta^*) - F_{\sigma_q(\ell)}^\delta(\bar{\mathbf{x}}_q) \right] \leq \frac{L\beta D^2}{K} + \frac{3LD(N + \sqrt{M_0})}{2K^{1/3}} + LC\sqrt{Q} + \frac{\beta QLD^2}{2K} \tag{116}$$

Proof of claim. Using Lemma 12 with $F_t = \bar{F}_{q,k-1}^\delta$, $\mathbf{x}_{t,k} = \bar{\mathbf{x}}_{q,k}$ and $\mathbf{d}_{t,k} = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{a}}_{q,k}^i$, we have

$$\begin{aligned}
 \bar{F}_{q,k-1}^\delta(\mathbf{x}_\delta^*) - \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k+1}) &\leq \left(1 - \frac{1}{K} \right) [\bar{F}_{q,k-1}^\delta(\mathbf{x}_\delta^*) - \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k})] \\
 &\quad + \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \left[\left\| \nabla \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| D + \langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}_\delta^* - \mathbf{v}_{q,k}^i \rangle \right] + \frac{\beta D^2}{2K^2} \tag{117}
 \end{aligned}$$

Similarly to the proof of Theorem 2 and using Claim 3, we note

$$\begin{aligned}
 &\mathbb{E} \left[\left(1 - \frac{1}{e} \right) \bar{F}_{q,0}^\delta(\mathbf{x}_\delta^*) - \bar{F}_{q,0}^\delta(\bar{\mathbf{x}}_q) \right] \\
 &\leq \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} \left[\left\| \nabla \bar{F}_{q,k-1}^\delta(\bar{\mathbf{x}}_{q,k}) - \tilde{\mathbf{a}}_{q,k}^i \right\| D \right] + \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} \left[\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}_\delta^* - \mathbf{v}_{q,k}^i \rangle \right] + \frac{\beta D^2}{2K} \\
 &\leq \frac{D}{K} \left(\beta D + \frac{3}{2} (N + \sqrt{M_0}) K^{2/3} \right) + \frac{1}{K} \cdot \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} \left[\langle \tilde{\mathbf{a}}_{q,k}^i, \mathbf{x}_\delta^* - \mathbf{v}_{q,k}^i \rangle \right] + \frac{\beta D^2}{2K} \tag{118}
 \end{aligned}$$

Thus, we can write

$$\begin{aligned}
 &\mathbb{E} \left[\sum_{q=1}^Q \sum_{\ell=1}^L \left(1 - \frac{1}{e} \right) F_{\sigma_q(\ell)}^\delta(\mathbf{x}_\delta^*) - F_{\sigma_q(\ell)}^\delta(\bar{\mathbf{x}}_q) \right] = \mathbb{E} \left[\sum_{q=1}^Q L \left(1 - \frac{1}{e} \right) \bar{F}_{q,0}^\delta(\mathbf{x}_\delta^*) - \bar{F}_{q,0}^\delta(\bar{\mathbf{x}}_q) \right] \\
 &\leq \frac{LD}{K} \left(\beta D + \frac{3}{2} (N + \sqrt{M_0}) K^{2/3} \right) + LC\sqrt{Q} + \frac{\beta QLD^2}{2K} \\
 &\leq \frac{L\beta D^2}{K} + \frac{3LD(N + \sqrt{M_0})}{2K^{1/3}} + LC\sqrt{Q} + \frac{\beta QLD^2}{2K} \tag{119}
 \end{aligned}$$

\square

Theorem 7. Let \mathcal{K} be a down-closed convex and compact set. We suppose the δ -interior \mathcal{K}' following Lemma 4. Let $Q = T^{2/9}$, $L = T^{7/9}$, $K = T^{2/3}$, $\delta = \frac{r}{\sqrt{d+2}} T^{-1/9}$ and $\rho_k = \frac{2}{(k+2)^{2/3}}$, $\eta_k = \frac{1}{K}$. Then the expected $(1 - \frac{1}{e})$ -regret is

upper bounded

$$\mathbb{E} [\mathcal{R}_T] \leq ZT^{8/9} + \frac{\beta D^2}{2} T^{1/9} + \frac{3}{2} D \frac{d(\sqrt{d}+2)}{r} P_{n,\lambda_2} T^{2/9} + \beta D^2 T^{3/9} \quad (120)$$

where we note $Z = (1 - \frac{1}{e}) \left(\sqrt{d} \left(\frac{R}{e} + 1 \right) + \frac{R}{r} \right) G \frac{r}{\sqrt{d+2}} + (2 - \frac{1}{e}) G \frac{r}{\sqrt{d+2}} + 2\beta + C$ and $P_{n,\lambda_2} = k_0 \cdot nB \max \left\{ \lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2 \right\} + 4^{1/3} \left(24n^2 \left(\frac{1}{\lambda_2-1} + 1 \right)^2 + 8n \left(\frac{1}{(\lambda_2-1)^2} + 2 \right) \right)^{1/2}$

Proof. Recall the values of N and M_0 from Lemma 15 and Lemma 16, we have

$$N = k_0 \cdot nB \frac{d}{\delta} \max \left\{ \lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2 \right\}$$

$$M_0 = 4^{2/3} \frac{d^2}{\delta^2} B^2 \left[24n^2 \left(\frac{1}{\lambda_2-1} + 1 \right)^2 + 8n \left(\frac{1}{(\lambda_2-1)^2} + 2 \right) \right]$$

Let $P_{n,\lambda_2} = k_0 \cdot nB \max \left\{ \lambda_2 \left(1 + \frac{2}{1-\lambda_2} \right), 2 \right\} + 4^{1/3} \left(24n^2 \left(\frac{1}{\lambda_2-1} + 1 \right)^2 + 8n \left(\frac{1}{(\lambda_2-1)^2} + 2 \right) \right)^{1/2}$. Then, one can easily see that $N + \sqrt{M_0} = \frac{d}{\delta} B P_{n,\lambda_2}$ where P_{n,λ_2} is a constant depending on n and λ_2 . For the next step, we set $\delta = \frac{r}{\sqrt{d+2}} T^{-1/9}$, then $\frac{d}{\delta} = \frac{d(\sqrt{d+2})}{r} T^{1/9}$, $Q = T^{2/9}$, $L = T^{7/9}$ and $K = T^{2/3}$. From the analysis in Theorem 4 of

[20], Lemma 4 and Claim 4, we have

$$\begin{aligned}
 \mathbb{E}[\mathcal{R}_{\mathcal{T}}] &\leq \left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) GT\delta^\gamma + \left(2 - \frac{1}{e}\right) GT\delta + 2BQK \\
 &\quad + \sum_{q=1}^Q \sum_{\ell=1}^L \left(1 - \frac{1}{e}\right) F_{\sigma_q(\ell)}^\delta(\mathbf{x}_\delta^*) - F_{\sigma_q(\ell)}^\delta(\bar{\mathbf{x}}_q) \\
 &\leq \left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) GT\delta^\gamma + \left(2 - \frac{1}{e}\right) GT\delta + 2BQK \\
 &\quad + \frac{L\beta D^2}{K} + \frac{3LD(N + \sqrt{M_0})}{2K^{1/3}} + LC\sqrt{Q} + \frac{\beta QLD^2}{2K} \\
 &\leq \left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) GT\delta^\gamma + \left(2 - \frac{1}{e}\right) GT\delta + 2BQK \\
 &\quad + \frac{L\beta D^2}{K} + \frac{3LD\frac{d}{\delta}P_{n,\lambda_2}}{2K^{1/3}} + LC\sqrt{Q} + \frac{\beta QLD^2}{2K} \\
 &\leq \left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) GT \frac{r}{\sqrt{d}+2} T^{-1/9} + \left(2 - \frac{1}{e}\right) GT \frac{r}{\sqrt{d}+2} T^{-1/9} \\
 &\quad + 2\beta T^{2/9} T^{2/3} + T^{7/9} \beta D^2 T^{-2/3} + \frac{3}{2} T^{7/9} D \frac{d(\sqrt{d}+2)}{r} T^{1/9} P_{n,\lambda_2} T^{-2/3} \\
 &\quad + T^{7/9} C T^{1/9} + \frac{\beta}{2} T^{2/9} T^{7/9} D^2 T^{-2/3} \\
 &\leq \left[\left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) G \frac{r}{\sqrt{d}+2} + \left(2 - \frac{1}{e}\right) G \frac{r}{\sqrt{d}+2} + C \right] T^{8/9} \\
 &\quad + \frac{\beta D^2}{2} T^{6/9} + \left[2\beta + \frac{3}{2} D \frac{d(\sqrt{d}+2)}{r} P_{n,\lambda_2} \right] T^{5/9} + \beta D^2 T^{4/9} \\
 &\leq \left[\left(1 - \frac{1}{e}\right) \left(\sqrt{d} \left(\frac{R}{e} + 1\right) + \frac{R}{r}\right) G \frac{r}{\sqrt{d}+2} + \left(2 - \frac{1}{e}\right) G \frac{r}{\sqrt{d}+2} + 2\beta + C \right] T^{8/9} \\
 &\quad + \frac{\beta D^2}{2} T^{1/9} + \frac{3}{2} D \frac{d(\sqrt{d}+2)}{r} P_{n,\lambda_2} T^{2/9} + \beta D^2 T^{3/9}
 \end{aligned} \tag{121}$$

□