

Optimal control problems governed by time-fractional diffusion equations with control constraint

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Abstract. A class of optimal control problems governed by linear fractional diffusion equation with control constraint is considered. We first establish some results on the existence of strong solution to the state equation and the existence of optimal solutions for the optimal control problem. Then we derive first-and second-order optimality conditions for locally optimal solutions to the problem.

Key words. Optimal control, Fractional diffusion equation, Solution existence, First-and second-order optimality conditions.

AMS Subject Classifications. 49K15 · 90C29

1 Introduction

Let Ω be an open and bounded set in \mathbb{R}^N with $N = 2, 3$ and boundary $\partial\Omega \in C^2$, $Q = [0, T] \times \Omega$ and $\Sigma = [0, T] \times \partial\Omega$, where $T > 0$. Let $D := H^2(\Omega) \cap H_0^1(\Omega)$. We consider the problem of finding a function $u \in L^\infty(Q)$ and the corresponding state $y \in L^p([0, T], D)$ with $1 < p < \infty$, which solve

$$\psi(y, u) = \int_Q L(t, x, y(t, x), u(t, x)) dt dx \rightarrow \min \quad (1)$$

s.t.

$$\frac{d^\alpha y}{dt^\alpha} = \Delta y + u \quad \text{in } Q \quad (2)$$

$$y = 0 \quad \text{on } \Sigma \quad (3)$$

$$y(0) = y_0 \quad \text{in } \Omega \quad (4)$$

$$a \leq u(t, x) \leq b \quad \text{a.e. } (t, x) \in Q, \quad (5)$$

where $0 < \alpha \leq 1$ and $\frac{d^\alpha}{dt^\alpha}$ is a fractional derivative operator in the sense of Caputo, which is given in Section 2, $L : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function, and $y_0 \in D$.

The system (2)-(4) is a time-fractional diffusion equation. Note that several physical systems are not truly modelled with differential equations of integer-order. So the fractional-order differential equations have been introduced in order to describe accurately these systems. A motivation for studying fractional diffusion equations comes from the fact that they describe efficiently anomalous diffusion on fractals such as physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials (see [2], [22] and references therein).

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In contrast with optimal control problems governed by differential equations with integer order (see [8], [14] and [32]), the class of optimal control problems governed by fractional differential equations have not been studied systematically yet. Recently, there have been some mathematicians studying optimal control problems governed by fractional differential equations. For the papers which have close connections to the present work, we refer the reader to [1], [4], [12], [17], [24] and [25]. Among them, O.P. Agrawal [1], considered optimal control problems governed by fractional differential equations (FDEs) without constraints. She gave first-order necessary conditions and some numerical examples for the problem. Kien et al. [17] studied problems governed by FDEs with control constraint. They gave first- and second-order optimality conditions for locally optimal solutions to the problem. G.M. Mophou et al. [12], [24] and [25] considered strongly convex optimal control problems governed by a time-fractional diffusion equations with Riemann–Liouville derivative and the cost functions of the type

$$J(u) = \frac{1}{2} \|y(u) - y_0\|_{L^2(Q)}^2 + \frac{\gamma}{2} \|u\|_{L^2(Q)}^2, \gamma > 0$$

for cases: control constraints, no constraint and state constraints, respectively. Then they established results on the existence of optimal solutions and first-order optimality conditions. Note that, this class of problems are convex so the second-order optimality conditions are fulfilled automatically. Recently Bahaa [4] has extended the results of [24] by considering the problem with control constraint. However, all of them mainly investigated first-order optimality conditions for the problem.

In this paper we consider problem (1)-(5) governed by time-fractional diffusion equation with Caputo derivative and with control constraint (5), where the cost function may not be convex. We shall prove the existence of solutions to the state equation and the existence of optimal solutions to (1)-(5). We then derive first- and second-order optimality conditions. We show that when $p = 2$ and $1 > \alpha > 1/2$, a theory of no-gap second-order optimality conditions is obtained. Optimality conditions are important for numerical methods of computing optimal solutions. Based on second-order sufficient conditions, we can give error estimate for approximate solutions. Nevertheless, to our best knowledge, so far there have been no result on the second-order optimality conditions for this class of problems.

The study of optimal control problems governed by fractional diffusion equations are quite complicated. One of the difficulties is that the regularity of the solution to the state equation is poor. It is known that, if the control function is Hölder continuous, then the state function is also Hölder continuous. Unfortunately, in optimal control problems, the control variables usually belong to L^∞ -spaces. So the solutions to the state equation only belong to L^p -spaces. Therefore, the cost function is difficult to be differentiable with respect to the state variable. Another difficulty for the problem is about compactness of bounded sequences in function spaces with Banach space values. To overcome these difficulties, we shall use a result from [15] in order to study regularity of solutions to the state equation. We also use a recent result on the compact embedding in [19] which is a version of the so-call Aubin’s Lemma for evolution equations to show that if a sequence is weakly convergent then it is strongly convergent. Besides, techniques of optimal control from [7], [16] and [17], nonlinear and variational analysis are also used to deal with the problem.

The paper is organized as follows. In Section 2, we present some auxiliary results on the fractional derivatives for functions with values in Banach space. Section 3 is devoted to results on the existence and regularity of solutions to the state equation and the existence of optimal solutions to the problem. Section 4 is destined for establishing first- and second-order optimality conditions.

2 Some auxiliary results on fractional derivatives

Let X be a Banach space. We denote by $L^p([0, T], X)$ the space of X -valued functions $u : [0, T] \rightarrow X$ such that $\|u\|_p := (\int_0^T \|u(t)\|_X^p dt)^{1/p} < +\infty$, where $1 \leq p \leq \infty$. Also we denote by $C([0, T], X)$ the space

of continuous functions $v : [0, T] \rightarrow X$, by $W^{1,p}([0, T], X)$ the space of absolutely continuous functions $y : [0, T] \rightarrow X$. The norm on $W^{1,p}([0, T], X)$ is defined by

$$\|y\|_{1,p} = \|y\|_0 + \|\dot{y}\|_p,$$

where $\|y\|_0 = \max_{t \in [0, T]} \|y(t)\|_X$ is the norm of y in $C([0, T], X)$.

Let $\theta > 0$ be a real number. The operator I_{0+}^θ defined on $L^1([0, T], X)$, by

$$(I_{0+}^\theta \phi)(t) = \frac{1}{\Gamma(\theta)} \int_0^t \frac{\phi(\tau)}{(t-\tau)^{1-\theta}} d\tau$$

is called *the left Riemann-Liouville fractional integral operator of order θ* . The operator I_{T-}^θ defined on $L^1([0, T], \mathbb{R})$, by

$$(I_{T-}^\theta \phi)(t) = \frac{1}{\Gamma(\theta)} \int_t^T \frac{\phi(\tau)}{(\tau-t)^{1-\theta}} d\tau$$

is called *the right Riemann-Liouville fractional integral operator of order θ* . For the case $\theta = 0$, we set $I_{0+}^0 = I_{T-}^0 = I$, the identity operator. Here the integrals are taken in Bochner's sense. By the same argument as in the proof of [11, Theorem 2.2], we can show that

$$(I_{0+}^{\theta_1} (I_{0+}^{\theta_2} \phi))(t) = (I_{0+}^{\theta_1 + \theta_2} \phi)(t) \quad \forall \phi \in L^1([0, T], X). \quad (6)$$

Let us denote by $I_{0+}^\theta(L^p([0, T], X))$ the space of summable functions $f : [0, T] \rightarrow X$ such that $f = I_{0+}^\theta \phi$ for some $\phi \in L^p([0, T], X)$. The space $I_{T-}^\theta(L^p)$ is defined similarly. We have the following characteristic property for $I_{0+}^\theta(L^p([0, T], X))$. Its proof is similar to the proof of [17, Proposition 2.1]

Proposition 2.1 $f \in I_{0+}^\theta(L^p([0, T], X))$ if and only if $I_{0+}^{1-\theta} f \in W^{1,p}([0, T], X)$.

Definition 2.1 Let $0 < \alpha < 1$ and $\phi \in C([0, T], X)$. Then each of the expressions

$$(D_{0+}^\alpha \phi)(t) := \frac{d}{dt} I_{0+}^{1-\alpha} \phi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\phi(\tau)}{(t-\tau)^\alpha} d\tau$$

and

$$(D_{T-}^\alpha \phi)(t) := -\frac{d}{dt} I_{T-}^{1-\alpha} \phi = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{\phi(\tau)}{(\tau-t)^\alpha} d\tau$$

is called a *left Riemann-Liouville fractional derivative* and a *right Riemann-Liouville fractional derivative*, respectively. The operator D_{0+}^α and D_{T-}^α is called the *left Riemann-Liouville fractional derivative operator* and the *right Riemann-Liouville fractional derivative operator*, respectively.

A sufficient condition for the existence of $D_{0+}^\alpha(\phi)$ and $D_{T-}^\alpha(\phi)$ is to require $\phi \in W^{1,1}([0, T], X)$ (see [11, Lemma 2.12]).

Proposition 2.2 Let $\alpha \in (0, 1)$. Then the following assertions are fulfilled.

(i) If $f \in L^1([0, T], X)$, then

$$D_{0+}^\alpha I_{0+}^\alpha f = f \quad \text{a.e.} \quad (7)$$

(ii) If $f \in I_{0+}^\alpha(L^1)$, then

$$I_{0+}^\alpha D_{0+}^\alpha f = f \quad \text{a.e.} \quad (8)$$

Proof. (i). Using (6) and the definition of D_{0+}^α , we have

$$D_{0+}^\alpha I_{0+}^\alpha f = D^1 I_{0+}^{1-\alpha} I_{0+}^\alpha f = D^1 I_{0+}^1 f = f.$$

(ii). Let $f = I_{0+}^\alpha \phi$ for some $\phi \in L^1([0, T], X)$. Then from (7), we have

$$I_{0+}^\alpha D_{0+}^\alpha f = I_{0+}^\alpha D_{0+}^\alpha I_{0+}^\alpha \phi = I_{0+}^\alpha \phi = f.$$

□

Definition 2.2 Let $0 < \alpha < 1$ and $\phi \in L^p([0, T], X)$ with $p \geq 1$. The operator $\frac{d^\alpha}{dt^\alpha}$ is defined by

$$\frac{d^\alpha}{dt^\alpha}\phi(t) := D_{0+}^\alpha(\phi - \phi(0)) = D_{0+}^\alpha\phi - \frac{\phi(0)}{(t-0)^\alpha\Gamma(1-\alpha)}$$

is called the left Caputo differential operator of order α , and the operator $\frac{\hat{d}^\alpha}{dt^\alpha}$ defined by

$$\frac{\hat{d}^\alpha}{dt^\alpha}\phi(t) := D_{T-}^\alpha(\phi - \phi(T)) = D_{T-}^\alpha\phi - \frac{\phi(T)}{(T-t)^\alpha\Gamma(1-\alpha)}$$

is called the right Caputo differential operator of order α provided all terms exist.

From the definition, we see that if $\phi(0) = 0$, then $\frac{d^\alpha}{dt^\alpha}\phi = D_{0+}^\alpha(\phi)$. Also, if $\phi(T) = 0$, then $\frac{\hat{d}^\alpha}{dt^\alpha}\phi = D_{T-}^\alpha(\phi)$.

Lemma 2.1 Suppose that $X = H$ is a Hilbert space, $f \in I_{T-}^\alpha(L^q([0, T], H))$ and $g \in I_{0+}^\alpha(L^p([0, T], H))$ with $p, q \geq 1$ and $1/p + 1/q \leq 1 + \alpha$. Then one has

$$\int_0^T (f, D_{0+}^\alpha g)_H dt = \int_0^T (g, D_{T-}^\alpha f)_H dt \quad (9)$$

Particularly, when $f(T) = 0$ and $g(0) = 0$, one has

$$\int_0^T (f, \frac{d^\alpha g}{dt^\alpha})_H dt = \int_0^T (g, \frac{\hat{d}^\alpha f}{dt^\alpha})_H dt. \quad (10)$$

Here $(\cdot, \cdot)_H$ denotes the scalar product in H .

Proof. We first claim that if $\phi_1 \in L^p([0, T], H)$ and $\phi_2 \in L^q([0, T], H)$ with $1/p + 1/q \leq 1 + \alpha$, then

$$\int_0^T (\phi_2(t), I_{0+}^\alpha \phi_1(t))_H dt = \int_0^T (\phi_1(t), I_{T-}^\alpha \phi_2(t))_H dt. \quad (11)$$

In fact, from the assumption we see that the integrals in both sides are defined. By Fubini's Theorem, we have

$$\begin{aligned} \int_0^T (\phi_2(t), I_{0+}^\alpha \phi_1(t))_H dt &= \int_0^T (\phi_2(t), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \phi_1(s) ds)_H dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t (\phi_2(t), \frac{1}{(t-s)^{1-\alpha}} \phi_1(s))_H ds dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^T ds \int_s^T (\phi_2(t), \frac{1}{(t-s)^{1-\alpha}} \phi_1(s))_H dt \\ &= \int_0^T (\phi_1(s), I_{T-}^\alpha \phi_2(s))_H ds. \end{aligned}$$

The claim is justified. We now prove (9). By the assumption, we have $f = I_{T-}^\alpha \phi_2$ and $g = I_{0+}^\alpha \phi_1$ for some $\phi_2 \in L^q([0, T], H)$ and $\phi_1 \in L^p([0, T], H)$. Then $\phi_2 = D_{T-}^\alpha(f)$. Using (11), we have

$$\int_0^T (f, D_{0+}^\alpha g)_H dt = \int_0^T (I_{T-}^\alpha \phi_2(t), \phi_1(t))_H dt = \int_0^T (\phi_2(t), I_{0+}^\alpha \phi_1)_H dt = \int_0^T (D_{T-}^\alpha f(t), g(t))_H dt.$$

We obtain formula (9). □

Proposition 2.3 Suppose that $f \in L^\infty([0, T], X)$ and $\alpha \in (0, 1)$. Then $I_{0+}^\alpha f \in C^\alpha([0, T], X)$ -the space of Hölder continuous functions with order α . Moreover, one has

$$\|I_{0+}^\alpha f\|_{C([0, T], X)} \leq \frac{T^\alpha}{\alpha\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)}. \quad (12)$$

Proof. Take any $t_0, t \in [0, T]$. Let $t > t_0$. Then we have

$$\begin{aligned} (I_{0+}^\alpha f)(t) - (I_{0+}^\alpha f)(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \int_0^{t_0} \frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_0} \left[\frac{f(\tau)}{(t-\tau)^{1-\alpha}} - \frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} \right] d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \end{aligned}$$

Hence

$$\begin{aligned} &\|(I_{0+}^\alpha f)(t) - (I_{0+}^\alpha f)(t_0)\|_X \\ &\leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_0^{t_0} \left| \frac{1}{(t-\tau)^{1-\alpha}} - \frac{1}{(t_0-\tau)^{1-\alpha}} \right| d\tau + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_{t_0}^t \frac{1}{(t-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_0^{t_0} \left[\frac{1}{(t_0-\tau)^{1-\alpha}} - \frac{1}{(t-\tau)^{1-\alpha}} \right] d\tau + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_{t_0}^t \frac{1}{(t-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} [(t-t_0)^\alpha + t_0^\alpha - t^\alpha] + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t-t_0)^\alpha \\ &\leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t-t_0)^\alpha + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t-t_0)^\alpha \end{aligned}$$

This implies that

$$\|(I_{0+}^\alpha f)(t) - (I_{0+}^\alpha f)(t_0)\|_X \leq K_1 |t - t_0|^\alpha.$$

Let $t < t_0$. Then we write

$$\begin{aligned} &(I_{0+}^\alpha f)(t_0) - (I_{0+}^\alpha f)(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} d\tau + \frac{1}{\Gamma(\alpha)} \int_t^{t_0} \frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \left[\frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} - \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \right] d\tau + \frac{1}{\Gamma(\alpha)} \int_t^{t_0} \frac{f(\tau)}{(t_0-\tau)^{1-\alpha}} d\tau. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\|(I_{0+}^\alpha f)(t_0) - (I_{0+}^\alpha f)(t)\|_X \\ &\leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_0^t \left| \frac{1}{(t_0-\tau)^{1-\alpha}} - \frac{1}{(t-\tau)^{1-\alpha}} \right| d\tau + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_t^{t_0} \frac{1}{(t_0-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_0^t \left[\frac{1}{(t-\tau)^{1-\alpha}} - \frac{1}{(t_0-\tau)^{1-\alpha}} \right] d\tau + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_t^{t_0} \frac{1}{(t_0-\tau)^{1-\alpha}} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} [t^\alpha - t_0^\alpha + (t_0-t)^\alpha] + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t_0-t)^\alpha \\ &\leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t_0-t)^\alpha + \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \frac{1}{\alpha} (t_0-t)^\alpha \end{aligned}$$

which implies that

$$\|(I_{0+}^\alpha f)(t_0) - (I_{0+}^\alpha f)(t)\|_X \leq K_2 |t - t_0|^\alpha$$

Consequently, $I_{0+}^\alpha \phi$ is Hölder continuous with order α . Also,

$$\|I_{0+}^\alpha f(t)\|_X \leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} ds = \frac{1}{\alpha \Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} t^\alpha \leq \frac{1}{\alpha \Gamma(\alpha)} \|f\|_{L^\infty([0, T], X)} T^\alpha$$

for all $t \in [0, T]$. Hence (12) is obtained. The proof of the proposition is completed. \square

Let $J = [0, T]$ and $g : J \rightarrow X$ be a X -valued function. We consider the equation of finding $y \in C(J, X)$ such that

$$\begin{cases} \frac{d^\alpha}{dt^\alpha} y(t) = g(t) & \text{a.e. } t \in J, \\ y(0) = y_0. \end{cases} \quad (13)$$

Proposition 2.4 Suppose that $g \in L^1(J, X)$ and $y \in C(J, X)$. Then y is a solution of equation (13) if and only if y is a solution of the following integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds. \quad (14)$$

Proof. (\Rightarrow). Suppose that y is a solution to (13). Then we have

$$g = \frac{d^\alpha y}{dt^\alpha} = \frac{d}{dt} I_{0+}^{1-\alpha}(y - y(0)) \in L^1([0, T], X).$$

Hence $I_{0+}^{1-\alpha}(y - y(0)) \in W^{1,1}([0, T], X)$. By Proposition 2.1, $(y - y(0)) \in I_{0+}^\alpha(L^1)$. By (8), we have

$$I_{0+}^\alpha D_{0+}^\alpha (y - y(0)) = y - y(0) = y - y_0.$$

We now act I_{0+}^α on both sides of (13) to get

$$y - y_0 = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds. \quad (15)$$

Hence y is a solution of equation (14).

(\Leftarrow). Suppose that $y \in C(J, X)$ and satisfies equation (14). Acting D_{0+}^α on both sides of (15) and using (7), we obtain

$$\frac{d^\alpha y}{dt^\alpha} = D_{0+}^\alpha (y - y_0) = D_{0+}^\alpha I_{0+}^\alpha g = g.$$

□

3 Solution existence

In this section, we investigate the existence of solutions to the state equation (2)-(4) and the existence of optimal solution to problem (1)-(5).

In the sequel, we will assume that $J = [0, T]$, $D = H^2(\Omega) \cap H_0^1(\Omega)$, $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. Then we have

$$D \hookrightarrow V \hookrightarrow H.$$

Besides, since $n = 2, 3$, the embedded theorem implies that $D \hookrightarrow C(\bar{\Omega})$ is compact. Given functions f and g on J , the convolution of f and g is denoted by $f * g$ which is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

We say y is a *strong solution* to (2)-(4) if $y \in C(J, H)$ and $\Delta y + u \in L^p(J, H)$ with $p \geq 1$ such that

$$\begin{cases} \frac{d^\alpha y}{dt^\alpha} = \Delta y + u & \text{a.e. } t \in J, \\ y(0) = y_0. \end{cases} \quad (16)$$

Here $y_0 \in D$ and $u \in L^p(J, H)$ are given.

Let us assume that $y \in C(J, H)$ is a solution to (16). Then $\Delta y + u \in L^p(J, H)$ with $p \geq 1$. By Proposition 2.4, y satisfies the integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Delta y(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds \quad (17)$$

or equivalently

$$y(t) = y_0 + k_{1-\alpha} * (\Delta y + u), \quad (18)$$

where

$$k_{1-\alpha}(t) := \frac{1}{\Gamma(\alpha)t^{1-\alpha}}. \quad (19)$$

Each solution of (18) is called a *mild solution* to (2)-(4). It is clear that any strong solution is a mild solution. We now establish formula representing solution of (18).

It is known that $\Delta : D \subset H \rightarrow H$ is a sectorial operator. Namely, the resolvent set $\rho(\Delta)$ contains the sector

$$S_\theta = \{\lambda \in C : \lambda \neq 0, |\arg \lambda| < \theta\}, \pi/2 < \theta < \pi,$$

and

$$\|R(\lambda, \Delta)\|_{L(L^2(\Omega))} \leq \frac{M}{\lambda} \quad \forall \lambda \in S_\theta.$$

Hence Δ is a generator of a uniformly bounded analytic semigroup $\{S(t), t \geq 0\}$ (see Theorem 2.4.1 and Proposition 2.4.2 in [21], [20, Theorem 3.1.2] and [26, Theorem 2.7, p. 211]). Besides, the following formula for the resolvent operator is valid:

$$R(\lambda, A) = (\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} S(t) dt \quad \forall t > 0. \quad (20)$$

In other words, $R(\lambda, \Delta)$ is the Laplace transform of $S(t)$.

Let us denote by z the Laplace transform of y , that is

$$z(s) = \mathcal{L}(y)(s) := \int_0^{+\infty} e^{-st} y(t) dt.$$

By taking Laplace transform both sides of (18) and using the property of Laplace transform for convolution (see [18, Theorem 1, p. 232]), we get

$$z(\lambda) = \frac{y_0}{\lambda} + \frac{1}{\lambda^\alpha} \Delta z(\lambda) + \frac{1}{\lambda^\alpha} \mathcal{L}(u)(\lambda).$$

This implies that

$$z(\lambda) = (\lambda^\alpha I - \Delta)^{-1} [\lambda^{\alpha-1} y_0] + (\lambda^\alpha I - \Delta)^{-1} [\mathcal{L}(u)(\lambda)]. \quad (21)$$

By taking the inverse Laplace transform, we obtain

$$y(t) = F(t)y_0 + \int_0^t E(t-s)u(s)ds,$$

where

$$F(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta, \delta}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I + \Delta)^{-1} d\lambda,$$

$$E(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta, \delta}} e^{\lambda t} (\lambda^\alpha I + \Delta)^{-1} d\lambda.$$

Here $\Gamma_{\theta, \delta}$ is a Hankel contour which is oriented with an increasing imaginary part. Namely,

$$\Gamma_{\theta, \delta} = \{\lambda \in C : |\lambda| = \delta, |\arg \lambda| < \theta\} \cup \{\lambda \in C : \lambda = \rho e^{\pm i\theta}, \rho \geq \delta\}.$$

We refer the reader to [15, Theorem 6.4] for properties of F and E .

There is another way to represent solution $y(t)$ of (18). Let ϕ_α be the one-sided probability density function such that

$$\int_0^\infty \phi_\alpha(\tau) d\tau = 1 \quad \text{and} \quad \int_0^\infty e^{-\lambda\tau} \phi_\alpha(\tau) d\tau = e^{-\lambda^\alpha}.$$

For the explicit formula of ϕ_α , we refer the reader to [23] and [29]. Moreover, from the explicit formula of ϕ_α in [23], we can show (see [27]) that

$$\int_0^\infty t^\mu \phi_\alpha(t) dt = \frac{\Gamma(-\mu/\alpha)}{\alpha\Gamma(-\mu)}. \quad (22)$$

Let us define

$$\zeta_\alpha(\tau) = \frac{1}{\alpha\tau^{1+1/\alpha}} \phi_\alpha\left(\frac{1}{\tau^{1/\alpha}}\right).$$

Then we can check that $\int_0^\infty \zeta_\alpha(\tau) d\tau = 1$. Moreover, using (22), we can show that

$$\int_0^\infty \theta \zeta_\alpha(\theta) d\theta = \frac{1}{\alpha\Gamma(\alpha)}.$$

Using (21), we have

$$\begin{aligned} z(\lambda) &= (\lambda^\alpha I - A)^{-1}[h(\lambda)] = \int_0^\infty e^{-\lambda^\alpha s} S(s) h(\lambda) ds \\ &= \int_0^\infty e^{-(\lambda s^{1/\alpha})^\alpha} S(s) h(\lambda) ds \\ &= \int_0^\infty \int_0^\infty e^{-\lambda s^{1/\alpha} \tau} \phi_\alpha(\tau) d\tau S(s) h(\lambda) ds \\ &= \int_0^\infty \int_0^\infty e^{-\lambda t} \frac{1}{s^{1/\alpha}} \phi_\alpha\left(\frac{t}{s^{1/\alpha}}\right) dt S(s) h(\lambda) ds \\ &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \frac{1}{s^{1/\alpha}} \phi_\alpha\left(\frac{t}{s^{1/\alpha}}\right) S(s) h(\lambda) ds \right] dt. \end{aligned}$$

Here $h(\lambda) := \lambda^{\alpha-1} y_0 + \mathcal{L}(u)(\lambda)$. By changing variable $s = \theta t^\alpha$, we have

$$z(\lambda) = \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \alpha t^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta t^\alpha) h(\lambda) d\theta \right] dt.$$

By defining

$$P(t) = \int_0^\infty \alpha t^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta t^\alpha) d\theta, \quad (23)$$

we get

$$\begin{aligned} z(\lambda) &= \mathcal{L}(P)(\lambda) [\lambda^{\alpha-1} y_0 + \mathcal{L}(u)(\lambda)] \\ &= \mathcal{L}\left(\left(\frac{1}{\Gamma(1-\alpha)t^\alpha}\right) * P\right)(\lambda) y_0 + \mathcal{L}(P * u)(\lambda). \end{aligned}$$

By acting the converse Laplace transform on both sides and using the convolution property of Laplace transform, we obtain

$$y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} P(s) y_0 ds + \int_0^t P(t-s) u(s) ds \quad (24)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \int_0^\infty \alpha s^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta s^\alpha) y_0 d\theta ds + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta (t-s)^\alpha) d\theta u(s) ds \quad (25)$$

$$\begin{aligned} &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \frac{1}{(1-\tau)^{\alpha} \tau^{1-\alpha}} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta (t\tau)^\alpha) y_0 d\theta d\tau \\ &+ t^\alpha \int_0^1 (1-\tau)^{\alpha-1} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta t^\alpha (1-\tau)^\alpha) u(t\tau) d\theta d\tau. \end{aligned} \quad (26)$$

Based on the above representation of $y(t)$, we have

Proposition 3.1 Let $y_0 \in D = H^2(\Omega) \cap H_0^1(\Omega)$. The following assertions are fulfilled:

(i) If $u \in L^p(J, H)$ with $p > 1/\alpha$, then equation (18) has a unique mild solution $y \in C(J, H)$ which is given by (26) and there exist constants $K > 0$ and $K' > 0$ such that

$$\|y\|_{C(J, H)} \leq K\|y_0\|_H + K'T^{\alpha-1/p}\|u\|_{L^p(J, H)}. \quad (27)$$

Moreover, $y(t) \in D$ for a.e. $t \in [0, T]$.

(ii) If $u \in C^\gamma(J, X)$ with $0 < \gamma < 1$, then equation (18) has a unique solution $y \in C(J, D) \cap C^\gamma(J, X)$ and there exist positive constants K, K_1 and K_2 such that

$$\|y\|_{C(J, D)} \leq K\|y_0\|_D + K_1\|u\|_{C^\gamma(J, X)} + K_2\|u\|_{C(J, X)}. \quad (28)$$

In this case y is also a strong solution to (2)-(4).

Proof. (i) Let us represent y by (24)-(26). We now show that $\lim_{t \rightarrow 0^+} y(t) = y_0$. In fact, it is clear that

$$\begin{aligned} \lim_{t \rightarrow 0^+} y(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \frac{1}{(1-\tau)^\alpha \tau^{1-\alpha}} \int_0^\infty \theta \zeta_\alpha(\theta) S(0) y_0 d\theta d\tau + 0 \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \frac{1}{(1-\tau)^\alpha \tau^{1-\alpha}} d\tau \int_0^\infty \theta \zeta_\alpha(\theta) y_0 d\theta \\ &= \frac{\alpha}{\Gamma(1-\alpha)} B(\alpha, 1-\alpha) \frac{1}{\alpha \Gamma(\alpha)} y_0 \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \frac{\Gamma(\alpha) \Gamma(1-\alpha)}{\Gamma(1)} \frac{1}{\alpha \Gamma(\alpha)} y_0 = y_0. \end{aligned}$$

Here the beta function is defined by

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}, \quad 0 < a, b < +\infty.$$

Note that $S(\tau)$ is uniformly bounded. Hence there exists a constant $K > 0$ such that

$$\|P(t)\|_H \leq K t^{\alpha-1} \quad \forall t > 0.$$

We now claim that $y \in C([0, T], H)$. Take any $t_0, t \in [0, T]$ and assume that $t_0 < t$. Then we have

$$\begin{aligned} \|y(t) - y(t_0)\|_H &= \left\| \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \frac{1}{(1-\tau)^\alpha \tau^{1-\alpha}} \int_0^\infty \theta \zeta_\alpha(\theta) [S(\theta(t\tau)^\alpha) - S(\theta(t_0\tau)^\alpha)] y_0 d\theta d\tau \right. \\ &\quad + t^\alpha \int_0^1 (1-\tau)^{\alpha-1} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta t^\alpha (1-\tau)^\alpha) u(t\tau) d\theta d\tau \\ &\quad \left. - t_0^\alpha \int_0^1 (1-\tau)^{\alpha-1} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta t_0^\alpha (1-\tau)^\alpha) u(t_0\tau) d\theta d\tau \right\| \\ &\leq \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \frac{1}{(1-\tau)^\alpha \tau^{1-\alpha}} \int_0^\infty \theta \zeta_\alpha(\theta) \|S(\theta(t\tau)^\alpha) - S(\theta(t_0\tau)^\alpha)\| \|y_0\|_H d\theta d\tau \\ &\quad + \left\| t^\alpha \int_0^1 (1-\tau)^{\alpha-1} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta t^\alpha (1-\tau)^\alpha) u(t\tau) d\theta d\tau \right. \\ &\quad \left. - t_0^\alpha \int_0^1 (1-\tau)^{\alpha-1} \int_0^\infty \theta \zeta_\alpha(\theta) S(\theta t_0^\alpha (1-\tau)^\alpha) u(t_0\tau) d\theta d\tau \right\| \\ &:= I(t) + II(t). \end{aligned} \quad (29)$$

Since $S(t)$ is continuous and uniformly bounded, the Dominated Convergence Theorem implies that

$I(t) \rightarrow 0$ as $t \rightarrow t_0$. To estimate $II(t)$, we write

$$\begin{aligned}
II(t) &= \left\| \int_0^t P(t-s)u(s)ds - \int_0^{t_0} P(t_0-s)u(s)ds \right\|_H \\
&= \left\| \int_0^{t_0} (P(t-s) - P(t_0-s))u(s)ds + \int_{t_0}^t P(t-s)u(s)ds \right\|_H \\
&= \left\| \int_0^{t_0} \left[\int_0^\infty \alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta(t-s)^\alpha) d\theta - \int_0^\infty \alpha(t_0-s)^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta(t_0-s)^\alpha) d\theta \right] u(s) ds \right. \\
&\quad \left. + \int_{t_0}^t P(t-s)u(s)ds \right\|_H \\
&= \left\| \int_0^{t_0} \left[\int_0^\infty \alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) (S(\theta(t-s)^\alpha) - S(\theta(t_0-s)^\alpha)) d\theta \right. \right. \\
&\quad \left. \left. + \int_0^{t_0} \int_0^\infty \alpha((t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}) \theta \zeta_\alpha(\theta) S(\theta(t_0-s)^\alpha) d\theta \right] u(s) ds \right. \\
&\quad \left. + \int_{t_0}^t P(t-s)u(s)ds \right\|_H \\
&\leq \|u\|_{L^p(J,H)} \left[\int_0^{t_0} \left(\int_0^\infty \alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|S(\theta(t-s)^\alpha) - S(\theta(t_0-s)^\alpha)\| d\theta \right)^q ds \right]^{1/q} \\
&\quad + \|u\|_{L^p(J,H)} \left[\int_0^{t_0} \left(\int_0^\infty |(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}| \theta \zeta_\alpha(\theta) \|S(\theta(t_0-s)^\alpha)\| d\theta \right)^q ds \right]^{1/q} \\
&\quad + \|u\|_{L^p(J,H)} \left[\int_{t_0}^t \|P(t-s)\|^q ds \right]^{1/q} \\
&\leq \|u\|_{L^p(J,H)} \left[\int_0^{t_0} \left(\int_0^\infty \alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|S(\theta(t-s)^\alpha) - S(\theta(t_0-s)^\alpha)\| d\theta \right)^q ds \right]^{1/q} \\
&\quad + \|u\|_{L^p(J,H)} \left[\int_0^{t_0} \left(\int_0^\infty |(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}| \theta \zeta_\alpha(\theta) \|S(\theta(t_0-s)^\alpha)\| d\theta \right)^q ds \right]^{1/q} \\
&\quad + \|u\|_{L^p(J,H)} K(t-t_0)^{(q(\alpha-1)+1)/q} \\
&:= II_1(t) + II_2(t) + II_3(t).
\end{aligned}$$

Since $p > 1/\alpha$, $q(\alpha-1)+1 > 0$. Hence $II_3(t) \rightarrow 0$ as $t \rightarrow t_0+$. By the continuity of S ,

$$\alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|S(\theta(t-s)^\alpha) - S(\theta(t_0-s)^\alpha)\| \rightarrow 0 \quad \text{as } t \rightarrow t_0+$$

and

$$\begin{aligned}
&\int_0^{t_0} \left(\int_0^\infty \alpha(t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|S(\theta(t-s)^\alpha) - S(\theta(t_0-s)^\alpha)\| d\theta \right)^q ds \\
&\leq 2K \int_0^{t_0} \left(\int_0^\infty \alpha(t_0-s)^{\alpha-1} \theta \zeta_\alpha(\theta) d\theta \right)^q ds \\
&= \frac{(2K)^q}{\Gamma(\alpha+1)^q} t_0^{q(\alpha-1)+1}.
\end{aligned}$$

The Dominated Convergence Theorem implies that $II_1(t) \rightarrow 0$ when $t \rightarrow t_0+$. By the same argument, we can show that $II_2(t) \rightarrow 0$ as $t \rightarrow t_0+$. Combining these we conclude that

$$\lim_{t \rightarrow t_0+} \|y(t) - y(t_0)\|_X = 0.$$

Repeating the procedure, we also have

$$\lim_{t \rightarrow t_0-} \|y(t) - y(t_0)\|_X = 0.$$

Hence y is continuous at t_0 . Since t_0 is arbitrary in $[0, T]$, we have $y \in C([0, T], H)$. To prove (27), we use (25) and the fact that $\|S(t)\| \leq K$ for all $t > 0$. Then we have

$$\begin{aligned}
\|y(t)\|_H &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} \int_0^\infty \alpha \theta \zeta_\alpha(\theta) \|S(\theta s^\alpha)\| \|y_0\|_H d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|S(\theta(t-s)^\alpha)\| \|u(s)\|_H d\theta ds \\
&\leq K \|y_0\|_H + \frac{K}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{q(\alpha-1)} ds \right]^{1/q} \|u\|_{L^p(J, H)} \\
&\leq K \|y_0\|_H + \frac{K}{\Gamma(\alpha)} T^{\alpha-1/p} \|u\|_{L^p(J, H)}.
\end{aligned}$$

This leads to (27).

(ii). Note that $S'(t) = \Delta S(t)$. Since $y_0 \in D$, $\Delta S(t)y_0 = S(t)\Delta y_0$. Therefore, we have from (25) that

$$\begin{aligned}
\Delta y(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \int_0^\infty \alpha s^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta s^\alpha) \Delta y_0 d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \Delta S(\theta(t-s)^\alpha) u(s) d\theta ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \int_0^\infty \alpha s^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta s^\alpha) \Delta y_0 d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \Delta S(\theta(t-s)^\alpha) [u(s) - u(t)] d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \Delta S(\theta(t-s)^\alpha) u(t) d\theta dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \int_0^\infty \alpha s^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta s^\alpha) \Delta y_0 d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \Delta S(\theta(t-s)^\alpha) [u(s) - u(t)] d\theta ds \\
&\quad + \int_0^\infty \zeta_\alpha(\theta) \int_0^t \frac{d}{ds} [S(\theta(t-s)^\alpha)] u(t) ds d\theta \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \int_0^\infty \alpha s^{\alpha-1} \theta \zeta_\alpha(\theta) S(\theta s^\alpha) \Delta y_0 d\theta ds \\
&\quad + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \Delta S(\theta(t-s)^\alpha) [u(s) - u(t)] d\theta ds \\
&\quad + u(t) + \int_0^\infty \zeta_\alpha(\theta) S(\theta t^\alpha) u(t) d\theta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\Delta y(t)\|_H &\leq K \|\Delta y_0\|_H + \int_0^t \int_0^\infty \alpha (t-s)^{\alpha-1} \theta \zeta_\alpha(\theta) \|\Delta S(\theta(t-s)^\alpha)\| \|u(s) - u(t)\|_H d\theta ds + (1+K) \|u(t)\|_H \\
&\leq K \|\Delta y_0\|_H + \int_0^t \int_0^\infty \alpha (t-s)^{-1} \zeta_\alpha(\theta) K' \|u(s) - u(t)\|_H d\theta ds + (1+K) \|u(t)\|_H \\
&\leq K \|\Delta y_0\|_H + \int_0^t \alpha (t-s)^{-1} K' \|u\|_{C^\gamma([0, T], H)} |t-s|^\gamma ds + (1+K) \|u(t)\|_H \\
&= K \|y_0\|_D + K' \|u\|_{C^\gamma([0, T], H)} \frac{t^\gamma}{\gamma} + (1+K) \|u(t)\|_H.
\end{aligned}$$

This implies that

$$\|y\|_{C(J, D)} \leq K \|y_0\|_D + K' \|u\|_{C^\gamma([0, T], H)} \frac{T^\gamma}{\gamma} + (1+K) \|u\|_{C(J, H)}.$$

Hence (28) is obtained. Since $\Delta y \in C(J, H)$, we have $\Delta y + u \in L^\infty(J, H)$. By Proposition 2.3, $y \in C^\alpha(J, H)$. By Proposition 2.4, y is a strong solution to (2)-(4). \square

Proposition 3.2 *Suppose that $y_0 \in D$ and $u \in L^{p'}([0, T], H)$ with $1 < p' < \infty$. Then equation (18) has a unique solution $y \in L^{p'}([0, T], D)$ such that $\frac{d^\alpha y}{dt^\alpha} \in L^{p'}([0, T], H)$ and*

$$\|y\|_{L^{p'}([0, T], D)} + \left\| \frac{d^\alpha y}{dt^\alpha} \right\|_{L^{p'}([0, T], H)} \leq C_1 \|u\|_{L^{p'}([0, T], H)} + C_2 \|\Delta y_0\|_H, \quad (30)$$

where C_1 and C_2 are positive constants which are independent of y, f and y_0 . In addition if $p' > 1/\alpha$, then $y \in C([0, T], H) \cap L^{p'}([0, T], D)$. In this case y is a strong solution to (2)-(4).

Proof. Define $v(t) = y(t) - y_0$. Then $v(0) = 0$, $y(t) = v(t) + y_0$ and $D_{0+}^\alpha v = \frac{d^\alpha v}{dt^\alpha}$. By a simple computation, we have $\frac{d^\alpha y}{dt^\alpha} = \frac{d^\alpha v}{dt^\alpha}$. Then equation (16) becomes

$$\begin{cases} \frac{d^\alpha v}{dt^\alpha} = \Delta v + (u + \Delta y_0) \\ v(0) = 0. \end{cases} \quad (31)$$

Since $u \in L^{p'}(J, H)$, we have $u + \Delta y_0 \in L^{p'}(J, H)$. By Theorem 6.11 in [15], equation (31) has a unique mild solution $v \in L^{p'}(J, D)$ with $\frac{d^\alpha v}{dt^\alpha} \in L^{p'}(J, H)$ and

$$\|v\|_{L^{p'}(J, D)} + \left\| \frac{d^\alpha v}{dt^\alpha} \right\|_{L^{p'}(J, H)} \leq C \|f + \Delta y_0\|_{L^{p'}(J, H)}.$$

It follows that equation (17) has a unique solution $y \in L^{p'}(J, D)$ and $\frac{d^\alpha y}{dt^\alpha} \in L^{p'}(J, H)$ such that

$$\|y\|_{L^{p'}(J, D)} + \left\| \frac{d^\alpha y}{dt^\alpha} \right\|_{L^{p'}(J, H)} \leq C \|f\|_{L^{p'}(J, H)} + (1+c)T^{\frac{1}{p'}} \|\Delta y_0\|_H,$$

where c is a positive constant which is independent of f and y_0 . If $p' > 1/\alpha$, then Proposition 3.1 implies that $y \in C(J, H)$. Finally, since $\Delta y + u \in L^{p'}(J, H)$, Proposition 2.4 implies that y is a strong solution to (2)-(4). The proof of the proposition is completed. \square

In the sequel, we shall denote by Φ the feasible set of problem (1)-(5). In order to establish a result for the existence of optimal solution to problem (1)-(5), we make the following assumption

(A1) $L : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, that is, for each $y, u \in \mathbb{R}$, $L(\cdot, \cdot, y, u)$ is integrable in (t, x) and for each $(t, x) \in [0, T] \times \Omega$, $L(t, x, \cdot, \cdot)$ is continuous. Moreover, there exist functions $\phi_1, \phi_2 \in L_1(Q)$, nonnegative numbers $a_1, b_1, \beta_1, \beta_2$ and positive numbers a_2, b_2 such that

$$\phi_1(t, x) - a_1|y|^p - b_1|u|^{\beta_1} \leq L(t, x, y, u) \leq \phi_2(t, x) + a_2|y|^p + b_2|u|^{\beta_2} \quad (32)$$

for all $(t, x, y, u) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}$.

(A2) $L(t, x, \cdot, \cdot)$ is convex in (y, u) for each fixed $(t, x) \in [0, T] \times \Omega$.

(A3) $L(t, x, y, \cdot)$ is convex in u for each fixed $(t, x, y) \in [0, T] \times \Omega \times \mathbb{R}$, $p > 1/\alpha$ and $p \leq 6$ whenever $N = 3$.

Theorem 3.1 *Suppose that (A1) and (A2) are satisfied or (A1) and (A3) are satisfied. Then problem (1)-(5) has an optimal solution.*

Proof. Let $\xi = \inf_{(y, u) \in \Phi} \psi(y, u)$. Then for each $(y, u) \in \Phi$, (y, u) satisfies equation (16) and inequality (30) is valid. It follows that

$$\begin{aligned} \|y\|_{L^p(Q)} + \left\| \frac{d^\alpha y}{dt^\alpha} \right\|_{L^p(J, H)} &\leq \|y\|_{L^p(J, D)} + \left\| \frac{d^\alpha y}{dt^\alpha} \right\|_{L^p(J, H)} \\ &\leq C_1 \|u\|_{L^p(J, H)} + C_2 \|\Delta y_0\|_H \\ &\leq C' \|u\|_{L^\infty(Q)} + C'_2 \|\Delta y_0\|_H. \end{aligned}$$

This and (32) imply that ξ is finite, that is, $\xi \in \mathbb{R}$.

Let $(y_n, u_n) \in \Phi$ such that $V = \lim_{n \rightarrow \infty} \psi(y_n, u_n)$. Since $a \leq u_n \leq b$, $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Therefore, we may assume that u_n converges in $\sigma(L^\infty(Q), L^1(Q))$, the weak star topology, to a function $\bar{u} \in L^\infty(Q)$. From the above inequality, we see that $\{y_n\}$ is bounded in $L^p(J, D)$, $\{\Delta y_n\}$ and $\{\frac{d^\alpha}{dt^\alpha} y_n\}$ is bounded in $L^p(J, H)$. Therefore, we may assume that y_n converges weakly to \bar{y} in $L^p(J, D)$ and so $\Delta y_n \rightharpoonup \Delta \bar{y}$ in $L^p(J, H)$. Let us claim that $(\bar{y}, \bar{u}) \in \Phi$. Since $(y_n, u_n) \in \Phi$, from (18) we have

$$y_n = y_0 + k_{1-\alpha} * (\Delta y_n + u_n).$$

Taking any $\phi \in L^q(J, H)$, we have

$$(y_n(t), \phi(t))_H = (y_0, \phi(t))_H + \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s)^{\alpha-1} \phi(t), \Delta y_n(s) + u_n(s))_H ds. \quad (33)$$

Integrating both sides of (33) on $[0, T]$ and using Fubini's theorem, we have

$$\begin{aligned} \int_0^T (y_n(t), \phi(t))_H dt &= \int_0^T (y_0, \phi(t))_H dt + \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t ((t-s)^{\alpha-1} \phi(t), \Delta y_n(s) + u_n(s))_H ds dt \\ &= \int_0^T (y_0, \phi(t))_H dt + \frac{1}{\Gamma(\alpha)} \int_0^T ds \int_s^T ((t-s)^{\alpha-1} \phi(t), \Delta y_n(s) + u_n(s))_H dt \\ &= \int_0^T (y_0, \phi(t))_H dt + \frac{1}{\Gamma(\alpha)} \int_0^T \left(\int_s^T (t-s)^{\alpha-1} \phi(t) dt, \Delta y_n(s) + u_n(s) \right)_H ds \\ &= \int_0^T (y_0, \phi(t))_H dt + \frac{1}{\Gamma(\alpha)} \int_0^T \left(\int_{s'+T}^T (t-s'-T)^{\alpha-1} \phi(t) dt, \Delta y_n(s'+T) + u_n(s'+T) \right)_H ds' \\ &= \int_0^T (y_0, \phi(t))_H dt - \frac{1}{\Gamma(\alpha)} \int_0^T \left(\int_0^{s'} (t'-s')^{\alpha-1} \phi(t'+T) dt', \Delta y_n(s'+T) + u_n(s'+T) \right)_H ds' \\ &= \int_0^T (y_0, \phi(t))_H dt - \int_0^T (k_{1-\alpha} * \phi(\cdot + T)(s'), \Delta y_n(s'+T) + u_n(s'+T))_H ds'. \end{aligned}$$

By a property for convolution (see [6, Theorem 4.15]), we see that the function $s' \mapsto k_{1-\alpha} * \phi(\cdot + T)(s')$ belongs to $L^q(J, H)$. Thus we have shown that

$$\int_0^T (y_n(t), \phi(t))_H dt = \int_0^T (y_0, \phi(t))_H dt - \int_0^T (k_{1-\alpha} * \phi(\cdot + T)(s'), \Delta y_n(s'+T) + u_n(s'+T))_H ds'.$$

By letting $n \rightarrow \infty$, we obtain

$$\int_0^T (\bar{y}(t), \phi(t))_H dt = \int_0^T (y_0, \phi(t))_H dt - \int_0^T (k_{1-\alpha} * \phi(\cdot + T)(s'), \Delta \bar{y}(s'+T) + \bar{u}(s'+T))_H ds'.$$

This implies that

$$\bar{y} = y_0 + k_{1-\alpha} * (\Delta \bar{y} + \bar{u}).$$

It remains to show that $a \leq \bar{u}(t, x) \leq b$ for a.e. $(t, x) \in Q$. In fact, the set

$$K := \{u \in L^p(Q) : a \leq \bar{u}(t, x) \leq b, \text{ a.e.}\}$$

is a closed convex set in $L^p(Q)$. Hence it is a weakly closed set. Since $u_n \in K$ and $u_n \rightharpoonup \bar{u}$ in $L^p(Q)$, we have $\bar{u} \in K$. In a summary, we have showed that $(y_n, u_n) \rightharpoonup (\bar{y}, \bar{u})$ in $L^p(J, D) \times L^p(Q)$ and $(\bar{y}, \bar{u}) \in \Phi$.

(i) If (A1) and (A2) are valid, then the functional $\psi(y, u)$ is sequentially lower semicontinuous on $L^p(Q) \times L^p(Q)$. Since $L^p(J, D) \times L^p(Q) \hookrightarrow L^p(Q) \times L^p(Q)$, we see that $(y_n, u_n) \rightharpoonup (\bar{y}, \bar{u})$ in $L^p(Q) \times L^p(Q)$. It follows that

$$\xi = \lim_{n \rightarrow \infty} \psi(y_n, u_n) \geq \psi(\bar{y}, \bar{u}).$$

Hence (\bar{y}, \bar{u}) is an optimal solution to problem (1)-(5).

(ii) Suppose that (A1) and (A3) are valid. We claim that $y_n \rightarrow \bar{y}$ strongly in $L^p(Q)$. In fact, it is known that the embeddings

$$D \hookrightarrow V \hookrightarrow H$$

are compact. Since $1/p < \alpha < 1$, we have

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_n(s)\|_D ds &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)q} ds \right)^{1/q} \|y_n\|_{L^p(J,D)} \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{1}{((\alpha-1)q+1)^{1/q}} t^{(\alpha-1)+1/q} \|y_n\|_{L^p(J,D)} \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{1}{((\alpha-1)q)^{1/q}} T^{\alpha-1/p} M := M_1. \end{aligned}$$

Hence

$$\sup_{t \in (0,T)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_n(s)\|_D ds \leq M_1.$$

Besides, we have $p > 1$ and $p > \frac{p}{1+p\alpha}$, and $\{D_{0+}^\alpha y_n\} = \{\frac{d^\alpha}{dt^\alpha} y_n\}$ is bounded in $L^2(J, H)$. By Theorem 4.1 in [19], we see that $\{y_n\}$ is a relatively compact set in $L^p(J, V)$. Therefore, we may assume that $z_k \rightarrow z$ strongly in $L^p(J, V)$. If $N = 2$, then $V \hookrightarrow L^\infty(\Omega)$. If $N = 3$, then $V \hookrightarrow L^6(\Omega)$. Since $1 < p \leq 6$, we have $L^p(J, V) \hookrightarrow L^p(J, L^6(\Omega)) \hookrightarrow L^p(Q)$. Thus we have shown that $y_n \rightarrow \bar{y}$ strongly in $L^p(Q)$ and $u_n \rightarrow \bar{u}$ in $L^p(Q)$. By (A3), the functional $\psi(y, u)$ is sequentially lower semicontinuous (see Theorem 3.23 in [9]). Hence we have

$$\xi = \lim_{n \rightarrow \infty} \psi(y_n, u_n) \geq \psi(\bar{y}, \bar{u})$$

and so (\bar{y}, \bar{u}) is an optimal solution to problem (1)-(5). The proof of the theorem is completed. \square

4 Optimality conditions

We now impose the following assumption for L which makes sure that ψ is of class C^2 around a feasible point (\bar{y}, \bar{u}) . Given $1 < p < +\infty$, we denote by q the conjugate number of p , that is, $q > 1$ and $1/p + 1/q = 1$.

(A4) $L : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to variable (y, u) and satisfies the property that for each $M > 0$, there exist a positive number k_{LM} and a function $r_M \in L^\infty(Q)$ such that

$$\begin{aligned} |L_y(t, x, y, u)| + |L_u(t, x, y, u)| &\leq k_{LM} (|y|^{p-1} + |u|) + r_M(t, x), \\ |L_u(t, x, y_1, u_1) - L_u(t, x, y_2, u_2)| &\leq k_{LM} (|y_1 - y_2| + |u_1 - u_2|), \\ |L_y(t, x, y_1, u) - L_y(t, x, y_2, u)| &\leq k_{LM} \sum_{j=0}^{[p-1]} |y_1 - y_2|^{p-1-j} |y_2|^j \end{aligned}$$

for all $u, u_1, u_2 \in \mathbb{R}$ satisfying $|u|, |u_i| \leq M$ and any $y_1, y_2 \in \mathbb{R}$. Also for each $M > 0$, there is a number $k_{LM} > 0$ such that

$$\begin{aligned} |L_{uu}(t, x, y_1, u_1) - L_{uu}(t, x, y_2, u_2)| &\leq k_{LM} (|y_1 - y_2| + |u_1 - u_2|), \\ |L_{yu}(t, x, y_1, u_1) - L_{yu}(t, x, y_2, u_2)| &\leq k_{LM} (\varepsilon |y_1 - y_2|^{p-1} + |u_1 - u_2|) \end{aligned}$$

(with $\varepsilon = 0$ if $p = 2$ and $\varepsilon = 1$ if $p > 2$) and

$$|L_{yy}(t, x, y_1, u) - L_{yy}(t, x, y_2, u)| \begin{cases} = 0 & \text{if } p = 2, \\ \leq k_{LM} \sum_{j=0}^{[p-2]} |y_1 - y_2|^{p-2-j} |y_2|^j & \text{if } p > 2 \end{cases} \quad (34)$$

for a.e. $x \in \Omega$, for all $u, u_i, y_i \in \mathbb{R}$ with $|u|, |u_i| \leq M$ and $i = 1, 2$, where $[p - 2]$ is the integer part of $p - 2$. Moreover, we require that $L(t, x, 0, 0) \in L^1(\Omega)$ and $L_{yy}(t, x, 0, u) \in L^\infty(Q)$.

Note that (A4) makes sure that the functional ψ is of class C^2 . The following example shows that L satisfies (A4).

Example 4.1 Let $p \geq 2$ and $p \in \mathbb{N}$. Then

$$L(t, x, y, u) := (y - y_0(t, x))^p + y^{p-1} + \dots + y^2 + yu + u^2 + u^4 + \dots + u^{2p}$$

satisfies assumption (A4), where $y_0 \in L^p(Q)$ is given.

Let us define the mapping $F : L^p(J, D) \times L^\infty(Q) \rightarrow L^p(J, X)$ by setting

$$F(y, u) = y - y_0 - k_{1-\alpha} * (\Delta y + u), \quad (35)$$

where $k_{1-\alpha}$ is given by (19). Put

$$K_\infty := \{u \in L^\infty(Q) \mid a \leq u(t, x) \leq b \text{ a.e. } t \in Q\} \quad (36)$$

$$K_p := \{u \in L^p(Q) \mid a \leq u(t, x) \leq b \text{ a.e. } t \in Q\}. \quad (37)$$

Obviously, $K_\infty = K_p$. Besides, we can check that \bar{y} is a strong solution to (2)-(4) corresponding to $\bar{u} \in L^\infty(Q)$ if and only if $\bar{y} \in L^p(J, D)$ and $F(\bar{y}, \bar{u}) = 0$. Therefore, problem (1)-(5) can be formulated in the following form: find $(\bar{y}, \bar{u}) \in L^p(J, D) \times L^\infty(Q)$ which solves

$$\psi(y, u) \rightarrow \inf \quad (38)$$

s.t.

$$F(y, u) = 0, \quad (39)$$

$$u \in K_\infty. \quad (40)$$

In this section we assume that $(\bar{y}, \bar{u}) \in \Phi$, that is, $F(\bar{y}, \bar{u}) = 0$ and $\bar{u} \in K_\infty$. The symbols

$$L[t, x], L_y[t, x], L_u[t, x], L_{yy}[t, x], L_{uu}[t, x], L_{yu}[t, x]$$

stand for

$$L(t, x, \bar{y}(t, x), \bar{u}(t, x)), L_y(t, x, \bar{y}(t, x), \bar{u}(t, x)), L_u(t, x, \bar{y}(t, x), \bar{u}(t, x))$$

and so on, respectively.

Proposition 4.1 *Suppose that assumption (A4) is valid. Then the mappings ψ and F are of class C^2 on $L^p(J, D) \times L^\infty(Q)$. Moreover, the following formulae are valid:*

$$\langle \psi_y(\bar{y}, \bar{u}), y \rangle = \int_Q L_y[t, x] y(t, x) dt dx, \quad \langle \psi_u(\bar{y}, \bar{u}), u \rangle = \int_Q L_u[t, x] u(t, x) dt dx,$$

$$\langle \psi_{yy}(\bar{y}, \bar{u}) y_1, y_2 \rangle = \int_Q L_{yy}[t, x] y_1(t, x) y_2(t, x) dt dx, \quad \langle \psi_{yu}(\bar{y}, \bar{u}) y, u \rangle = \int_Q L_{yu}[t, x] y(t, x) u(t, x) dt dx,$$

$$\langle \psi_{uu}(\bar{y}, \bar{u}) u, v \rangle = \int_Q L_{uu}[t, x] u(t, x) v(t, x) dt dx$$

and

$$F_y(\bar{y}, \bar{u}) y = y - k_{1-\alpha} * (\Delta y),$$

$$F_u(\bar{y}, \bar{u}) u = -k_{1-\alpha} * (u),$$

$$F_{yy}(\bar{y}, \bar{u})[y_1, y_2] = 0, \quad F_{uu}(\bar{y}, \bar{u})[u, v] = 0, \quad F_{yu}(\bar{y}, \bar{u})[y, u] = 0,$$

for all $y \in L^p(J, D)$ and $u, v \in L^\infty(Q)$.

Proof. By the same argument as in the proof of Proposition A in [16], we can show that ψ is of class C^2 on $L^p(Q) \times L^\infty(Q)$. Since $L^p(J, D) \hookrightarrow L^p(Q)$, we see that ψ is of class C^2 on $L^p(J, D) \times L^\infty(Q)$. The differentiability of F follows from the linear property of F . \square

Fix any $u \in L^\infty(Q)$. By Proposition 3.2, the equation $F(y, u) = 0$ has a unique solution $y = y(u) \in L^p(J, D)$. Let us define the mapping $G : L^\infty(Q) \rightarrow L^p(J, D)$ by setting $G(u) = y(u)$. The mapping G is said to be a control-to-state mapping. By (24), we have

$$G(u) = k_\alpha * Py_0 + P * u.$$

It is clear that $G(u)$ is linear and so it is of class C^∞ . Namely, we have $G'(u)v = P * v$ and $G''(u)[v, v] = 0$.

Let us define the mapping $f(u) = \psi(G(u), u)$. Then by a simple argument, we can show that (\bar{y}, \bar{u}) is a locally optimal solution of (38)-(40) if and only if \bar{u} is a locally optimal solution of the problem:

$$\begin{cases} f(u) = \psi(G(u), u) \rightarrow \min \\ u \in K_\infty. \end{cases} \quad (41)$$

The following proposition gives some properties of control-to-state mapping G .

Proposition 4.2 *The mapping $G : L^\infty(Q) \rightarrow L^p(J, D)$ has the following properties.*

(i) *If $v \in L^\infty(Q)$ and $z_v = G'(\bar{u})v$, then z_v is a unique solution of two equivalent equations:*

$$z_v = k_{1-\alpha} * (\Delta z_v + v) \quad (42)$$

and

$$\frac{d^\alpha z_v}{dt} = \Delta z_v + v, \quad z_v(0) = 0. \quad (43)$$

(ii) *For every $v_1, v_2 \in L^\infty$, $G''(\bar{u})[v_1, v_2] = 0$*

Proof. For all $u \in L^\infty(Q)$, we have $F(G(u), u) = 0$. By differentiating both sides in u , we get

$$F_y(\bar{y}, \bar{u})G'(\bar{u}) + F_u(\bar{y}, \bar{u}) = 0.$$

This means

$$F_y(\bar{y}, \bar{u})G'(\bar{u})v + F_u(\bar{y}, \bar{u})v = 0 \quad \forall v \in L^\infty(Q).$$

By Proposition 4.1, we get

$$z_v - k_{1-\alpha} * (\Delta z_v) - k_{1-\alpha} * v = 0.$$

Hence z_v is a solution of (42) and (43). Assertion (ii) follows from the linear property of $G(u)$. \square

Proposition 4.3 *The function f is of class C^2 around \bar{u} and for any $v, v_1, v_2 \in L^\infty(Q)$ the following formulae are valid:*

$$\langle f'(\bar{u}), v \rangle = \int_Q (L_u[t, x] - \varphi(t, x))v(t, x)dxdt \quad (44)$$

and

$$f''(\bar{u})[v_1, v_2] = \int_Q (L_{yy}[t, x]z_{v_1}z_{v_2} + L_{yu}[t, x]z_{v_1}v_2 + L_{uy}[t, x]z_{v_1}v_2 + L_{uu}[t, x]v_1v_2)dt dx, \quad (45)$$

where $\varphi \in L^q(J, D)$ is a unique mild solution of the adjoint equation

$$\frac{\hat{d}^\alpha \varphi}{dt^\alpha} = \Delta \varphi - L_y[\cdot, \cdot], \quad \varphi(T) = 0. \quad (46)$$

Proof. We first show that equation (46) has a unique solution. Indeed, by changing variable $\zeta(t) = \varphi(T - t)$, equation (46) is equivalent to the equation

$$\frac{\hat{d}^\alpha}{dt^\alpha} \varphi(T - t) = \Delta \zeta(t) - L_y[T - t, \cdot], \quad \zeta(0) = 0. \quad (47)$$

Since $\varphi(T) = 0$, we have $\frac{\hat{d}^\alpha \varphi}{dt^\alpha} = D_{T-}^\alpha \varphi$ and

$$\begin{aligned} D_{T-}^\alpha \varphi(T - t) &= \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_{T-t}^T \frac{\varphi(\tau)}{(\tau - T + t)^\alpha} d\tau = -\frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_t^0 \frac{\varphi(T - s)}{(t - s)^\alpha} ds \\ &= \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{\varphi(T - s)}{(t - s)^\alpha} ds = D_{0+}^\alpha \zeta(t) = \frac{d^\alpha}{dt^\alpha} \zeta(t). \end{aligned}$$

Hence equation (47) is equivalent to

$$\frac{d^\alpha}{dt^\alpha} \zeta(t) = \Delta \zeta(t) - L_y[T - t, \cdot], \quad \zeta(0) = 0. \quad (48)$$

By assumption (H1), $L_y[\cdot, \cdot] \in L^q(J, L^\infty(\Omega)) \hookrightarrow L^q(J, H)$. By Proposition 3.2, equation (48) has a unique mild solution $\zeta \in L^q(J, D)$ with $\frac{d^\alpha}{dt^\alpha} \zeta \in L^q(J, H)$. Consequently, equation (46) has a unique mild solution $\varphi(t) = \zeta(T - t) \in L^q(J, D)$. We now have $f'(u) = \psi_y(G(u), u)G'(u) + \psi_u(G(u), u)$. Hence

$$\langle f'(u), v \rangle = \psi_y(G(u), u)G'(u)v + \psi_u(G(u), u)v. \quad (49)$$

By taking $u = \bar{u}$ and putting $z = z_v = G'(\bar{u})v$, we get

$$\langle f'(\bar{u}), v \rangle = \psi_y(\bar{y}, \bar{u})G'(\bar{u})v + \psi_u(\bar{y}, \bar{u})v = \int_Q (L_y[t, x]z(t, x) + L_u[t, x]v(t, x)) dt dx. \quad (50)$$

Note that $z = z_v$ satisfies (43) and φ is a mild solution of (46). Hence

$$\varphi = I_{T-}^\alpha (\Delta \varphi - L_y[\cdot, \cdot]). \quad (51)$$

Since $\Delta \varphi - L_y[\cdot, \cdot] \in L^q(J, H)$, we see that $\varphi \in I_{T-}^\alpha (L^q)$. Hence Lemma 2.1 is applicable for φ and $z = z_v \in I_{0+}^\alpha (L^p)$. By applying operator D_{T-}^α on both sides of (51), we get

$$D_{T-}^\alpha \varphi = \Delta \varphi - L_y[\cdot, \cdot]$$

and so

$$L_y[\cdot, \cdot] = \Delta \varphi - D_{T-}^\alpha \varphi.$$

Besides, for each $t \in J$, $\varphi(t), z(t, \cdot) \in H^2(\Omega) \cap H_0^1(\Omega)$. By Lemma 1.5.3.2 in [13], we have

$$\int_\Omega \Delta \varphi(t, x) z(t, x) dx = \int_\Omega \varphi(t, x) \Delta z(t, x) dx.$$

Using this, (43), (46) and (9), we obtain

$$\begin{aligned} \langle f'(\bar{u}), v \rangle &= \int_0^T \int_\Omega (-D_{T-}^\alpha \varphi + \Delta \varphi) z(t, x) + L_u[t, x] v(t, x) dt dx \\ &= \int_0^T \int_\Omega (-D_{T-}^\alpha \varphi z(t, x) dx dt + \int_0^T \int_\Omega \Delta \varphi(t, x) z(t, x) dx dt + \int_0^T \int_\Omega L_u[t, x] v(t, x) dx dt \\ &= -\int_0^T (-D_{T-}^\alpha \varphi(t), z(t, \cdot))_H dt + \int_0^T \int_\Omega \varphi(t, x) \Delta z(t, x) dx dt + \int_0^T \int_\Omega L_u[t, x] v(t, x) dx dt \\ &= -\int_0^T (D_{0+}^\alpha z(t), \varphi(t, \cdot))_H dt + \int_0^T \int_\Omega \varphi(t, x) \Delta z(t, x) dx dt + \int_0^T \int_\Omega L_u[t, x] v(t, x) dx dt \\ &= \int_0^T \left(-\frac{d^\alpha z(t, \cdot)}{dt^\alpha} + \Delta z(t, \cdot), \varphi(t, \cdot) \right)_H dt + \int_0^T \int_\Omega L_u[t, x] v(t, x) dx dt \\ &= \int_0^T (-v, \varphi)_H dt + \int_0^T \int_\Omega L_u[t, x] v(t, x) dx dt \\ &= \int_Q (L_u[t, x] - \varphi(t, x)) v(t, x) dx dt. \end{aligned}$$

Hence (44) is obtained. Next, we have from (49) that

$$\begin{aligned} f''(u)v &= (\psi_{yy}(G(u), u)G'(u) + \psi_{yu}(G(u), u))G'(u)v + \psi_y(G(u), u)G''(u)v \\ &\quad + (\psi_{uy}(G(u), u)G'(u) + \psi_{uu}(G(u), u))v. \end{aligned}$$

Replacing $u = \bar{u}$, $v = v_1$ and acting on v_2 , we obtain

$$\begin{aligned} f''(u)[v_1, v_2] &= \psi_{yy}(\bar{y}, \bar{u})[z_{v_1}, z_{v_2}] + \psi_{yu}(\bar{y}, \bar{u})[z_{v_1}, v_2] + 0 \\ &\quad + \psi_{uy}(\bar{y}, \bar{u})[z_{v_1}, v_2] + \psi_{uu}(\bar{y}, \bar{u})[v_1, v_2] \\ &= \int_Q (L_{yy}[t, x]z_{v_1}z_{v_2} + L_{yu}[t, x]z_{v_1}v_2 + L_{uy}[t, x]z_{v_1}v_2 + L_{uu}[t, x]v_1v_2) dt dx \end{aligned}$$

which is (45). The proof of the proposition is complete. \square

In the sequel, we shall use the following sets.

$$\begin{aligned} Q_a &= \{(t, x) \in Q : \bar{u}(t, x) = a\}, \\ Q_b &= \{(t, x) \in Q : \bar{u}(t, x) = b\}, \\ Q_{ab} &= \{(t, x) \in Q : a < \bar{u}(t, x) < b\}. \end{aligned}$$

The following theorem is a result on the first-order necessary optimality conditions.

Theorem 4.1 *Suppose that assumption (A4) is valid. If $(\bar{y}, \bar{u}) \in \Phi$ is a locally optimal solution of problem (1)-(5), then there exist a function $\varphi \in L^q(J, D)$ with $\frac{d^\alpha \varphi}{dt^\alpha} \in L^q(J, H)$ and a function $e \in L^q(J, L^\infty(\Omega))$ such that the following conditions are fulfilled:*

(i) φ is a mild solution of the adjoint equation

$$\frac{d^\alpha \varphi}{dt^\alpha} = \Delta \varphi - L_y[\cdot, \cdot], \quad \varphi(T, \cdot) = 0; \quad (52)$$

(ii) (the stationary condition in u)

$$L_u[t, x] - \varphi(t, x) = e(t, x) \quad \text{a.e. } (t, x) \in Q;$$

(iii) (the complimentary condition) $e(t, x)$ has the property that

$$e(t, x) \begin{cases} \geq 0 & \text{a.e. } (t, x) \in Q_a, \\ \leq 0 & \text{a.e. } (t, x) \in Q_b, \\ 0 & \text{a.e. } (t, x) \in Q_{ab}. \end{cases}$$

Proof. It is clear that (\bar{y}, \bar{u}) is a locally optimal solution of (1)-(5) if and only if \bar{u} is a locally optimality solution of problem (41). By definition, there exists an open ball $B(\bar{u}, \epsilon) \subset L^\infty(Q)$ so that

$$f(u) \geq f(\bar{u}) \quad \forall u \in B(\bar{u}, \epsilon) \cap K_\infty.$$

Fix any $v \in K_\infty$. Then for $\lambda > 0$ small enough, we have $v_\lambda = \bar{u} + \lambda(v - \bar{u}) \in B(\bar{u}, \epsilon) \cap K_\infty$. Hence

$$f(\bar{u} + \lambda(v - \bar{u})) \geq f(\bar{u}).$$

This implies that

$$\frac{f(\bar{u} + \lambda(v - \bar{u})) - f(\bar{u})}{\lambda} \geq 0.$$

By letting $\lambda \rightarrow 0+$, we obtain

$$\langle f'(\bar{u}), v - \bar{u} \rangle \geq 0 \quad \forall v \in K_\infty. \quad (53)$$

By Proposition 3.2, equation (52) has a unique solution $\varphi \in C(J, H) \cap L^q(J, D)$. Hence (i) is valid. From Proposition 4.3 and (53), we have

$$\int_Q (L_u[t, x] - \varphi(t, x))(v(t, x) - \bar{u}(t, x)) dt dx \geq 0 \quad \forall v \in K_\infty. \quad (54)$$

By putting $L_u[t, x] - \varphi(t, x) = e(t, x)$, we see that $e \in L^q(J, L^\infty(\Omega))$ which satisfies (ii) and (54) becomes

$$\int_Q e(t, x)(v(t, x) - \bar{u}(t, x)) dt dx \geq 0 \quad \forall v \in K_\infty. \quad (55)$$

Using variational arguments as in the proof of [17, Theorem 4.1], we can show that e satisfies (iii). The theorem is proved. \square

To deal with second-order optimality conditions, we need the so-called critical cone as follows.

Definition 4.1 A couple $(y, v) \in L^p(J, D) \times L^p(Q)$ is said to be a critical direction to problem (1)-(5) at (\bar{y}, \bar{u}) if the following conditions are fulfilled:

$$(c_1) \int_Q (L_y[t, x]y(t, x) + L_u[t, x]v(t, x)) dt dx \leq 0;$$

$$(c_2) y = k_{1-\alpha} * (\Delta y + v);$$

$$(c_3) v(t, x) \geq 0 \text{ for a.e. } (t, x) \in Q_a \text{ and } v(t, x) \leq 0 \text{ for a.e. } (t, x) \in Q_b.$$

The set of critical directions at (\bar{y}, \bar{u}) is denoted by $\mathcal{C}_p[(\bar{y}, \bar{u})]$ which is a closed convex cone containing $(0, 0)$.

We now take a couple $(y, v) \in \mathcal{C}(\bar{y}, \bar{u})$ such that $v \in L^\infty(Q)$. Then (c₂) is equivalent to saying that $y = z_v = G'(\bar{u})v$. Moreover, if (\bar{y}, \bar{u}) is a locally optimal solution, then

$$\int_Q (L_y[t, x]y(t, x) + L_u[t, x]v(t, x)) dt dx = 0.$$

Indeed, it is clear that $e(t, x)v(t, x) \geq 0$ for a.e. $(t, x) \in Q$. Since $y = z_v$ we have from Proposition 4.3 and (50) that

$$0 \geq \int_Q (L_y[t, x]y(t, x) + L_u[t, x]v(t, x)) dt dx = \int_Q (L_u[t, x] - \varphi(t, x))v(t, x) dt dx = \int_Q e(t, x)v(t, x) dt \geq 0. \quad (56)$$

Let $\mathcal{L}(y, u, \varphi, e)$ be the Lagrange function associated with the problem, which is given by

$$\mathcal{L}(y, u, \varphi, e) = \int_Q [L(t, x, y, u) + y(\frac{\hat{d}^\alpha \varphi}{dt^\alpha} - \Delta \varphi) - \varphi u - eu] dt dx. \quad (57)$$

Then conditions (i) and (ii) of Theorem 4.1 is equivalent to the condition

$$\nabla_{(y, u)} \mathcal{L}(\bar{y}, \bar{u}, \varphi, e) = 0. \quad (58)$$

We have the following result on second-order necessary optimality conditions.

Theorem 4.2 Suppose that (A4) is valid, (\bar{y}, \bar{u}) is a locally optimal solution of (1)-(5) and (φ, e) satisfies the conclusion of Theorem 4.1. Then

$$\nabla_{(y, u)}^2 \mathcal{L}(\bar{y}, \bar{u}, \varphi, e)[(y, v), (y, v)] = \int_Q (L_{yy}[t, x]y^2 + L_{yu}[t, x]yv + L_{uy}[t, x]yv + L_{uu}[t, x]v^2) dt dx \geq 0 \quad (59)$$

for all $(y, v) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$, where $p \geq 2$.

Proof. We divide the proof into two steps.

Step1. Proving (59) for the case $(y, v) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$ where $v \in L^\infty(Q)$.

Take any $(y, v) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$ with $v \in L^\infty(Q)$. By (c₂), $y = z_v$. For each $0 < \rho < b - a$, we define

$$v_\rho(t, x) = \begin{cases} 0 & \text{if } a < \bar{u}(t, x) < a + \rho \text{ or } b - \rho < \bar{u}(t, x) < b \\ v(t) & \text{otherwise} \end{cases}$$

and $y_\rho = z_{v_\rho}$. Then by the same argument as in the proof of [17, Theorem 4.3], we can show that $(y_\rho, u_\rho) \in \mathcal{C}_p[(\bar{y}, \bar{u})]$ and $v_\rho \rightarrow v$ strongly in $L^p(Q)$. For $\lambda \in (0, \rho/\|v\|_\infty)$, we have $\bar{u} + \lambda v_\rho \in K_\infty$. Since \bar{u} is an optimal solution, for $\lambda > 0$ small enough, we get

$$0 \leq \frac{f(\bar{u} + \lambda v_\rho) - f(\bar{u})}{\lambda} = \lambda f'(\bar{u})v_\rho + \frac{\lambda^2}{2} f''(\bar{u} + \theta \lambda v_\rho)[v_\rho, v_\rho],$$

where $0 \leq \theta \leq 1$. By Proposition 4.3,

$$f'(\bar{u})v_\rho = \int_Q e(t, x)v_\rho(t, x)dtdx \leq 0.$$

It follows that

$$0 \leq f''(\bar{u} + \theta \lambda v_\rho)[v_\rho, v_\rho].$$

By letting $\lambda \rightarrow 0+$, we obtain $0 \leq f''(\bar{u})[v_\rho, v_\rho]$. This and Proposition 4.3 imply

$$0 \leq \int_Q [L_{yy}[t, x]y_\rho^2 + 2L_{yu}[t, x]y_\rho v_\rho + L_{uu}[t, x]v_\rho^2]dtdx \quad (60)$$

Now we have

$$\begin{aligned} y &= k_{1-\alpha} * (\Delta y + v) \\ y_\rho &= k_{1-\alpha} * (\Delta y_\rho + v_\rho) \end{aligned}$$

Hence

$$y - y_\rho = k_{1-\alpha} * (\Delta(y - y_\rho) + (v - v_\rho)).$$

By Proposition 3.2, there exist positive constants C and C' such that

$$\|y_\rho - y\|_{L^p(J, D)} + \left\| \frac{d^\alpha}{dt^\alpha} (y_\rho - y) \right\|_{L^p(J, H)} \leq C \|v - v_\rho\|_{L^p(J, H)} \leq C' \|v - v_\rho\|_{L^p(Q)}. \quad (61)$$

This implies that $\|y_\rho - y\|_{L^p(J, D)} \rightarrow 0$ as $\rho \rightarrow 0+$. Since $L^p(J, D) \hookrightarrow L^p(J, L^\infty(\Omega))$, we also have $\|y_\rho - y\|_{L^p(J, L^\infty(\Omega))} \rightarrow 0$. From assumption (A4), we see that $L_{yy}[\cdot, \cdot], L_{yu}[\cdot, \cdot], L_{uy}[\cdot, \cdot] \in L^q(J, L^\infty)$. We now claim that

$$\int_Q [L_{yy}[t, x]y_\rho^2] \rightarrow \int_Q L_{yy}[t, x]y^2 dtdx.$$

In fact, using (34) with $y_1 = 0$ and $y_2 = \bar{y}$, we have

$$\begin{aligned} & \left| \int_Q [L_{yy}[t, x]y_\rho^2] - \int_Q L_{yy}[t, x]y^2 dtdx \right| = \left| \int_Q L_{yy}[t, x](y_\rho - y)(y_\rho + y) dtdx \right| \\ & \leq C'' \int_0^T \|L_{yy}[\cdot, \cdot]\|_\infty \|y_\rho - y\|_\infty \|y_\rho + y\|_\infty dt \\ & \leq C'' \left[\int_0^T \|L_{yy}[\cdot, \cdot]\|_\infty^{\frac{p}{p-2}} dt \right]^{1-2/p} \left[\int_0^T \|y_\rho - y\|_\infty^p dt \right]^{1/p} \left[\int_0^T \|y_\rho + y\|_\infty^p dt \right]^{1/p} \\ & \leq C'' \left[\int_0^T (\|\bar{y}\|_\infty^{p-2} + \|L_{yy}[\cdot, \cdot]\|_{L^\infty(Q)}^{\frac{p}{p-2}}) dt \right]^{1-2/p} \left[\int_0^T \|y_\rho - y\|_\infty^p dt \right]^{1/p} \left[\int_0^T \|y_\rho + y\|_\infty^p dt \right]^{1/p} \\ & \leq M \|y_\rho - y\|_{L^p(J, L^\infty(\Omega))} \|y_\rho + y\|_{L^p(J, L^\infty(\Omega))} \rightarrow 0. \end{aligned}$$

The claim is justified. Similarly, we also have

$$\int_Q [2L_{yu}[t, x]y_\rho v_\rho + L_{uu}[t, x]v_\rho^2] dt dx \rightarrow \int_Q [2L_{yu}[t, x]yv + L_{uu}[t, x]v^2] dt dx.$$

By letting $\rho \rightarrow 0+$, we obtain from (60) that

$$0 \leq \int_Q [L_{yy}[t, x]y^2 + 2L_{yu}[t, x]yv + L_{uu}[t, x]v^2] dt dx.$$

Hence (59) is valid.

Step 2. Proving (59) for any $(y, v) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$.

Next we take $(y, v) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$. Then from (c_2) of $\mathcal{C}_p[(\bar{x}, \bar{u})]$, we have $y = z_v$. For each $n \geq 1$, we define

$$v_k(t, x) = P_{[-n, n]}(v(t, x)) = \min\{\max\{-n, v(t, x)\}, +n\},$$

where $P_{[-n, n]}(\xi)$ denotes the metric projection of ξ on $[-n, n]$ in \mathbb{R} . It is clear that $v_n \in L^\infty(Q)$ and $v_n(t, x) \rightarrow v(t, x)$ a.e. $t \in [0, 1]$. By the nonexpansive property of metric projection, we have

$$|v_n(t, x)| = |P_{[-n, n]}(v(t, x)) - P_{[-n, n]}(0)| \leq |v(t, x)|.$$

The Dominated Convergence Theorem implies that $v_n \rightarrow v$ in $L^p(Q)$ and in $L^2(Q)$. Let us put $y_n = z_{v_n}$. We now claim that $(y_n, v_n) \in \mathcal{C}_p[(\bar{x}, \bar{u})]$. Indeed, condition (c_2) is obvious. To verify condition (c_3) , we note that if $\bar{u}(t, x) = a$, then $v(t, x) \geq 0$. Hence $v_n(t, x) = \min\{v(t, x), n\} \geq 0$. If $\bar{u}(t, x) = b$, then $v(t, x) \leq 0$. Hence $v_n(t, x) = \max\{-n, v(t, x)\} \leq 0$. To check (c_1) , we have

$$\begin{aligned} & \int_Q (L_y[t, x]y_n(t, x) + L_u[t, x]v_n(t, x)) dt dx = \int_Q e(t, x)v_n(t, x) dt dx \\ &= \int_{Q_a} e(t, x)v_n(t, x) dt + \int_{Q_b} e(t, x)v_n(t, x) dt dx = \int_{Q_a} e(t, x) \min\{v(t, x), n\} dt dx + \int_{Q_b} e(t, x) \max\{-n, v(t, x)\} dt dx \\ &\leq \int_{Q_a} e(t, x)v(t, x) dt dx + \int_{Q_b} e(t, x)v(t, x) dt dx \leq \int_Q e(t, x)v(t, x) dt dx \leq 0. \end{aligned}$$

Therefore, the claim is justified. Now we have

$$\begin{aligned} y &= k_{1-\alpha} * (\Delta y + v) \\ y_n &= k_{1-\alpha} * (\Delta y_n + v_n). \end{aligned}$$

Hence $y - y_n = k_{1-\alpha} * (\Delta(y - y_n) + (v - v_n))$. By Proposition 3.2, there exist positive constants C and C' such that

$$\|y_n - y\|_{L^p(J, D)} + \left\| \frac{d^\alpha}{dt^\alpha} (y_n - y) \right\|_{L^p(J, H)} \leq C \|v - v_n\|_{L^p(J, H)} \leq C' \|v - v_n\|_{L^p(Q)}. \quad (62)$$

This implies that $\|y_n - y\|_{L^p(J, D)} \rightarrow 0$ as $n \rightarrow \infty$. Since $L^p(J, D) \hookrightarrow L^p(J, L^\infty(\Omega))$, we also have $\|y_n - y\|_{L^p(J, L^\infty(\Omega))} \rightarrow 0$. By Step 1, we have

$$\int_Q (L_{yy}[t, x]y_n^2 + L_{yu}[t, x]y_n v_n + L_{uy}[t, x]y_n v_n + L_{uu}[t, x]v_n^2) dt dx \geq 0.$$

By letting $n \rightarrow \infty$ and using similar arguments as in Step 1 with noting that $y_n \rightarrow y$, $u_n \rightarrow u$, we obtain

$$\int_Q (L_{yy}[t, x]y^2 + L_{yu}[t, x]yv + L_{uy}[t, x]yv + L_{uu}[t, x]v^2) dt dx \geq 0.$$

The proof of the theorem is completed. \square

For the remaining part of this section, we give second-order sufficient optimal conditions for locally optimal solution to (1)-(5). To do this, we need some tools of variational analysis.

Let K be a nonempty and closed subset in a Banach space X and $\bar{w} \in K$. The sets

$$\begin{aligned} T^b(K; \bar{w}) &:= \{h \in X \mid \forall t_n \rightarrow 0^+, \exists h_n \rightarrow h, \bar{w} + t_n h_n \in K \quad \forall n \in \mathbb{N}\}, \\ T(K, \bar{w}) &:= \{h \in X \mid \exists t_n \rightarrow 0^+, \exists h_n \rightarrow h, \bar{w} + t_n h_n \in K \quad \forall n \in \mathbb{N}\}, \end{aligned}$$

are called *the adjacent tangent cone* and *the contingent cone* to K at \bar{w} , respectively. It is well-known that when K is convex, then

$$T^b(K; \bar{w}) = T(K; \bar{w}) = \overline{\text{cone}(K - \bar{w})},$$

where

$$\text{cone}(K - \bar{w}) := \{\lambda(h - \bar{w}) \mid h \in K, \lambda > 0\}.$$

Let K_p be defined by (37). Then from [16, Lemma 2.4], we have

$$T^b(K_p, \bar{u}) = \{v \in L^p : v(t, x) \text{ satisfies } (c_3)\}.$$

Hence condition (c_3) in $\mathcal{C}_p[(\bar{y}, \bar{u})]$ is equivalent to saying that $v \in T^b(K_p, \bar{u})$.

Below, we set $Y = L^p(J, D)$ and $U = L^\infty(Q)$. Given $\epsilon > 0$, the symbols $B(\bar{y}, \epsilon)_Y$ is a ball in Y with center at $\bar{y} \in Y$ and radius $\epsilon > 0$.

We now have

Theorem 4.3 *Suppose (A4), $p \geq 2$, $1 > \alpha > 1/2$ and (φ, e) satisfies the conclusion of Theorem 4.1. Furthermore, suppose that the following conditions are satisfied:*

(i) $L_{yy}[\cdot, \cdot], L_{yu}[\cdot, \cdot] \in L^\infty(Q)$ and there exists a number $\Lambda > 0$ such that

$$L_{uu}(t, x, y, u) \geq \Lambda \quad \forall (t, x, y, u) \in Q \times \mathbb{R} \times \mathbb{R}. \quad (63)$$

(ii) *(the strictly second-order condition)*

$$\nabla_{(y,u)}^2 \mathcal{L}(\bar{y}, \bar{u}, \varphi, e)[(z, v), (z, v)] > 0 \quad \forall (z, v) \in \mathcal{C}_2[(\bar{y}, \bar{u})] \setminus \{(0, 0)\}. \quad (64)$$

Then there exist positive constants ϱ and ϵ such that

$$\psi(y, u) \geq \psi(\bar{y}, \bar{u}) + \varrho \|u - \bar{u}\|_2^2 \quad \forall (y, u) \in B_Y(\bar{y}, \epsilon) \times B_U(\bar{u}, \epsilon).$$

Proof. On the contrary, we suppose the conclusion were false. Then, we could find sequences $\{(y_n, u_n)\} \subset \Phi$ and $\{\varrho_n\} \subset \text{int } \mathbb{R}_+$ such that $y_n \rightarrow \bar{y}$ in Y , $u_n \rightarrow \bar{u}$ in U , $\varrho_n \rightarrow 0$ and

$$\psi(y_n, u_n) < \psi(\bar{y}, \bar{u}) + \varrho_n \|u_n - \bar{u}\|_2^2. \quad (65)$$

If $u_n = \bar{u}$, then we can show that $y_n = \bar{y}$. It follows that $\psi(\bar{y}, \bar{u}) < \psi(\bar{y}, \bar{u}) + 0$ which is absurd. Therefore, we can assume that $u_n \neq \bar{u}$ for all $n \geq 1$.

Define $t_n = \|u_n - \bar{u}\|_2$, $z_n = \frac{y_n - \bar{y}}{t_n}$ and $v_n = \frac{u_n - \bar{u}}{t_n}$. Then $t_n \rightarrow 0^+$ and $\|v_n\|_2 = 1$. By the reflexivity of $L^2(Q)$, we may assume that $v_n \rightharpoonup v$. From the above, we have

$$\psi(y_n, u_n) - \psi(\bar{y}, \bar{u}) \leq t_n^2 \varrho_n \leq o(t_n^2). \quad (66)$$

We claim that z_n converges to some z in $L^2(J, D)$. In fact, since $(y_n, u_n) \in \Phi$, we have

$$\begin{aligned} y_n(t) &= y_0 + k_{1-\alpha} * (\Delta y_n + u_n), \\ \bar{y} &= y_0 + k_{1-\alpha} * (\Delta \bar{y} + \bar{u}). \end{aligned}$$

This implies that

$$z_n = k_{1-\alpha} * (\Delta z_n + v_n), \quad z_n(0) = 0.$$

By Proposition 3.2, there exists constant $C > 0$ such that

$$\|z_n\|_{L^2(J,D)} + \left\| \frac{d^\alpha}{dt^\alpha} z_n \right\|_{L^2(J,H)} \leq C \|v_n\|_{L^2(J,H)} = C \|v_n\|_{L^2(Q)}.$$

Hence $\{z_n\}$ is bounded in $L^2(J,D)$. Without loss of generality, we may assume that $z_n \rightharpoonup z$ in $L^2(J,D)$.

Let us claim that if $2 > \frac{1}{\alpha}$, then $z_n \rightarrow z$ strongly in $L^2(J,V)$. In fact, it is known that the embeddings

$$D \hookrightarrow V \hookrightarrow H$$

are compact. For $\alpha > 1/2$, we have

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_k(s)\|_D ds &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)2} ds \right)^{1/2} \|z_k\|_{L^2(J,D)} \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{1}{(\alpha-1/2)^{1/2}} t^{\alpha-1/2} \|z_k\|_{L^p(J,D)} \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{1}{(\alpha-1/2)^{1/2}} T^{\alpha-1/2} M := M'. \end{aligned}$$

Hence

$$\sup_{t \in (0,T)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_n(s)\|_D ds \leq M'.$$

Besides, we have $2 > \frac{2}{1+2\alpha}$, and $\{D_{0+}^\alpha z_n\} = \{\frac{d^\alpha}{dt^\alpha} z_n\}$ is bounded in $L^2(J,H)$. By Theorem 4.1 in [19], we see that $\{z_n\}$ is a relatively compact set in $L^2(J,V)$. Therefore, we may assume that $z_n \rightarrow z$ strongly in $L^2(J,V)$. The claim is justified.

By repeating the procedure in the the proof of Theorem 3.1 we can show that

$$z = k_{1-\alpha} * (\Delta z + v). \quad (67)$$

The remaining part of the proof will be divided into two steps.

Step 1. Showing that $(y, v) \in \mathcal{C}_2[(\bar{y}, \bar{u})]$.

By a Taylor expansion, we have from (66) that

$$\psi_y(\bar{y}, \bar{u}) z_n + \psi_u(\bar{y}, \bar{u}) v_n + \frac{o(t_n)}{t_n} \leq \frac{o(t_n^2)}{t_n}. \quad (68)$$

Note that $L_u[\cdot, \cdot] \in L^\infty(Q)$ and $\psi_u(\bar{y}, \bar{u}): L^2(Q) \rightarrow \mathbb{R}$ is a continuous linear mapping, where

$$\langle \psi_u(\bar{y}, \bar{u}), u \rangle := \int_Q L_u[t, x] u(t, x) dt dx \quad \forall u \in L^2(Q).$$

By [6, Theorem 3.10], $\psi_u(\bar{y}, \bar{u})$ is weakly continuous on $L^2(Q)$. On the other hand, $L_y[\cdot, \cdot] \in L^q(J, H) \subset (L^p(J, D))^*$. Hence $\lim_{k \rightarrow \infty} \int_Q L_y[t, x] z_k(t, x) dt dx = \int_Q L_y[t, x] z(t, x) dt dx$. Letting $k \rightarrow \infty$ in (68), we get

$$\psi_y(\bar{y}, \bar{u}) z + \psi_u(\bar{y}, \bar{u}) v = \int_Q (L_y[t, x] z(t, x) + L_u[t, x] v(t, x)) dt dx \leq 0. \quad (69)$$

Hence condition (c_1) in $\mathcal{C}_2[(\bar{y}, \bar{u})]$ is valid. By (67), condition (c_2) is fulfilled. For (c_3) , we note that $u_n - \bar{u} \in K_\infty - \bar{u}$. This implies that

$$v_n \in \frac{1}{t_n} (K_\infty - \bar{u}) \subset T^b(K_\infty; \bar{u}),$$

where $T^b(K_\infty; \bar{u})$ is the tangent cone to K_∞ at \bar{u} in $L^2([0, 1], \mathbb{R})$. Note that $K_\infty = K_2$. Since $T^b(K_\infty; \bar{u})$ is convex and closed in L^2 , it is weakly closed. Passing to the limit, we obtain $v \in T^b(K_\infty; \bar{u})$. Hence condition (c_3) of $\mathcal{C}_2[(\bar{y}, \bar{u})]$ is fulfilled.

Step 2. Showing that $(z, v) = 0$.

Note that $\langle e, u_n - \bar{u} \rangle \geq 0$ and

$$\begin{aligned} y_n &= y_0 + k_{1-\alpha} * (\Delta y_n + u_n) \\ \bar{y} &= y_0 + k_{1-\alpha} * (\Delta \bar{y} + \bar{u}). \end{aligned}$$

This implies that

$$(y_n - \bar{y}) = k_{1-\alpha} * (\Delta(y_n - \bar{y}) + u_n - \bar{u}) = 0.$$

Hence

$$\frac{d^\alpha}{dt^\alpha}(y_n - \bar{y}) - \Delta(y_n - \bar{y}) - (u_n - \bar{u}) = 0.$$

This and Lemma 2.1 yield

$$\begin{aligned} & \int_Q [(y_n - \bar{y}) D_{*T}^\alpha \varphi - \varphi(\Delta(y_n - \bar{y}) - \varphi(u_n - \bar{u}))] dt dx \\ &= \int_Q \varphi(t) [\frac{d^\alpha}{dt^\alpha}(y_n - \bar{y}) - \Delta(y_n - \bar{y}) - (u_n - \bar{u})] dt dx = 0. \end{aligned}$$

Combining these with (66) and definition of Lagrange function (57), we have

$$\begin{aligned} \mathcal{L}(y_n, u_n, \varphi, e) - \mathcal{L}(\bar{y}, \bar{u}, \varphi, e) &= \psi(y_n, u_n) - \psi(\bar{y}, \bar{u}) - \langle e, u_n - \bar{u} \rangle \\ &\leq \psi(y_n, u_n) - \psi(\bar{y}, \bar{u}) \leq o(t_n^2). \end{aligned}$$

From this, (58) and a second-order Taylor expansion, we get

$$\frac{t_n^2}{2} \nabla_{(y,u)}^2 \mathcal{L}(\bar{y}, \bar{u}, \varphi, e)[(z_n, v_n), (z_n, v_n)] + o(t_n^2) \leq o(t_n^2),$$

or, equivalently,

$$\nabla_{(y,u)}^2 \mathcal{L}(\bar{y}, \bar{u}, \varphi, e)[(z_n, v_n), (z_n, v_n)] \leq \frac{o(t_n^2)}{t_n^2}. \quad (70)$$

This means

$$\int_Q (L_{yy}[t, x] z_n^2 + 2L_{yu}[t, x] z_n v_n + L_{uu}[t, x] v_n^2) dt dx \leq \frac{o(t_n^2)}{t_n^2}. \quad (71)$$

Let us claim that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_Q (L_{yy}[t, x] z_n^2 + 2L_{yu}[t, x] z_n v_n + L_{uu}[t, x] v_n^2) dt dx \\ & \geq \int_Q (L_{yy}[t, x] z^2 + 2L_{yu}[t, x] z v + L_{uu}[t, x] v^2) dt dx. \end{aligned}$$

Indeed, since $L_{uu}[t, x] \geq \Lambda > 0$, the functional $u \mapsto \int_Q L_{uu}[t, x] u^2 dt dx$ is convex and so it is sequentially lower semicontinuous. Hence

$$\lim_{n \rightarrow \infty} \int_Q L_{uu}[t, x] v_n^2 dt dx \geq \int_Q L_{uu}[t, x] v^2 dt dx.$$

Since $z_n \rightarrow z$ in $L^2(J, V)$, $\|z_n + z\|_{L^2(J, V)} \leq M$ for some constant $M > 0$. Using assumption $L_{yy}[\cdot, \cdot] \in L^\infty(Q)$ in (i), we have

$$\begin{aligned} \left| \int_Q L_{yy}[t, x] (z_n^2 - z^2) dt dx \right| &\leq \int_0^T \int_\Omega |L_{yy}[t, x]| |z_n(t, x) - z(t, x)| |z_n(t, x) + z(t, x)| dx dt \\ &\leq \int_0^T \|L_{yy}[t, \cdot]\|_{L^\infty(\Omega)} \|z_n(t, \cdot) - z(t, \cdot)\|_{L^2(\Omega)} \|z_n(t, \cdot) + z(t, \cdot)\|_{L^2(\Omega)} \\ &\leq \|L_{yy}[\cdot, \cdot]\|_{L^\infty(Q)} \| \|z_n(\cdot, \cdot) - z(\cdot, \cdot)\|_{L^2(Q)} \|z_n(\cdot, \cdot) + z(\cdot, \cdot)\|_{L^2(Q)} \\ &\leq \|L_{yy}[\cdot, \cdot]\|_{L^\infty(Q)} \| \|z_n(\cdot, \cdot) - z(\cdot, \cdot)\|_{L^2(Q)} M. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \int_Q L_{yy}[t, x] z_n^2(t, x) dt dx = \int_Q L_{yy}[t, x] z^2(t, x) dt dx.$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} \int_Q L_{yu}[t, x] z_n(t, x) v_n(t, x) dt dx = \int_Q L_{yu}[t, x] z(t, x) v(t, x) dt dx.$$

Hence the claim is justified. By letting $n \rightarrow \infty$, we obtain from (71) that

$$\nabla_{(y,u)}^2 \mathcal{L}(\bar{y}, \bar{u}, \varphi, e)[(z, v), (z, v)] \leq 0.$$

Combining this with (64), we conclude that $(z, v) = (0, 0)$.

Finally, from (63), (71) and $\|v_n\|_{L^2(Q)} = 1$, we have

$$\begin{aligned} & \int_Q (L_{yy}[t, x] z_n^2 + 2L_{yu}[t, x] z_n v_n) dt dx + \Lambda \\ & \leq \int_Q (L_{yy}[t, x] z_n^2 + 2L_{yu}[t, x] z_n v_n + L_{uu}[t, x] v_n^2) dt dx \leq \frac{o(t_n^2)}{t_n^2}. \end{aligned}$$

By letting $n \rightarrow \infty$ and using $(z, v) = (0, 0)$, we obtain $\Lambda \leq 0$ which is absurd. Therefore, the theorem is proved. \square

Remark 4.1 When $0 < \alpha \leq 1/2$, the embedded theorem is not applicable and the proof of Theorem 4.2 is collapsed. Therefore, establishing second-order sufficient conditions for this case is an open problem.

When $1/2 < \alpha < 1$ and $p = 2$, Theorem 4.1 and Theorem 4.2 give no-gap second-order conditions on the common critical cone $\mathcal{C}_2[(\bar{y}, \bar{u})]$.

Acknowledgments

The research of the first author was partially supported by Vietnam Academy of Science and Technology under grant number QTRU 01-01/21-22. The second author was supported by Taiwan MOST grant 111-2811-M-110-017.

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