

ON PROJECTIVE VARIETIES OF GENERAL TYPE WITH MANY GLOBAL K-FORMS

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ABSTRACT. We prove the following results: (1) for any nonsingular projective 3-folds X of general type with $\chi(\mathcal{O}_X) \neq 2, 3$, the canonical volume $\text{Vol}(X)$ has the optimal lower bound $\frac{1}{420}$; (2) for nonsingular projective 3-folds (resp. 4-folds) X of general type with $h^{2,0}(X) \geq 108 \cdot 42^3 + 4$ (resp. with sufficiently large $h^{2,0}(X)$), the 3-canonical map (resp. 5-canonical map) is stably birational; (3) for any nonsingular projective n -fold X of general type with $q(X) > n \geq 4$, the canonical stability index $r_s(X)$ is upper bounded by the $(n-1)$ -th canonical stability index r_{n-1} .

1. Introduction

We work over an algebraically closed field of characteristic zero. Given a nonsingular projective variety V of general type, it has been of great importance to calculate the canonical stability index $r_s(V)$. For any $n \in \mathbb{Z}_{>0}$, both the n -th canonical stability index

$$r_n := \sup_W \{r_s(W) \mid W \text{ is a smooth projective } n\text{-fold of general type}\}$$

and the n -th minimal volume

$$v_n := \inf_W \{\text{Vol}(W) \mid W \text{ is a smooth projective } n\text{-fold of general type}\}$$

are key global quantities of birational geometry. It is known that $r_1 = 3$ and, by Bombieri [3], $r_2 = 5$. Iano-Fletcher's example in [27] shows $r_3 \geq 27$, $v_3 \leq \frac{1}{420}$ and, by Chen-Chen [9, 10, 11] and [16], $r_3 \leq 57$ and $v_3 \geq \frac{1}{1680}$. For any $n \geq 4$, the theorem of Hacon-McKernan [26], Takayama [45] and Tsuji [47] tells $r_n < +\infty$. However, no effective upper bound is known beside the fact that $r_n > 2^{2^{\frac{n-2}{2}}}$ due to those interesting examples found by Esser-Totaro-Wang [21].

According to [11], the untameable 3-folds of general type are those admitting global 2-forms. So, in the first part of this paper, we prove the Noether type of inequality between the canonical volume and the Hodge number $h^{0,k} = h^{k,0}$:

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Theorem 1.1. (=Theorem 2.1) *Fix two integers n and k with $n > 0$ and $0 \leq k \leq n$. There exist positive numbers $a_{n,k}$ and $b_{n,k}$ such that the inequality*

$$\text{vol}(X) \geq a_{n,k} h^0(X, \Omega_X^k) - b_{n,k}$$

holds for every smooth projective n -fold X of general type.

The constants $a_{n,k}$ and $b_{n,k}$ are related to minimal volumes of varieties of general type of dimensions $\leq n - 1$. When n is small, these numbers are explicit. These inequalities also suggest that the pluricanonical systems on varieties with many global k -forms should behave well. Indeed, as an interesting application, we show the following result which improves [9, Theorem A(iii)]:

Theorem 1.2. (=Corollary 2.7) *Let X be a nonsingular projective 3-fold of general type. Then*

$$\text{Vol}(X) \geq \frac{1}{420}$$

provided that $\chi(\mathcal{O}_X) \neq 2, 3$.

The second part of this paper devotes to studying explicit birational geometry. Let us recall the following known results:

- When $\dim(V) = 3$ and $p_g(V) \geq 4$, $r_s(V) \leq 5 (= r_2^+)$ by [13, Theorem 1.2]; when $\dim(V) = 3$ and $\text{vol}(V) > 12^3$, $r_s(V) \leq 5 (= r_5)$ by Todorov [46, Theorem 1.2] and [15, Theorem 1.1];
- There are constants $K(4)$, $L(4)$ and $L(5)$. When $\dim(V) = 4$ and $\text{vol}(V) \geq K(4)$ (resp. $p_g(V) \geq L(4)$), $r_s(V) \leq r_3$ (resp. $\leq r_3^+$) by [18, Theorem 1.4, Theorem 1.5]; When $\dim(V) = 5$ and $p_g(V) \geq L(5)$, $r_s(V) \leq r_4^+$ by [18, Theorem 1.5]; (see [18, P. 2044] for the definition of r_n^+)
- When $\dim(V) = n \geq 4$, $r_s(V) \leq \max\{r_{n-1}, (n-1)r_{n-2} + 2\}$ by Lacini [37, Theorem 1.3].

We shall consider 3-folds and 4-folds with many global two forms and varieties in any dimension with many global 1-forms. It turns out that the pluricanonical system of these varieties behaves very well.

Theorem 1.3. *Let X be any nonsingular projective 3-fold of general type with $h^{2,0}(X) \geq 108 \cdot 18^3 + 4$. Then the m -canonical map $\varphi_{m,X}$ is birational for all $m \geq 3$.*

Note that $\chi(\mathcal{O}_X) = 1 + h^{2,0}(X) - q(X) - p_g(X)$. Hence we can also provide an alternative form of Theorem 1.3 as follows.

Theorem 1.4. *Let X be any nonsingular projective 3-fold of general type with $\chi(\mathcal{O}_X) \geq 108 \cdot 18^3 + 5$. Then $\varphi_{m,X}$ is birational for all $m \geq 3$.*

A typical 3-fold of general type with arbitrarily large $\chi(\mathcal{O})$ can be constructed as follows.

Example 1.5. Let C_i be a hyperelliptic curve of genus $g_i > 1$ for $i = 1, 2, 3$. We denote by τ_i the hyperelliptic involution on C_i and let $f_i : C_i \rightarrow \mathbb{P}^1$ be the hyperelliptic quotient for $i = 1, 2, 3$. We then have $f_{i*}\mathcal{O}_{C_i} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(g_i + 1))$.

Let $X := (C_1 \times C_2 \times C_3)/\langle \tau_1 \times \tau_2 \times \tau_3 \rangle$ be the diagonal quotient and let $f : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the natural morphism. Then X has finitely many singular points, which are isolated terminal quotient singularities. Let $\tau : V \rightarrow X$ be a desingularization. Since X has rational singularities, $\mathbf{R}\tau_*\mathcal{O}_V = \mathcal{O}_X$. Considering the composition of morphisms

$$g : V \xrightarrow{\tau} X \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

we have

$$\begin{aligned} \mathbf{R}g_*\mathcal{O}_V &= \mathbf{R}f_*\mathbf{R}\tau_*\mathcal{O}_V = f_*\mathcal{O}_X \\ &= ((f_{1*}\mathcal{O}_{C_1}) \boxtimes (f_{2*}\mathcal{O}_{C_2}) \boxtimes (f_{3*}\mathcal{O}_{C_3}))^{\langle \tau_1 \times \tau_2 \times \tau_3 \rangle} \\ &= (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}) \\ &\quad \bigoplus (\mathcal{O}_{\mathbb{P}^1}(-(g_1 + 1)) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-(g_2 + 1)) \boxtimes \mathcal{O}_{\mathbb{P}^1}) \\ &\quad \bigoplus (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-(g_2 + 1)) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-(g_3 + 1))) \\ &\quad \bigoplus (\mathcal{O}_{\mathbb{P}^1}(-(g_1 + 1)) \boxtimes \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-(g_3 + 1))). \end{aligned}$$

We then see that $h^1(V, \mathcal{O}_V) = h^3(V, \mathcal{O}_V) = 0$ and $h^2(V, \mathcal{O}_V) = g_1g_2 + g_1g_3 + g_2g_3$. Moreover, $\text{vol}(V) = \text{vol}(K_X) = \frac{1}{8}\text{vol}(C_1 \times C_2 \times C_3) = (g_1 - 1)(g_2 - 1)(g_3 - 1)$.

If g_1 is large, so is $\text{vol}(V)$. Let $g_2 = 2$, then V has a genus 2 fibration, $|2K_V|$ cannot be birational. Therefore, $r_s(V) = 3$ which means the statement in Theorem 1.3 is sharp.

We can extend Theorem 1.3 to dimension 4.

Theorem 1.6. *There exists a constant $M(4)$ such that, for any non-singular projective 4-fold X of general type with $h^0(X, \Omega_X^2) \geq M(4)$, $\varphi_{m,X}$ is birational for all $m \geq 5$.*

The next two examples show that the statement in Theorem 1.6 is sharp as well.

Example 1.7. Let Y be a minimal irregular threefold of general type with $q(Y) = 1$ such that the general fiber of the Albanese morphism of Y is an $(1, 2)$ -surface (namely, the minimal model of this fiber has invariants $(K^2, p_g) = (1, 2)$). Let $V = Y \times C$, where C is a smooth projective curve of any genus $g \geq 2$. Then, as g is sufficiently large, V is a fourfold with sufficiently large $h^{2,0}(V)$, but $|4K_V|$ cannot induce a birational map since $r_s(S) = 5$. Thus, when g is large enough, $r_s(V) = 5$.

Example 1.8. We still denote by S a minimal $(1, 2)$ -surface. Let S' be a smooth minimal surface of general type with $p_g(S') = g$. Let $V = S \times S'$. Then, when g is large enough, so is $h^{2,0}(V)$, but $|4K_V|$ cannot induce a birational map. Hence $r_s(V) = 5$.

It is interesting to mention that, though constructions in Examples 1.7 and 1.8 are simple, those varieties with similar structures are exactly the most difficult cases while proving Theorem 1.6.

Observing that two digits “3” and “5” in the statements of Theorem 1.3 and Theorem 1.6 can be understood as r_{n-2} where $n = \dim(X) = 3, 4$, one might put the higher dimensional analog as an interesting conjecture (see Section 7)! However the next example shows that, unlike in dimension 3, one cannot replace the condition “ $h^{2,0}(X) \gg 0$ ” with the condition “ $\chi(\mathcal{O}_X) = \chi(\omega_X) \gg 0$ ” even in dimension 4. Thus the higher dimensional analog of Theorem 1.4 seems to be impossible. At least it fails in dimension 4.

Example 1.9. Let C be a smooth projective curve and let L be a very ample line bundle on C with $d = \deg(L) \gg 1$. Let $\mu : Y \rightarrow \mathbb{P} := \mathbb{P}(1, 3, 4, 5, 14)$ be a resolution of the weighted projective space of dimension 4 such that $|\mu^*\mathcal{O}_{\mathbb{P}}(28)| = |M| + E$, where the effective \mathbb{Q} -divisor E is the fixed part and the mobile part $|M|$ is base point free. Let $V \subset C \times Y$ be a general hypersurface of $|L \boxtimes M|$. Then V is a smooth 4-fold and $\omega_V = ((K_C \otimes L) \boxtimes (K_Y \otimes M))|_V$. Let $f : V \rightarrow C$ be the natural fibration. One sees that $R^i f_* \omega_V = 0$ for $i = 1, 2$, since a general fiber of f is birational to a hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 5, 14)$. Moreover, an easy computation shows that $f_* \omega_V = K_C \otimes L$ and $R^3 f_* \omega_V = K_C$. Hence, $\chi(\omega_V) = d \gg 0$, $h^{2,0}(V) = 0$ and that $|13K_V|$ cannot induce a birational map of V . In a word, $r_s(V) > 13 > r_2 = 5$.

Finally we study irregular varieties of general type and prove the following theorem:

Theorem 1.10. (=Theorem 7.2) *Let X be a smooth projective variety of general type of dimension $n \geq 4$. Assume that $q(X) > n$. Then $|mK_X|$ induces a birational map for all $m \geq r_{n-1}$.*

1.1. Notions and notations.

A variety X is an integral separated scheme of finite type over an algebraic closed field of characteristic 0. We will always work on normal projective varieties. Let D_1 and D_2 be two \mathbb{Q} -Weil divisors on a normal variety X . We say that $D_1 \geq D_2$ if $D_1 - D_2$ is an effective \mathbb{Q} -Weil divisor. We say that $D_1 \geq_{\mathbb{Q}} D_2$ if there exists a positive integer M such that $M(D_1 - D_2)$ is a Cartier divisor and is linearly equivalent to an effective Cartier divisor. We write $D_1 \sim_{\mathbb{Q}} D_2$ if $M(D_1 - D_2)$ is a principal Cartier divisor for M sufficiently large and divisible. We

say a \mathbb{Q} -Cartier divisor D on a normal projective variety X is psuedo-effective if for any ample Cartier divisor H on X and any rational number $b > 0$, $D + bH$ is a big \mathbb{Q} -Cartier divisor.

Let X be a normal variety and D an effective \mathbb{Q} -Weil divisor on X . Let $\mu : X' \rightarrow X$ be a log resolution of (X, D) . We may write

$$K_{X'} = \mu^*(K_X + D) + \sum_E a(E; X, D)E,$$

where E runs over all the distinct prime divisors of X' and $a(E; X, D) \in \mathbb{Q}$. We call $a(E; X, D)$ the discrepancy of E with respect to (X, D) . We say that (X, D) is log canonical at $x \in X$ if $a(E; X, D) \geq -1$ for each E such that $x \in \mu(E)$. Let $E \subset X'$ be a prime divisor with discrepancy -1 . We say that $\mu(E)$ is a log canonical center or lc center of (X, D) if (X, D) is log canonical at a general point of $\mu(E)$. A log canonical center which is minimal with respect to the inclusion is called a minimal log canonical center. Assume (X, D) is log canonical at $x \in X$ and let C_1 and C_2 be two lc centers of (X, D) containing x . By [31, Proposition 1.5], each irreducible component of $C_1 \cap C_2$ containing x is also a lc center of (X, D) . In particular, the minimal lc center of (X, D) at x is well-defined. Let E_1, \dots, E_m be the divisors with discrepancy ≤ -1 of (X, D) . Then $\mu(E_1 \cup \dots \cup E_m)$ is called the non-klt locus of (X, D) , usually denoted by $\text{Nklt}(X, D)$. When X is smooth, we denote by $\mathcal{J}(D) = \mathcal{J}(X, D) = \mu_* \mathcal{O}_{X'}(\sum_E [-a(E; X, D)E])$ the multiplier ideal of D (see [39, Section 9]). Then it is clear that $\text{Nklt}(X, D)$ is the support of the subscheme of X defined by $\mathcal{J}(D)$. Let D be an effective \mathbb{Q} -divisor on a smooth variety X . We denote by $\text{lct}(X; D)$ the maximal positive rational number t such that (X, tD) is log canonical at each point of X . We call $\text{lct}(X; D)$ the log canonical threshold of D .

Let \mathcal{F} be a torsion-free coherent sheaf of rank r on a smooth variety X . Let $j : U \subset X$ be the locus where \mathcal{F} is locally free. Then, $\text{codim}_X(X \setminus U) \geq 2$. We write $\det \mathcal{F}$ to be the unique Cartier divisor on X , which extends $\wedge^r \mathcal{F}$ on U . We also denote by \mathcal{F}^{**} the reflexive hull $j_*(j^* \mathcal{F})$ of \mathcal{F} .

We say that a set \mathfrak{X} of varieties is birationally bounded if there is a projective morphism between schemes, say $\tau : \mathcal{X} \rightarrow T$, where T is of finite type, such that for every element $X \in \mathfrak{X}$, there is a closed point $t \in T$ and a birational equivalence $X \dashrightarrow Z_t$.

We usually denote by ϵ a sufficiently small positive rational number.

2. INEQUALITIES AMONG BIRATIONAL INVARIANTS

2.1. General inequalities.

According to Hacon-McKernan [26], Takayama [45] and Tsuji [47], given any positive rational number M , the set of smooth projective general type n -folds whose canonical volumes are upper bounded by M is birationally bounded. Denote by ν_n the minimal volume among

all smooth projective n -folds of general type. By the MMP and Birkar-Cascini-Hacon-McKernan [2], any variety of general type has a minimal model.

Theorem 2.1. *Fix two integers n and k with $n > 0$ and $0 \leq k \leq n$. There exist positive numbers $a_{n,k}$ and $b_{n,k}$ such that the inequality*

$$\mathrm{vol}(X) \geq a_{n,k} h^0(X, \Omega_X^k) - b_{n,k}$$

holds for every smooth projective n -fold X of general type.

When $k = n$, we have $h^n(X, \mathcal{O}_X) = h^0(X, K_X) = p_g(X)$ and hence Theorem 2.1 is a generalization of the Noether type inequality (see Chen-Jiang [18, Corollary 5.1]). When $n = 2$, we have Debarre's inequality (see [19]): $\mathrm{vol}(X) \geq 2p_g + 2(q(X) - 4)$.

Proof. By Lemma 2.2 below, there exists a subsheaf \mathcal{F} of Ω_X^k such that $h^0(X, \det \mathcal{F}) \geq \frac{h^0(X, \Omega_X^k)}{\binom{n}{k}}$. We may replace \mathcal{F} by its saturation in Ω_X^k and denote by \mathcal{Q} the corresponding quotient bundle. Set $H := \det \mathcal{F}$ and $L := \det \mathcal{Q}$. Then

$$\binom{n-1}{k-1} K_X \sim \det(\Omega_X^k) \sim H + L.$$

By Campana and Paun [7, Theorem 1.2], we know that L is pseudo-effective.

Modulo birational modifications, we may assume that $|H|$ is base point free. We consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X_{\min} \\ \downarrow \varphi_H & & \\ \mathbb{P}(H^0(X, H)), & & \end{array}$$

where X_{\min} is the minimal model of X and φ_H is the morphism induced by the linear system $|H|$. Denote by $\varphi_H : X \xrightarrow{f} \Gamma \xrightarrow{s} \mathbb{P}(H^0(X, H))$ the Stein factorization of φ_H and let $d = \dim \Gamma$. Let F be a general fiber of f .

Take $d-1$ general hyperplane sections H_1, \dots, H_{d-1} of $\mathbb{P}(H^0(X, H))$. Let $W = s^*(H_1) \cap \dots \cap s^*(H_{d-1})$ and $X_W = f^{-1}(W)$. Then the induced morphism $f_W := f|_{X_W} : X_W \rightarrow W$ is a fibration from a smooth projective variety X_W of dimension $n-d+1$ to a smooth projective curve. Let $a \geq 1$ be the degree of s^*H_1 on W . Note that $a \geq h^0(X, H) - d$.

Then, by Kawamata's restriction theorem (see [32]), for each $m \geq 2$,

$$|am(K_{X_W} + \frac{1}{a}H|_{X_W})|_F = |amK_F|.$$

Repeatedly applying Kawamata's restriction theorem, one gets, for $m \geq 2$,

$$\begin{aligned}
& |ma(K_X + (d-1 + \frac{1}{a})H)|_F \tag{2.1} \\
&= |ma(K_X + \varphi_H^* H_1 + \cdots + \varphi_H^* H_{d-1} + \frac{1}{a}H)|_F \\
&= |ma(K_{X_W} + \frac{1}{a}H|_{X_W})|_F \\
&= |maK_F|.
\end{aligned}$$

We take a rational number $0 < \epsilon \ll 1$ and consider the \mathbb{Q} -divisor $(d-1 + \frac{1}{a})L + \epsilon K_X$. Since L is pseudo-effective and K_X is big,

$$M((d-1 + \frac{1}{a})L + \epsilon K_X)$$

is effective for sufficiently large and divisible integer M . Therefore,

$$\begin{aligned}
& |M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)K_X| \\
&= |M(K_X + (d-1 + \frac{1}{a})H) + M((d-1 + \frac{1}{a})L + \epsilon K_X)| \\
&\supseteq |M(K_X + (d-1 + \frac{1}{a})H)| + D,
\end{aligned}$$

where $D \in |M((d-1 + \frac{1}{a})L + \epsilon K_X)|$ is an effective divisor. Restricting on F , by (2.1), we get

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)K_X|_F \supseteq |MK_F| + D|_F.$$

Modulo a further birational modification to π , we may assume that $\theta : F \rightarrow F_{min}$ is a morphism onto one of its minimal model. Note that the free part of

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)K_X|$$

is

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)\pi^*K_{X_{min}}|.$$

Thus

$$\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)\pi^*K_{X_{min}}|_F \geq_{\mathbb{Q}} \theta^*K_{F_{min}}.$$

We finally conclude that

$$\begin{aligned}
\text{vol}(X) &= (\pi^* K_{X_{\min}}^n) \geq \frac{1}{\binom{n-1}{k-1}} \cdot ((L+H) \cdot \pi^* K_{X_{\min}}^{n-1}) \\
&\geq \frac{1}{\binom{n-1}{k-1}^d} \cdot (H^d \cdot \pi^* K_{X_{\min}}^{n-d}) \geq \frac{a}{\binom{n-1}{k-1}^d} \cdot ((\pi^* K_{X_{\min}})|_F)^{n-d} \\
&\geq \frac{a}{\binom{n-1}{k-1}^d} \cdot \frac{\text{vol}(F)}{\left(\binom{n-1}{k-1}(d-1+\frac{1}{a})+1\right)^{n-d}} \\
&\geq \frac{h^0(X, H) - d}{\binom{n-1}{k-1}^d} \cdot \frac{\text{vol}(F)}{\left(\binom{n-1}{k-1}d+1\right)^{n-d}} \\
&\geq \frac{\frac{h^0(X, \Omega_X^k)}{\binom{n}{k}} - d}{\binom{n-1}{k-1}^d} \cdot \frac{\text{vol}(F)}{\left(\binom{n-1}{k-1}d+1\right)^{n-d}} \\
&\geq \frac{\nu_{n-d}}{\left(\binom{n-1}{k-1}d+1\right)^{n-d} \binom{n-1}{k-1}^d \binom{n}{k}} \cdot \left(h^{k,0}(X) - d \binom{n}{k}\right). \quad (2.2)
\end{aligned}$$

□

Lemma 2.2. *Let \mathcal{E} be a torsion-free sheaf of rank r over a projective variety X . Assume that $h^0(X, \mathcal{E}) > 0$. There exists a torsion-free subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $h^0(X, \det \mathcal{F}) \geq \frac{h^0(X, \mathcal{E})}{r}$.*

Proof. We run induction on r . When $r = 1$, it is a trivial statement. We now assume that $\text{rank}(\mathcal{E}) = r > 1$. We may assume that the evaluation

$$ev : H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$$

is generically surjective. Otherwise, the image \mathcal{E}' of the evaluation map is of rank $\leq r-1$ and we replace \mathcal{E} by \mathcal{E}' .

We then take $s_1, \dots, s_{r-1} \in H^0(X, \mathcal{E})$, which generate a subspace W of $H^0(X, \mathcal{E})$, and the evaluation map $ev|_W : W \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ is injective. We then take the wedge product:

$$\begin{aligned}
\phi_W &: H^0(X, \mathcal{E}) \rightarrow H^0(X, \det \mathcal{E}) \\
s &\rightarrow s_1 \wedge \cdots \wedge s_{r-1} \wedge s.
\end{aligned}$$

If $h^0(X, \det \mathcal{E}) \geq \frac{h^0(X, \mathcal{E})}{r}$, we are done. Otherwise,

$$\dim \ker \phi_W > \frac{r-1}{r} \cdot h^0(X, \mathcal{E}).$$

Let $W' := \ker \phi_W$. We now consider

$$ev|_{W'} : W' \otimes \mathcal{O}_X \rightarrow \mathcal{E}.$$

Since for each $s \in W'$, s is linearly dependent with s_1, \dots, s_{r-1} at a general point of X , the image of $ev|_{W'}$ is a subsheaf $\mathcal{E}'' \subset \mathcal{E}$ of rank

$r - 1$. Note that $H^0(X, \mathcal{E}'') \supset W'$. Thus, by induction, \mathcal{E}'' contain a subsheaf \mathcal{F} with $h^0(X, \det \mathcal{F}) \geq \frac{h^0(X, \mathcal{E}'')}{r-1} > \frac{h^0(X, \mathcal{E})}{r}$. \square

The linear bound (2.2) is probably far from being optimal, but to the authors' knowledge, this is the first explicit inequality between canonical volumes and intermediate Hodge numbers in high dimensions.

Corollary 2.3. *Let X be a 3-fold of general type,*

$$\text{vol}(X) \geq \begin{cases} \frac{h^{2,0}(X)-9}{24}, & h^{2,0}(X) \leq 13 \\ \frac{h^{2,0}(X)-3}{54}, & h^{2,0}(X) \geq 14, \end{cases}$$

and

$$\text{vol}(X) \geq \begin{cases} \frac{\chi(\mathcal{O}_X)-10}{24}, & \chi(\mathcal{O}_X) \leq 14 \\ \frac{\chi(\mathcal{O}_X)-4}{54}, & \chi(\mathcal{O}_X) \geq 15. \end{cases}$$

Proof. We note that $\nu_1 = 2$ and $\nu_2 = 1$. Hence, by (2.2), when $n = 3$ and $k = 2$, we have

$$\text{vol}(X) \geq \min\left\{\frac{h^{2,0}(X) - 9}{24}, \frac{h^{2,0}(X) - 6}{30}, \frac{h^{2,0}(X) - 3}{54}\right\}.$$

Since $\chi(\mathcal{O}_X) = 1 - q(X) + h^{2,0}(X) - p_g(X)$, we have $h^{2,0}(X) \geq \chi(\mathcal{O}_X) - 1$. \square

2.2. A stronger inequality between vol and q .

Here we deduce a stronger inequality between the canonical volume and the irregularity via the Albanese morphism using generic vanishing theory. One may compare it with various Severi inequalities (see, for instance, [29]).

We first recall some results from generic vanishing. For a coherent sheaf \mathcal{F} on an abelian variety A , we define the i -th cohomological support locus

$$V^i(\mathcal{F}) := \{P \in \text{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes P) \neq 0\}.$$

We say that \mathcal{F} is a GV sheaf if $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) \geq i$ for each $i \geq 1$. Following [42], we define the generic vanishing index

$$gv(\mathcal{F}) := \min_{1 \leq i \leq \dim A} \{\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) - i\}$$

for a GV sheaf \mathcal{F} . Note that, if $V^i(\mathcal{F}) = \emptyset$, we let $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) = \infty$. The main result of [42] states that, if \mathcal{F} is a GV sheaf on A with $gv(\mathcal{F}) < \infty$, $\chi(\mathcal{F}) \geq gv(\mathcal{F})$. Given a morphism $f : X \rightarrow A$ from a smooth projective variety X to an abelian variety, the higher direct images $R^i f_* \omega_X$ are GV for each $i \geq 0$ (see [25]). Moreover, by Green-Lazarsfeld [24] and Simpson [44], $V^j(R^i f_* \omega_X)$ is a union of torsion translates of abelian subvarieties of $\text{Pic}^0(A)$ for each $i, j \geq 0$.

The following lemma is a kind of geometric version of Lemma 2.2 for $h^{1,0}(X)$.

Lemma 2.4. *Let $f : Z \rightarrow A$ be a morphism from a smooth projective variety Z to an abelian variety A . Assume that f is generically finite onto its image and $f(Z) \not\subseteq_{\neq} A$ generates A . Then there exists a quotient between abelian varieties $q_B : A \rightarrow B$ with connected fibers such that, when taking the Stein factorization of $q_B \circ f : Z \rightarrow B$:*

$$\begin{array}{ccc} Z & \xrightarrow{f} & A \\ q_Z \downarrow & & \downarrow q_B \\ Z_B & \xrightarrow{f_B} & B, \end{array}$$

$f_B(Z_B) \not\subseteq_{\neq} B$ generates B , any smooth model Z'_B of Z_B is of general type, and

$$\chi(\omega_{Z'_B}) \geq \frac{\dim A - \dim Z}{\dim Z}.$$

Proof. Let $n = \dim Z$ and $g = \dim A$. We run induction on n . When $n = 1$, the conclusion follows from the assumption that $f(Z)$ generates A . We then assume that $n \geq 2$.

Note that $f_*\omega_Z$ is a GV sheaf and, since f is generically finite, $R^i f_*\omega_Z = 0$. We consider $gv(f_*\omega_Z)$. Since $H^n(A, f_*\omega_Z \otimes P) = H^n(Z, \omega_Z \otimes f^*P)$ for each $P \in \text{Pic}^0(A)$,

$$V^n(f_*\omega_Z) = \ker(f^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(Z))$$

consists of finitely many points. In particular, $gv(f_*\omega_Z) < \infty$. If $gv(f_*\omega_Z) \geq \frac{g-n}{n}$, we conclude from Pareschi-Popa [42] that $\chi(\omega_Z) \geq \frac{g-n}{n}$.

We then assume that $gv(f_*\omega_Z) = k < \frac{g}{n} - 1$ and

$$\text{codim}_{\text{Pic}^0(A)} V^{i_0}(f_*\omega_Z) - i_0 = k$$

for some $1 \leq i_0 \leq n$. Since $n \geq 2$ and $\dim V^n(f_*\omega_Z) = 0$, we see that $1 \leq i_0 \leq n - 1$. Pick an irreducible component W of $V^{i_0}(f_*\omega_Z)$ of codimension $i_0 + k$, then W must be of the form $Q + \widehat{C}$ where $\widehat{C} \subset \text{Pic}^0(A)$ is an abelian subvariety and $Q \in \text{Pic}^0(A)$ is a torsion point. We then consider the dual quotient $q_C : A \rightarrow C := \text{Pic}^0(\widehat{C})$. After taking further necessary birational modification to $q_C \circ f : Z \rightarrow C$, we obtain the Stein factorization: $Z \xrightarrow{q_C} Z_C \xrightarrow{f_C} C$, where Z_C may be assumed smooth.

We claim that $\dim Z_C \leq n - i_0$. Indeed, since $Q + \widehat{C} \subset V^{i_0}(f_*\omega_Z)$, for general $P \in \widehat{C}$,

$$H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^*P)) = H^{i_0}(A, f_*\omega_Z \otimes Q \otimes q_C^*P) \neq 0.$$

On the other hand, by Kollár's splitting (see [34, the main theorem]),

$$\begin{aligned} H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^*P)) &\cong \bigoplus_{0 \leq j \leq i_0} H^j(Z_C, R^{i_0-j}q_{C*}(\omega_Z \otimes f^*Q) \otimes f_C^*P) \\ &\cong \bigoplus_{0 \leq j \leq i_0} H^j(C, f_{C*}R^{i_0-j}q_{C*}(\omega_Z \otimes f^*Q) \otimes P). \end{aligned}$$

By Hacon's theorem (see [25]), all sheaves $f_{C*}R^{i_0-j}q_{C*}(\omega_Z \otimes f^*Q)$ are GV on C for $0 \leq j \leq i_0$. Thus

$$H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^*P)) \simeq H^0(C, f_{C*}R^{i_0}q_{C*}(\omega_Z \otimes f^*Q) \otimes P) \neq 0.$$

This implies that $R^{i_0}q_{C*}(\omega_Z \otimes f^*Q) \neq 0$ and, by Kollár's theorem [33, Theorem 2.1], $\dim Z - \dim Z_C \geq i_0$.

We then have

$$\begin{aligned} \frac{\dim C - \dim Z_C}{\dim Z_C} &\geq \frac{g - i_0 - k}{n - i_0} - 1 \geq \frac{g - n - k}{n - i_0} \\ &> \frac{g - n - \frac{g}{n} + 1}{n - 1} = \frac{g}{n} - 1. \end{aligned}$$

Since $f_C(Z_C) \subsetneq C$ generates C and, by induction, there exists a further quotient $q_{CB} : C \rightarrow B$ with connected fibers between abelian varieties such that for the Stein factorization $Z_C \rightarrow Z_B \rightarrow B$ of $q_{CB} \circ f_C$, any smooth model of Z_B or its image in B is of general type, and

$$\chi(\omega_{Z'_B}) \geq \frac{\dim C - \dim Z_C}{\dim Z_C} > \frac{g}{n} - 1.$$

□

Given any smooth projective n -fold X of general type, for the case $k = 1$, Theorem 2.1 gives

$$\text{vol}(X) \geq \min_{1 \leq d \leq n} \frac{\nu_{n-d}}{n(d+1)^{n-d}} (q(X) - dn),$$

which can be greatly improved as follows.

Theorem 2.5. *Let $n > 0$ and $1 \leq d \leq n$. Set $\lambda_n := \min_{1 \leq d \leq n} \frac{\nu_{n-d}}{(n-d)!}$. The inequality*

$$\text{vol}(X) \geq 2(n-1)! \lambda_n (q(X) - n)$$

holds for any nonsingular projective n -fold X of general type.

Proof. We may assume that $q(X) \geq n + 1$. Let $a_X : X \rightarrow A_X$ be the Albanese morphism of X . Taking the Stein factorization of a_X ,

$$a_X : X \xrightarrow{h} Z \xrightarrow{f} A_X.$$

Modulo further birational modifications, we may assume that Z is smooth. Let $1 \leq \dim Z = m \leq n$. By taking further birational modifications and applying Lemma 2.4, we get the following commutative

diagram:

$$\begin{array}{ccc}
 X & & \\
 \downarrow h & \searrow a_X & \\
 Z & \xrightarrow{f} & A_X \\
 \downarrow h_B & & \downarrow \\
 Z_B & \xrightarrow{f_B} & B,
 \end{array}$$

where Z_B is smooth of general type, $\chi(\omega_{Z_B}) \geq \frac{q(X)-m}{m} \geq \frac{q(X)-n}{n}$, and h_B is a fibration. Let $d = \dim Z_B \leq m$ and let F be a general fiber of h_B .

By the Severi inequality (see [1] and [51]), $\text{vol}(Z_B) \geq 2d! \chi(\omega_{Z_B}) \geq 2d! \frac{q(X)-n}{n}$.

We have

$$\text{vol}(X) \geq \frac{n!}{d!(n-d)!} \text{vol}(Z_B) \text{vol}(F)$$

by [50, Theorem 7.1]. Therefore

$$\text{vol}(X) \geq 2(n-1)! \cdot \frac{\nu_{n-d}}{(n-d)!} (q(X) - n).$$

□

Remark 1. By considering the product, we see that $\nu_d \leq 2d\nu_{d-1}$. Thus $\frac{\nu_d}{d!} \leq 2 \frac{\nu_{d-1}}{(d-1)!}$. It is natural to expect that $\nu_d < \nu_{d-1}$ when $d \geq 2$. If this is the case, we would have

$$\text{vol}(X) \geq 2\nu_{n-1}(q(X) - n)$$

for any smooth projective n -fold ($n \geq 3$) of general type.

2.3. Volumes of threefolds of general type.

We excavate information from the method of Corollary 2.3 to study the volume of 3-folds when $h^{2,0}(X)$ is small. This subsection is devoted to proving the following two results.

Theorem 2.6. *Let X be a smooth projective threefold of general type with $h^{2,0}(X) \geq 3$. Then $\text{vol}(X) \geq \frac{1}{224}$.*

Corollary 2.7. *Let X be a smooth projective threefold of general type with $\chi(\mathcal{O}_X) \neq 2$ or 3. Then $\text{vol}(X) \geq \frac{1}{420}$.*

Corollary 2.7 improves an earlier result of Chen-Chen [9, Theorem A] where the condition is $\chi(\mathcal{O}_X) \leq 1$.

In order to treat the situation with $\chi(\mathcal{O}_X) \geq 2$, one necessarily has $h^{2,0}(X) = h^2(\mathcal{O}_X) \geq 1$. Since $h^{2,0}(X) = h^0(X, \Omega_X^2) \geq 1$, it is natural to consider the evaluation map

$$\text{ev}_2 : H^0(X, \Omega_X^2) \rightarrow \Omega_X^2.$$

Lemma 2.8. *Assume that there exists a coherent subsheaf \mathcal{F} of Ω_X^2 such that $h^0(\det \mathcal{F}) \geq 2$. Then $\text{vol}(X) \geq \frac{1}{18}$.*

Proof. We may assume that \mathcal{F} is saturated with $h^0(\det \mathcal{F}) \geq 2$. Let \mathcal{Q} be the corresponding quotient sheaf. We have $2K_X \sim \det(\Omega_X^2) \sim \det \mathcal{F} + \det \mathcal{Q}$, where $\det \mathcal{Q}$ is pseudo-effective by [7]. We then consider the map $\varphi_{\mathcal{F}}$ induced by $|\det \mathcal{F}|$ and, by repeatedly apply Kawamata's extension theorem, get an estimation the lower bound of $\text{vol}(X)$ similar to the proof of (2.2).

Precisely, if the image of $\varphi_{\mathcal{F}}$ is a curve and, modulo a further birational modification over X , let S be a irreducible component of a general member of $|\det \mathcal{F}|$, we have $\text{vol}(X) \geq \frac{1}{18} \text{vol}(S) \geq \frac{1}{18}$. If the image of $\varphi_{\mathcal{F}}$ is a surface, then $\text{vol}(X) \geq \frac{1}{8}$ by [11, Table A2]. If $\varphi_{\mathcal{F}}$ is generically finite, we simply have $\text{vol}(X) \geq \frac{1}{4}$. \square

Proposition 2.9. *Let X be a threefold of general type. Assume that $h^{2,0}(X) \geq 3$, then either $\text{vol}(X) \geq \frac{1}{18}$ or $P_2(X) \geq 1$. In the latter case, $\text{vol}(X) \geq \frac{1}{224}$.*

Proof. If $h^{2,0}(X) \geq 4$, let \mathcal{F} be the image of the evaluation map ev_2 . By Lemma 2.2, $h^0(\det \mathcal{F}) \geq 2$. The statement follows from Lemma 2.8.

Assume that $h^{2,0}(X) = 3$. If the image of the evaluation map is of rank ≤ 2 , by Lemma 2.2, there exists a subsheaf \mathcal{F} of Ω_X^2 with $h^0(\det \mathcal{F}) \geq 2$. Then by Lemma 2.8, we have $\text{vol}(X) \geq \frac{1}{18}$. If the evaluation map is generically surjective, we have $P_2(X) = h^0(\det \Omega_X^2) \geq 1$. By [9, (3.10)], we know that

$$P_4(X) + P_5(X) + P_6(X) \geq 3P_2(X) + P_3(X) + P_7(X).$$

Since $P_2(X) \geq 1$, we have $P_7(X) \geq P_5(X)$ and $P_6(X) \geq P_4(X)$. Thus $2P_6(X) \geq 3P_2(X) \geq 3$. We have $P_6(X) \geq 2$. Denote by $\delta(X) = \min\{m \in \mathbb{Z} \mid P_m(X) \geq 2\}$. Then $\delta(X) \leq 6$. By [11, Theorem 4.3], one has $\text{vol}(X) \geq \frac{1}{224}$. \square

Proposition 2.9 directly implies Theorem 2.6.

Proof of Corollary 2.7. By [9, Theorem A], it suffices to study the case $\chi(\mathcal{O}_X) \geq 4$.

If $p_g(X) \geq 1$, by [11, Corollary 1.7] and [14, Theorem 1.4], $\text{vol}(X) \geq \frac{1}{75}$. If $q(X) \geq 1$, by [29, Theorem 1.5], $\text{vol}(X) \geq \frac{3}{8}$.

If $p_g(X) = q(X) = 0$ and $\chi(\mathcal{O}_X) \geq 4$, we have $h^{2,0}(X) \geq 3$ and the statement follows directly from Theorem 2.6. We are done. \square

3. PROOF OF THEOREM 1.3

Let V be a nonsingular projective 3-fold of general type. We show that $|3K_V|$ induces a birational map under the condition that $h^0(V, \Omega_V^2) \geq 108 \cdot 18^3 + 4$. The method naturally works for all $|mK_V|$ with $m \geq 4$.

By Corollary 2.3, we have $\text{vol}(V) > 2 \cdot 18^3$. Applying Fujita's approximation (see [39, Subsection 11.4]), we write $K_V \sim_{\mathbb{Q}} A + E$, where E is an effective \mathbb{Q} -divisor and A is an ample \mathbb{Q} -divisor such that $0 < \text{vol}(V) - \text{vol}(A) \ll 1$.

We now apply the method of cutting non-klt locus in Hacon-McKernan [26], Takayama [45] and Tsuji [47], which is also exploited in Todorov [46] and in our previous work [18, Subsection 4.2].

Pick very general points $x, y \in V$. There exists an effective \mathbb{Q} -divisor $D_1 \sim_{\mathbb{Q}} t_1 K_V$ with $t_1 < 3\sqrt[3]{\frac{2}{\text{vol}(V)}} + \epsilon < \frac{1}{6}$, where $0 < \epsilon \ll 1$ such that (V, D_1) is log canonical but not klt at x , and that (V, D_1) is not klt at y . Modulo a small perturbation, we may also assume that the non-klt locus of (V, D_1) , passing through x , is the minimal log canonical center V_1 .

3.1. The case with $\dim V_1 = 1$.

We apply Takayama's induction to conclude that there exists a divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $t_2 \leq t_1 + \frac{2}{\text{vol}_{V|V_1}(K_V)} + \epsilon$, (X, D_2) is log canonical at x , $\{x\}$ is an isolated component of $\text{Nklt}(X, D_2)$ at x , and (X, D_2) is not klt at y . Moreover, by Takayama [45, Theorem 4.5], we know that $\text{vol}_{V|V_1}(K_V + D_1) \geq \text{vol}(\overline{V_1})$, where $\overline{V_1}$ is the normalization of V_1 . Thus

$$t_2 \leq t_1 + \frac{2(1+t_1)}{2g(\overline{V_1})-2} + \epsilon \leq 1 + 2t_1 + \epsilon.$$

Since $t_1 < \frac{1}{6}$, we can choose $t_2 < 2$.

We now conclude by using Nadel vanishing. Indeed, since x and y are very general, both x and y are not contained in the support of E . Thus we still have $x, y \in \text{Nklt}(V, D_2 + (2-t_2)E)$ and $\{x\}$ is an isolated component of $\text{Nklt}(V, D_2 + (2-t_2)E)$. Consider the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_V(3K_V) \otimes \mathcal{J}(D_2 + (2-t_2)E) &\rightarrow \mathcal{O}_V(3K_V) \\ &\rightarrow \mathcal{O}_V(3K_V) \otimes (\mathcal{O}_V/\mathcal{J}(D_2 + (2-t_2)E)). \end{aligned}$$

Since $2K_V - D_2 - (2-t_2)E \sim_{\mathbb{Q}} (2-t_2)A$ is ample, $H^1(V, \mathcal{O}_V(3K_V) \otimes \mathcal{J}(D_2 + (2-t_2)E)) = 0$ by Nadel vanishing. Thus $|3K_V|$ separates x and y .

3.2. The case with $\dim V_1 = 2$ and $\text{vol}(V_1) \geq 128$.

Similarly, there exists a divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that (X, D_2) is log canonical at x , $V_2 \subsetneq V_1$ is the minimal log canonical center of (X, D_2)

at x and (X, D_2) is not klt at y , where

$$\begin{aligned} t_2 &\leq t_1 + 2\sqrt{\frac{2}{\text{vol}_{V|V_1}(K_V)}} + \epsilon \\ &\leq t_1 + 2(1+t_1)\sqrt{\frac{2}{\text{vol}(V_1)}} + \epsilon. \end{aligned}$$

Thus, if $\text{vol}(V_1) \geq 128$, since $t_1 < \frac{1}{6}$, $t_2 < \frac{1}{2}$, the statement follows from the argument in Subsection 3.1.

3.3. The case with $\dim V_1 = 2$ and $\text{vol}(V_1) \leq 127$. We apply a result of Todorov in [46, Lemma 3.2] to spread the minimal log canonical centers into a family. More precisely, there exists a smooth projective threefold \tilde{V} with the following diagram:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\pi} & V \\ & \downarrow f & \\ & C, & \end{array}$$

where

- (i) $f : \tilde{V} \rightarrow C$ is a surjective morphism to a smooth projective curve whose general fiber F is a smooth projective surface of volume ≤ 127 ;
- (ii) π is generically finite;
- (iii) for $v \in \tilde{V}$ general, let F_v be the fiber of f passing through v and $z = \pi(v)$, then $\pi|_{F_v} : F_v \rightarrow \pi(F_v)$ is birational onto its image and there exists an effective \mathbb{Q} -divisor $D_v \sim_{\mathbb{Q}} t_1 K_V$ such that $\pi(F_v)$ is the minimal log canonical center of (V, D_v) at z .

3.3.1. The subcase with $\deg \pi = m \geq 2$. For a general point z of V , the pre-image $\pi^{-1}(z)$ lies on m distinct fibers of f and we denote by S_z the set of these fibers. We also observe that, for such a general $z \in V$ and for any $v \in \pi^{-1}(z)$, z is a smooth point of D_v . In fact, locally F_v maps onto D_v . The following argument is due to Todorov [46, Lemma 3.3].

If $S_x \neq S_y$, we may take $F_1 \in S_x$, $F_2 \in S_x \setminus S_y$, and $F_3 \in S_y \setminus S_x$. Let $D'_x, D''_x \sim_{\mathbb{Q}} t_1 K_V$ be the corresponding effective \mathbb{Q} -divisors such that $\pi(F_1)$ and $\pi(F_2)$ are respectively the minimal log canonical center of (V, D'_x) and (V, D''_x) at x . Let $D_y \sim_{\mathbb{Q}} t_1 K_V$ be the effective \mathbb{Q} -divisors such that $\pi(F_3)$ is the minimal log canonical center of (V, D_y) at y . Note that $(V, D'_x + D''_x + D_y)$ is not klt at both x and y . We set

$$c := \max\{t \mid (V, t(D'_x + D''_x) + D_y) \text{ is log canonical at } x\}.$$

Then $c \in (0, 1]$ is a rational number. Moreover, since $x \notin D_y$ and x is a smooth point of D'_x and D''_x , the minimal log canonical center of

$(V, c(D'_x + D''_x) + D_y)$ at x is contained in $\pi(F_1) \cap \pi(F_2)$ and, hence, is of dimension ≤ 1 . It is also clear that $(V, c(D'_x + D''_x) + D_y)$ is not klt at y .

If $S_x = S_y$, we take $F_i \in S_x = S_y$ for $i = 1, 2$ and similarly denote by D' and D'' the corresponding effective \mathbb{Q} -divisors. Both x and y are smooth point of D' and D'' . Let

$$c = \max\{\text{lct}_x(V, D' + D''), \text{lct}_y(V, D' + D'')\}.$$

Similarly, $0 < c \leq 1$ is a rational number. After switching x and y , we may assume that $(V, c(D' + D''))$ is log canonical at x . Then the minimal log canonical center of $(V, c(D' + D''))$ is again contained in $\pi(F_1) \cap \pi(F_2)$.

In conclusion, if $\deg \pi \geq 2$, there exists an effective \mathbb{Q} -divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that (V, D_2) is log canonical at x , $V_2 \subsetneq V_1$ is the minimal log canonical center of (X, D_2) at x and (X, D_2) is not klt at y , where $0 < t_2 \leq 3t_1 < \frac{1}{2}$. The statement also follows from the argument in Subsection 3.1

3.3.2. The subcase with π being birational. Since π is birational, we may simply assume that $V = \tilde{V}$. Hence we have a fibration $f : V \rightarrow C$ such that a general fiber F of f has its canonical volume $\text{vol}(F) \leq 127$.

We now apply the assumption that $h^1(\omega_V) = h^0(\Omega_V^2) \geq 108 \cdot 18^3 + 4$. We have

$$h^1(V, \omega_V) = h^0(C, R^1 f_* \omega_V) + h^1(C, f_* \omega_V)$$

by Leray's spectral sequence. Note that $h^1(C, f_* \omega_V) = h^0(C, (f_* \omega_{V/C})^*)$ by Serre duality. Moreover, $f_* \omega_{V/C}$ is a locally free sheaf of rank $p_g(F)$. Since $\text{vol}(F) \leq 127$, by Noether inequality $p_g(F) \leq 250$. Moreover, by Fujita's theorem (see [22]), $f_* \omega_{V/C}$ is nef. By considering the Harder-Narasimhan filtration of $f_* \omega_{V/C}$, we see that

$$h^0(C, (f_* \omega_{V/C})^*) \leq p_g(F) \leq 250.$$

Therefore $h^0(C, R^1 f_* \omega_V)$ is very large. In particular,

$$q(F) = \text{rank}(R^1 f_* \omega_V) > 0.$$

We may run the relative minimal model program for $f : V \rightarrow C$. After resolving the finitely many terminal singularities of the relative minimal model, we may assume that a general fiber F of f is a minimal surface. Since F is irregular, $p_g(F) \geq q(F) \geq 1$. Hence the linear system $|2K_F|$ is base point free (see [4, Chapter VII. Theorem 7.4]) and $|3K_F|$ induces a birational morphism of F (see [4, Proposition 7.3]). By the main theorem of Kawamata [32], the restriction map

$$|m(K_X + F)| \rightarrow |mK_F|$$

is surjective for $m \geq 2$. Fix a general divisor $G \in |M(K_V + F)|$ for M sufficiently large. We also write $K_V \sim_{\mathbb{Q}} A + E$, where A is an ample \mathbb{Q} -divisor and E is an effective \mathbb{Q} -divisor.

Since $\text{vol}(F) \leq 127$ and $\text{vol}(V) > 2 \cdot 18^3$, there exists an effective \mathbb{Q} -divisor D such that $D \sim_{\mathbb{Q}} \lambda K_V$ and $D = F + D'$, where D' is also an effective \mathbb{Q} -divisor and $\lambda^{-1} \approx \frac{\text{vol}(V)}{3\text{vol}(F)} > \frac{2 \cdot 18^3}{3 \cdot 127} > 91$. (see, for instance, [18, the last paragraph of Page 2055]).

Fix two general fibers F_1 and F_2 of f , we introduce an effective \mathbb{Q} -divisor

$$Z := 4D + \frac{2 - 4\lambda - \epsilon}{M}G + (4\lambda - 4 + \epsilon)F + \epsilon E.$$

Note that Z is \mathbb{Q} -effective. We denote by $\mathcal{J}(Z)$ the multiplier ideal of Z . Since $Z \sim_{\mathbb{Q}} (2 - \epsilon)K_V - 2F + \epsilon E$ and thus $2K_V - F_1 - F_2 - Z \sim_{\mathbb{Q}} \epsilon A$, by the Nadel vanishing theorem,

$$H^1(V, \mathcal{O}_V(3K_V - F_1 - F_2) \otimes \mathcal{J}(Z)) = 0.$$

Thus the restriction map

$$\begin{aligned} & H^0(V, \mathcal{O}_V(3K_V) \otimes \mathcal{J}(Z)) \\ & \rightarrow H^0(F_1, \mathcal{O}_{F_1}(3K_{F_1}) \otimes \mathcal{J}(Z)|_{F_1}) \bigoplus H^0(F_2, \mathcal{O}_{F_2}(3K_{F_2}) \otimes \mathcal{J}(Z)|_{F_2}) \end{aligned}$$

is surjective.

By the restriction theorem ([39, Theorem 9.5.1]), $\mathcal{J}(F_1, Z|_{F_1}) \subset \mathcal{J}(Z)|_{F_1}$. Since M can be sufficiently large, $|MK_{F_1}|$ is base point free, and ϵ can be sufficiently small, $\mathcal{J}(F_1, Z|_{F_1}) = \mathcal{J}(F_1, 4D|_{F_1})$.

When $\text{vol}(F) \geq 3$, since $D|_{F_1} \sim_{\mathbb{Q}} \lambda K_{F_1}$, we finish the proof by Lemma 3.1.

When $\text{vol}(F) = 2$, we have $p_g(F) = q(F) = 1$. We may choose λ such that

$$\lambda^{-1} \approx \frac{\text{vol}(V)}{3\text{vol}(F)} > \frac{2 \cdot 18^3}{3 \cdot 2} = 1944.$$

We finish the proof by Lemma 3.2. \square

Lemma 3.1. *Let F be a minimal surface of general type with $K_F^2 \geq 3$. The linear system*

$$|3K_F \otimes \mathcal{J}(\mu D_F)|$$

induces a birational map for any effective \mathbb{Q} -divisor $D_F \sim_{\mathbb{Q}} K_F$ and rational number $0 < \mu < 2 - \sqrt{3}$.

Proof. It is convenient to apply the \mathbb{Q} -divisor method on surfaces. Let $\sigma : \tilde{F} \rightarrow F$ be a log resolution of (F, D_F) . Then $\mathcal{J}(\mu D_F) = \sigma_* \mathcal{O}_{\tilde{F}}(K_{\tilde{F}/F} - \lfloor \mu \sigma^* D_F \rfloor)$.

Hence σ_* induces an isomorphism:

$$H^0(\tilde{F}, K_{\tilde{F}} + \lceil 2\sigma^* K_F - \sigma^*(\mu D_F) \rceil) \cong H^0(F, 3K_F \otimes \mathcal{J}(\mu D_F)).$$

Recall the following theorem of Langer (see [38]), for a nef \mathbb{Q} -divisor $Q := 2\sigma^* K_F - \sigma^*(\mu D_F)$ on the surface \tilde{F} , if $Q^2 > 8$ and $(Q \cdot C) > \frac{4}{1 + \sqrt{1 - \frac{8}{Q^2}}}$ for any curve C passing through a very general point of \tilde{F} , then $|K_{\tilde{F}} + \lceil Q \rceil|$ induces a birational map of \tilde{F} .

Now we have $Q^2 = (2\sigma^*K_F - \sigma^*(\mu D_F))^2 > 3(K_F^2) \geq 9$. Thus it suffices to verify that

$$(Q \cdot C) > 3 = \frac{4}{1 + \sqrt{1 - \frac{8}{9}}} \geq \frac{4}{1 + \sqrt{1 - \frac{8}{Q^2}}}$$

for any curve C passing through a very general point of \tilde{F} . This is the case since, by Chen-Chen [11, Lemma 2.5], we always have $(\sigma^*K_F \cdot C) \geq 2$. \square

Lemma 3.2. *Assume that F is a minimal surface of general type with $K_F^2 = 2$ and $p_g(F) = q(F) = 1$. Then $\mathcal{J}(F, \mu D_F) = \mathcal{O}_F$ for any effective \mathbb{Q} -divisor $D_F \sim_{\mathbb{Q}} K_F$ and any rational number $\mu: 0 < \mu < \frac{1}{26}$.*

Proof. We have $h^0(F, 2K_F) = 3$.

Let $\sigma : F \rightarrow F_0$ be the contraction onto the canonical model of F . Then $K_F = \sigma^*K_{F_0}$. We denote by $H_0 \sim K_{F_0}$ the ample Cartier divisor on F_0 . By the birational transformation rule (see [39, Theorem 9.2.33]), it suffices to show that $\text{lct}(F_0; D_{F_0}) \geq \frac{1}{26}$ for any $D_{F_0} \sim_{\mathbb{Q}} H_0$.

We apply Kollár's method (see the appendix of [17]). Since $H_0^2 = 2$ and $h^0(F_0, K_{F_0} + H_0) = 3$, we have

$$3 \geq \text{mcd}(\text{lct}(F_0; \frac{1}{2}D_{F_0})),$$

for any $D_{F_0} \sim_{\mathbb{Q}} H_0$ (see [17, Proposition A.3 and Remark A.7]). By [17, Proposition A.4], $\text{lct}(F_0; \frac{1}{2}D_{F_0}) \geq \frac{1}{13}$. \square

4. EXTENSION THEOREMS

We prove two extension theorems in this section. Both are crucial ingredients in the proof of Theorem 1.6. We hope that they are useful in other contexts.

4.1. The first theorem.

Theorem 4.1. *Given a birationally bounded set \mathfrak{X} of varieties of general type of dimension n . Let $f : V \rightarrow S$ be a fibration from a smooth projective variety V onto a smooth projective surface S , whose general fiber F is birationally equivalent to an element of \mathfrak{X} , and $h : S \rightarrow C$ a fibration from S onto a smooth projective curve C . Let $g = f \circ h : V \rightarrow C$ be the composition of fibrations and denote by Y a general fiber of g . For any integer $m \geq 2$, there exists a constant M depending only on m and \mathfrak{X} such that either of the following statements holds for any V with $K_V \geq_{\mathbb{Q}} MY$:*

- (1) *the restriction map $H^0(V, mK_V) \rightarrow H^0(Y, mK_Y)$ is surjective;*
- (2) *the restriction map $H^0(V, mK_V) \rightarrow H^0(F, mK_F)$ is surjective.*

only on m and \mathfrak{X} , such that whenever $\text{vol}(Y) > M_1$ and $0 < \eta < \epsilon_1$, for any effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} K_Y$, the restriction map

$$H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(|(m-1-\eta)K_Y| + \eta D)) \rightarrow H^0(F, \mathcal{O}_F(mK_F))$$

is surjective, where F is a general fiber of f_Y .

Proof. After suitable birational modifications, we denote by $r : Y \rightarrow Y_0$ (resp. $r_F : F \rightarrow F_0$) the contraction morphism onto a minimal model Y_0 (resp. F_0). We write $K_Y = r^*K_{Y_0} + E_Y$ and $K_F = r_F^*K_{F_0} + E_F$. Note that E_Y and E_F are effective \mathbb{Q} -divisors and for N sufficiently large and divisible, $|NK_{Y_0}|$ and $|NK_{F_0}|$ are base point free and we have

$$\begin{aligned} |NK_Y| &= r^*|NK_{Y_0}| + NE_Y, \\ |NK_F| &= r_F^*|NK_{F_0}| + NE_F. \end{aligned} \quad (4.2)$$

Let $\mu : \tilde{Y} \rightarrow Y$ be a log resolution of $(Y, E_Y + D)$, let \tilde{F} be the strict transform of F , and let $\mu_F : \tilde{F} \rightarrow F$ be the induced morphism. We have the following commutative diagram

$$\begin{array}{ccccc} \tilde{F} & \xrightarrow{\mu_F} & F & \xrightarrow{r_F} & F_0 \\ \downarrow & & \downarrow & & \\ \tilde{Y} & \xrightarrow{\mu} & Y & \xrightarrow{r} & Y_0 \\ & \searrow & \downarrow f & & \\ & & Z & & \end{array}$$

Define

$$\begin{aligned} \mathcal{J} &:= \mathcal{J}(|(m-1-\eta)K_Y| + \eta D) \\ &= \mu_*\mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}/Y} - [(m-1-\eta)\mu^*E_Y + \eta\mu^*D]). \end{aligned}$$

Then

$$\begin{aligned} &H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}) \\ &\cong H^0(\tilde{Y}, K_{\tilde{Y}} + (m-1)\mu^*K_Y - [(m-1-\eta)\mu^*E_Y + \eta\mu^*D]). \end{aligned}$$

We consider the restriction map:

$$\begin{aligned} &H^0(\tilde{Y}, K_{\tilde{Y}} + (m-1)\mu^*K_Y - [(m-1-\eta)\mu^*E_Y + \eta\mu^*D]) \\ &\xrightarrow{\Psi} H^0(\tilde{F}, K_{\tilde{F}} + (m-1)\mu_F^*K_F - [(m-1-\eta)\mu^*E_Y + \eta\mu^*D]|_{\tilde{F}}) \\ &\subset H^0(\tilde{F}, mK_{\tilde{F}}). \end{aligned}$$

Since F is birationally equivalent to an element in \mathfrak{X} , $\text{vol}(F) \leq N_{\mathfrak{X}}$. Thus, by considering the asymptotic Riemann-Roch (see, for instance, [18, the proof of Theorem 3.7]), one has $\tau K_Y \geq_{\mathbb{Q}} F$ for any number $\tau > \frac{N_{\mathfrak{X}} \cdot \dim Y}{\text{vol}(Y)}$. In particular, by Kawamata's extension theorem, for N sufficiently large and divisible, there exists an effective divisor $V_N \sim N\tau K_F$ on F such that

$$|N(1+\tau)K_Y||_F \supset |NK_F| + V_N. \quad (4.3)$$

Combining (4.2) and (4.3), we have

$$\begin{aligned} (1 + \tau)r^*K_{Y_0}|_F &\sim_{\mathbb{Q}} r_F^*K_{F_0} + Z_1, \\ (1 + \tau)E_Y|_F &\sim_{\mathbb{Q}} E_F + Z_2, \\ \tau r^*K_{Y_0} &\sim_{\mathbb{Q}} F + Z_3, \end{aligned} \quad (4.4)$$

where Z_1 and Z_2 are effective \mathbb{Q} -divisors on F such that $Z_1 + Z_2 \sim_{\mathbb{Q}} \tau K_F$ and Z_3 is an effective \mathbb{Q} -divisor on Y . Note that

$$Z_3|_F \sim_{\mathbb{Q}} \tau r^*K_{Y_0}|_F = \frac{\tau}{1 + \tau} r_F^*K_{F_0} + \frac{\tau}{1 + \tau} Z_1.$$

We may and will choose ϵ_1 sufficiently small and M_1 sufficiently large such that $\epsilon_1 + \frac{N_{\mathfrak{X}} \dim Y}{M_1} < 1$ and choose a rational number τ such that $0 < \tau - \frac{N_{\mathfrak{X}} \dim Y}{M_1} \ll 1$. Then

$$\begin{aligned} &(m - 1)\mu^*K_Y - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor - \tilde{F} \\ &\geq_{\mathbb{Q}} (m - 1 - \eta)(r \circ \mu)^*K_{Y_0} - \mu^*F \\ &\geq_{\mathbb{Q}} (m - 1 - \eta - \tau)(r \circ \mu)^*K_{Y_0} \end{aligned}$$

is big. Define

$$\mathcal{J}' := \mathcal{J}(\|(m - 1)\mu^*K_Y - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor - \tilde{F}\|).$$

By Nadel vanishing (see [39, Theorem 11.2.12]),

$$H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}} + (m - 1)\mu^*K_Y - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor - \tilde{F}) \otimes \mathcal{J}') = 0.$$

Thus the image of the restriction map Ψ contains the subspace

$$H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m - 1)\mu_F^*K_F - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}}).$$

We want to show that

$$\begin{aligned} &H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m - 1)\mu_F^*K_F - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}}) \\ &= H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m - 1)\mu_F^*K_F)) \xrightarrow{\sim} H^0(F, mK_F). \end{aligned}$$

We now study the sheaf $\mathcal{J}'|_{\tilde{F}}$. By (4.4), we have

$$\begin{aligned} &(m - 1)\mu^*K_Y - \lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor - \tilde{F} \\ &\sim_{\mathbb{Q}} (m - 1 - \eta)(r \circ \mu)^*K_{Y_0} + V' - \tilde{F} \\ &\sim_{\mathbb{Q}} (m - 1 - \eta - \tau)(r \circ \mu)^*K_{Y_0} + V'', \end{aligned}$$

where $V' = \{(m - 1 - \eta)\mu^*E_Y + \eta\mu^*D\}$ and $V'' = V' + \mu^*(Z_3)$.

Since K_{Y_0} is semiample, $\mathcal{J}' \supset \mathcal{J}(\|V''\|) \supset \mathcal{J}(V'')$. By the restriction theorem of asymptotic multiplier ideals (see [39, Theorem 11.2.1]), we see that $\mathcal{J}'|_{\tilde{F}} \supset \mathcal{J}(\tilde{F}, V''|_{\tilde{F}})$.

We then have

$$\begin{aligned} &\mathcal{O}_{\tilde{F}}(-\lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}} \\ &\supset \mathcal{J}(\lfloor (m - 1 - \eta)\mu^*E_Y + \eta\mu^*D \rfloor|_{\tilde{F}} + \mu^*(Z_3)|_{\tilde{F}}). \end{aligned}$$

By (4.4), we have

$$\begin{aligned}
& ((m-1-\eta)\mu^*E_Y + \eta\mu^*D)|_{\tilde{F}} + Z_3|_{\tilde{F}} \\
\sim_{\mathbb{Q}} & \frac{m-1-\eta}{1+\tau}\mu_F^*(E_F + Z_2) + \frac{\tau}{1+\tau}\mu_F^*r_F^*K_{F_0} + \frac{\tau}{1+\tau}\mu_F^*Z_1 + \eta\mu^*D|_{\tilde{F}} \\
\sim_{\mathbb{Q}} & \frac{m-1-\eta-\tau}{1+\tau}\mu_F^*E_F + \frac{m-1-\eta-\tau}{1+\tau}\mu_F^*Z_2 + \frac{\tau}{1+\tau}\mu_F^*(r_F^*K_{F_0} + E_F) \\
& + \frac{\tau}{1+\tau}\mu_F^*(Z_1 + Z_2) + \eta\mu^*D|_{\tilde{F}}.
\end{aligned}$$

We observe that

$$\begin{aligned}
& \frac{m-1-\eta-\tau}{1+\tau}\mu_F^*Z_2 + \frac{\tau}{1+\tau}\mu_F^*(r_F^*K_{F_0} + E_F) + \frac{\tau}{1+\tau}\mu_F^*(Z_1 + Z_2) + \eta\mu^*D|_{\tilde{F}} \\
\sim_{\mathbb{Q}} & \frac{m-1-\eta-\tau}{1+\tau}\mu_F^*Z_2 + (\tau + \eta)\mu_F^*K_F \\
\leq_{\mathbb{Q}} & \frac{m-1-\eta-\tau}{1+\tau}\tau\mu_F^*K_F + (\tau + \eta)\mu_F^*K_F \\
= & \frac{m\tau + \eta}{1+\tau}\mu_F^*K_F,
\end{aligned}$$

and

$$\frac{m-1-\eta-\tau}{1+\tau}\mu_F^*E_F = (m-1 - \frac{m\tau + \eta}{1+\tau})\mu_F^*E_F.$$

Set $\rho = \frac{m\tau + \eta}{1+\tau}$ and choose an effective \mathbb{Q} -divisor $D_F \sim_{\mathbb{Q}} K_F$ so that

$$\begin{aligned}
& \frac{m-1-\eta-\tau}{1+\tau}\mu_F^*Z_2 + \frac{\tau}{1+\tau}\mu_F^*(r_F^*K_{F_0} + E_F) + \frac{\tau}{1+\tau}\mu_F^*(Z_1 + Z_2) + \eta\mu^*D|_{\tilde{F}} \\
\leq_{\mathbb{Q}} & \rho\mu_F^*D_F.
\end{aligned}$$

We then have

$$\begin{aligned}
& \mathcal{O}_{\tilde{F}}(-[(m-1-\eta)\mu^*E_Y + \eta\mu^*D]|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}} \\
\supset & \mathcal{J}(\tilde{F}, (m-1-\rho)\mu_F^*E_F + \rho\mu_F^*D_F) \\
= & \mathcal{J}(\|(m-1-\rho)\mu_F^*K_F\| + \rho\mu_F^*D_F).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m-1)\mu_F^*K_F - [(m-1-\eta)\mu^*E_Y + \eta\mu^*D]|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}}) \\
\supset & H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m-1)\mu_F^*K_F) \otimes \mathcal{J}(\|(m-1-\rho)\mu_F^*K_F\| + \rho\mu_F^*D_F)).
\end{aligned}$$

By the birational transformation rule (see [39, Theorem 9.2.33]),

$$\begin{aligned}
& \mu_{F*}(\mathcal{O}_{\tilde{F}}(K_{\tilde{F}/F}) \otimes \mathcal{J}(\|(m-1-\rho)\mu_F^*K_F\| + \rho\mu_F^*D_F)) \\
= & \mathcal{J}(F, \|(m-1-\rho)K_F\| + \rho D_F).
\end{aligned}$$

Therefore it follows that

$$\mu_{F*}\text{Im}(\Psi) \supset H^0(F, \mathcal{O}_F(mK_F) \otimes \mathcal{J}(F, \|(m-1-\rho)K_F\| + \rho D_F)).$$

By [18, Theorem 3.5], there exists a positive constant $\epsilon_{\mathfrak{X}}$, depending only on m and \mathfrak{X} , such that for each $0 < \epsilon < \epsilon_{\mathfrak{X}}$ and $D_F \sim_{\mathbb{Q}} K_F$,

$$H^0(F, \mathcal{O}_F(mK_F) \otimes \mathcal{J}(F, \|(m-1-\epsilon)K_F\| + \epsilon D_F)) = H^0(F, mK_F).$$

Thus we may choose M_1 and ϵ_1 such that $\epsilon_1 + \frac{m \cdot \dim Y \cdot N_{\mathfrak{X}}}{M_1} < \epsilon_{\mathfrak{X}}$. Then, for $0 < \tau - \frac{N_{\mathfrak{X}} \cdot \dim Y}{M_1} \ll 1$ and $0 < \eta < \epsilon_1$, we have $\rho = \frac{m\tau + \eta}{1 + \tau} < \epsilon_1 + \frac{m \cdot \dim Y \cdot N_{\mathfrak{X}}}{M_1} < \epsilon_{\mathfrak{X}}$ and hence the restriction map Ψ is surjective. \square

4.2. The second theorem.

Theorem 4.3. *Let $f : V \rightarrow S$ be a fibration from a smooth projective variety of general type to a smooth projective variety S and let F be a general fiber of f . Assume that there exists an effective \mathbb{Q} -divisor Δ on S such that $K_S + \Delta$ is big and nef, and*

- (1) *for any big divisor H on S , $0 < \epsilon \ll 1$ and $N \in \mathbb{Z}_{>0}$ sufficiently large and divisible, the restriction map*

$$H^0(V, N(K_V - f^*(K_S + \Delta)) + N\epsilon f^*H) \rightarrow H^0(F, NK_F)$$

is surjective;

- (2) *there exists a rational number $a > 0$ such that, for a general point $s \in S$, there exists an effective \mathbb{Q} -divisor $D_s \sim_{\mathbb{Q}} a(K_S + \Delta)$ such that (S, D_s) is log canonical at s and s is a lc center of (S, D_s) .*

Then the restriction map

$$H^0(V, mK_V) \rightarrow H^0(F, mK_F)$$

is surjective for $m \geq \lceil a + \epsilon \rceil + 1$.

Proof. Let $f_0 : V_0 \rightarrow S$ be a f -minimal model. After necessary birational modifications of V , we may and do assume that we have a birational morphism $\rho : V \rightarrow V_0$. Write $K_V = \rho^*K_{V_0} + \sum_{i=1}^n a_i E_i$. We also fix $K_V \sim_{\mathbb{Q}} A + E$, where E is an effective \mathbb{Q} -divisor and A is an ample \mathbb{Q} -divisor. Modulo further birational modifications, we may assume that $\sum_{i=1}^n E_i + E$ has SNC support and its restriction on a general fiber F of f also has SNC support. We may assume that E_i is f -horizontal for $1 \leq i \leq n_0$ and E_i is f -vertical for $n_0 + 1 \leq i \leq n$. Since V_0 is a f -minimal model, for sufficiently large and divisible N ,

$$\begin{aligned} |NK_F| &= |N\rho^*K_{V_0}|_F + N\left(\sum_{i=1}^n a_i E_i\right)|_F \\ &= |N\rho^*K_{V_0}|_F + N\left(\sum_{i=1}^{n_0} a_i E_i\right)|_F, \end{aligned} \quad (4.5)$$

where $N(\sum_{i=1}^n a_i E_i)|_F = N(\sum_{i=1}^{n_0} a_i E_i)|_F$ is the fixed part of $|NK_F|$ and the mobile part $|N\rho^*K_{V_0}|_F$ is base point free.

For $s \in S$ general, by assumption, there exists an effective \mathbb{Q} -divisor $D_s \sim_{\mathbb{Q}} a(K_S + \Delta)$ such that (S, D_s) is log canonical at s and s is a lc center. By Tie breaking (see, for instance, [36]), after a small perturbation of D_s (thus $D_s \sim_{\mathbb{Q}} (a + \eta)(K_S + \Delta)$, where $0 < \eta \ll 1$), we may assume that (S, D_s) is log canonical at s , s is an isolated

component of $\text{Nklt}(S, D_s)$, and there is a unique exceptional divisor with discrepancy -1 over s . Let F be the fiber of f over s .

Let $\rho_S : S' \rightarrow S$ be a log resolution of (S, D_s) . Then we have

$$K_{S'} + E_S + E'_S = \rho_S^*(K_S + D_s) + E''_S,$$

where E_S, E'_S and E''_S are effective \mathbb{Q} -divisors with no common component, E''_S is ρ_S -exceptional, E_S is the unique exceptional divisor with discrepancy -1 over s , and each component of E'_S meeting E_S has coefficient < 1 .

Since $(V, \text{Supp}(\sum_{i=1}^n E_i + E))$ is log smooth over an open neighborhood of $s \in S$, we may take a log resolution of $(V, f^*D_s + \sum_{i=1}^n E_i + E)$, which is isomorphic to $V \times_S S'$ over an open neighborhood of $F = f^{-1}(s)$. Namely, there exists an open neighborhood U of s such that $(f \circ \rho_V)^{-1}(U) \simeq f^{-1}(U) \times_U \rho_S^{-1}(U)$. We have

$$\begin{array}{ccccc} V' & \xrightarrow{\rho_V} & V & \xrightarrow{\rho} & V_0 \\ \downarrow f' & & \downarrow f & \swarrow f_0 & \\ S' & \xrightarrow{\rho_S} & S & & \end{array}$$

which summarizes all morphisms. Then we have

$$K_{V'} + E_V + E'_V = \rho_V^*(K_V + f^*D_s) + E''_V,$$

where E_V, E'_V , and E''_V are effective \mathbb{Q} -divisors without common component, E''_V is ρ_V -exceptional, E_V is the unique exceptional divisor with discrepancy -1 over F , each component of E'_V meeting E_V has coefficient < 1 , and E'_V is f' -vertical.

We then have

$$K_{V'} + E_V + E'_V + \epsilon \rho_V^* E = \rho_V^*(K_V + f^*D_s + \epsilon E) + E''_V.$$

Since $F \not\subseteq \text{Supp}(E)$, the coefficient of each component of $E'_V + \epsilon \rho_V^* E$ meeting E_V is < 1 , while $0 < \epsilon \ll 1$.

For $0 < \epsilon \ll 1$ and the sufficiently large and divisible M , let $b = m - 1 - \epsilon - (a + \eta) > 0$ and let

$$G \in |M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta))|$$

be a general element. By the first assumption, for the sufficiently large and divisible M , the restriction map

$$\begin{aligned} & H^0(V, M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta))) \\ \rightarrow & H^0(F, M(m - 1 - \epsilon)K_F) \end{aligned} \quad (4.6)$$

is surjective. Thus we may write

$$\begin{aligned} & |M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta))| \\ = & |L_M| + M(m - 1 - \epsilon) \sum_{i=1}^n a_i E_i, \end{aligned}$$

where $|L_M|$ is a linear system with $\text{Bs}|L_M| \cap F = \emptyset$ by (4.5) and (4.6). After shrinking U , we may assume that the base locus on $|L_M|$ is contained in $f^{-1}(S \setminus U)$. Hence, we write $G = G_1 + M(m-1-\epsilon) \sum_{i=1}^n a_i E_i$, where $(V, \text{Supp}(\sum_{i=1}^m E_i + E + G_1))$ is log smooth over U and $G_1|_F$ is linearly equivalent to an irreducible smooth divisor which does not contain any component of $\rho_V(\text{Supp}(E'_V + \epsilon \rho_V^* E)) \cap F$. After blowing-up centers in $(f \circ \rho_V)^{-1}(S \setminus U)$, we may assume that ρ_V is a log resolution of $(V, G + f^* D_s + E + \sum_{i=1}^n E_i)$.

We now study the multiplier ideal $\mathcal{J} := \mathcal{J}(\frac{1}{M}G + f^* D_s + \epsilon E)$. We have

$$\begin{aligned} & K_{V'} + E_V + E'_V + \epsilon \rho_V^* E + \frac{1}{M} \rho_V^* G_1 + (m-1-\epsilon) \sum_{i=1}^m a_i \rho_V^* E_i \\ &= \rho_V^* (K_V + \frac{1}{M} G + f^* D_s + \epsilon E) + E''_V. \end{aligned}$$

Note that $\rho_V^* E_i = \tilde{E}_i + E'_i$, where $\tilde{E}_i = \rho_{V^*}^{-1}(E_i)$ is the strict transform of E_i and E'_i does not meet E_V for $1 \leq i \leq n_0$ and $\rho_V^* E_i$ does not meet E_V for $n_0 + 1 \leq i \leq m$. Let b_i be the coefficient of \tilde{E}_i in $\rho_V^* E$ for $1 \leq i \leq m_0$.

Let $D_{V'} = E'_V + \epsilon \rho_V^* E + \frac{1}{M} \rho_V^* G_1 + (m-1-\epsilon) \sum_{i=1}^n a_i \rho_V^* E_i$. We observe that the coefficient of \tilde{E}_i in $D_{V'}$ is $(m-1-\epsilon)a_i + \epsilon b_i$ for $1 \leq i \leq n_0$. The coefficient of other component of $D_{V'}$, meeting E_V , is < 1 .

By definition,

$$\mathcal{J} = \rho_{V*} \mathcal{O}_{V'}(\lceil E''_V - E_V - D_{V'} \rceil).$$

After shrinking U , we may assume that for any component of $D_{V'}$ not meeting E_V , its image in S is contained in the complement of U .

We claim that, for $0 < \epsilon \ll 1$,

$$\mathcal{J}|_{f^{-1}(U)} = \mathcal{I}_F \cdot \mathcal{O}_{f^{-1}(U)}(-D),$$

where $0 \leq D \leq \sum_{i=1}^{n_0} \lfloor m a_i \rfloor E_i$ and \mathcal{I}_F is the ideal sheaf of F .

Indeed,

$$\lceil E''_V - E_V - D_{V'} \rceil = -E_V - \sum_{i=1}^{m_0} c_i \tilde{E}_i - Z_1 + Z_2,$$

where Z_1 and Z_2 are effective divisors without common components, Z_2 is ρ_V -exceptional, each component of Z_1 does not meet E_V .

Thus

$$\mathcal{J}|_{f^{-1}(U)} \tag{4.7}$$

$$= \rho_{V*} \mathcal{O}_{\rho_V^{-1}(f^{-1}(U))}(-E_V - \sum_{i=1}^{n_0} c_i \tilde{E}_i)$$

$$= \mathcal{I}_F \cdot \mathcal{O}_{f^{-1}(U)}(-\sum_{i=1}^{n_0} c_i E_i) \tag{4.8}$$

where $c_i = \lfloor (m-1-\epsilon)a_i + \epsilon b_i \rfloor$. Since $0 < \epsilon \ll 1$, we have $(m-1-\epsilon)a_i + \epsilon b_i < ma_i$ and, hence, $c_i \leq \lfloor ma_i \rfloor$. Set $D = \sum_{i=1}^{n_0} c_i E_i$. The claim is proved.

Note that $(m-1)K_V - (\frac{1}{M}G + f^*D_s + \epsilon E) \sim_{\mathbb{Q}} \epsilon A$. By Nadel vanishing, we have $H^1(V, \mathcal{O}_V(mK_V) \otimes \mathcal{J}) = 0$.

Moreover, let \mathcal{Q} be the quotient sheaf $\mathcal{O}_V(-D)/\mathcal{J}$. By (4.7), we see that

$$\mathcal{Q} = \mathcal{O}_F(-D) \oplus \mathcal{Q}^1$$

where $\text{Supp}(\mathcal{Q}^1) \subset V \setminus f^{-1}(U)$. By considering the short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_V(-D) \rightarrow \mathcal{Q} \rightarrow 0,$$

we have the surjective map:

$$H^0(V, mK_V - D) \rightarrow H^0(F, mK_F - D|_F) \oplus H^0(\mathcal{Q}^1(mK_V)).$$

Since $D \leq \sum_{i=1}^{n_0} \lfloor ma_i \rfloor E_i$ and $\sum_{i=1}^{n_0} \lfloor ma_i \rfloor E_i|_F$ is contained in the fixed part of $|mK_F|$, we conclude that the restriction map

$$H^0(V, mK_V) \rightarrow H^0(F, mK_F)$$

is surjective. \square

Corollary 4.4. *Let $f : V \rightarrow S$ be a fibration from a smooth projective variety of general type onto a smooth projective surface S and let F be a general fiber of f . Assume that there exists an effective \mathbb{Q} -divisor Δ on S such that $K_S + \Delta$ is big and nef, and*

- (1) *for any big divisor H on S , $0 < \epsilon \ll 1$ and any sufficiently large and divisible number N , the restriction map*

$$H^0(V, N(K_V - f^*(K_S + \Delta)) + N\epsilon f^*H) \rightarrow H^0(F, NK_F)$$

is surjective;

- (2) *$(K_S + \Delta)^2 = \alpha > 0$ and $((K_S + \Delta) \cdot C) = \beta > 0$ for any curve $C \subset S$ passing through very general points of S .*

Then the restriction map

$$H^0(V, mK_V) \rightarrow H^0(F, mK_F)$$

is surjective for $m \geq \lceil \frac{2}{\alpha} + \frac{1}{\beta} + \epsilon \rceil + 1$.

Proof. It is a direct application of Theorem 4.3. We just need to apply [35, Example-Theorem 6.3], which states that, for a very general point $s \in S$, there exists an effective \mathbb{Q} -divisor $D_s \sim_{\mathbb{Q}} (\frac{2}{\alpha} + \frac{1}{\beta})(K_S + \Delta)$ such that (S, D_s) is log canonical at s and s is a lc center of (S, D_s) . \square

5. THE PROOF OF THEOREM 1.6

The proof of Theorem 1.6 follows the same strategy as that of Theorem 1.3. A very difficult case appears, however, when the 4-fold V admits a fibration $f : V \rightarrow S$ onto a surface, whose general fibers are surfaces with small birational invariants. The usual inductive argument is not sufficient to show the birationality of $\Phi_{|5K_V|}$. We need to

apply Theorem 4.1, as well as a great deal of extra arguments, to deal with this difficult case.

First we remark that, by Theorem 2.1, $\text{vol}(V) \gg 0$ since $h^0(\Omega_V^2) \gg 0$. Let $x, y \in V$ be two very general points of V .

We take an effective \mathbb{Q} -divisor $D_1 \sim_{\mathbb{Q}} t_1 K_V$ such that $t_1 < 4\sqrt[4]{\frac{2}{\text{vol}(V)}} + \epsilon$, $x, y \in \text{Nklt}(V, D_1)$, and after switching x and y , we may assume that (V, D_1) is log canonical at x . Let V_1 be the minimal log canonical center of (V, D_1) through x .

5.1. The case with $\dim V_1 = 3$.

We have two ways, according to the value of $\text{vol}(V_1)$, to cut down the log canonical center through x .

If $\text{vol}(V_1) \gg 0$, by Takayama's method, there exists an effective \mathbb{Q} -divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that

$$t_2 < t_1 + 3(1 + t_1)\sqrt[3]{\frac{2}{\text{vol}(V_1)}} + \epsilon,$$

$x, y \in \text{Nklt}(V, D_2)$, and the minimal log canonical center of (V, D_2) through x is a closed subvariety $V_2 \subsetneq V_1$. Note that $0 < t_1 < t_2 \ll 1$ in this case. The problem is reduced to Subcase 5.2 and Subcase 5.3.

If $\text{vol}(V_1)$ is upper bounded, we still apply Todorov's inductions and spread the centers V_1 into a family in the similar way to that of Subsection 3.3. We have the following diagram:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\pi} & V \\ \downarrow f & & \\ C, & & \end{array}$$

where $\pi : \tilde{V} \rightarrow V$ is a generically finite and surjective morphism onto V , \tilde{V} is nonsingular projective, f is a fibration onto a smooth curve C such that a general fiber F is birational onto V_1 via π . There exists an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} t_1 K_V$ such that $D = V_1 + D'$, where D' is an effective \mathbb{Q} -divisor and $V_1 \notin \text{Supp}(D')$.

Assume that $\deg \pi \geq 2$. We conclude as in Subsection 3.3.1 that there exists an effective \mathbb{Q} -divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $t_2 \leq 3t_1$, $x, y \in \text{Nklt}(V, D_2)$, and that the minimal log canonical center of (V, D_2) through x is a closed subvariety $V_2 \subsetneq V_1$. Note that $0 < t_2 < 3t_1 \ll 1$ under the condition of Theorem 1.6. The problem is also reduced to Subcase 5.2 and Subcase 5.3.

Assume that π is birational. We simply identify \tilde{V} with V to save symbols. Note that

$$h^2(V, \omega_V) = h^0(V, \Omega_V^2) \gg 0$$

and

$$h^2(V, \omega_V) = h^1(C, R^1 f_* \omega_V) + h^0(C, R^2 f_* \omega_V).$$

Both $R^1 f_* \omega_V$ and $R^2 f_* \omega_V$ are torsion-free sheaves on C by Kollár's theorem ([33]). Moreover, $R^1 f_* \omega_{V/C}$ is weakly positive (see, for instance, [48] or [43, Theorem 1.4]) and, hence, nef on C .

Since $\text{vol}(F)$ is bounded, $\text{rank}(R^1 f_* \omega_V) = h^{2,0}(F)$, hence by Serre duality, $h^1(C, R^1 f_* \omega_V) = h^0(C, (R^1 f_* \omega_{V/C})^*) \leq h^{2,0}(F)$. Hence $h^0(C, R^2 f_* \omega_V)$ is sufficiently large. In particular, we have $R^2 f_* \omega_V \neq 0$. Thus F is an irregular 3-fold of general type. By the main theorem of [8], $|5K_F|$ induces a birational map of F .

Since $\text{vol}(V) \gg 0$ and fibers of f are birationally bounded, by [18, Theorem 3.4], the restriction map

$$H^0(V, 5K_V) \rightarrow H^0(F_1, 5K_{F_1}) \oplus H^0(F_2, 5K_{F_2})$$

is surjective for any two different general fibers F_1 and F_2 of f . Thus $|5K_V|$ induces a birational map of V .

5.2. The case with $\dim V_2 = 2$ and $0 < t_2 \ll 1$.

After discussions in the previous subsection, we may assume that there exists an effective \mathbb{Q} -divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $0 < t_2 \ll 1$, $x, y \in \text{Nklt}(V, D_2)$, and that the minimal log canonical center V_2 of (V, D_2) through x is of dimension 2. Modulo a small perturbation, we may and do assume that $\text{Nklt}(V, D_2) = V_2$ locally around x .

Let \tilde{V}_2 be a smooth model of V_2 . By Takayama's induction, there exists an effective \mathbb{Q} -divisor $D_3 \sim_{\mathbb{Q}} t_3 K_V$ such that

$$0 < t_3 < t_2 + 2(1 + t_2) \sqrt{\frac{2}{\text{vol}(\tilde{V}_2)}} + \epsilon,$$

$x, y \in \text{Nklt}(V, D_3)$, and that the minimal log canonical center V_3 of (V, D_3) through x is a proper subset of V_2 .

If $\text{vol}(\tilde{V}_2) \geq 4$, $t_3 < \sqrt{2} + (1 + \sqrt{2})t_2 + \epsilon$. The problem is reduced to Subcase 5.3, noting that $0 < t_3 - \sqrt{2} \ll 1$.

If $\text{vol}(\tilde{V}_2) \leq 3$, the bound for t_3 is not good enough for our purpose. We still apply Todorov's method to discuss the following exclusive situations:

- 5.2.1. There exists an effective \mathbb{Q} -divisor $D_3 \sim_{\mathbb{Q}} t_3 K_V$ such that $0 < t_3 \leq 3t_2$, $x, y \in \text{Nklt}(V, D_3)$, and that the minimal log canonical center V_3 of (V, D_3) through x is a proper subset of V_2 , for which the problem is reduced to Subcase 5.3 with $0 < t_3 \ll 1$;
- 5.2.2. (\star) Modulo birational modifications, there exists a fibration $f : V \rightarrow S$ onto a smooth projective surface S such that a general fiber F of f is a surface whose canonical volume is ≤ 3 , that there exists an effective \mathbb{Q} -divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $x, y \in \text{Nklt}(V, D_2)$, (V, D_2) is log canonical at x (after possibly switching x and y), and that the minimal log canonical center at x is the fiber F_x of f through x , F_x is an irreducible component of $\text{Nklt}(V, D_2)$. The statement follows from Theorem 6.1.

5.3. The case with $\dim V_3 \leq 1$, $0 < t_2 \ll 1$ and $0 < t_3 - \sqrt{2} \ll 1$.

So far, except Situation (\star) (i.e., 5.2.2.), we always have the following:

we already have a rational number $0 < t_2 \ll 1$; there exists an effective \mathbb{Q} -divisor $D_3 \sim_{\mathbb{Q}} t_3 K_V$ such that

$$0 < t_3 \leq \max\{3t_2, \sqrt{2} + (1 + \sqrt{2})t_2 + \epsilon\}$$

($0 < \epsilon \ll 1$), $x, y \in \text{Nklt}(V, D_3)$, and that the minimal log canonical center V_3 of (V, D_3) through x is of dimension ≤ 1 .

By Takayama's induction, there exists an effective \mathbb{Q} -divisor $D_4 \sim_{\mathbb{Q}} t_4 K_V$ such that

$$0 < t_4 \leq 2t_3 + 1 + \epsilon,$$

$x, y \in \text{Nklt}(V, D_4)$, and that $\{x\}$ is an isolated component of $\text{Nklt}(V, D_4)$. Since $0 < t_2 \ll 1$ and $2\sqrt{2} + 1 < 4$, we conclude by Nadel vanishing, as in Subsection 3.1, that $|5K_V|$ separates x and y . Hence we have finished the proof of Theorem 1.6.

6. PROOF FOR SUBCASE 5.2.2.

We deal with Subcase (\star) in this section.

Theorem 6.1. *Under Assumption (\star) (i.e., Subcase 5.2.2), $|5K_V|$ induces a birational map of V .*

The proof of Theorem 6.1 is elaborate. Recall that we have a fibration $f : V \rightarrow S$ onto a smooth projective surface S . We start with the following simple reduction.

Lemma 6.2. *Under Assumption (\star) (i.e., Subcase 5.2.2), if x and y are in different fibers of f , $|5K_V|$ separates x and y .*

Proof. By assumption, each component of $\text{Nklt}(V, D_2)$ containing y does not contain x . It is then easier to cut down the log canonical centers.

Note that

$$\dim \text{Im}(H^0(V, mK_V) \rightarrow H^0(F_x, mK_{F_x})) \sim \frac{\text{vol}_{V|F_x}(K_X)}{2} m^2 + O(m)$$

for sufficiently large integers m . Hence we can choose an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} K_V$ such that F_x is not contained in the support of D and that, by Takayama [45, Theorem 4.5],

$$\text{mult}_x(D|_{F_x}) > \sqrt{\text{vol}_{V|F_x}(K_X)} - \epsilon \geq \frac{\sqrt{\text{vol}(K_F)}}{1 + t_2} - \epsilon.$$

Let t'_3 be the maximal rational number such that $(V, D_2 + t'_3 D)$ is log canonical at x . It is known that $t'_3 \leq \frac{2}{\text{mult}_x(D|_{F_x})} < \frac{2(1+t_2)}{\sqrt{\text{vol}(K_F)}} + \epsilon$.

Let $D_3 = D_2 + t'_3 D \sim_{\mathbb{Q}} t_3 K_V$ and $t_3 = t_2 + t'_3$. Then the minimal log canonical center V_3 of (V, D_3) is a proper subset of F_x and there

exists an irreducible component of $\text{Nklt}(V, D_3)$ containing y and not containing x .

Note that $t_3 < 2 + 3t_2 + \epsilon$. If $V_3 = \{x\}$, since there exists an irreducible component of $\text{Nklt}(V, D_3)$ containing y and not containing x , we can take a small perturbation of D_3 such that x is an isolated component of $\text{Nklt}(X, D_3)$ and $y \in \text{Nklt}(X, D_3)$. Note $t_3 < 4$. We can conclude the statement by Nadel vanishing.

If V_3 is a curve, we denote by \overline{V}_3 its normalization. Similarly, there exists an effective \mathbb{Q} -divisor $D_4 \sim_{\mathbb{Q}} t_4 K_V$ such that

$$t_4 < t_3 + \frac{1 + t_3}{\text{vol}(\overline{V}_3)} + \epsilon \leq \frac{3}{2}t_3 + \frac{1}{2} + \epsilon < \frac{7}{2} + \frac{9}{2}t_2 + \epsilon,$$

$\{x\}$ is a log canonical center of (V, D_4) , (V, D_4) is not klt at y , and there exists an irreducible component of $\text{Nklt}(V, D_4)$ containing y , while not containing x . Since $t_2 \ll 1$ and then $t_4 < 4$, we conclude the statement as before. \square

We then focus on proving that $|5K_V|_F$ induces a birational map for the general fiber F of f .

6.1. The case with $\kappa(S) = 2$ or $\kappa(S) \leq 1$ and $p_g(S) \gg 0$.

Lemma 6.3. *Under Assumption (\star) (i.e., Subcase 5.2.2), if S is of general type, $|5K_V|$ induces a birational map of V .*

Proof. By Lemma 6.2, it suffices to show that $|5K_V|_F = |5K_F|$, since $|5K_F|$ induces a birational map of F . By simply blowing down S , we may assume that S is minimal. Set $\Delta = 0$ and we will apply Corollary 4.4. Note that $\text{vol}(S) \geq 1$ and $(K_S \cdot C) \geq 1$ for each irreducible curve C through a general point of S . Moreover, $f_*\mathcal{O}_V(NK_{V/S})$ is weakly positive by Viehweg's theorem (see [48]) and hence condition (1) in Corollary 4.4 is also satisfied. \square

Now we assume that $\kappa(S) \leq 1$. By Hodge symmetry, Serre duality and the assumption that $h^0(V, \Omega_V^2) \gg 0$, we see that $h^2(V, \omega_V) \gg 0$. Thus we have

$$h^0(S, R^2 f_* \omega_V) + h^1(S, R^1 f_* \omega_V) + h^2(S, f_* \omega_V) = h^2(V, \omega_V) \gg 0.$$

We observe that $h^2(S, f_* \omega_V) \leq 3$. Indeed, since $\text{vol}(F) \leq 3$, we have $p_g(F) \leq 3$ by the Noether inequality. We take a general member H in a very ample linear system of S . Let $f_H : V_H \rightarrow H$ be the induced morphism, where $V_H := f^{-1}(H)$. By Kollár [33, Theorem 2.1], we have the short exact sequence:

$$0 \rightarrow f_* \omega_V \rightarrow f_* \omega_V(H) \rightarrow f_{H*} \omega_{V_H} \rightarrow 0$$

and $H^2(S, f_* \omega_V(H)) = 0$. Thus we have the surjective map

$$H^1(H, f_{H*} \omega_{V_H}) \rightarrow H^2(V, f_* \omega_V).$$

We have already seen that $h^1(H, f_{H*}\omega_{V_H}) \leq p_g(F) \leq 3$ using the semi-positivity of $f_{H*}(\omega_{V_H/H})$.

By Kollár [34], we know that $R^2f_*\omega_V = \omega_S$ and hence

$$h^0(S, R^2f_*\omega_V) = h^0(S, \omega_S) = p_g(S).$$

Lemma 6.4. *Under Assumption (\star) (i.e., Subcase 5.2.2), if $p_g(S) \gg 0$, then $|5K_V|$ induces a birational map of V*

Proof. Since $\kappa(S) \leq 1$, we have $\kappa(S) = 1$. We may assume that S is minimal and denote by $I_S : S \rightarrow C$ the Iitaka fibration of S . By the canonical bundle formula, $K_S = I_S^*(H + B)$, where $H = I_{S*}\omega_S$ and B is an effective \mathbb{Q} -divisor on C depending on the singular fibers of I_S . We also have $p_g(S) = h^0(S, K_S) = h^0(C, H)$. Hence $\deg H \gg 0$.

Let $g : V \xrightarrow{f} S \xrightarrow{I_S} C$ be the composition of fibrations and let Y be a general fiber of g . Since a general fiber of I_S is an elliptic curve, Y is an irregular threefold of general type.

By Viehweg's weak positivity (see, for instance, [48]), we know that $K_{V/S} + \epsilon K_V$ is big. Thus $(1 + \epsilon)K_V \geq_{\mathbb{Q}} f^*K_S \geq_{\mathbb{Q}} g^*H$. By Theorem 4.1, one of the restriction maps:

$$H^0(V, 5K_V) \rightarrow H^0(Y, 5K_Y),$$

$$H^0(V, 5K_V) \rightarrow H^0(F, 5K_F)$$

is surjective. In the latter case, we are done since $|5K_F|$ induces a birational map of F . In the former case, we conclude by the main theorem of [8]. \square

6.2. The case with $\kappa(S) \leq 1$ and $p_g(S)$ being upper bounded.

From now on within this section, we assume that $\kappa(S) \leq 1$ and $p_g(S)$ is upper bounded. As discussed above, the condition of Theorem 1.6 forces $h^1(S, R^1f_*\omega_V) \gg 0$. Note that $R^1f_*\omega_X \neq 0$ implies that F is an irregular surface. Moreover, since $\text{vol}(F) \leq 3$, we have $p_g(F) = q(F) = 1$ and $2 \leq \text{vol}(F) \leq 3$ by Debarre's inequality that $\text{vol}(F) \geq 2p_g(F)$ (see [19]). Let $g : V \rightarrow X$ be the relative Albanese morphism of f . Note that g exists by [5, Théorème 1] or [23, Theorem 2]. We have the commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ & \searrow f & \downarrow h \\ & & S. \end{array}$$

By the definition of the relative Albanese morphism, for $s \in S$ general, let $g_s : V_s \rightarrow X_s$ be the fibers of $g : V \rightarrow X$ over s , then g_s is the Albanese morphism of V_s . Since $q(V_s) = 1$ and the Albanese morphism of V_s is a fibration. Thus a general fiber of h is a genus 1 curve and a general fiber of g is connected. Modulo birational modifications, we may also assume that the 3-fold X is nonsingular and projective.

By Kollár [34, Theorem 3.4], $R^1 f_* \omega_V \cong h_* R^1 g_* \omega_V \oplus R^1 h_* g_* \omega_V$. Moreover, since $h^1(X_s, g_{s*} \omega_{V_s}) = 0$ by the fact that $q(V_s) = 1$, we conclude that $R^1 h_* g_* \omega_V = 0$. Thus

$$h^1(S, h_* \omega_X) = h^1(S, h_* R^1 g_* \omega_V) = h^1(S, R^1 f_* \omega_V) \gg 0.$$

Therefore X is a 3-fold with many global two forms, i.e. $h^0(X, \Omega_X^2) \gg 0$. By the easy addition (see [41, Corollary 2.3]), we also have $\kappa(X) \leq \dim S = 2$.

Next we follow Campana and Peternell's results in [6] to trace 2-forms on non-general type 3-folds to study the structure of $h : X \rightarrow S$ and to finish the proof of Theorem 1.6. We organize the argument according to the value of $\kappa(X)$.

6.2.1. *The case with $\kappa(X) = -\infty$.* The global holomorphic 2-forms on X are induced from the maximal rationally connected quotient of X . More precisely, by Campana-Peternell [6, Theorem 3.1], there exists a morphism $h' : X \rightarrow S'$ onto a minimal smooth projective surface S' with $\kappa(S') \geq 0$ such that $H^0(X, \Omega_X^2) \cong H^0(S', K_{S'})$ has sufficiently large dimension. In particular, h and h' are not birational to each other. Since a general fiber E of h is an elliptic curve, the family of $h'(E)$ covers S' . Thus $\kappa(S') = 1$ and $E' \rightarrow h'(E')$ is an isogeny, since otherwise S' couldn't have non-negative Kodaira dimension. Denote by $I' : S' \rightarrow C$ the Iitaka fibration of S' . Note that I' is simply the morphism contracting the family of $h'(E)$. Thus we also have an induced morphism

$$\begin{aligned} I : S &\rightarrow C \\ s &\rightarrow I' \circ h'(h^{-1}(s)), \end{aligned}$$

so that we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ f \downarrow & \searrow h & \searrow h' \\ S & & S' \\ & \searrow I & \swarrow I' \\ & & C \end{array}$$

Since $p_g(S') \gg 0$, there exists a line bundle H on C such that $\deg H \gg 0$ and $K_{S'} \geq_{\mathbb{Q}} I'^* H$. We then again have

$$(1 + \epsilon)K_V \geq_{\mathbb{Q}} (h' \circ g)^* K_{S'} \geq (I' \circ h' \circ g)^* H.$$

Let Y be a general fiber of $I \circ f$. Then, by Theorem 4.1, one of the restriction maps:

$$\begin{aligned} H^0(V, 5K_V) &\rightarrow H^0(Y, 5K_Y), \\ H^0(V, 5K_V) &\rightarrow H^0(F, 5K_F) \end{aligned}$$

is surjective. Note that Y is again an irregular threefold of general type, because I' is an elliptic fibration. We then conclude as in the proof of Lemma 6.4.

6.2.2. *The case with $\kappa(X) = 0$.* By Campana-Peternell [6, Theorem 3.2], we see $h^0(X, \Omega_X^2) \leq 3$, contradicting to our assumption $h^0(X, \Omega_X^2) \gg 0$.

6.2.3. *The case with $\kappa(X) = 1$.* Let $I_X : X \rightarrow C$ be the Iitaka fibration of X .

A general fiber of I_X is birational either to an abelian surface, or to a hyperelliptic surface, or to a K3 surface, or to a Kummer surface. For a general point $p \in C$, let X_p be the corresponding fiber of I_X . We have exact sequences

$$0 \rightarrow \Omega_X^2(-X_p) \rightarrow \Omega_X^2 \rightarrow \Omega_X^2|_{X_p} \rightarrow 0$$

and

$$0 \rightarrow \Omega_{X_p} \rightarrow \Omega_X^2|_{X_p} \rightarrow \Omega_{X_p}^2 \rightarrow 0.$$

In all cases, we have

$$h^0(\Omega_X^2|_{X_p}) \leq h^0(\Omega_{X_p}) + h^0(\Omega_{X_p}^2) \leq 3.$$

Let $N = \lfloor \frac{h^0(\Omega_X^2)-1}{3} \rfloor$. For p_1, \dots, p_N be general points of C and let X_{p_i} be the corresponding fiber over p_i . Considering

$$0 \rightarrow \Omega_X^2(-\sum_{i=1}^N X_{p_i}) \rightarrow \Omega_X^2 \rightarrow \bigoplus_{i=1}^N \Omega_X^2|_{X_{p_i}} \rightarrow 0,$$

we have $h^0(\Omega_X^2(-\sum_{i=1}^N X_{p_i})) > 0$.

Let $\mathcal{L} = \mathcal{O}_X(L)$ be the saturation of $I_X^* \Omega_C \subset \Omega_X$ and denote by \mathcal{F} the torsion-free quotient. Note that $\mathcal{L} = I_X^* \omega_C \otimes \mathcal{O}_X(E)$, where E is an effective divisor supported on the singular fibers of I_X . We have the exact sequence:

$$0 \rightarrow \mathcal{L} \rightarrow \Omega_X \rightarrow \mathcal{F} \rightarrow 0,$$

$$0 \rightarrow K_X \otimes \mathcal{F}^* \rightarrow \Omega_X^2 \rightarrow \det \mathcal{F} \otimes \mathcal{I}_Z \rightarrow 0,$$

where $Z \subset X$ is a subscheme of codimension ≥ 2 .

If $h^0(X, \det \mathcal{F}(-\sum_{i=1}^N X_{p_i})) > 0$, we have

$$K_X \sim L + \det \mathcal{F} \geq I_X^*(K_C) + \sum_{i=1}^N X_{p_i}$$

where $K_C + \sum_{i=1}^N X_{p_i}$ is an ample divisor on C of degree $\geq N - 2$.

If $h^0(X, K_X \otimes \mathcal{F}^*(-\sum_{i=1}^N X_{p_i})) > 0$, then we have a non-trivial map

$$\mathcal{F} \rightarrow K_X \otimes \mathcal{O}_X(-\sum_{i=1}^N X_{p_i}).$$

Since any torsion-free quotient of \mathcal{F} is pseudo-effective by the main result of [40], we have

$$K_X = \sum_{i=1}^N X_{p_i} + P$$

where P is an pseudo-effective divisor.

In both cases, $(1 + \epsilon)K_V = \epsilon K_V + K_{V/X} + K_X \geq_{\mathbb{Q}} (I_X \circ g)^* H_C$ where H_C is an ample divisor of degree $\geq N - 2$ on the curve C .

We now consider the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ f \downarrow & \searrow h & \searrow I_X \\ S & & C. \end{array}$$

If the fibration $I_X \circ g : X \rightarrow C$ does not factor through f , a general fiber F of f dominates C . Let $g_F := (I_X \circ g)|_F : F \rightarrow C$ be the induced morphism. We then have

$$(1 + \epsilon)K_F = (1 + \epsilon)K_V|_F \succeq g_F^* H_C.$$

It is then clear that $(1 + \epsilon)\text{vol}(F) \geq \deg H_C \geq N - 2$, which is a contradiction since $\text{vol}(F) \leq 3$ and $N \gg 0$.

Thus $I_X \circ g$ factors as $I_X \circ g : V \xrightarrow{g} X \xrightarrow{h} S \xrightarrow{f_S} C$. Let Y be a general fiber of $I_X \circ g$ and let Z be the corresponding fiber of I_X . Note that

$$0 \ll h^1(S, h_* \omega_X) = h^1(C, I_{X*} \omega_X) + h^0(C, R^1 f_{S*} h_* \omega_X).$$

Since $I_{X*} \omega_{X/C}$ is a nef line bundle on C , $h^1(C, I_{X*} \omega_X) \leq 1$. Hence we have $h^0(C, R^1 f_{S*} h_* \omega_X) \gg 0$ and thus

$$R^1 I_{X*} \omega_X = R^1 f_{S*} h_* \omega_X \oplus f_{S*} R^1 h_* \omega_X$$

is non-zero. Therefore $q(Z) > 0$ and Y is an irregular threefold of general type.

Since $N \gg 0$, by Theorem 4.1, either

$$H^0(V, 5K_V) \rightarrow H^0(F, 5K_F)$$

is surjective or

$$H^0(V, 5K_V) \rightarrow H^0(Y, 5K_Y)$$

is surjective. We finish the proof as before.

6.2.4. *The case with $\kappa(X) = 2$.* In this case, $h : X \rightarrow S$ is exactly the Iitaka fibration of X . Since $h^1(S, h_* \omega_X) \gg 0$, X has sufficiently many non- h -vertical holomorphic 2 forms in the terminology of [6, Definition 1.9]. Indeed, the Hodge pairing between the conjugate of h -vertical holomorphic 2-forms and forms of $H^1(S, f_* \omega_X) (\subset H^1(X, K_X))$ is zero.

Note that $h : X \rightarrow S$ is a genus 1 curve fibration. There are two cases to discuss depending on whether or not the j -invariant of h is constant.

If the j -invariant is constant, let E be a general fiber of h . After a Galois base change $\hat{S} \rightarrow S$ and birational modifications, we get a trivial elliptic fibration $\hat{h} : \hat{X} \rightarrow \hat{S}$, i.e. \hat{h} is birational to the projection $\hat{S} \times E \rightarrow \hat{S}$. Let $\pi : \hat{X} \rightarrow X$ be the generically finite cover, which is a Galois cover over an open dense subset of X with Galois group G . Note that, since \hat{S} is of general type, G acts on \hat{S} and E diagonally. We may and do assume that G acts on \hat{S} and E faithfully, hence $G \hookrightarrow \text{Aut}(E)$. Then X is birationally equivalent to the diagonal quotient $(\hat{S} \times E)/G$. For simplicity, we assume that \hat{S} is minimal and identify X with the diagonal quotient $(\hat{S} \times E)/G$. Then X is a normal threefold with quotient singularities and the quotient morphism π is quasi-étale. It is known that the neutral component $\text{Aut}^0(E)$ of the automorphism group $\text{Aut}(E)$ of the elliptic curve E is isomorphic to E and $\text{Aut}(E)/\text{Aut}^0(E)$ is either \mathbb{Z}_2 , or \mathbb{Z}_3 , or \mathbb{Z}_6 . Set $G_0 = G \cap \text{Aut}^0(E)$ and $\overline{G} := G/G_0$. Note that G_0 acts trivially on $H^0(E, \Omega_E)$ and $|\overline{G}| \leq 6$. Let $\overline{X} := (\hat{S} \times E)/G_0$ and $\overline{S} := \hat{S}/G_0$. Note that \overline{X} is smooth and \overline{S} is normal with quotient singularities. We have the commutative diagram:

$$\begin{array}{ccccc}
 & & & & V \\
 & & & & \downarrow g \\
 \hat{S} \times E & \longrightarrow & \overline{X} & \xrightarrow{\overline{\pi}_X} & X \\
 \downarrow \hat{h} & & \downarrow \overline{h} & & \downarrow h \\
 \hat{S} & \longrightarrow & \overline{S} & \xrightarrow{\overline{\pi}_S} & S
 \end{array}
 \quad \begin{array}{l} \\ \\ \\ \\ \curvearrowright f \end{array}
 \tag{6.1}$$

Since X has many non- h -vertical global 2-forms, \overline{X} has many non- \overline{h} -vertical global 2-forms. Moreover, since the space of non- \hat{h} -vertical global 2-forms on $\hat{S} \times E$ is $\hat{h}^*H^0(\hat{S}, \Omega_{\hat{S}}) \otimes p_E^*H^0(E, \Omega_E)$, where $p_E : \hat{S} \times E \rightarrow E$ is the second projection and G_0 acts trivially on $H^0(E, \Omega_E)$, we conclude that the space of non- \overline{h} -vertical global 2-forms on \overline{X} is

$$(\hat{h}^*H^0(\hat{S}, \Omega_{\hat{S}}) \otimes p_E^*H^0(E, \Omega_E))^{G_0} = \hat{h}^*H^0(\overline{S}, \tilde{\Omega}_{\overline{S}}) \otimes p_{\overline{E}}^*H^0(\overline{E}, \Omega_{\overline{E}}),$$

where $\overline{E} = E/G_0$ is also an elliptic curve, $\overline{h} : \overline{S} \rightarrow \overline{E}$ is the natural morphism, and $\tilde{\Omega}_{\overline{S}} = i_*(\Omega_U)$, where $i : U \subset \overline{S}$ is the smooth locus of \overline{S} . In particular, we see that $q(\overline{S}) = h^0(\overline{S}, \tilde{\Omega}_{\overline{S}}) \gg 0$.

If the image of the Albanese morphism of \overline{S} is a curve \overline{D} . Then \overline{D} is a smooth projective curve of genus equal to $q(\overline{S}) \gg 0$. Let $D = \overline{D}/\overline{G}$

and $\bar{\pi}_D : \bar{D} \rightarrow D$ be the quotient morphism. We have

$$\begin{array}{ccc}
 & & V \\
 & & \downarrow \\
 \bar{X} & \xrightarrow{\bar{\pi}_X} & X \\
 \downarrow & & \downarrow h \\
 \bar{S} & \xrightarrow{\bar{\pi}_S} & S \\
 \downarrow & & \downarrow h_D \\
 \bar{D} & \xrightarrow{\bar{\pi}_D} & D
 \end{array}
 \quad \begin{array}{l}
 \curvearrowright \\
 f_D
 \end{array}
 \tag{6.2}$$

where h_D and f_D are the naturally composited morphisms. By the ramification formula, we write $K_{\bar{D}} = \bar{\pi}_D^*(K_D + \sum_i(1 - \frac{1}{n_i})p_i)$, where p_i are the branched loci of \bar{D} and n_i is the ramification index over p_i . Note that $\deg_D(K_D + \sum_i(1 - \frac{1}{n_i})p_i) = \frac{1}{|\bar{G}|}(2g(\bar{D}) - 2) \gg 0$. Let $\Delta := \sum_i(1 - \frac{1}{n_i})p_i$. Recall that the ramification divisor $R(h_D) := \sum_p(h_D^*p - (h_D^*p)_{\text{red}})$, where p goes through each point of D (see [20, Notation 2.7]). Since the horizontal morphisms in (6.2) are \bar{G} -quotients, we see that $h_D^*\Delta \leq R(h_D)$. Hence $f_D^*\Delta \leq R(f_D)$. By [20, Corollary 4.5], $K_{V/D} - f_D^*\Delta$ is pseudo-effective. Thus, $(1 + \epsilon)K_V \geq_{\mathbb{Q}} f_D^*(K_D + \Delta)$. Let Y be a general fiber of f_D . Since V (or X) has many non- f_D -vertical (non- h -vertical) global holomorphic 2-forms, Y is an irregular 3-fold of general type. We now apply Theorem 4.1 to conclude that either $H^0(V, 5K_V) \rightarrow H^0(Y, 5K_Y)$ or $H^0(V, 5K_V) \rightarrow H^0(F, 5K_F)$ is surjective.

If the Albanese image of \bar{S} is a surface but $K_{\bar{S}}$ is not big, we have a similar picture as before. Indeed, in this case $\kappa(\bar{S}) = 1$ and let $\bar{S} \rightarrow \bar{D}$ be the Iitaka fibration of \bar{S} . Then \bar{G} also acts naturally on \bar{D} and $g(\bar{D}) = q(\bar{S}) - 1 \gg 0$. Then we prove the statement exactly similar to the last paragraph.

We then assume that the Albanese image of \bar{S} is a surface and $K_{\bar{S}}$ is big. In this case, any smooth model of \bar{S} is also of general type. Thus, modulo birational modifications, we may assume that \bar{X} and \bar{S} are smooth and \bar{S} is a minimal surface. We write $K_{\bar{S}} = \bar{\pi}_S^*(K_S + \sum_{D_i}(1 - \frac{1}{n_i})D_i)$, where D_i are branched divisors and n_i is the corresponding ramification index. Let $\Delta_S := \sum_{D_i}(1 - \frac{1}{n_i})D_i$. In particular, $K_S + \Delta_S$ is big and nef. Moreover, $\text{vol}(\bar{S}) = (K_{\bar{S}}^2) \geq 2q(\bar{S}) - 4 \gg 0$ by Debarre's inequality and thus $((K_S + \Delta_S)^2) = \frac{1}{|\bar{G}|}\text{vol}(\bar{S}) \gg 0$. On the other hand, let C be an irreducible curve passing through a very general point of S ,

$$((K_S + \Delta_S) \cdot C) = \frac{1}{|\bar{G}|}(K_{\bar{S}} \cdot \bar{\pi}_S^*C) \geq \frac{1}{3}.$$

We then deduce the surjectivity of the restriction map $H^0(V, 5K_V) \rightarrow H^0(F, 5K_F)$ by applying Corollary 4.4. It suffices to verify the assumption (1) thereof. We go back to the commutative diagram (6.1). Let \bar{V} be a resolution of $\bar{S} \times_S V$. We then have

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\bar{\pi}_V} & V \\ \downarrow & & \downarrow \\ \bar{X} & \xrightarrow{\bar{\pi}_X} & X \\ \downarrow & & \downarrow \\ \bar{S} & \xrightarrow{\bar{\pi}_S} & S \end{array} \quad \begin{array}{l} \curvearrowleft \\ \\ \curvearrowright \end{array} \quad \begin{array}{l} \\ \\ f \end{array}$$

By the weak positivity of $\bar{f}_* \omega_{\bar{V}/\bar{S}}^{\otimes N}$, for any effective big \mathbb{Q} -divisor H on S and any sufficiently large and divisible integer N ,

$$H^0(\bar{V}, N(K_{\bar{V}/\bar{S}} + \bar{f}^* \bar{\pi}_S^* H)) \rightarrow H^0(F, NK_F)$$

is surjective. Note that the restriction map is \bar{G} -equivariant and the space $H^0(F, NK_F)$ is \bar{G} -invariant. Thus

$$\begin{aligned} & H^0(\bar{V}, N(K_{\bar{V}/\bar{S}} + \bar{f}^* \bar{\pi}_S^* H))^{\bar{G}} \\ &= H^0(V, N(K_V - f^*(K_S + \Delta_S) + f^* H)) \rightarrow H^0(F, NK_F) \end{aligned}$$

is also surjective.

So far, we have finished the discussion when the j -invariant of h is constant.

Finally, if the j -invariant of h is non-constant, by [6, Corollary 5.4], there exists a fibration $f_S : S \rightarrow C$ onto a smooth projective curve C and another smooth projective surface S' , and an elliptic fibration $f_{S'} : S' \rightarrow C$ such that $S \times_C S'$ is smooth and X is birational to $S \times_C S'$. Replacing $S \times_C S'$ with X and let $h' : X \rightarrow S'$ be the projection. We have the commutative diagram:

$$\begin{array}{ccccc} & & V & & \\ & & \downarrow g & & \\ & & X & & \\ & f & \swarrow & \searrow h' & \\ & & S & & S' \\ & & \swarrow f_S & \searrow f_{S'} & \\ & & C & & \end{array}$$

By [6, (5.5)], a non- h -vertical holomorphic global 2 form on X belong to the space

$$h^* H^0(S, \Omega_S) \wedge h'^* H^0(S', \Omega_{S'}) \oplus h'^* H^0(S', K_{S'}). \quad (6.3)$$

Note that $f_{S'}$ is an elliptic fibration with non-constant j -invariant. Thus, $H^0(S', \Omega_{S'}) = f_{S'}^* H^0(C, \Omega_C)$. Hence non-h-vertical 2-forms of X can only come from the second summand of (6.3) and so $h^0(S', K_{S'}) \gg 0$.

We still denote by Y a general fiber of $f_S \circ f : V \rightarrow C$. Note that Y is an irregular 3-fold of general type, since $f_{S'}$ is an elliptic fibration.

Since $h^0(S', K_{S'}) \gg 0$, S' has Kodaira dimension 1 and $f_{S'}$ is the Iitaka fibration of S' . In particular, there exists an ample line bundle H on C with $h^0(C, H) \gg 0$ such that $K_{S'} \geq_{\mathbb{Q}} f_{S'}^* H$. Then $(1+\epsilon)K_V \geq_{\mathbb{Q}} (f_{S'} \circ h' \circ g)^* H$ and we conclude the statement of the theorem as before.

We have proved Theorem 6.1.

7. PROOF OF THEOREM 7.2 AND OPEN QUESTIONS

It is well-known, since the work of Kollár in [33], that the study of pluricanonical systems of varieties of general type with two linearly independent global top forms can be reduced to the study of pluricanonical systems of varieties of lower dimensions (see, for instance, Kollár [33, Corollary 4.8] and [13, 11]).

7.1. Varieties with global 1-forms.

The pluricanonical systems of varieties with many holomorphic 1-forms have also been studied by many authors (see, for instance, [12, 28, 8]). We are inclined to ask the following question:

Question 7.1. Let X be an irregular variety of general type of dimension $n \geq 4$. Does $|mK_X|$ induce a birational map for each $m \geq r_{n-1}$?

When $n = 2$, the statement is due to Bombieri [3]; when $n = 3$, the affirmative answer to Question 7.1 was recently given in Chen-Chen-Chen-Jiang [8].

We have here a partial answer to this question in any dimension as follows:

Theorem 7.2. *Let X be a smooth projective variety of general type of dimension $n \geq 4$. Assume that either $q(X) > n$ or the Albanese image of X is a proper subvariety of the Albanese variety. Then $|mK_X|$ induces a birational map for all $m \geq r_{n-1}$.*

Proof. Let $a_X : X \rightarrow A_X$ be the Albanese morphism of X . By assumption, $a_X(X) \subsetneq A_X$ generates A_X . By Ueno's theorem (see for instance [41, Theorem 3.7]), there exists a fibration $q_B : A_X \rightarrow B$ between abelian varieties such that any smooth model of $q_B \circ a_X(X)$ is of general type. Let $X \xrightarrow{h} Z \xrightarrow{t} q_B \circ a_X(X)$ be the Stein factorization of $q_B \circ a_X$. After birational modifications, we may assume that Z is a

smooth projective variety. We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \downarrow h & & \downarrow q_B \\ Z & \xrightarrow{t} & B. \end{array}$$

Note that Z is of maximal Albanese dimension. We denote by $n_1 = \dim Z$. We know that $|mK_Z|$ induces a birational map of Z for each $m \geq 3$ (see [12, 28]). Let F be a general fiber of h . Then, $0 \leq \dim F = n - n_1 \leq n - 1$. Let a be the canonical stability index of F . Then $1 \leq a \leq r_{n-n_1} \leq \nu_{n-1}$.

We first show that $|mK_X|$ separates two general points on different fibers of h for each $m \geq \max\{5, a\}$. Because $|3K_Z|$ induces a birational map of Z , it suffices to show that $mK_X - 3h^*K_Z$ is effective. We write $mK_X - 3h^*K_Z = K_X + (m-1)K_{X/Z} + (m-4)h^*K_Z$. Note that by Viehweg's weak positivity, the Iitaka model of $(m-1)K_{X/Z} + (m-4)h^*K_Z$ dominates Z . Let $D = (m-1)K_{X/Z} + (m-4)h^*K_Z$. We apply once again Viehweg's weak positivity with the generic restriction theorem (see [39, Theorem 11.2.8]) to conclude that $\mathcal{J}(\|D\|)|_F = \mathcal{J}(\|(m-1)K_F\|)$. Thus

$$h_*(\mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(\|D\|))$$

is a torsion-free sheaf on Z of rank equal to $h^0(F, \mathcal{O}_F(mK_F) \otimes \mathcal{J}(\|(m-1)K_F\|)) = p_m(F) > 0$. By Kollár's vanishing,

$$H^i(Z, h_*(\mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(\|D\|)) \otimes t^*P) = 0$$

for each $i \geq 1$ and $P \in \text{Pic}^0(B)$. Thus,

$$t_*h_*(\mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(\|D\|))$$

is IT^0 on B and it has a non-zero global section. Thus $mK_X - 3h^*K_Z$ is effective.

We then show that $|mK_X|_F$ induces a birational map of F for $m \geq \max\{n_1 + 2, a\}$. Since a smooth model of $t(Z)$ is of general type, by a result of Griffiths and Harris (see, for instance, [41, Theorem 3.9]), the canonical map of a smooth model of $t(Z)$ is generically finite. Thus the canonical map of Z is also generically finite. Let $\phi : Z \dashrightarrow Z_1 \subset \mathbb{P}^N$ be the canonical map of Z . Let $z \in Z$ be a general point such that ϕ induces an étale map between an open neighborhood of z with an open neighborhood of $\phi(z) \in Z_1$. Let H_1, \dots, H_{n_1} be n_1 general hyperplane of \mathbb{P}^N through $\phi(z)$. Then, $(Z', \sum_{i=1}^{n_1} H_i|_{Z'})$ is log canonical in an open neighborhood of $\phi(z)$ and $\phi(z)$ is a log canonical center of the pair. Let D_i be the corresponding divisor of K_Z for $1 \leq i \leq n_1$. Thus $(Z, \sum_{i=1}^{n_1} D_i)$ is log canonical in an open neighborhood of z and z is a log canonical center of this pair. Thus, by Theorem 4.3,

$$H^0(X, mK_X) \rightarrow H^0(F, mK_F)$$

is surjective for each $m \geq n_1 + 2$.

We then see that $|mK_X|$ induces a birational map of X , for each $m \geq \max\{n + 2, r_{n-1}\}$. It suffices to see that $r_{n-1} \geq n + 2$. It is well known that $r_5 \geq r_4 \geq r_3 \geq 27$. When $n \geq 7$, by [21, Theorem 1.1], $r_{n-1} \geq 2^{2^{\frac{n-3}{2}}} > n + 2$. \square

7.2. Open questions on varieties with many global k -forms.

Theorem 1.3 and Theorem 1.6 suggest that the pluricanonical system of n -folds of general type with many global two forms behave similarly to $(n - 2)$ -folds of general type. We have the following very bold conjecture:

Conjecture 7.3. For any $n \geq 5$, there exists a constant $M(n)$ such that, for every smooth projective n -fold X of general type with $h^{2,0}(X) \geq M(n)$, $|mK_X|$ induces a birational map for all $m \geq r_{n-2}$.

One can even ask a very general question about varieties with many global k -forms. Let

$$r_n^k := \sup\{r_s(W) \mid W \text{ is a smooth projective } n\text{-fold} \\ \text{of general type with } h^{k,0}(W) > 0\}.$$

Question 7.4. For any $n \geq 4$, does there exist a constant $M(n)$ such that, for every smooth projective n -fold X of general type with $h^{k,0}(X) \geq M(n)$, $|mK_X|$ induces a birational map for each

$$m \geq \max\{r_{n-k}, r_{n-k+1}^1, \dots, r_{n-1}^{k-1}\}?$$

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