Constructing a family of conformally flat scalar field models

Pantelis S. Apostolopoulos¹

Abstract. Using purely geometrical methods we present a mechanism to solve the scalar field equations of motion (non-minimally coupled with gravity) in a spherically symmetric background. We found that the *full* set of spacetimes, which are of Petrov type O (conformally flat) and admit a gradient Conformal Vector Field, can be determined completely. It is shown that the full group of scalar field equations reduced to a single equation that depends only on the distance $w = r^2 - t^2$ leaving the metric function (equivalently the functional form of the scalar field or the potential) freely chosen. Depending on the structure of the metric or the potential V (as a function of ϕ) a solution can be found either analytically or via numerical integration. We provide physically sound examples and prove that (Anti)-de Sitter fits this scheme. We also reconstruct a recently found solution [1] representing an expanding scalar bubble with metric that has a singularity and corresponds to what is termed as Anti-de Sitter crunch.

¹Department of Environment, Ionian University Mathematical Physics and Computational Statistics Research Laboratory Panagoula 29100, Island of Zakynthos, Greece

E-mail: papostol@ionio.gr

Conformal symmetries have been the subject of various studies during the last three decades (see e.g. [2], [3]). In the majority of the cases, the main reason to investigate the existence of conformal symmetries in General Relativity was the reduction of the complexity of the resulting system of partial differential equations (pdes) in order to locate, more easily, an exact solution of the Einstein Field Equations (EFEs). However, as the generality of the underlying geometry is increased the symmetry pdes and the equation of motion become progressively highly non-linear and often lead to models without a clear physical meaning. On the other hand, there are sufficiently enough cases where physically sound models admit a conformal symmetry (proper or not) that represents an inherent constituent of their kinematical and dynamical structure. The most well known example of this situation is the isotropic, homogeneous and conformally flat Friedmann-Lemaître cosmological model which admits a 9dimensional Lie algebra of *proper* Conformal Vector Fields (CVFs) [4]. In addition it has been shown that proper CVFs are of particular interest to construct viable astrophysical models [5]. [6], [7] and at the same time it has been established the significant role of self-similar spacetimes, admitting a proper Homothetic Vector Field (HVF), since they represent the past and future (equilibrium) states for a vast number of evolving vacuum and γ -law perfect fluid models [8], [9].

Throughout this paper, the following conventions have been used: the spacetime signature is assumed (-, +, +, +), lower Latin letters denote spacetime indices a, b, ... = 0, 1, 2, 3 and we use geometrized units such that $8\pi G = c = 1$.

The existence of a CVF X implies that under the infinitesimal transformation generated by X, the spacetime metric g_{ab} satisfies:

$$\mathcal{L}_{\mathbf{X}}g_{ab} = 2\psi g_{ab} \tag{1}$$

where \mathcal{L} is the Lie derivative along **X** and $\psi(\mathbf{X})$ denotes the conformal factor representing the scale deformation of the spacetime geometry.

The above general condition (proper CVF) specializes to a Killing Vector Field (KVF) $(\psi(\mathbf{X}) = 0)$, to a Homothetic Vector Field (HVF) $(\psi(\mathbf{X}) = \text{const.} \neq 0)$ and to a Special Conformal Killing Vector (SCKV) when $\psi_{;ab} = 0$ (where ";" stands for the covariant derivative w.r.t metric g_{ab}).

The simplest case of a spacetime geometry admitting a maximum of 15 CVFs, is the Minkowski spacetime with metric, in Cartesian coordinates, of the form:

$$ds_{\rm FLAT}^2 = -d\tau^2 + dx^2 + dy^2 + dz^2.$$
 (2)

The complete Lie Algebra of CVFs for the metric (2) has been determined and consists of a subalgebra of 10 KVFs, 1 proper HVF and 4 SCKVs [10].

For the purposes of the present work, it is convenient to transform the metric (2) and the CVFs in a form such that the geometry is foliated by spherically symmetric 2d hypersurfaces i.e. with constant and positive (+1) curvature. We exploit the coordinate transformation $(\tau, x, y, z) \hookrightarrow (t, r, \varphi, \vartheta)$:

$$\tau(t, r, \varphi, \vartheta) = t, \quad x(t, r, \varphi, \vartheta) = r \sin \varphi \sin \vartheta \tag{3}$$

Constructing a family of conformally flat scalar field models

$$y(t, r, \varphi, \vartheta) = r \cos \varphi \sin \vartheta, \quad z(t, r, \varphi, \vartheta) = r \cos \vartheta$$
 (4)

and the Minkowski metric is written

$$ds_{\rm FLAT}^2 = -dt^2 + dr^2 + r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \tag{5}$$

whereas the CVFs take the form[‡]:

$$\mathbf{X}_{1} = \partial_{t}, \quad \mathbf{X}_{2} = \sin\vartheta\sin\varphi\partial_{r} + \frac{\cos\vartheta\sin\varphi}{r}\partial_{\vartheta} + \frac{\cos\varphi}{\sin\vartheta}\partial_{\varphi},$$
$$\mathbf{X}_{3} = \sin\vartheta\cos\varphi\partial_{r} + \frac{\cos\vartheta\cos\varphi}{r}\partial_{\vartheta} - \frac{\sin\varphi}{\sin\vartheta}\partial_{\varphi}$$
$$\mathbf{X}_{4} = \cos\vartheta\partial_{r} - \frac{\sin\vartheta}{r}\partial_{\vartheta} \qquad (6)$$
$$\mathbf{X}_{5} = -\partial_{\varphi}, \quad \mathbf{X}_{6} = -\cos\varphi\partial_{\vartheta} + \cot\vartheta\sin\varphi\partial_{\varphi}$$

$$\mathbf{X}_7 \;=\; \sin arphi \partial_{artheta} + \cot artheta \cos arphi \partial_{arphi}$$

$$\mathbf{X}_{8} = r \sin \vartheta \sin \varphi \partial_{t} + t \sin \vartheta \sin \varphi \partial_{r} + \frac{t \cos \vartheta \sin \varphi}{r} \partial_{\vartheta} + \frac{t \cos \varphi}{r \sin \vartheta} \partial_{\varphi}$$
$$\mathbf{X}_{9} = r \sin \vartheta \cos \varphi \partial_{t} + t \sin \vartheta \cos \varphi \partial_{r} + \frac{t \cos \vartheta \cos \varphi}{r} \partial_{\vartheta} - \frac{t \sin \varphi}{r \sin \vartheta} \partial_{\varphi}$$

$$\mathbf{X}_{10} = r\cos\vartheta\partial_t + t\cos\vartheta\partial_r - \frac{t\sin\vartheta}{r}\partial_\vartheta \tag{7}$$

$$\mathbf{X}_{11} = t\partial_t + r\partial_r \tag{8}$$

 $\mathbf{X}_{12} = \left(r^2 + t^2\right)\partial_t + 2tr\partial_r$

 $\mathbf{X}_{13} = 2rt\sin\vartheta\sin\varphi\partial_t + \left(r^2 + t^2\right)\sin\vartheta\sin\varphi\partial_r +$

$$+\frac{(t^2-r^2)\cos\vartheta\sin\varphi}{r}\partial_\vartheta+\frac{(t^2-r^2)\cos\varphi}{r\sin\vartheta}\partial_\varphi$$

 $\mathbf{X}_{14} = 2rt\sin\vartheta\cos\varphi\partial_t + \left(r^2 + t^2\right)\sin\vartheta\cos\varphi\partial_r +$

$$+\frac{(t^2-r^2)\cos\vartheta\cos\varphi}{r}\partial_{\vartheta} + \frac{(r^2-t^2)\sin\varphi}{r\sin\vartheta}\partial_{\varphi}$$
$$\mathbf{X}_{15} = 2rt\cos\vartheta\partial_t + (r^2+t^2)\cos\vartheta\partial_r + \frac{(r^2-t^2)\sin\vartheta}{r}\partial_{\vartheta}.$$
(9)

 \ddagger We recall that the vectors $\mathbf{X}_1 - \mathbf{X}_4$ correspond to translations, $\mathbf{X}_5 - \mathbf{X}_7$ to spatial rotations, $\mathbf{X}_8 - \mathbf{X}_{10}$ to spacetime rotations (boosts), \mathbf{X}_{11} represents the generator of the homothety and the vectors $\mathbf{X}_{12} - \mathbf{X}_{15}$ are the SCKVs.

3

We select the background geometry to be spherically symmetric and Petrov type O (*conformally* flat). Under these restrictions the metric of the spacetime is:

$$ds^{2} = \mathcal{C}(t,r)^{2} \left[-dt^{2} + dr^{2} + r^{2} \left(d\vartheta^{2} + \sin^{2} \vartheta d\varphi^{2} \right) \right]$$
(10)

where $\mathcal{C}(t,r)$ is a smooth function of its arguments.

It is seen that the spacetime given in (10) shares common features with the standard Friedmann-Lemaître-Robertson-Walker (FLRW) metric like the conformal flatness and the spherically symmetric foliation therefore can be, in principle, matched (in some appropriate limits) with the FLRW geometry. Due to the conformal flatness, the spacetime (10) admits a 12-dimensional Lie Algebra of *proper* CVFs which can be used, in general, to determine the general solution of the null geodesic equation. In fact the existence of a proper CVF **X** implies that there is a constant of motion along null geodesics $(n^a n_a = 0, n_{a;b}n^b = 0)$ [4]:

$$(X_a n^a)_{;b} n^b = X_{a;b} n^a n^b + X_a n^a_{;b} n^b = \psi g_{ab} n^a n^b = 0.$$
(11)

The next step of the simplification setup is to demand that one of the CVFs is a gradient vector field $X_a = f_{;a}$ for some smooth function f which is equivalent to demand that its bivector $F_{ab}(\mathbf{X}) = \frac{1}{2}(X_{a;b} - X_{b;a}) = X_{[a;b]}$ vanishes. This heuristic assumption can be justified from the fact that FLRW spacetime also admits the gradient CVF \mathbf{X}_1 from which the null geodesic equation is solved. From (8) and (10) we obtain:

$$F_{ab}\left(\mathbf{X}_{11}\right) = 0 \iff t\mathcal{C}_{,r} + r\mathcal{C}_{,t} = 0.$$
(12)

The general solution of the linear pde (12) implies that the metric function depends on the quantity $r^2 - t^2$ hence $C(t, r) = C(r^2 - t^2)$. The aforementioned functional structure of C(t, r) indicates that is also invariant under the group of spacetime rotations (boosts) therefore we can easily deduce:

$$\mathcal{L}_{\mathbf{X}_8} g_{ab} = \mathcal{L}_{\mathbf{X}_9} g_{ab} = \mathcal{L}_{\mathbf{X}_{10}} g_{ab} = 0 \tag{13}$$

and the spacetime (10) admits (apart from the spatial rotations that manifest the spherical symmetry of the geometry) 3 extra KVFs thus a 6-dimensional Lie Algebra of isometries.

In standard scalar field theories, non-minimally coupled with gravity, the effective action S of the system without any matter contributions has the form (without loss of generality, we have ignored surface terms which are not alter the resulting field equations) [11], [12]:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}R - \frac{1}{2}\phi_{;a}\phi^{;a} - \frac{1}{2}\xi R\phi^2 - V(\phi) \right]$$
(14)

where R is the curvature scalar, ϕ represents the (massless) scalar field with potential $V(\phi)$ and ξ is the coupling constant.

Variation of (14) with respect to a general background metric g_{ab} gives the scalar field equations:

$$(1 - \xi \phi^2) G_{ab} = T_{ab}^{\text{Scalar}} \equiv (1 - 2\xi) \phi_{;a} \phi_{;b} + \left(2\xi - \frac{1}{2}\right) (\phi_{;d} \phi^{;d}) g_{ab} - 2\xi \phi \phi_{;ab} + 2\xi \phi \phi_{;d}^{;d} g_{ab} - V g_{ab}$$
(15)

where G_{ab} is the Einstein tensor and T_{ab}^{Scalar} is the effective energy-momentum tensor of the scalar field contribution.

The conservation of T_{ab}^{Scalar} implies the corresponding Klein-Gordon equation:

$$\phi^{;a}_{;a} - \xi R\phi - \frac{dV}{d\phi} = 0. \tag{16}$$

Obviously, the complexity of the system of equations (15) and (16) does not permit us to analyze a scalar field configuration in full generality without making some simplification assumptions. Nevertheless we will see that the geometric setup we choose, lead to the *complete set* of solutions with sound physical interest and, furthemore, will verify the significance (in certain cases) of the existence of a proper CVF.

Restricting our analysis to the spacetime (10) we note that the isometries of the underlying geometry are inherited from the dynamics [13], [14] $\mathcal{L}_{\mathbf{KVF}}G_{ab} = 0 = \mathcal{L}_{\mathbf{KVF}}T_{ab}^{\text{Scalar}}$ which implies that ϕ and $V(\phi)$ scale $\sim (r^2 - t^2)$. Setting $w = r^2 - t^2$ the field equations (15) become:

$$B_{0}^{0} = 0 = V(w) \mathcal{C}(w)^{4} - 2\mathcal{C}(w)^{2} \left\{ 4r^{2}\xi\phi(w)\phi(w)'' + \phi(w)' \left[\left(4\xi r^{2} - r^{2} - t^{2} \right)\phi(w)' + 6\xi\phi(w) \right] \right\} - 4\mathcal{C}(w) \left\{ 2r^{2}\mathcal{C}(w)'' \left[\xi\phi(w)^{2} - 1 \right] + \mathcal{C}(w)' \left[2\xi \left(r^{2} - 3t^{2} \right)\phi(w)\phi(w)' + 3 \left(\xi\phi(w)^{2} - 1 \right) \right] \right\} + 4 \left(r^{2} + 3t^{2} \right) \mathcal{C}(w)'^{2} \left[\xi\phi(w)^{2} - 1 \right]$$
(17)

$$B_{1}^{0} = 0 = \mathcal{C}(w)^{2} \left[2\xi\phi(w)'\phi(w)'' + (2\xi - 1)\phi(w)'^{2} \right] + 2\mathcal{C}(w) \left[\mathcal{C}(w)'' \left(\xi\phi(w)^{2} - 1\right) - 2\xi\phi(w)\phi(w)'\mathcal{C}(w)' \right] + 4\mathcal{C}(w)'^{2} \left[1 - \xi\phi(w)^{2} \right]$$
(18)

$$B_{2}^{2} = 0 = V(w) \mathcal{C}(w)^{4} + 2\mathcal{C}(w)^{2} \{4t^{2}\xi\phi(w)\phi(w)'' - -\phi(w)' [(t^{2} + r^{2} - 4\xit^{2})\phi(w)' + 6\xi\phi(w)]\} + 4\mathcal{C}(w) \{2t^{2}\mathcal{C}(w)'' [\xi\phi(w)^{2} - 1] - -\mathcal{C}(w)' [2\xi (3r^{2} - t^{2})\phi(w)\phi(w)' + 3 (\xi\phi(w)^{2} - 1)]\} + 4 (t^{2} + 3r^{2}) \mathcal{C}(w)'^{2} [1 - \xi\phi(w)^{2}]$$
(19)

$$B_{3}^{3} = B_{4}^{4} = 0 = V(w) \mathcal{C}(w)^{4} - 2\mathcal{C}(w)^{2} \left\{ 4 \left(r^{2} - t^{2} \right) \xi \phi(w) \phi(w)'' + \phi(w)' \left[\left(r^{2} - t^{2} \right) (4\xi - 1) \phi(w)' + 6\xi \phi(w) \right] \right\} - 4\mathcal{C}(w) \left\{ 2 \left(r^{2} - t^{2} \right) \mathcal{C}(w)'' \left[\xi \phi(w)^{2} - 1 \right] + \mathcal{C}(w)' \left[2\xi \left(r^{2} - t^{2} \right) \phi(w) \phi(w)' + 3 \left(\xi \phi(w)^{2} - 1 \right) \right] \right\} + 4 \left(r^{2} - t^{2} \right) \mathcal{C}(w)'^{2} \left[\xi \phi(w)^{2} - 1 \right]$$
(20)

where a prime "'" denotes differentiation w.r.t. $w = r^2 - t^2$.

From eq. (18) it follows

$$2\xi\phi(w)\mathcal{C}(w)^{2}\phi(w)'' + (2\xi - 1)\phi(w)'^{2}\mathcal{C}(w)^{2} + +2\mathcal{C}(w)\left[\mathcal{C}(w)''\left(\xi\phi(w)^{2} - 1\right) - 2\xi\phi(w)\phi(w)'\mathcal{C}(w)'\right] + 4\mathcal{C}(w)'^{2}\left(1 - \xi\phi(w)^{2}\right) = 0.(21)$$

Replacing $\mathcal{C}(w)''$ to the remaining field equations (17), (19), (20) we end up with only one equation namely:

$$V(w) = 2\{\mathcal{C}(w)^{2} \phi(w)' [w\phi(w)' + 6\xi\phi(w)] + 6\mathcal{C}(w)\mathcal{C}(w)' [2\xi w\phi(w)'\phi(w) + \xi\phi(w)^{2} - 1] + 6w\mathcal{C}(w)'^{2} [\xi\phi(w)^{2} - 1]\}\mathcal{C}(w)^{-4}.$$
(22)

It is straightforward to prove that the pair of equations (21) and (22) completely determine the family of solutions of the system (17)-(20). We note that (as expected), with the aid of (22), the conservation equation (16) is satisfied identically thus (22) represents a first integral of the Klein-Gordon equation.

Let us consider first, for simplicity, the case of the (Anti)-de Sitter spacetime with metric:

$$ds^{2} = \frac{c_{2}^{2}}{(c_{1} + w)^{2}} \left[-dt^{2} + dr^{2} + r^{2} \left(d\vartheta^{2} + \sin^{2} \vartheta d\varphi^{2} \right) \right]$$
(23)

where c_1 and $c_2 > 0$ are arbitrary constants and $\mathcal{C}(w) = \frac{c_2}{c_1 + w}$. We verify that

$$R^{a}_{b} = \frac{12c_{1}}{c_{2}^{2}}\delta^{a}_{b} = \frac{R}{4}\delta^{a}_{b}.$$
(24)

Here, R is the Ricci scalar and the c_1 controls the sign character of the spacetime curvature which is directly related with the cosmological constant of the (Anti)-de Sitter spacetime. Then, for minimally coupled scalar field $\xi = 0$, eq. (21) implies that $\phi(w) = \text{const.}$ and the potential $V(w) = \frac{12c_1}{c_2^2}$.

We must note that the system of equations (21) and (22), although depends (in this particular geometry) only on the distance $w = r^2 - t^2$, its solution can not be obtained always in closed form, so that must be derived case by case (possibly numerically). In addition the form of the potential in physically relevant models must feature a false vacuum, as well as a true vacuum or a region in which the potential is unbounded from below. We apply this scheme to a more complicated setup and reconstruct a recently found solution [1] for an expanding scalar bubble within a spherically symmetric geometry. The metric has a singularity and corresponds to what is termed as AdS crunch. We try solutions (up to some integration constants) of the form:

$$\mathcal{C}(w) = \left[1 - \frac{c_1^2}{\left(1 + w/c_2^2\right)^2}\right]^{1/2}$$
(25)

where c_1 and c_2 are again arbitrary constants. We observe that the main characteristic of the spacetime (25) is the existence of a curvature singularity that occurs when $\mathcal{C}(w) \to 0$ as $w \to \pm c_2^2 (c_1 \mp 1)$. In addition, an apparent horizon "protects" the spacetime from the singularity because there are two null geodesic vectors, say n^a and m^a , such that $n^a m_a = -1$ (representing ingoing and outgoing null geodesics) and $n^a_{;a}m^k_{;k} = 0$ (see e.g. [15], [16], [17], [18]). We refer the reader to [1] for a discussion regarding the structure of the spacetime (25).

Substituting eq. (25) in (21) the general solution for the scalar field (again for $\xi = 0$) reads:

$$\phi(w) = c_3 \tanh^{-1} \frac{c_1}{1 + w/c_2^2} \tag{26}$$

provided that $c_3 = \sqrt{6}$. Finally, we compute the potential from eq. (22):

$$V(w) = c_4 \frac{c_1^4 \left(1 + w/c_2^2\right)^{-4}}{\left[c_1^2 \left(1 + w/c_2^2\right)^{-2} - 1\right]^2} \quad \text{or} \quad V(\phi) = c_4 \sinh^4 \frac{\phi}{c_3}.$$
(27)

As a final application let us consider the case of a conformally coupled ($\xi = 1/6$) scalar field with potential proposed recently in [12]:

$$V(\phi) = -\frac{\lambda}{4}\phi^4 + \frac{R}{4}$$
(28)

where $\lambda, R > 0$ are constants.

After standard but tedious calculations, we find that the unique exact solution of the system of equations (21), (22) and (28) is:

$$\phi(w) = \frac{D_3 \left(Rw + 48 \right)}{D_3^2 w + 288\lambda}.$$
(29)

$$\mathcal{C}\left(w\right) = \frac{1}{1 + \frac{R}{48}w}\tag{30}$$

where D_3 is a constant of integration. It is straightforward to show that the spacetime (30) corresponds to the de Sitter model with constant scalar curvature R.

In the present article we presented a mechanism to produce compatible scalar field spacetimes in standard gravity using geometrical methods. We observed that, depending on the structure of a given metric function C(w) or a potential V (as a function of ϕ), a solution can be found either analytically or via numerical integration. Although the family of spacetimes (10), (21), (22) still have some sort of simplicity (e.g. being conformally and, in certain cases, asymptotically flat like the spacetime (25)) however they keep their intrinsic generality and correspond to geometries with sound physical interest. They also indicate an eventually *close connection* between these classes of models and the existence of a *gradient* CVF, so far underestimated.

Acknowledgments

The author wishes to thank Nikos Tetradis for very useful discussions.

- A. Strumia and N. Tetradis, JHEP 09 (2022), 203 doi:10.1007/JHEP09(2022)203 [arXiv:2207.00299 [hep-ph]].
- [2] G. S. Hall, Symmetries and Curvature Structure in General Relativity (World Scientific Lecture Notes in Physics: Volume 46, 2004).
- [3] K. L. Duggal and R. Sharma, Symmetries of Spacetimes and Riemannian Manifolds (Kluwer, Academic Publishers 1999).
- [4] R. Maartens and S. D. Maharaj, Class. Quant. Grav. 3 (1986) 1005.
- [5] L. Herrera, J. Jiménez, L. Leal, J. Ponce de León, M. Esculpi and V. Galina, J. Math. Phys. 25 (1984) no.11, 3274 doi:10.1063/1.526075
- [6] R. Maartens, Causal thermodynamics in relativity, [arXiv:astro-ph/9609119 [astro-ph]].
- [7] L. Herrera, A. Di Prisco and J. Ospino, Universe 8 (2022) no.6, 296 doi:10.3390/universe8060296
 [arXiv:2206.02143 [gr-qc]].

- 8
- [8] J. Wainwright and G. F. R. Ellis (Eds), Dynamical Systems in Cosmology (Cambridge University Press, Cambridge 1997).
- [9] A. A. Coley, Dynamical Systems and Cosmology, (Kluwer, Academic Publishers 2003).
- [10] Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics (Amsterdam, North Holland 1977).
- [11] E. Winstanley, Found. Phys. 33 (2003), 111-143 doi:10.1023/A:1022871809835 [arXiv:gr-qc/0205092 [gr-qc]].
- [12] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010), 451-497 doi:10.1103/RevModPhys.82.451 [arXiv:0805.1726] [gr-qc]]; Ν. Ohta. Phys. Rev. Lett. 91 (2003),061303 doi:10.1103/PhysRevLett.91.061303 [arXiv:hep-th/0303238 [hep-th]]; N. Ohta, Prog. Theor. Phys. 110 (2003), 269-283 doi:10.1143/PTP.110.269 [arXiv:hep-th/0304172 [hep-th]]; N. Tetradis, [arXiv:2302.12132 [hep-ph]].
- [13] H. Stephani, D. Kramer, M. A. A. H. MacCallum, C. Hoenselaers and E. Herlt, Exact solutions of Einstein's field equations, (Second Edition, Cambridge University Press, Cambridge 2003).
- [14] L. P. Eisenhart, *Riemannian geometry*, (Princeton University Press, Princeton 1960).
- [15] S. W. Hawking and G. F. R. Ellis, The large scale structure of spacetime (Cambridge University Press, 1973 Cambridge).
- [16] A. Krasiński, Inhomogeneous cosmological models, (Cambridge University Press, 1997 Cambridge).
- [17] C. Hellaby and A. Krasinski, Phys. Rev. D 66 (2002) 084011 [arXiv:gr-qc/0206052].
- [18] P. S. Apostolopoulos, Class. Quant. Grav. 34 (2017) no.9, 095013 doi:10.1088/1361-6382/aa66df [arXiv:1611.04569 [gr-qc]].