**A transference result for Lebesgue spaces with $\mathbf{A}_{\infty}$ weights and its applications**

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**Abstract** In this work we obtain a transference theorem for Lebesgue spaces with $A_{\infty}$ weights, namely, starting from some uniform-norm inequalities it is possible to obtain similar inequalities in Lebesgue spaces with $A_{\infty}$ weights. This transference technic allows us to obtain some weighted norm inequalities easily. Also transference result gives possibility to use fractional difference operators in weighted Lebesgue spaces easier than the classical known one. We can obtain some norm-like inequalities easily as a consequence. Some important approximation inequalities of approximation by integral functions of finite degree can be obtained with a different proof.

**Key Words** Muckenhoupt weight, Steklov operator, Integral functions of finite degree, Fractional difference operator, Best approximation.

**2020 Mathematics Subject Classifications** 46E30; 42B20; 42B25; 42B35.

# 1 Introduction and main results

## 1.1 Preliminary Definitions

We can give some preliminary definitions to state main results. A function $\omega:\mathbb{R}^{d}\to[0,\infty]$ will be called weight if $\omega$ is a measurable and positive function almost everywhere (a.e.) on $\mathbb{R}^{d}$. Define $\langle\omega\rangle_{A}:=\int_{A}\omega(t)dt$ for $A\subset\mathbb{R}^{d}$. For a weight $\omega$ on $\mathbb{R}^{d}$, we denote by $L_{p,\omega}$, $0<p\leq\infty$ the class of Lebesgue measurable functions $f:\mathbb{R}^{d}\to\mathbb{R}$ such that

$$\left\Vert f\right\Vert_{p,\omega}\equiv\left(\int_{\mathbb{R}^{d}}\left\vert f\left(x\right)\right\vert^{p}\omega\left(x\right)dx\right)^{1/p}<\infty,\quad(0<p<\infty),$$

$$\left\Vert f\right\Vert_{\infty,\omega}\equiv\left\Vert f\right\Vert_{\infty}\equiv esssup_{x\in\mathbb{R}^{d}}\left\vert f\left(x\right)\right\vert,\quad\left(p=\infty\right).$$

We set $L_{p}\equiv L_{p,1}$ and $\left\Vert f\right\Vert_{p}\equiv\left\Vert f\right\Vert_{p,1}$ and $\oint_{A}\omega(t)dt\equiv\left(\left\vert A\right\vert^{-1}\right)\int_{A}\omega(t)dt$ where $|A|$ denotes the Lebesgue measure of a set $A\subset\mathbb{R}^{d}$.

For $1<p<\infty$ and $(1/p)+(1/p')=1$ we set $\omega':=\omega^{1-p'}$ for a weight $\omega$. A weight $\omega$ satisfies the Muckenhoupt's condition $A_{p}$, $1\leq p<\infty$, if

$$\left[\omega\right]_{1}\equiv\sup_{Q\in\mathbb{J}}(\left\vert Q\right\vert^{-1}\left\langle\omega\right\rangle_{Q}esssup_{x\in Q}(\omega\left(x\right)^{-1}))<\infty,\quad\left(p=1\right),\tag{1}$$

$$\left[\omega\right]_{p}\equiv\sup_{Q\in\mathbb{J}}\left\vert Q\right\vert^{-p}\left\langle\omega\right\rangle_{Q}\left\langle\omega'\right\rangle_{Q}^{p-1}<\infty,\quad(1<p<\infty)\tag{2}$$

with some finite constants independent of $Q$, where $\mathbb{J}$ is the class of cubes in $\mathbb{R}^d$ with sides parallel to coordinate axes .

Define $A_\infty := \cup_{1\leq p<\infty} A_p$. It is well known that a characterization of weights $\omega$ in the class $A_\infty$ is

$$[\omega]_\infty \equiv \sup_Q \int_Q M\left[\omega\left(y\right)\chi_Q\left(y\right)\right]dy < \infty$$

where $Q$ is any cube in $\mathbb{R}^d$ and $M$ is the Hardy-Littlewood maximal function. Let $\mathbb{S}_0$ be the set of integrable simple functions defined on $\mathbb{R}^d$.

**Definition 1** *Let $u, x \in \mathbb{R}^d$, $\omega \in A_\infty$ and $f \in \mathbb{S}_0$. (i) Define weighted Steklov mean*

$$S_{u,\omega} f\left(x\right) \equiv \left(\langle\omega\rangle_{[-1/2,1/2]^d}\right)^{-1} \int_{[-1/2,1/2]^d} f\left(x+u+t\right)\omega\left(t\right)dt.$$

*(ii) Define*

$$\mathcal{R}_{u,\omega} f\left(x\right) \equiv \sum_{k=0}^{1} \tfrac{1}{2^k} \tfrac{\left(S_{u,\omega}\right)^k f(x)}{\|S_{u,\omega}\|^k_{\mathcal{B}(L_{p,\omega},L_{p,\omega})}}, \quad \left(S_{u,\omega}\right)^0 f \equiv f,$$

*where $\|T\|_{\mathcal{B}(U,V)}$ is the operator norm of a bounded operator $T : U \to V$.*

**Definition 2** *For given $\omega \in A_\infty$, $p \in [1,\infty)$ we define the class $\mathcal{Z}\left(p,\omega\right) \equiv \{g \in L_{p',\omega} : \|g\|_{p',\omega} = 1\}$ where as usual $p' \equiv p/(p-1)$ for $p \in (1,\infty)$ and $1' \equiv \infty$.*

**Definition 3** *Let $\omega \in A_\infty$, $p \in (0,\infty)$, $f \in L_{p,\omega}$. (a) For $p \in [1,\infty)$ we define*

$$F_f \equiv F_f\left(u,G,p,\omega\right) \equiv \int_{\mathbb{R}^d} \mathcal{R}_{u,\omega} f\left(x\right)|G(x)|\,\omega\left(x\right)dx, \quad u \in \mathbb{R}^d, \tag{3}$$

*with $G \in \mathcal{Z}\left(p,\omega\right)$.*

*(b) Let $p \in (0,1)$. Since, there is $a_0 \equiv e^{2^{11+d}[\omega]_\infty} > 1$ (see [17, p.786]) such that, we obtain $\omega \in A_a$ with $a \equiv a_0 + 0,01$. Then, one can get a $q \in (0,p)$ such that $\omega \in A_{p/q}$ (Take for example any $q$ less than $p/a_0$). Now, set $r \equiv p/q$ and define*

$$F_f \equiv F_f\left(u,G,r,\omega\right) \equiv \int_{\mathbb{R}^d} \left(\mathcal{R}_{u,\omega} f\left(x\right)\right)^q |G(x)|\,\omega\left(x\right)dx, \quad u \in \mathbb{R}^d \tag{4}$$

*with $G \in \mathcal{Z}\left(r,\omega\right)$.*

Let $\mathcal{C}(\mathbb{R}^d)$ be the class of bounded, uniformly continuous functions defined on $\mathbb{R}^d$ and $\|f\|_{\mathcal{C}(\mathbb{R}^d)} := \sup\left\{|f\left(t\right)| : t \in \mathbb{R}^d\right\}$ for $f \in \mathcal{C}(\mathbb{R}^d)$.

**Remark 4** *Note that, by Theorem 14, $F_f \in \mathcal{C}(\mathbb{R}^d)$ for $\omega \in A_\infty$, $p \in (0,\infty)$, and $f \in L_{p,\omega}$.*

## 1.2 Main results

To obtain a weighted norm inequality of the following type

$$\|f\|_{p,\omega} \leq c\,\|g\|_{p,\omega} \tag{5}$$

for $0 < p < \infty$, $\omega \in A_\infty$, $f \in L_{p,\omega}$, we define an intermediate function as in Definition 3

$$F_f : \mathbb{R}^d \to \mathcal{C}\left(\mathbb{R}^d\right), \quad u \mapsto F_f(u)$$

having properties

$$\|f\|_{p,\omega} \leq c\,\|F_f(\cdot)\|_{\mathcal{C}(\mathbb{R}^d)} \text{ and } \|F_g(\cdot)\|_{\mathcal{C}(\mathbb{R}^d)} \leq c\,\|g\|_{p,\omega}$$

for some positive constants. Now, if the following uniform norm estimate

$$\|F_f(\cdot)\|_{\mathcal{C}(\mathbb{R}^d)} \leq c\,\|F_g(\cdot)\|_{\mathcal{C}(\mathbb{R}^d)}$$

holds, then, we obtain desired weighted norm inequality (5).

All constants $c > 0$ will be some positive number such that, they depend on the main parameters in question, and change in each occurrences.

Main theorem of this work is the following transference result.

**Theorem 5** *Let $\mathcal{F}$ be a family of couples $(f,g)$ of nonnegative functions, $d \in \mathbb{N}$, $p \in (0,\infty)$, $\omega \in A_\infty$ and $f \in L_{p,\omega}$. Suppose that the following uniform-norm estimate holds*

$$\|F_f(\cdot, G, p, \omega)\|_{\mathcal{C}(\mathbb{R}^d)} \leq c\,\|F_g((\cdot, G^*, p, \omega))\|_{\mathcal{C}(\mathbb{R}^d)} \tag{6}$$

*for any $G, G^* \in \mathcal{Z}(p/q,\omega)$ where $q := 1$ for $p \in [1,\infty)$ and $q \in (0,p)$ for $p \in (0,1)$. Then, weighted norm inequality*

$$\|f\|_{p,\omega} \leq c\,\|g\|_{p,\omega}, \quad (f,g) \in \mathcal{F},$$

*holds with a positive constant $c := c(q,\omega,p,d)$.*

As a corollary of Theorem 5 we can easily obtain norm inequalities with a suitable fractional difference operator $(E - V_\delta)^r$ with $\delta > 0$ in weighted Lebesgue spaces $L_{p,\omega} \equiv L^p(\omega(x)\,dx)$ where $E$ is the identity operator and $V_\delta$ is a suitable translation operator in $L_{p,\omega}$ with $\omega \in A_\infty$.

Theorem 5 gives several important norm-like inequalities. For example, one can consider reverse sharp Marchaud inequality: If $p \in (1,\infty)$, $\omega \in A_\infty$, $f \in L_{p,\omega}$, then, there are $m \in \mathbb{N}$ and $a > 1$ such that

$$\|(E - V_\delta)^r f\|_{p,\omega}^s \geq c \sum_{j=0}^{m} 2^{-j2rs} \left\|(E - V_{2^j\delta})^{r+1} f\right\|_{p,\omega}^s, \tag{7}$$

holds for $s := \max\{2, a\}$ with a positive constant $c := c(s,\omega,p,d)$ depending only on $s, \omega, p, d$.

To obtain inequalities of type (7) with the classical extrapolation theorem seem to be not possible. Note also that, classical direct proof of (7)-type inequality ([13, Theorem 2.1]) is also hard to overcome for weighted spaces.

To give proof of inequality (7) we need to obtain a version of main Theorem 5.

**Definition 6** *Let $B$ be a Banach space with norm $\|\cdot\|_B$. For an $m \in \mathbb{N}$ and $s \in (0,\infty)$, we define*

$$\|f\|_{l_s^m(B)} \equiv \left\|(f_j)_{j=0}^m\right\|_{l_s^m(B)} \equiv \left(\sum\nolimits_{j=0}^m \|f_j\|_B^s\right)^{1/s}.$$

**Theorem 7** *Let $\mathcal{F}$ be a family of couples $(f,g)$ of nonnegative functions, $d \in \mathbb{N}$, $p \in (1,\infty)$ and $\omega \in A_\infty$. Suppose that there exists an $a \in (1,\infty)$ such that, for any $G, G^* \in \mathcal{Z}(p/q,\omega)$,*

$$\|F_f(\cdot,G,p,\omega)\|_{L_a} \leq c\left\|F_{g_j}(\cdot,G^*,p,\omega)\right\|_{l_s^m(L_a)}, \quad (f,g_j) \in \mathcal{F}, \tag{8}$$

*holds for some $s > 2$, and positive $c := c(m,a,s,d)$ provided the left hand side (8) is finite, where $q := 1$ for $p \in [1,\infty)$ and $q \in (0,p)$ for $p \in (0,1)$. Then*

$$\|f\|_{p,\omega} \leq c\,\|g_j\|_{l_s^m(L_{p,\omega})}, \quad (f,g_j) \in \mathcal{F}, \tag{9}$$

*holds with a positive $c := c(m,a,s,p,\omega,d)$ when the left-hand side (9) is finite.*

Note that, other versions of main Theorem 5 is also possible for other versions of weighted norm inequalities.

On the other hand, by using Theorem 5, many of the basic inequalities of approximation by entire functions of finite degree can be obtained by extrapolation argument as an alternative proof. See proof of Theorem 26.

To prove main properties of intermediate functions $F_f$ we need some preliminary observations related to (weighted) Steklov averages.

**Definition 8** *Define Steklov mean, for $u, x \in \mathbb{R}^d$, $1 \leq p < \infty$, $\omega \in A_p$ and $f \in L_{p,\omega}$, as*

$$S_u f(x) \equiv \int_{[-1/2,1/2]^d} f(x+u+t)\,dt.$$

**Theorem 9** *We suppose that $1 \leq p < \infty$, $\omega \in A_p$ and $f \in L_{p,\omega}$. In this case, for any $u \in \mathbb{R}^d$, there holds*

$$\|S_u f\|_{p,\omega} \leq 3^{2d+1/p}[\omega]_p^{1/p}\|f\|_{p,\omega}.$$

**Remark 10** *(a) By theorem 18.3 of [38], the class $\mathbb{S}_0$ of integrable simple functions defined on $\mathbb{R}^d$, is a dense subset of $L_{p,\omega}$ with $0 < p < \infty$, and $\omega \in A_\infty$.*

*(b) By Theorems 18.3, 19.37 and Observation 7.5 of [38], we can observe that the class $C_c \equiv C_c(\mathbb{R}^d)$ of continuous functions of compact support, is a dense subset of $L_{p,\omega}$ with $0 < p < \infty$, and $\omega \in A_\infty$. Note that, Theorem 18.3 of [38] is proved for $1 \leq p < \infty$ but the same proof holds also for $0 < p < 1$.*

Now, using Theorem 9 we can prove the following result.

**Theorem 11** *We suppose that $0 < p < \infty$ and $\omega \in A_\infty$. In this case, for any $u \in \mathbb{R}^d$, and $f \in \mathbb{S}_0$, there holds*

$$\left\| S_{u,\omega} f \right\|_{p,\omega} \le c \left\| f \right\|_{p,\omega} \tag{10}$$

*with a positive constant $c = c(d, p, \omega)$.*

**Remark 12** *It is clear from its definition that $\mathcal{R}_{u,\omega} |f| \ge |f|$.*

Now, using Theorem 11 we can prove the following result.

**Theorem 13** *We suppose that $0 < p < \infty$ and $\omega \in A_\infty$. In this case, for any $u \in \mathbb{R}^d$, and $f \in \mathbb{S}_0$, there holds*

$$\left\| \mathcal{R}_{u,\omega} f \right\|_{p,\omega} \le 4^{1/\min\{1,p\}} \left\| f \right\|_{p,\omega}. \tag{11}$$

**Theorem 14** *Let $0 < p < \infty$, $\omega \in A_\infty$, and $f \in L_{p,\omega}$. In this case, the function $F_f$ defined in (3) or (4) is bounded, uniformly continuous function on $\mathbb{R}^d$.*

## 2 Applications on operators

We can give several corollaries that can be obtained easily by using Theorem 5.

We define the following operators.

**Definition 15** *For $\delta > 0$, $x, u \in \mathbb{R}^d$ and locally integrable functions $f : \mathbb{R}^d \to \mathbb{R}$, we define operators*

$$S_{\delta,u} f(x) \equiv \int_{[-\delta/2,\delta/2]^d} f(x + u + s)\, ds, \tag{12}$$

$$V_\delta f(x) \equiv S_{\delta,0} f(x) \equiv \int_{[-\delta/2,\delta/2]^d} f(x + s)\, ds, \tag{13}$$

$$Z_\delta f(x) \equiv \int_{[\delta/2,\delta]^d} f(x + t)\, dt, \tag{14}$$

$$\mathcal{B}_\delta f(x) \equiv \int_{[0,\delta]^d} f(x + v)\, dv. \tag{15}$$

**Theorem 16** *We suppose that $1 < p < \infty$, $\omega \in A_\infty$, $u, v \in \mathbb{R}^d$, and $\delta \in (0, \infty)$. Let an operator $\Gamma$ represents operator $f \to S_{\delta,v} f$. In this case, for $f \in L_{p,\omega}$, there hold properties*

$$S_{u,\omega}(\Gamma f) = \Gamma(S_{u,\omega} f), \qquad \mathcal{R}_{u,\omega}(\Gamma f) = \Gamma(\mathcal{R}_{u,\omega} f), \tag{16}$$

$$F_{\Gamma f} = \Gamma(F_f), \qquad \left\| \Gamma f \right\|_{p,\omega} \le c \left\| f \right\|_{p,\omega}, \tag{17}$$

*with a positive constant $c = c(d, p, \omega)$.*

As a corollary of Theorem 16 and Theorem 30 we have the following result.

**Corollary 17** *We suppose that* $0 < p < \infty$, $\omega \in A_\infty$ *and* $v \in \mathbb{R}^d$, $\delta \in (0, \infty)$. *Let an operator* $\Gamma$ *represents operator* $f \to S_{\delta,v} f$. *In this case, for any* $f \in L_{p,\omega}$ ($f \in L_{p,\omega} \cap \mathbb{S}_0$ *when* $0 < p < 1$), *there holds*

$$\|\Gamma f\|_{p,\omega} \le c \|f\|_{p,\omega},$$

*with a positive constant* $c = c(d, p, \omega)$.

Proof of the following two results for several operators are the same with Theorem 16 and Corollary 17.

**Theorem 18** *We suppose that* $1 < p < \infty$, $\omega \in A_\infty$, $u, v \in \mathbb{R}^d$, *and* $\delta \in (0, \infty)$. *If we replace the operator* $\Gamma$ *in Theorem 16, by one of the following operators* $f \to S_v f$, $f \to S_{v,\omega} f$, $f \to \mathcal{R}_{v,\omega} f$, $f \to V_v f$, $f \to Z_v f$, *or* $f \to \mathcal{B}_v f$ *then, conclusion of Theorem 16 is remain valid.*

**Corollary 19** *We suppose that* $0 < p < \infty$, $\omega \in A_\infty$, $u, v \in \mathbb{R}^d$, *and* $\delta \in (0, \infty)$. *If we replace the operator* $\Gamma$ *in Corollary 17, by one of the following operators* $f \to S_v f$, $f \to S_{v,\omega} f$, $f \to \mathcal{R}_{v,\omega} f$, $f \to V_v f$, $f \to Z_v f$, *or* $f \to \mathcal{B}_v f$, *then, conclusion of Corollary 17 is remain valid.*

Fractional Difference Operator can be defined as follows.

**Definition 20** *Let* $\delta, k \in (0, \infty)$ *and define difference* $(E - V_\delta)^k$ *of fractional order* $k$ *at* $x \in \mathbb{R}^d$ *with step* $\delta$, *by*

$$(E - V_\delta)^k f(x) \equiv \sum\nolimits_{s=0}^{\infty} (-1)^s C_s^k (V_\delta)^s f(x) \tag{18}$$

*where* $C_0^k \equiv 1$, *and* $C_s^k \equiv \Pi_{n=1}^s \frac{k-n+1}{n}$ *are binomial coefficients.*

**Theorem 21** *We suppose that* $0 < p < \infty$ *and* $\omega \in A_\infty$. *In this case, for any* $\delta \in (0, \infty)$, *and* $f \in \mathbb{S}_0$, *there holds*

$$\left\| (E - V_\delta)^k f \right\|_{p,\omega} \le c \|f\|_{p,\omega}, \tag{19}$$

*with a positive constant* $c := c(s, \omega, p, d)$ *depending only on* $s, \omega, p$.

Theorem 5 gives several important norm-like inequalities. For example, one can consider weighted reverse sharp Marchaud inequality:

**Theorem 22** *If* $r \in \mathbb{N}$, $\omega \in A_\infty$, $p \in (1, \infty)$, $\delta \in (0, \infty)$, *and* $f \in L_{p,\omega}$, *then, there are* $m \in \mathbb{N}$ *and* $a > 1$ *such that*

$$\|(E\text{-}V_\delta)^r f\|_{p,\omega}^s \ge c \sum_{j=0}^{m} 2^{-j2rs} \left\| (E\text{-}V_{2^j\delta})^{r+1} f \right\|_{p,\omega}^s, \tag{20}$$

*holds for* $s := \max\{2, a\}$ *with a constant* $c > 0$ *depending only on* $s, \omega, p, d$.

## 3 Applications on approximation by exponential type functions

**Definition 23** *Let* $X := L^p(\mathbb{R}^d)$ *or* $L_{p,\omega}$ *or* $\mathcal{C}(\mathbb{R}^d)$.

*(i) We define* $\mathcal{G}_\sigma(X)$ *as the class of entire function of exponential type* $\sigma > 0$ *that belongs to* $X$*, namely, "*$g \in \mathcal{G}_\sigma(X)$ *iff* $supp\hat{g}(\mathbf{y}) \subset \{\mathbf{y} : |\mathbf{y}| \leq \sigma\}$ *and* $g \in X$*" where* $\hat{g}$ *is the Fourier transform of* $g$.

*We set* $\mathcal{G}_\sigma(p) := \mathcal{G}_\sigma(L^p(\mathbb{R}^d))$, $\mathcal{G}_\sigma(p,\omega) := \mathcal{G}_\sigma(L_{p,\omega})$, *and* $\mathcal{G}_\sigma(\mathcal{C}) := \mathcal{G}_\sigma(\mathcal{C}(\mathbb{R}^d))$.

*(ii) The best approximation in* $L_{p,\omega}$ *by functions of exponential type is given by*

$$A_\sigma(f)_X := \inf_g\{\|f-g\|_X : g \in \mathcal{G}_\sigma(X)\}. \tag{21}$$

*Let* $A_\sigma(f)_p := A_\sigma(f)_{L^p(\mathbb{R}^d)}$, $A_\sigma(f)_{p,\omega} := A_\sigma(f)_{L_{p,\omega}}$, *and* $A_\sigma(f)_{\mathcal{C}} := A_\sigma(f)_{\mathcal{C}(\mathbb{R}^d)}$.

**Definition 24** *Let* $\sigma > 0$, $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^d)$,

$$\vartheta_\sigma(t) := \frac{1}{\sigma^d}\prod_{j=1}^{d}\frac{\cos(\sigma t_j) - \cos(2\sigma t_j)}{t_j^2}, \quad t \in \mathbb{R}^d,$$

*and*

$$J(f,\sigma)(x) = \frac{1}{\pi^d}\int_{\mathbb{R}^d}\vartheta_\sigma(x-u)f(u)\,du, \quad x \in \mathbb{R}^d,$$

*be the de là Valèe Poussin operator ([25, pp. 304-306; (11)]).*

**Theorem 25** *([25, pp. 304-306]) It is known that, if* $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, *then,*

*(i)* $J(f,\sigma) \in \mathcal{G}_{2\sigma}(p)$,

*(ii)* $J(g_\sigma,\sigma) = g_\sigma$ *for any* $g_\sigma \in \mathcal{G}_\sigma(p)$,

*(iii)* $\|J(f,\sigma)\|_{L_p(\mathbb{R}^d)} \leq c\|f\|_{L_p(\mathbb{R}^d)}$.

**Theorem 26** *We suppose that* $0 < p < \infty$ *and* $\omega \in A_\infty$*. In this case, for any* $\sigma \in (0,\infty)$*, and* $f \in L_{p,\omega}$ *(*$f \in L_{p,\omega} \cap \mathbb{S}_0$ *when* $0 < p < 1$*), there holds*

$$A_\sigma(f)_{p,\omega} \leq c\left\|\left(I - V_{1/\sigma}\right)^r f\right\|_{p,\omega} \tag{22}$$

*with some constant* $c > 0$ *depending* $p, \omega, d$ *only.*

After the results of S. N. Bernstein [8, 1912], some systematic studies on approximation by exponential functions of degree$\leq \sigma$ for $d = 1$ or $d > 1$, continued by A. F. Timan [32], N. I. Akhieser [2], S. M. Nikolski [25], I. I. Ibragimov [18], H. Triebel [35], P. L. Butzer, H. J. Schmeisser and W. Sickel [30], R. M. Trigub and E. S. Belinsky [36]. These reference books contain several inequalities of exponential functions of degree$\leq \sigma$ in spaces $L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. Some other works also include results of approximation by exponential functions of degree$\leq \sigma$. See for example, [4], [6], [10], [12], [13], [14], [15], [16], [22], [24], [26], [27], [31], [28], [34], [33], [37]. For periodic $\omega \in A_p$, $1 < p < \infty$ and periodic $f \in L_{p,\omega}$, $(d = 1)$ some results on trigonometric approximation are known. See e.g. [1], [3], [5], [7], [23], [19], [20], [39].

## 4 Proofs

Suppose that $Q(x,\varepsilon)$ denotes the cube with center $x$ and sidelenght $2\varepsilon$.

**Definition 27** *([11, Def. 4.4.2; pp: 115-116]) (a) A family $\Psi$ of measurable sets $U\subset\mathbb{R}^d$ is called locally $N$-finite ($N\in\mathbb{N}$) if*

$$\sum\nolimits_{U\in\Psi}\chi_U(x)\leq N$$

*almost everywhere in $\mathbb{R}^d$ where $\chi_U$ is the characteristic function of the set $U$.*

*(b) A family $\Psi$ of open bounded sets $U\subset\mathbb{R}^d$ is locally 1-finite if and only if the sets $U\in\Psi$ are pairwise disjoint.*

**Definition 28** *Suppose that $B$ is a Banach space on $\mathbb{R}^d$ with norm $\|\cdot\|_B$. We set, for $f\in B$, $r\in\mathbb{N}$ and $\delta>0$,*

$$\inf_{\Delta^r g\in B}\left\{\|f-g\|_B+\delta^r\|\Delta^r g\|_B\right\}\equiv K_{\Delta^r}(f,\delta^r,B),$$

*where $\Delta f\equiv f_{x_1x_1}+...+f_{x_dx_d}$ is Laplace transform and $\Delta^r$ is $r$th iterate of $\Delta$.*

**Lemma 29** *([38, Theorem 16.14]) Let $1<p<\infty$, $\omega$ be a weight, $f\in L_{p,\omega}$ and $g\in L_{p',\omega}$. In this case, Hölder's inequality*

$$\int_{\mathbb{R}^d}|f(x)g(x)|\,\omega(x)\,dx\leq\|f\|_{p,\omega}\|g\|_{p',\omega} \tag{23}$$

*holds.*

**Theorem 30** *([9]) Let $\mathcal{F}$ be a family of couples $(f,g)$ of nonnegative functions and $d\in\mathbb{N}$. Suppose that, for some $p_0\in(0,\infty)$ and for every weight $\omega\in A_\infty$ there holds inequality*

$$\int_{\mathbb{R}^d}f(x)^{p_0}\,\omega(x)\,dx\leq c\int_{\mathbb{R}^d}g(x)^{p_0}\,\omega(x)\,dx,\quad(f,g)\in\mathcal{F}, \tag{24}$$

*provided the left hand side is finite. Then, for all $p\in(0,\infty)$ and all $\omega\in A_\infty$,*

$$\int_{\mathbb{R}^d}f(x)^{p}\,\omega(x)\,dx\leq c\int_{\mathbb{R}^d}g(x)^{p}\,\omega(x)\,dx,\quad(f,g)\in\mathcal{F}, \tag{25}$$

*holds when the left-hand side is finite.*

**Proof of Theorem 9.** Let $\Psi$ be 1-finite family of open bounded cubes $Q_i$ of $\mathbb{R}^d$ having Lebesgue measure 1 and with sides parallel to coordinate axes, such that $(\cup_i Q_i)\cup A=\mathbb{R}^d$ for some null-set $A$. Since $u\in\mathbb{R}^d$ there exists $m\in\mathbb{Z}^d$ such that $m\leq u<(m+2)$. Let $Q+m$ be translation of the cube $Q$ by vector $m$. We set $(Q_i+m)^{\pm}:=(Q_{i-1}\cup Q_i\cup Q_{i+1})+m$. Then

$$\|S_uf\|_{p,\omega}^p=\sum_{Q_i\in\Psi}\int_{Q_i}\left|\oint_{[-1/2,1/2]^d}f(x+u+t)dt\right|^p\omega(x)dx$$

$$\leq \sum_{Q_i\in\Psi}\int_{Q_i}\left[\ \oint_{Q(x+u,1/2)}\omega^{\frac{1}{p}}(t)\left|f(t)\right|\omega^{\frac{-1}{p}}(t)dt\right]^p\omega(x)dx$$

$$\leq \sum_{Q_i\in\Psi}\int_{Q_i}\left[\left(\ \oint_{Q(x+u,1/2)}\omega(t)\left|f(t)\right|^p dt\right)^{\frac{1}{p}}\left(\ \oint_{Q(x+u,1/2)}\omega^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}\right]^p\omega(x)dx$$

$$\leq \sum_{Q_i\in\Psi}\int_{Q_i}\ \int_{Q(x+u,1/2)}\omega(t)\left|f(t)\right|^p dt\left(\ \oint_{Q(x+u,1/2)}\omega^{\frac{-p'}{p}}(t)dt\right)^{\frac{p}{p'}}\omega(x)dx$$

$$\leq 3^{2dp}\sum_{Q_i\in\Psi}\ \oint_{(Q_i+m)^{\pm}}\omega(x)dx\left(\ \oint_{(Q_i+m)^{\pm}}\omega^{\frac{-1}{p-1}}(t)dt\right)^{p-1}\int_{(Q_i+m)^{\pm}}\omega(t)\left|f(t)\right|^p dt$$

$$\leq 3^{2dp}\left[\omega\right]_p\sum_{Q_i\in\Psi}\ \int_{(Q_i+m)^{\pm}}\left|f(t)\right|^p\omega(t)dt$$

$$\leq 3^{2dp}\left[\omega\right]_p\sum_{Q_i\in\Psi}\left\{\int_{Q_{i-1}+m}+\int_{Q_i+m}+\int_{Q_{i+1}+m}\right\}\left|f(t)\right|^p\omega(t)dt$$

$$\leq 3^{2dp}\left[\omega\right]_p\int_{\mathbb{R}^d}\left|f(t)\right|^p\omega(t)\left\{\sum_{Q_i\in\Psi}\left(\chi_{Q_{i-1}+m(t)}+\chi_{Q_i+m(t)}+\chi_{Q_{i+1}+m(t)}\right)\right\}dt$$

$$\leq 3^{2dp+1}\left[\omega\right]_p\left\|f\right\|_{p,\omega}^p.$$

For $p=1$ we find

$$\left\|S_u f\right\|_{1,\omega}=\sum_{Q_i\in\Psi}\int_{Q_i}\left|\ \oint_{[-1/2,1/2]^d}f(x+u+t)dt\right|\omega(x)dx$$

$$\leq \sum_{Q_i\in\Psi}\int_{Q_i}\ \int_{Q(x+u,1/2)}\omega(t)\left|f(t)\right|\frac{1}{\omega(t)}dt\omega(x)dx$$

$$\leq 3^d\sum_{Q_i\in\Psi}\frac{1}{\left|(Q_i+m)^{\pm}\right|}\int_{(Q_i+m)^{\pm}}\omega(x)dx\left(\underset{t\in(Q_i+m)^{\pm}}{esssup}\frac{1}{\omega(t)}\right)\int_{(Q_i+m)^{\pm}}\left|f(t)\right|\omega(t)dt$$

$$\leq 3^d\left[\gamma\right]_1\sum_{Q_i\in\Psi}\left\{\int_{Q_{i-1}+m}+\int_{Q_i+m}+\int_{Q_{i+1}+m}\right\}\left|f(t)\right|\omega(t)dt$$

$$= 3^d [\gamma]_1 \int\limits_{\mathbb{R}^d} |f(t)| \omega(t) \left\{ \sum_{Q_i \in Q} \chi_{Q_{i-1}+m}(t) + \sum_{Q_i \in Q} \chi_{Q_i+m}(t) + \sum_{Q_i \in Q} \chi_{Q_{i+1}+m}(t) \right\} dt$$

$$\leq 3^{d+1} [\gamma]_1 \int\limits_{\mathbb{R}^d} |f(t)| \omega(t) dt = 3^{d+1} [\gamma]_1 \|f\|_{1,\omega},$$

as required. ■

**Proof of Theorem 11.** Let $\omega \in A_\infty$. (a) First, we consider the case $p \in (1, \infty)$. Suppose that $f \in L_{p,\omega}$. Then, there is $a_0 \equiv e^{2^{11+d}[\omega]_\infty} > 1$ (see [17, p.786]) such that, for $\tilde{p} > a_0$, we have $\omega \in A_{\tilde{p}}$. Setting $a \equiv a_0 + 0,01$ we obtain $\omega \in A_a$.

(1°) If $a \leq p'$, then, $\omega \in A_{p'}$ and, hence, $\omega^{1-p} \in A_p$. By Theorem 9, for any $u \in \mathbb{R}^d$, there holds $S_u : L_{p',\omega} \hookrightarrow L_{p',\omega}$ and $S_u : L_{p,\omega^{1-p}} \hookrightarrow L_{p,\omega^{1-p}}$. Now, following step by step the proof of Theorem 1.1 of [21, p.369] of Jawerth, we find that $S_{u,\omega} : L_{p,\omega} \hookrightarrow L_{p,\omega}$ for any $u \in \mathbb{R}^d$.

(2°) If $a > p'$, then, $\omega \in A_a$ and, hence, $\omega^{1-a'} \in A_{a'}$. By Theorem 9, for any $u \in \mathbb{R}^d$, there holds $S_u : L_{a,\omega} \hookrightarrow L_{a,\omega}$ and $S_u : L_{a',\omega^{1-a'}} \hookrightarrow L_{a',\omega^{1-a'}}$. Again, following step by step the proof of Theorem 1.1 of [21, p.369] of Jawerth, we find that $S_{u,\omega} : L_{a',\omega} \hookrightarrow L_{a',\omega}$ for any $u \in \mathbb{R}^d$. Since $S_{u,\omega} : L_{\infty,\omega} \hookrightarrow L_{\infty,\omega}$ and $S_{u,\omega} : L_{a',\omega} \hookrightarrow L_{a',\omega}$, using Marcinkiewicz interpolation theorem, for any $p \in (a', \infty)$ we get $S_{u,\omega} : L_{p,\omega} \hookrightarrow L_{p,\omega}$ for any $u \in \mathbb{R}^d$. Namely, for $a > p'$, we have $S_{u,\omega} : L_{p,\omega} \hookrightarrow L_{p,\omega}$ for any $u \in \mathbb{R}^d$, as desired.

(b) We consider the case $\omega \in A_\infty$, $p \in (0, \infty)$ and $f \in L_{p,\omega}$. This case follows from (a) and extrapolation result Theorem 30. ■

**Proof of Theorem 13.** Let $0 < p < \infty$, $\omega \in A_\infty$ and $p^* \equiv \min\{1, p\}$. Then,

$$\|\mathcal{R}_{u,\omega} f\|_{p,\omega}^{p^*} = \left\| \sum_{k=0}^{1} \tfrac{1}{2^k c^k} (S_{u,\omega})^k f \right\|_{p,\omega}^{p^*} \leq 2 \sum_{k=0}^{1} \tfrac{1}{2^{kp^*} c^{kp^*}} \left\| (S_{u,\omega})^k f \right\|_{p,\omega}^{p^*} \leq 4 \|f\|_{p,\omega}^{p^*}.$$

■

**Proof of Theorem 14.** (a) By Remark 10(b), $C_c$ is a dense subset of $L_{p,\omega}$. First we consider the case $0 < p < 1$ and prove that $F_H(u)$ is bounded and uniformly continuous on $\mathbb{R}^d$ for functions $H \in C_c$, where $q, a, r$ and $G$ is from Definition 3 with $G \in L_{r',\omega}$ and $\|G\|_{r',\omega} = 1$. Boundedness of $F_H(\cdot)$ is easy consequence of the Hölder's inequality (23) and Theorem 13. Indeed:

$$|F_H(u)| \leq \int_{\mathbb{R}^d} |\mathcal{R}_{u,\omega} f(x)|^q |G(x)| \omega(x) dx \leq \|\mathcal{R}_{u,\omega} f\|_{p,\omega}^q \|G\|_{r',\omega} < \infty.$$

On the other hand, note that $H$ is uniformly continuous on $\mathbb{R}^d$, see e.g. Lemma 23.42 of [38, pp.557-558] for $d = 1$. Take $\varepsilon > 0$ and $u_1, u_2, x \in \mathbb{R}^d$. Then, for this $\varepsilon$, there exists a $\delta \equiv \delta(\varepsilon) > 0$ such that

$$|H(u_1 + x) - H(u_2 + x)| \leq \frac{\varepsilon^{1/q}}{2^q \left(1 + \langle\omega\rangle_{\text{supp}H}\right)}$$

when $|u_1 - u_2| < \delta$. Then,

$$|F_H(u_1)\text{-}F_H(u_2)| \leq \int_{\mathbb{R}^d} |\mathcal{R}_{u_1,\omega}H(x)^q\text{-}\mathcal{R}_{u_2,\omega}H(x)^q|\,|G(x)|\,\omega(x)\,dx$$

$$\leq 2^{q-1}\int_{\mathrm{supp}H} |\mathcal{R}_{u_1,\omega}H(x)\text{-}\mathcal{R}_{u_2,\omega}H(x)|^q\,|G(x)|\,\omega(x)\,dx$$

$$\leq \frac{2^{q-1}\varepsilon}{2^q\left(1+\langle\omega\rangle_{\mathrm{supp}H}\right)}\int_{\mathrm{supp}H}|G(x)|\,\omega(x)\,dx \leq \frac{\varepsilon\,\langle\omega\rangle_{\mathrm{supp}H}}{2\left(1+\langle\omega\rangle_{\mathrm{supp}H}\right)}\|G\|_{r',\omega} < \varepsilon.$$

Thus conclusion of Theorem 14 follows on $C_c$.

For the case $f \in L_{p,\omega}$ there exists an $H \in C_c$ so that

$$\|f-H\|_{p,\omega} < \frac{\xi^{1/q}}{2^{1/q}\left(1+2^q4^{q/p}\right)^{1/q}}$$

for any $\xi > 0$. Therefore

$$|F_f(u_1)-F_f(u_2)| \leq |F_f(u_1)-F_H(u_1)| +$$

$$+|F_H(u_1)-F_H(u_2)| + |F_H(u_2)-F_f(u_2)|$$

$$\leq 2^{q-1}\int_{\mathbb{R}^d}|\mathcal{R}_{u_1,\omega}f(x)\text{-}\mathcal{R}_{u_1,\omega}H(x)|^q\,|G(x)|\,\omega(x)\,dx + \frac{\xi}{2} +$$

$$+2^{q-1}\int_{\mathbb{R}^d}|\mathcal{R}_{u_2,\omega}H(x)\text{-}\mathcal{R}_{u_2,\omega}f(x)|^q\,|G(x)|\,\omega(x)\,dx$$

$$\leq 2^{q-1}\|\mathcal{R}_{u_1,\omega}(f-H)\|_{p,\omega}^q + 2^{q-1}\|\mathcal{R}_{u_2,\omega}(f-H)\|_{p,\omega}^q + \frac{\xi}{2}$$

$$\leq 2^q4^{q/p}\|f-H\|_{p,\omega}^q + \frac{\xi}{2} \leq 2^q4^{q/p}\frac{\xi}{2\left(1+2^q4^{q/p}\right)} + \frac{\xi}{2} \leq \frac{\xi}{2}+\frac{\xi}{2} = \xi.$$

As a result we have $F_f \in \mathcal{C}(\mathbb{R}^d)$. In the case $1 \leq p < \infty$, proof of $F_f \in \mathcal{C}(\mathbb{R}^d)$ is the same with minor modification of above proof. ■

**Proof of Theorem 5.** Let $0 < p < \infty$, $\omega \in A_\infty$, and $0 \leq f, g \in L_{p,\omega}$. If $\|g\|_{p,\omega} = \|f\|_{p,\omega}$ or $\|g\|_{p,\omega} = \|f\|_{p,\omega} = 0$, then, result (25) is obvious. So we assume that $\|g\|_{p,\omega}, \|f\|_{p,\omega} > 0$ and $\|g\|_{p,\omega} \neq \|f\|_{p,\omega}$. Case (1°): Let $1 \leq p < \infty$ and we define, for $g \in L_{p,\omega}$, function

$$F_g(u, G_0, p, \omega) = \int_{\mathbb{R}^d}\mathcal{R}_{u,\omega}g(x)\,|G_0(x)|\,\omega(x)\,dx, \quad u \in \mathbb{R}^d$$

with $G_0 \in \mathcal{Z}(p,\omega)$ and use this to obtain

$$\|F_g\|_{\mathcal{C}(\mathbb{R}^d)} = \left\|\int_{\mathbb{R}^d}\mathcal{R}_{u,\omega}g(x)\,|G_0(x)|\,\omega(x)\,dx\right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$\leq \sup_{u\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{R}_{u,\omega} g(x)| \, |G_0(x)| \, \omega(x) \, dx \leq \sup_{u\in\mathbb{R}^d} \|\mathcal{R}_{u,\omega} g\|_{p,\omega} \|G_0\|_{p',\omega} \leq c \|g\|_{p,\omega} .$$

On the other hand, for any $\varepsilon > 0$ (see e.g. Theorem 18.4 of [38]) we can choose appropriately an $G_\varepsilon \in \mathcal{Z}(p,\omega)$ satisfying

$$\int_{\mathbb{R}^d} f(x) \, |G_\varepsilon(x)| \, \omega(x) \, dx \geq \|f\|_{p,\omega} - \varepsilon,$$

and one can find

$$\|F_f(u, G_\varepsilon, p, \omega)\|_{\mathcal{C}(\mathbb{R}^d)} \geq |F_f(0, G_\varepsilon, p, \omega)| \geq \int_{\mathbb{R}^d} \mathcal{R}_{0,\omega} f(x) \, |G_\varepsilon(x)| \, \omega(x) \, dx$$

$$\geq \int_{\mathbb{R}^d} f(x) \, |G_\varepsilon(x)| \, \omega(x) \, dx = \|f\|_{p,\omega} - \varepsilon.$$

By hypothesis, we get, for any $\varepsilon > 0$,

$$\|f\|_{p,\omega} - \varepsilon \leq \|F_f(\cdot, G_\varepsilon, p, \omega)\|_{\mathcal{C}(\mathbb{R}^d)} \leq c \|F_g(\cdot, G_0, p, \omega)\|_{\mathcal{C}(\mathbb{R}^d)} \leq C \|g\|_{p,\omega} . \quad (26)$$

Now taking as $\varepsilon \to 0+$ we find $\|f\|_{p,\omega} \leq C \|g\|_{p,\omega}$. In the general case $f, g \in L_{p,\omega}$ we get $\|f\|_{p,\omega} \leq 2C \|g\|_{p,\omega}$.

Case (2°): Case $p \in (0,1)$ can be obtained using the same procedure given in the Case (1°) with small modifications.

$$\|F_g\|_{\mathcal{C}(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} (\mathcal{R}_{u,\omega} g(x))^q \left| \tilde{G}_0(x) \right| \omega(x) \, dx \right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$\leq \sup_{u\in\mathbb{R}^d} \|\mathcal{R}_{u,\omega} g\|^q_{p,\omega} \left\| \tilde{G}_0 \right\|_{r',\omega} \leq c \|g\|^q_{p,\omega} .$$

On the other hand, for any $\varepsilon > 0$ and appropriately chosen $\tilde{G}_\varepsilon \in \mathcal{Z}(r,\omega)$ with

$$\int_{\mathbb{R}^d} f(x)^q \left| \tilde{G}_\varepsilon(x) \right| \omega(x) \, dx \geq \|f\|^q_{p,\omega} - \varepsilon,$$

one can find

$$\left\| F_f\left(\cdot, \tilde{G}_\varepsilon, p, \omega\right) \right\|_{\mathcal{C}(\mathbb{R}^d)} \geq |F_f(0)| \geq \int_{\mathbb{R}^d} (\mathcal{R}_{0,\omega} f(x))^q \, |G(x)| \, \omega(x) \, dx$$

$$\geq \int_{\mathbb{R}^d} f(x)^q \, |G(x)| \, \omega(x) \, dx \geq \|f\|^q_{p,\omega} - \varepsilon.$$

Then by hypothesis, for any $\varepsilon > 0$,

$$\|f\|^q_{p,\omega} - \varepsilon \leq \left\| F_f\left(\cdot, \tilde{G}_\varepsilon, p, \omega\right) \right\|_{\mathcal{C}(\mathbb{R}^d)} \leq c \left\| F_g\left(\cdot, \tilde{G}_0, p, \omega\right) \right\|_{\mathcal{C}(\mathbb{R}^d)} \leq C \|g\|^q_{p,\omega} .$$

If we take $\varepsilon \to 0+$, then we obtain desired result $\|f\|_{p,\omega} \leq C \|g\|_{p,\omega}$. For the general case $f, g \in L_{p,\omega}$ we get $\|f\|_{p,\omega} \leq 2C \|g\|_{p,\omega}$. ■

**Proof of Theorem 16.** Equalities in (16) is follow from definitions:

$$S_{u,\omega}S_{\delta,v}f(\cdot)=(\langle\omega\rangle_{[-1/2,1/2]^d})^{-1}\int\limits_{[-1/2,1/2]^d}(S_{\delta,v}f)(\cdot+u+t)\,\omega(t)\,dt$$

$$=(\langle\omega\rangle_{[-1/2,1/2]^d})^{-1}\int\limits_{[-1/2,1/2]^d}\int\limits_{[-\delta/2,\delta/2]^d}f(\cdot+u+t+v+s)\,ds\omega(t)\,dt$$

$$=\int\limits_{[-\delta/2,\delta/2]^d}(\langle\omega\rangle_{[-1/2,1/2]^d})^{-1}\int\limits_{[-1/2,1/2]^d}f(\cdot+u+t+v+s)\,\omega(t)\,dtds$$

$$=\int_{[-\delta/2,\delta/2]^d}S_{u,\omega}f(\cdot+v+s)\,ds=S_{\delta,v}S_{u,\omega}f(\cdot)\,. \tag{27}$$

Second equality in (16) is follow from (27). Equality in (17) follows from (16) and (3). Now we give the proof of inequality in (17). Since $F_{S_{\delta,v}}=S_{\delta,v}F_f$, we get by (26) that

$$\|S_{\delta,u}f\|_{p,\omega}\leq\left\|F_{S_{\delta,v}}\right\|_{\mathcal{C}(\mathbb{R}^d)}=\|S_{\delta,v}F_f\|_{\mathcal{C}(\mathbb{R}^d)}\leq\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}\leq c\,\|f\|_{p,\omega}\,.$$

■

**Proof of Theorem 21.** ($1^\circ$) Let $p\in(1,\infty)$. Since $F_{V_\delta f}=V_\delta(F_f)$, we get $F_{(V_\delta)^s f}=(V_\delta)^s(F_f)$ for any $s\in\mathbb{N}$. For any $N\in\mathbb{N}$, we have

$$F_{\sum_{s=0}^N(-1)^sC_s^k(V_\delta)^sf}=\sum\nolimits_{s=0}^N(-1)^sC_s^k(V_\delta)^s(F_f)\,,$$

$$\left\|F_{\sum_{s=0}^N(-1)^sC_s^k(V_\delta)^sf}\right\|_{\mathcal{C}(\mathbb{R}^d)}=\left\|\sum\nolimits_{s=0}^N(-1)^sC_s^k(V_\delta)^s(F_f)\right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$\leq\sum\nolimits_{s=0}^N\left|C_s^k\right|\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}\leq\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}+\sum\nolimits_{s=1}^\infty\frac{c(k)}{s^{1+k}}\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$<c\,\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}$$

by $\left|C_s^k\right|\leq c_ks^{-1-k}$ ([29, p.14, (1.51)]). Now using Corollary 19 and Theorem 5 we obtain

$$\left\|(E-V_\delta)^kf\right\|_{p,\omega}=\lim_{N\to\infty}\left\|\sum\nolimits_{s=0}^N(-1)^sC_s^k(V_\delta)^sf\right\|_{p,\omega}$$

$$\leq\lim_{N\to\infty}\left\|F_{\sum_{s=0}^N(-1)^sC_s^k(V_\delta)^sf}\right\|_{\mathcal{C}(\mathbb{R}^d)}=\lim_{N\to\infty}\left\|\sum\nolimits_{s=0}^N(-1)^sC_s^k(V_\delta)^s(F_f)\right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$\leq 2^k\|F_f\|_{\mathcal{C}(\mathbb{R}^d)}\leq c\,\|f\|_{p,\omega}\,.$$

($2^\circ$) For general case $p\in(0,\infty)$, we use ($1^\circ$) and Theorem 30 to finish proof.

■

**Proof of Theorem 7.** Let $1<p<\infty$, $\omega\in A_\infty$, and $(f,g_j)\in\mathcal{F}$ with $f,g_j\in L_{p,\omega}$. If $\|g_j\|_{l_s^m(L_{p,\omega})}=\|f\|_{p,\omega}=0$, then, result (9) is obvious. So we assume that $\|g_j\|_{l_s^m(L_{p,\omega})},\|f\|_{p,\omega}>0$. Since $\omega\in A_\infty$, there is $a_0\equiv e^{2^{11+d}[\omega]_\infty}>1$ ([17, p.786]) such that, we obtain $\omega\in A_a$ with $a\equiv a_0+0,01$. Let $\Psi$ be 1-finite family of open bounded cubes $Q_i$ of $\mathbb{R}^d$ having Lebesgue measure 1, such that $(\cup_i Q_i)\cup A=\mathbb{R}^d$ for some null-set $A$. Then,

$$\left\|F_{g_j}\right\|_{L_a}^a=\int_{\mathbb{R}^d}\left|F_{g_j}(u)\right|^a du=\sum_{Q_i\in\Psi}\int_{Q_i}\left|F_{g_j}(u)\right|^a du$$

$$\leq\sum_{Q_i\in\Psi}\int_{Q_i}c^a\|g_j\|_{p,\omega}^a\chi_{Q_i}(u)\,du=c^a\|g_j\|_{p,\omega}^a\sum_{Q_i\in\Psi}\int_{Q_i}\chi_{Q_i}(u)\,du=c^a\|g_j\|_{p,\omega}^a.$$

In this case

$$\|F_f\|_{L_a}\leq c\left\|F_{g_j}\right\|_{l_s^m(L_a)}=c\left(\sum\nolimits_{j=0}^m\left\|F_{g_j}\right\|_{L_a}^s\right)^{1/s}$$

$$=c\left(\sum\nolimits_{j=0}^m\|g_j\|_{p,\omega}^s\right)^{1/s}=c\|g\|_{l_s^m(L_{p,\omega})}.$$

On the other hand,

$$\|F_f\|_{L_a}=\left(\int_{\mathbb{R}^d}|F_f(u)|^a du\right)^{1/a}\geq\left(\int_{[0,1]^d}|F_f(u)|^a du\right)^{1/a}\geq\|f\|_{p,\omega}.$$

Combining these inequalities we get

$$\|f\|_{p,\omega}\leq\|F_f\|_{L_a}\leq c\left\|F_{g_j}\right\|_{l_s^m(L_a)}\leq c\|g\|_{l_s^m(L_{p,\omega})}. \tag{28}$$

■

**Proof of Theorem 22.** Let $r\in\mathbb{N}$, $\omega\in A_\infty$, $p\in(1,\infty)$, $\delta\in(0,\infty)$, and $f\in L_{p,\omega}$. Then there is $a_0\equiv e^{2^{11+d}[\omega]_\infty}>1$ ([17, p.786]) such that, we obtain $\omega\in A_a$ with $a\equiv a_0+0,01$. Then there exist $m\in\mathbb{N}$ such that

$$\left\|(E\text{-}T_\delta)^{2r}(F_f)\right\|_{L_a}^s\geq c\sum_{j=0}^m 2^{-j2rs}K_{\Delta^r}\left(F_f,\left(2^j\delta\right)^{2r+2},L_a\right)^s,$$

for $s\equiv\max\{a,2\}$. On the other hand, we know that

$$\|(E-V_\delta)^r(F_f)\|_{L_a}\approx K_{\Delta^r}\left(F_f,\delta^{2r},L_a\right)\approx\left\|(E-T_\delta)^{2r}(F_f)\right\|_{L_a}.$$

As a consequence,

$$\|(E\text{-}V_\delta)^r(F_f)\|_{L_a}^s\geq c\sum_{j=0}^m 2^{-j2rs}\left\|(E\text{-}V_\delta)^{r+1}(F_f)\right\|_{L_a}^s.$$

From the last inequality and Theorem 7 we obtain (20). ■
**Proof of Theorem 26.** It is enough to proof

$$A_{2\sigma}(f)_{p(\cdot)} \le c \left\| \left(I - V_{1/(2\sigma)}\right)^r f \right\|_{p(\cdot)}. \tag{29}$$

Let $g_\sigma$ be an exponential type entire function of degree $\le \sigma$, belonging to $\mathcal{C}(\mathbb{R}^d)$, as the best approximation of $F_f \in \mathcal{C}(\mathbb{R}^d)$. Since $F_{J(f,\sigma)} = J(F_f, \sigma)$ and $J(g_\sigma, \sigma) = g_\sigma$, there holds

$$A_{2\sigma}(f)_{p,\omega} \le \|f - J(f,\sigma)\|_{p,\omega} \le c \left\| F_{f-J(f,\sigma)} \right\|_{\mathcal{C}(\mathbb{R}^d)} = c \left\| F_f - F_{J(f,\sigma)} \right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$= c \|F_f - J(F_f,\sigma)\|_{\mathcal{C}(\mathbb{R}^d)} = c \|F_f - g_\sigma + g_\sigma - J(F_f,\sigma)\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$= c \|F_f - g_\sigma + J(g_\sigma,\sigma) - J(F_f,\sigma)\|_{\mathcal{C}(\mathbb{R}^d)} = c \|F_f - g_\sigma + J(g_\sigma - F_f,\sigma)\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$\le c(A_\sigma(F_f)_{\mathcal{C}(\mathbb{R}^d)} + cA_\sigma(F_f)_{\mathcal{C}(\mathbb{R}^d)}) = cA_\sigma(F_f)_{\mathcal{C}(\mathbb{R}^d)}.$$

Therefore

$$A_{2\sigma}(f)_{p,\omega} \le cA_\sigma(F_f)_{\mathcal{C}(\mathbb{R}^d)} \le c \left\| \left(I - T_{\frac{1}{2\sigma}}\right)^{2r} (F_f) \right\|_{\mathcal{C}(\mathbb{R}^d)} \le c \left\| \left(I - V_{\frac{1}{2\sigma}}\right)^{r} (F_f) \right\|_{\mathcal{C}(\mathbb{R}^d)}$$

$$= c \left\| F_{\left(I - V_{1/(2\sigma)}\right)^r f} \right\|_{\mathcal{C}(\mathbb{R}^d)} \le c \left\| \left(I - V_{1/(2\sigma)}\right)^r f \right\|_{p,\omega}.$$

■

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