Effective exponential bounds on the prime gaps

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Abstract:

Over the last 50 years a large number of effective exponential bounds on the first Chebyshev function $\vartheta(x)$ have been obtained. Specifically we shall be interested in effective exponential bounds of the form

$$|\vartheta(x) - x| < a\, x\, (\ln x)^b\, \exp\left(-c\,\sqrt{\ln x}\right); \qquad (x \geq x_0).$$

Herein we shall convert these effective bounds on $\vartheta(x)$ into effective exponential bounds on the prime gaps $g_n = p_{n+1} - p_n$. Specifically we shall establish a number of effective exponential bounds of the form

$$\frac{g_n}{p_n} < \frac{2a\, (\ln p_n)^b\, \exp\left(-c\,\sqrt{\ln p_n}\right)}{1 - a\, (\ln p_n)^b\, \exp\left(-c\,\sqrt{\ln p_n}\right)}; \qquad (x \geq x_*);$$

and

$$\frac{g_n}{p_n} < 3a\, (\ln p_n)^b\, \exp\left(-c\,\sqrt{\ln p_n}\right); \qquad (x \geq x_*);$$

for some effective computable x_*. It is the explicit presence of the exponential factor, with known coefficients and known range of validity for the bound, that makes these bounds particularly interesting.

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Contents

1 Introduction

The last 50 years have seen the development of a large number of fully effective exponential bounds on the first Chebyshev function $\vartheta(x)$ — bounds of the form:

$$|\vartheta(x) - x| < a\, x\, (\ln x)^b\, \exp\left(-c\sqrt{\ln x}\right); \qquad (x \geq x_0). \tag{1.1}$$

See references [1–5]. Here $a > 0$ always, while typically $b \geq 0$, and $c > 0$ always. The special case $b = 0$ corresponds to effective bounds of the de la Valle Poussin form [6, 7]. For some widely applicable effective bounds of this form see Table I. (An elementary computation is required for the numerical coefficients in the Schoenfeld [1] and Trudgian [2] bounds.)

Table 1. Some widely applicable effective bounds on the first Chebyshev function $\vartheta(x)$.

a	b	c	x_0	Source
0.2196138920	1/4	0.3219796502	101	Schoenfeld [1]
1/4	1/4	1/4	31	relaxed version of Schoenfeld [1]
0.2428127763	1/4	0.3935970880	149	Trudgian [2]
1/4	1/4	1/3	43	relaxed version of Trudgian [2]
9.220226	3/2	0.8476836	2	Fiori–Kadiri–Swidinsky [4]
9.40	1.515	0.8274	2	Johnston–Yang [3]
0.3510691792	0	1/4	101	Visser [7]; based on Schoenfeld [1]
0.2748124978	0	1/4	149	Visser [7]; based on Trudgian [2]
0.4242102935	0	1/3	149	Visser [7]; based on Trudgian [2]
295	0	1/2	2	Visser [7]; based on FKS [4]
385	0	1/2	2	Visser [7]; based on JY [3]
1	0	1/4	2	Visser [7]
1	0	1/3	3	Visser [7]
1/2	0	1/4	29	Visser [7]
1/2	0	1/3	41	Visser [7]

For some asymptotically more stringent effective bounds, but valid on more restricted regions see Table II, (based on reference [3]), and Table III, (based on reference [7]). See also the extensive tabulations in reference [5].

Table 2. Asymptotically stringent bounds on the first Chebyshev function $\vartheta(x)$ valid on restricted regions [3].

a	b	c	x_0
8.87	1.514	0.8288	exp(3000)
8.16	1.512	0.8309	exp(4000)
7.66	1.511	0.8324	exp(5000)
7.23	1.510	0.8335	exp(6000)
7.00	1.510	0.8345	exp(7000)
6.79	1.509	0.8353	exp(8000)
6.59	1.509	0.8359	exp(9000)
6.73	1.509	0.8359	exp(10000)
23.14	1.503	0.8659	$\exp(10^5)$
38.58	1.502	1.0318	$\exp(10^6)$
42.91	1.501	1.0706	$\exp(10^7)$
44.42	1.501	1.0839	$\exp(10^8)$
44.98	1.501	1.0886	$\exp(10^9)$
45.18	1.501	1.0903	$\exp(10^{10})$

Table 3. More asymptotically stringent bounds on the first Chebyshev function $\vartheta(x)$ of the de la Valle Poussin form valid on restricted regions [7]. (Based on reference [3].)

a	b	c	x_0
357	0	1/2	exp(3000)
320	0	1/2	exp(4000)
295	0	1/2	exp(5000)
274	0	1/2	exp(6000)
263	0	1/2	exp(7000)
252	0	1/2	exp(8000)
243	0	1/2	exp(9000)
249	0	1/2	exp(10000)
644	0	1/2	$\exp(10^5)$
348	0	1/2	$\exp(10^6)$
312	0	1/2	$\exp(10^7)$
301	0	1/2	$\exp(10^8)$
298	0	1/2	$\exp(10^9)$
297	0	1/2	$\exp(10^{10})$
1642333	0	1	$\exp(10^6)$
165152	0	1	$\exp(10^7)$
101831	0	1	$\exp(10^8)$
87551	0	1	$\exp(10^9)$
83063	0	1	$\exp(10^{10})$

Herein we shall show how to convert these effective bounds on $\vartheta(x)$ into effective bounds on the prime gaps $g_n = p_{n+1} - p_n$. Specifically we shall establish both

$$\frac{g_n}{p_n} < \frac{2a\,(\ln p_n)^b\,\exp\left(-c\,\sqrt{\ln p_n}\right)}{1 - a\,(\ln p_n)^b\,\exp\left(-c\,\sqrt{\ln p_n}\right)}; \qquad (x \geq x_*); \tag{1.2}$$

and

$$\frac{g_n}{p_n} < 3a\,(\ln p_n)^b\,\exp\left(-c\,\sqrt{\ln p_n}\right); \qquad (x \geq x_*); \tag{1.3}$$

for some effective computable x_*. In all cases it is the presence of the exponential factor that is central to making these bounds interesting and relatively stringent.

2 Strategy

Let us now develop some effective bounds on prime gaps $g_n = p_{n+1} - p_n$, starting from effective bounds on the first Chebyshev function of the form

$$|\vartheta(x) - x| < a\,x\,(\ln x)^b\,\exp\left(-c\,\sqrt{\ln x}\right); \qquad (x \geq x_0). \tag{2.1}$$

For convenience rewrite our bound on the first Chebyshev function in the form

$$|\vartheta(x) - x| < x\,f(x); \qquad (x \geq x_0). \tag{2.2}$$

Here $f(x) = a(\ln x)^b \exp\left(-c\sqrt{\ln x}\right)$ is easily verified to be monotone decreasing for $x > x_{peak} = \exp([2b/c]^2)$, where it takes on the value $f_{peak} = a(2b/c)^{2b}\exp(-2b)$.

Define

$$x_* = \max\left\{x_0,\ \exp\left(\left[\frac{2b}{c}\right]^2\right)\right\}. \tag{2.3}$$

Then in the range $x \geq x_*$ the inequality (2.1) is valid with $f'(x) \leq 0$. This will be the primary range of interest for the following computations. Note that in the limit $b \to 0$, appropriate to effective bounds of the de la Valle Poussin form, one has

$$x_* \to \max\{x_0,\ 1\} = x_0. \tag{2.4}$$

Let us now take any $\epsilon \in (0,1)$ and consider the inequality

$$\vartheta(p_{n+1} - \epsilon) - (p_{n+1} - \epsilon) > -(p_{n+1} - \epsilon)\,f(p_{n+1} - \epsilon); \tag{2.5}$$

Thence

$$p_{n+1} < \vartheta(p_n) + \epsilon + (p_{n+1} - \epsilon)\,f(p_{n+1} - \epsilon). \tag{2.6}$$

But since this holds for all $\epsilon \in (0,1)$ we can in particular consider the limit $\epsilon \to 0$

and so deduce

$$p_{n+1} \leq \vartheta(p_n) + p_{n+1}\, f(p_{n+1}). \tag{2.7}$$

On the other hand from

$$\vartheta(p_n) - p_n < p_n\, f(p_n); \tag{2.8}$$

we deduce

$$p_n > \vartheta(p_n) - p_n\, f(p_n); \tag{2.9}$$

Thence we can bound the prime gaps as

$$g_n < p_{n+1}\, f(p_{n+1}) + p_n\, f(p_n). \tag{2.10}$$

We now have two options:

- If $f_{peak} \leq 1$ then use $p_{n+1} = p_n + g_n$, and the fact that $f(x)$ is monotone decreasing in the range of interest, to deduce

  $$g_n < (2p_n + g_n)\, f(p_n), \tag{2.11}$$

  Rearranging, and using the fact that $f(x) < 1$ in the range of interest, we see

  $$\frac{g_n}{p_n} < \frac{2\, f(p_n)}{1 - \ f(p_n)}; \qquad (f_{peak} \leq 1). \tag{2.12}$$

- If $f_{peak} > 1$ it is more useful to use the standard Bertrand–Chebyshev theorem $p_{n+1} < 2p_n$, and the fact that $f(x)$ is monotone decreasing in the range of interest, to deduce

  $$\frac{g_n}{p_n} < 3\, f(p_n); \qquad (f_{peak} \text{ arbitrary}). \tag{2.13}$$

We can summarize this in a simple Lemma.

Lemma: *If one has somehow established a bound of the form*

$$|\vartheta(x) - x| < a\, x\, (\ln x)^b\, \exp\left(-c\sqrt{\ln x}\right); \qquad (x \geq x_0); \tag{2.14}$$

as in Tables I, II, and III above, then defining

$$x_* = \max\left\{x_0,\ \exp\left(\left[\frac{2b}{c}\right]^2\right)\right\}, \tag{2.15}$$

for the prime gap $g_n = p_{n+1} - p_n$ one has the bounds

$$\frac{g_n}{p_n} < \frac{2a\,(\ln p_n)^b\;\exp\left(-c\,\sqrt{\ln p_n}\right)}{1 - a\,(\ln p_n)^b\;\exp\left(-c\,\sqrt{\ln p_n}\right)}; \qquad (x \geq x_*;\; f_{peak} \leq 1); \tag{2.16}$$

$$\frac{g_n}{p_n} < 3a\,(\ln p_n)^b\;\exp\left(-c\,\sqrt{\ln p_n}\right); \qquad (x \geq x_*;\; f_{peak}\ \textit{arbitrary}); \tag{2.17}$$

These bounds certainly hold for $x \geq x_*$, but if x_* is sufficiently small one might be able to widen the range of applicability to some $x \geq x_{**}$, with $x_{**} \leq x_*$, by explicit computation.

3 Effective bounds on the prime gaps

3.1 Widely applicable bounds

For some widely applicable bounds of the form

$$\frac{g_n}{p_n} < \frac{2a\,(\ln p_n)^b\;\exp\left(-c\,\sqrt{\ln p_n}\right)}{1 - a\,(\ln p_n)^b\;\exp\left(-c\,\sqrt{\ln p_n}\right)}; \qquad (p_n \geq x_{**};\; f_{peak} \leq 1); \tag{3.1}$$

consider Table IV below. For any collection of coefficients $\{a, b, c\}$ one first calculates x_{peak} and checks that $f_{peak} < 1$. From that and x_0 one determines x_*. Finally, for x_* sufficiently small, one determines x_{**} by explicit computation.

Table 4. Some widely applicable effective bounds on the relative prime gap g_n/p_n. Compare with parts of Table I.

a	b	c	x_0	x_{peak}	f_{peak}	x_*	x_{**}
0.2196138920	1/4	0.3219796502	101	11.15042039	0.1659905476	101	11
1/4	1/4	1/4	31	54.59815003	0.2144409711	55	11
0.2428127763	1/4	0.3935970880	149	5.021606990	0.1659905476	149	11
1/4	1/4	1/3	43	9.487735836	0.1857113288	43	11
0.3510691792	0	1/4	101	1	0.3510691792	101	2
0.2748124978	0	1/4	149	1	0.2748124978	149	11
0.4242102935	0	1/3	149	1	0.4242102935	149	2
1	0	1/4	2	1	1	2	2
1	0	1/3	3	1	1	3	2
1/2	0	1/4	29	1	1/2	29	2
1/2	0	1/3	41	1	1/2	41	2

3.2 Some intermediate strength bounds

Now consider some intermediate strength bounds, (now trading off the range of applicability versus tightness of the bound), based on the Fiori–Kadiri–Swidinsky [4] and Johnston–Yang [3] results. Consider the coefficients presented in Table V, applied to bounds of the form

$$\frac{g_n}{p_n} < 3a\,(\ln p_n)^b\ \exp\left(-c\,\sqrt{\ln p_n}\right); \qquad (x \geq x_*;\ \ f_{peak} \text{ arbitrary}); \tag{3.2}$$

For any collection of coefficients $\{a, b, c\}$ one first calculates x_{peak}, (and also verifies $f_{peak} > 1$). From that and x_0 one determines x_*, which is sometimes distressingly large. Finally one determines x_{**} by direct computation. Unfortunately the resulting bounds, while widely applicable, are not particularly stringent.

Table 5. Some intermediate strength effective bounds on the relative prime gap g_n/p_n. Based on Fiori–Kadiri–Swidinsky [4] and Johnston–Yang [3].
Compare with parts of Table I.

a	b	c	x_0	x_{peak}	f_{peak}	x_*	x_{**}
9.220226	3/2	0.8476836	2	275108.1632	20.34794437	275109	2
9.40	1.515	0.8274	2	667160.3762	23.19042582	667161	2
295	0	1/2	2	1	295	2	2
385	0	1/2	2	1	385	2	2

3.3 Asymptotically stringent bounds

Finally, based on Tables II and III, consider asymptotically stringent bounds of the form

$$\frac{g_n}{p_n} < 3a\,(\ln p_n)^b\ \exp\left(-c\,\sqrt{\ln p_n}\right); \qquad (x \geq x_*;\ f_{peak} \text{ arbitrary}); \tag{3.3}$$

For any collection of coefficients $\{a, b, c\}$ one first calculates x_{peak}. From that and x_0 one determines x_*.

- For all entries in Table II it is easy to verify that $x_{peak} = \exp([2b/c]^2) \ll x_0$, (and for that matter, $f_{peak} > 1$). Thence for all entries in Table II one has $x_* = x_0$. Since x_* is truly enormous direct computation of x_{**} is hopeless. In short, the effective bounds on $\vartheta(x)$ given in terms of the parameters $\{a, b, c, x_0\}$ of Table II directly imply effective bounds (3.3) on g_n/p_n in terms of the same parameters $\{a, b, c, x_0\}$.

- For all entries in Table III, since they are all of de la Valle Poussin form, (that is, $b = 0$), it is trivial to verify that $x_{peak} = \exp([2b/c]^2) = 1$, (and for that matter, $f_{peak} = a > 1$). Thence for all entries in Table III one trivially has $x_* = x_0$. Since x_* is truly enormous direct computation of x_{**} is hopeless. In short, the effective bounds on $\vartheta(x)$ given in terms of the parameters $\{a, b, c, x_0\}$ of Table III directly imply effective bounds (3.3) on g_n/p_n in terms of the same parameters $\{a, b, c, x_0\}$.

4 Conclusions

We have developed a number of effective bounds on the prime gaps g_n/p_n. Some of these effective bounds could in principle have been deduced almost 50 years ago. Others rely on recent numerical work from the previous decade. In the interests of clarity, let me quote a few explicit examples:

$$\frac{g_n}{p_n} < \frac{\frac{1}{2}(\ln p_n)^{1/4}\exp\left(-\sqrt{\ln p_n}/3\right)}{1-\frac{1}{4}(\ln p_n)^{1/4}\exp\left(-\sqrt{\ln p_n}/3\right)}; \qquad (p_n \geq 2); \tag{4.1}$$

$$\frac{g_n}{p_n} < \frac{\exp\left(-\sqrt{\ln p_n}/3\right)}{1-\frac{1}{2}\exp\left(-\sqrt{\ln p_n}/3\right)}; \qquad (p_n \geq 2); \tag{4.2}$$

$$\frac{g_n}{p_n} < 885\exp\left(-\sqrt{\ln p_n}/2\right); \qquad (p_n \geq 2); \tag{4.3}$$

and the asymptotically tighter result

$$\frac{g_n}{p_n} < 4926999\exp\left(-\sqrt{\ln p_n}\right); \qquad (p_n \geq \exp(10^6)). \tag{4.4}$$

In all cases it is the presence of the exponential factor that is central to making these bounds interesting and relatively stringent.

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