Best constants in bipolar L^p- Hardy-type Inequalities *

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Abstract

In this work we prove sharp L^p versions of the multipolar Hardy inequalities in [8, 10], in the case of a bipolar potential and $p \geq 2$. Our results are sharp and minimizers do exist in the energy space. New features appear when $p > 2$ compared to the linear case $p = 2$ at the level of criticality of the p-Laplacian $-\Delta_p$ perturbed by a singular Hardy bipolar potential.

1 Introduction

In the last decades, motivated by problems in quantum mechanics, there has been a consistent interest in Schrödinger operators with multi-singular potentials and their applications to spectral theory and partial differential equations. It is well-known that qualitative properties of such operators are related to the validity of Hardy-type inequalities. A significant work has been done in the L^2-setting for linear Hamiltonians of the form $H := -\Delta - W$ where W denotes a potential with n quadratic singular poles a_i, with $i = \overline{1, n}$, in the Euclidian space $\mathbb{R}^N$, $N \geq 3$. The most studied cases focus especially on $W^{(1)} = \sum_{i=1}^{n} \frac{\lambda_i}{|x-a_i|^2}$, $\lambda_i \in \mathbb{R}$, and $W^{(2)} = \sum_{1\leq i<j\leq n} \frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2}$. Various Hardy-inequalities were proved for $W^{(1)}$ and applied then to the well-posedness and asymptotic behaviour to some nonlinear elliptic equations in a series of papers by Felli-Terracini (e.g. [12], [13], [11]) and more recently by Canale et al. in [5, 6, 7], in the context of evolution problems involving Kolmogorov operators. As far as we know, the potential $W^{(2)}$ was firstly

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analyzed by Bosi-Dolbeault-Esteban in [4] in order to determine lower bounds for the spectrum of some Schrödinger and Dirac-type operators and later on by Cazacu et. al in [8, 10]. More precisely, in [10] the authors proved the following multipolar Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{n^2} \int_{\mathbb{R}^N} W^{(2)} |u|^2 dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \tag{1.1}$$

with $N \geq 3$ and $n \geq 2$, where the constant $\frac{(N-2)^2}{n^2}$ in (1.1) is sharp and not achieved in the energy space $\mathcal{D}^{1,2}(\mathbb{R}^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}} |\nabla u|^2 \, dx < \infty \right\}$. The main goal here is to extend (1.1) to the L^p-setting. As far as we know such a result has not been obtained yet. For a one singular potential of the form $W^{(3)} = 1/|x|^p$ there is a huge variety of functional inequalities and applications around the celebrated L^p Hardy inequality which holds for any $N > 1$ and $1 \leq p < N$

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \tag{1.2}$$

where the constant $\left(\frac{N-p}{p}\right)^p$ is sharp and not achieved (see, e.g. [2]). In addition, no reminder terms can be added to the right hand side of (1.2). We recall the following definition from [9]:

Definition 1.1. *We say that $-\Delta_p$ is a subcritical operator if and only if there exists a non-negative potential $V \in L^1_{loc}(\mathbb{R}^N)$, $V \neq 0$, such that $-\Delta_p \geq V|\cdot|^{p-2} \cdot$ in the sense of L^2 quadratic forms, that is,*

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^N} V|u|^p \, dx, \ \ \forall u \in W^{1,p}(\mathbb{R}^N).$$

Otherwise, we say that $-\Delta_p$ is critical.

In view of it is known that the nonlinear operator $-\Delta_p \cdot - \left(\frac{N-p}{p}\right)^p \frac{|\cdot|^{p-2}\cdot}{|x|^p}$ is critical. For a more extensive work on variants of L^p−Hardy inequalities one can consult for instance [2], [3] and references therein.
Our paper is structured as follows. In section 2 we state the main results and make some comments. In sections 3 and 4 we give the proofs of the main theorems, namely Theorem 2.2 and Theorem 2.3, respectively.

2 Main results

To fix the hypotheses, let us consider $N \geq 3$, $1 < p < N$ and two points $a_1, a_2 \in \mathbb{R}^N$ arbitrarily fixed. Also, we denote by a the medium point on the segment $[a_1, a_2]$, that is $a := \frac{a_1+a_2}{2}$. We introduce the bipolar potentials V_1 and V_2 defined in $\mathbb{R}^N$ by

$$V_1(x) = \frac{|a_1 - a_2|^2 |x-a|^{p-2}}{|x-a_1|^p |x-a_2|^p} \tag{2.1}$$

respectively,

$$V_2(x) = \frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\Big[|x-a_1|^2|x-a_2|^2 - \big((x-a_1)\cdot(x-a_2)\big)^2\Big]. \tag{2.2}$$

Notice that both V_1 and V_2 are non-negative, by Cauchy-Schwarz inequality. The potentials V_1 and V_2 blow-up at the singular poles a_1 and a_2. It is interesting that our new potentials involve also the medium point a. When $p > 2$, V_1 degenerates at a, while the same is true for V_2 for $p > 4$. Also, V_1 blows-up at a for $p < 2$. On the other hand, there is no $\gamma \in \mathbb{R}$ such that the limit $\lim_{x\to a} V_2(x)/|x-a|^\gamma$ is finite. This can be justified by computing the limit across different directions (e.g. on the line segment $[a_1, a_2]$ and on the mediator of the segment).

Let μ_1 and μ_2 be the constants depending on N and p, defined as

$$\mu_1 = \frac{p-1}{4}\left(\frac{N-p}{p-1}\right)^p, \quad \mu_2 = \frac{p-2}{2}\left(\frac{N-p}{p-1}\right)^{p-1}. \tag{2.3}$$

Finally, denote by V the following potential

$$V := \mu_1 V_1 + \mu_2 V_2. \tag{2.4}$$

Note that

Proposition 2.1. *It holds that $V \geq 0$ for any $1 < p < N$.*

For the sake of completeness we give a proof Proposition 2.1 in the Appendix.

We also define the functional space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as

$$\mathcal{D}^{1,p}(\mathbb{R}^N) := \Big\{u \in \mathcal{D}'(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}} |\nabla u|^p \, dx < \infty\Big\}. \tag{2.5}$$

Now we are in position to state our main results.

Theorem 2.2. *Let $N \geq 3$, $1 < p < N$ and V as in* (2.4)*. For any $u \in C_c^\infty(\mathbb{R}^N)$ it holds*

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^N} V|u|^p \, dx. \tag{2.6}$$

Moreover, for $2 \leq p < N$ the constant 1 is sharp in (2.6) *and it is achieved in the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by the minimizers of the form*

$$\phi(x) = \lambda|x-a_1|^{\frac{p-N}{2(p-1)}}|x-a_2|^{\frac{p-N}{2(p-1)}}, \lambda \in \mathbb{R}$$

unless the case $p = 2$ when the best constant is not achieved.

When restricted to the potential V_1 we have

Theorem 2.3. *For any $N \geq 3$ and $2 \leq p < N$, the following inequality holds*

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \mu_1 \int_{\mathbb{R}^N} V_1 |u|^p \, dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \tag{2.7}$$

Moreover, for any $2 \leq p < N$ the constant μ_1 is sharp in (2.7)*. but not achieved in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.*

Let us emphasize some of the properties regarding our new potentials V_1 and V_2 and discuss how these new results generalize the work of previous authors.

Remark 2.4. *1) Notice that the potential integrals V_1, $V_2 \in L^1_{loc}(\mathbb{R}^N)$, for any $1 < p < N$.*

2) The structure of the potential V_1 change significantly when $1 < p < N$ compared to the case $p = 2$ due to a degeneracy (singularity) which appears at the middle point a.

3) The inequality (2.6) *is an improvement of* (2.7) *when $p > 2$, they coincide when $p = 2$ with the result in* (1.1)*, while* (2.7) *becomes stronger than* (2.6) *when $1 < p < 2$.*

4) Theorem 2.3 establishes a clear generalization of (1.1) *since $V_1 = W^{(2)}$ and $\mu_1 = \frac{(N-2)^2}{4}$ when $n = p = 2$.*

As a consequence of Theorems 2.2-2.3 we have a surprising result

Corollary 2.5. *The operator $-\Delta_p \cdot -\mu_1 V_1 |\cdot|^{p-2} \cdot$ is subcritical when $p > 2$ in opposite with the case $p = 2$ when it becomes critical.*

3 Proof of Theorem 2.2

The proof of the first part of Theorem 2.2 relies partially on an adaptation of the method of supersolutions introduced by Allegretto and Huang in [1]. To be more specific, we apply the following general result

Proposition 3.1. *Let $N \geq 3$, $1 < p < \infty$. If there exists a function $\phi > 0$ such that $\phi \in C^2(\mathbb{R}^n \setminus \{a_1, \ldots, a_n\})$ and*

$$-\Delta_p \phi \geq \mu V \phi^{p-1}, \quad \forall x \in \mathbb{R}^N \setminus \{a_1, ..., a_n\}, \tag{3.1}$$

where $V > 0$, with $V \in L^1_{loc}(\mathbb{R}^N)$, is a given multi-singular potential with the poles $a_1, \ldots, a_n$, then

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \mu \int_{\mathbb{R}^N} V |u|^p \, dx, \ \ \forall u \in C_c^\infty(\mathbb{R}^N).$$

The proof of Proposition 3.1 is a trivial consequence of Theorem 2.2 proven in [1].

3.1 Determination of the triplet (μ, ϕ, V) in (3.1)

In order to prove inequality (2.6) in Theorem 2.2 it is enough to show that, using notations introduced above, $(1, \phi, \mu_1 V_1 + \mu_2 V_2)$ is an admissible triplet in Proposition 3.1. Up to some technicalities, this could be checked by direct computations. However, in the following we will explain how we reach to this triplet. Therefore, we want to find a function $\phi > 0$, a constant μ and a potential V, with singularities in the points a_1 and a_2, depending on N and p, which satisfy the identity

$$-\frac{\Delta_p \phi}{\phi^{p-1}} = \mu V, \quad a.e.\text{for } x \in \mathbb{R}^N \setminus \{a_1, a_2\}.$$

Inspired by the paper [10] in the case $p = 2$, for the general case $1 < p < N$ we follow, up to some point, the same strategy by considering the functions

$$\phi_i = |x - a_i|^\beta, i = 1, 2,$$

where β is negative, aimed to depend on N and p that will be precised later. We introduce

$$\phi = \phi_1 \phi_2. \tag{3.2}$$

We compute the p-Laplacian of ϕ in (3.2) in several steps. First, we note that

$$\nabla \phi = \left(\frac{\nabla \phi_1}{\phi_1} + \frac{\nabla \phi_2}{\phi_2} \right) \phi.$$

To simplify the notation, denote

$$v := \frac{\nabla \phi_1}{\phi_1} + \frac{\nabla \phi_2}{\phi_2}. \tag{3.3}$$

Using the definition of the p-Laplacian operator, we obtain

$$\begin{aligned}
\Delta_p \phi &= \mathtt{div}\left(\left|\nabla \phi\right|^{p-2} \nabla \phi \right) \\
&= \mathtt{div}\left(\phi^{p-1} |v|^{p-2} v \right) \\
&= \nabla\left(\phi^{p-1} \right) |v|^{p-2} v + \phi^{p-1} \nabla\left(|v|^{p-2} \right) v + \phi^{p-1} |v|^{p-2} \mathtt{div}(v).
\end{aligned}$$

Hence, we denote

$$-\frac{\Delta_p \phi}{\phi^{p-1}} =: V,$$

where

$$V = -\left[\nabla\left(|v|^{p-2} \right) \cdot v + |v|^{p-2} \mathtt{div}(v) + (p-1)|v|^p \right] \tag{3.4}$$

Next, we compute explicitly the three terms in (3.4). The expression of v in (3.3) is given by

$$v = \left(\frac{x - a_1}{|x - a_1|^2} + \frac{x - a_2}{|x - a_2|^2} \right). \tag{3.5}$$

Taking the modulus, we get

$$|v| = 2|\beta|\frac{|x-a|}{|x-a_1||x-a_2|},$$

where $a = \frac{a_1+a_2}{2}$ is the medium point on the segment $[a_1, a_2]$. The gradient in (3.4) becomes

$$\begin{aligned}
\nabla\Big(|v|^{p-2}\Big) &= (p-2)|v|^{p-3}\,\nabla\big(|v|\,\big) \\
&= (p-2)|v|^{p-3}\,2\big|\beta\big|\left[\frac{x-a}{|x-a||x-a_1||x-a_2|} - \frac{|x-a|\,(x-a_1)}{|x-a_1|^3|x-a_2|} - \frac{|x-a|\,(x-a_2)}{|x-a_1||x-a_2|^3}\right] \\
&= (p-2)|v|^{p-2}\left[\frac{x-a}{|x-a|^2} - \frac{x-a_1}{|x-a_1|^2} - \frac{x-a_2}{|x-a_2|^2}\right]. \\
&= (p-2)|v|^{p-2}\left[\frac{x-a}{|x-a|^2} - \frac{v}{\beta}\right]. \qquad (3.6)
\end{aligned}$$

The second term in (3.4) yields to

$$\mathtt{div}(v) = \beta(N-2)\left[\frac{1}{|x-a_1|^2} + \frac{1}{|x-a_2|^2}\right]. \qquad (3.7)$$

Using (3.6), (3.7) and (3.4), the potential reduces to

$$\begin{aligned}
V = -|v|^{p-2}\Bigg[&(p-2)\left(\frac{x-a}{|x-a|^2} - \frac{1}{\beta}v\right)\cdot v \\
&+ \beta(N-2)\sum_{i=1}^{2}\frac{1}{|x-a_i|^2} + 4(p-1)\beta^2\frac{|x-a|^2}{|x-a_1|^2|x-a_2|^2}\Bigg] \\
= -|v|^{p-2}&\left[T_1 + T_2 + T_3\right], \qquad (3.8)
\end{aligned}$$

where we denoted

$$\begin{aligned}
T_1 &= (p-2)\left(\frac{x-a}{|x-a|^2} - \frac{1}{\beta}v\right)\cdot v, \\
T_2 &= \beta(N-2)\sum_{i=1}^{2}\frac{1}{|x-a_i|^2}, \\
T_3 &= 4(p-1)\beta^2\frac{|x-a|^2}{|x-a_1|^2|x-a_2|^2}.
\end{aligned}$$

Next, we rearrange the expression of V in (3.8). To simplify the computations we employ the notations $v_1 := x - a_1$ and $v_2 := x - a_2$. Hence, $x - a = \frac{v_1+v_2}{2}$ and then we obtain

$$
\begin{aligned}
|v|^{p-2} &= 2^{p-2}\left|\beta\right|^{p-2} \frac{|x-a|^{p-2}}{|x-a_1|^{p-2}|x-a_2|^{p-2}} \\
&= \left|\beta\right|^{p-2} \frac{|v_1+v_2|^{p-2}}{|v_1|^{p-2}|v_2|^{p-2}}.
\end{aligned} \tag{3.9}
$$

The first term in (3.8) becomes

$$
\begin{aligned}
T_1 &= (p-2)\left(\frac{x-a}{|x-a|^2} - \frac{1}{\beta}v\right)\cdot v \\
&= (p-2)\frac{v_1+v_2}{2}\frac{4}{|v_1+v_2|^2}\beta\left(\frac{v_1}{|v_1|^2} + \frac{v_2}{|v_2|^2}\right) - 4(p-2)\beta\frac{|v_1+v_2|^4}{4|v_1|^2|v_2|^2} \\
&= (p-2)\beta\frac{2(v_1+v_2)(v_1|v_2|^2 + v_2|v_1|^2) - |v_1+v_2|^2}{|v_1|^2|v_2|^2|v_1+v_2|^2}
\end{aligned} \tag{3.10}
$$

The second term in (3.8) reads to

$$
\begin{aligned}
T_2 &= \beta(N-2)\sum_{i=1}^{2}\frac{1}{|x-a_i|^2} \\
&= \beta(N-2)\frac{|v_1|^2 + |v_2|^2}{|v_1|^2|v_2|^2}.
\end{aligned} \tag{3.11}
$$

The third term in (3.8) is

$$
\begin{aligned}
T_3 &= 4(p-1)\beta^2\frac{|x-a|^2}{|x-a_1|^2|x-a_2|^2} \\
&= (p-1)\beta^2\frac{|v_1+v_2|^2}{|v_1|^2|v_2|^2}.
\end{aligned} \tag{3.12}
$$

From (3.9), (3.10), (3.11) and (3.12) we get successively

$$
\begin{aligned}
V = &-\left|\beta\right|^{p-2}\beta\frac{|v_1+v_2|^{p-2}}{|v_1|^{p-2}|v_2|^{p-2}} \times \frac{1}{|v_1|^2|v_2|^2|v_1+v_2|^2}\Big[(N-2)(|v_1|^2 + |v_2|^2)|v_1+v_2|^2 + \\
&+ (p-1)\beta|v_1+v_2|^4 + 2(p-2)(v_1+v_2)(v_1|v_2|^2 + v_2|v_1|^2) - (p-2)|v_1+v_2|^4\Big] \\
= &\left|\beta\right|^{p-1}\frac{|v_1+v_2|^{p-4}}{|v_1|^p|v_2|^p}\Big[\Big((p-1)\beta + N - p\Big)(|v_1|^4 + |v_2|^4) + 2\Big(2(p-1)\beta + N - p\Big)v_1\cdot v_2(|v_1|^2 + |v_2|^2) \\
&+ 2\Big((p-1)\beta + N + p - 4\Big)|v_1|^2|v_2|^2 + 4\Big((p-1)\beta + 2 - p\Big)(v_1\cdot v_2)^2\Big]
\end{aligned} \tag{3.13}
$$

In order to get rid of the cross term in the last identity of (3.13) we choose $\beta = \frac{p-N}{2(p-1)}$ which implies

$$\phi = \phi_1\phi_2 = |x-a_1|^{\frac{p-N}{2(p-1)}}|x-a_2|^{\frac{p-N}{2(p-1)}}. \tag{3.14}$$

Then, using notations above, we get the following form of the potential V:

$$\begin{aligned}
V &= \left(\frac{N-p}{2(p-1)}\right)^{p-1}\frac{|v_1+v_2|^{p-4}}{|v_1|^p|v_2|^p}\Bigg[\left(\frac{N-p}{2}\right)(|v_1|^4+|v_2|^4)+ \\
&\qquad + \Big((N-p+4(p-2)\Big)|v_1|^2|v_2|^2 + \Big(2(p-N)+4(2-p)\Big)(v_1\cdot v_2)^2\Bigg] \\
&= \left(\frac{N-p}{2(p-1)}\right)^{p-1}\frac{|v_1+v_2|^{p-4}}{|v_1|^p|v_2|^p}\Bigg[\left(\frac{N-p}{2}\right)(|v_1|^4+|v_2|^4)+(N-p)|v_1|^2|v_2|^2 \\
&\qquad -2(N-p)(v_1\cdot v_2)^2 + 4(p-2)\Big(|v_1|^2|v_2|^2-(v_1\cdot v_2)^2\Big)\Bigg] \\
&= (p-1)\left(\frac{N-p}{2(p-1)}\right)^{p}\frac{|v_1+v_2|^{p-2}|v_1-v_2|^2}{|v_1|^p|v_2|^p} \\
&\qquad + 4(p-2)\left(\frac{N-p}{2(p-1)}\right)^{p-1}\frac{|v_1+v_2|^{p-4}}{|v_1|^p|v_2|^p}\Big[|v_1|^2|v_2|^2-(v_1\cdot v_2)^2\Big]
\end{aligned} \tag{3.15}$$

Undoing the notations v_1 and v_2 in (3.15) we finally obtain

$$\begin{aligned}
V &= \frac{p-1}{4}\left(\frac{N-p}{p-1}\right)^{p}\frac{|a_1-a_2|^2|x-a|^{p-2}}{|x-a_1|^p|x-a_2|^p} + \frac{p-2}{2}\left(\frac{N-p}{p-1}\right)^{p-1}\times \\
&\qquad \times\frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\Big[|x-a_1|^2|x-a_2|^2-\big((x-a_1)\cdot(x-a_2)\big)^2\Big]
\end{aligned} \tag{3.16}$$

By (2.1)-(2.3) we can write

$$V = \mu_1 V_1 + \mu_2 V_2$$

and ϕ in (3.14) verifies the identity

$$-\frac{\Delta_p\phi}{\phi^{p-1}} = V, \text{ in } \mathbb{R}^N\setminus\{a_1,a_2\}.$$

The proof of (2.6) is complete now, by 3.1.

3.2 Sharpness of inequality (2.6)

We want to show that, for $2<p<N$, the constant $\mu=1$ is sharp in inequality

$$\int_{\mathbb{R}^N}|\nabla u|^p\,dx \ge \mu\int_{\mathbb{R}^N}V|u|^p\,dx,$$

for $u \in C_c^\infty(\mathbb{R}^N)$ and it is actually attained in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by the function ϕ in (3.14). We show that ϕ satisfies the identity:

$$\int_{\mathbb{R}^N} \left|\nabla\phi\right|^p \, dx = \int_{\mathbb{R}^N} V\left|\phi\right|^p \, dx, \tag{3.17}$$

which proves both of the facts stated above. This is done using integration by parts, but we need to check the integrability of $|\nabla\phi|^p$.

Proposition 3.2. *For $N \geq 3$, $2 < p < N$ and ϕ in* (3.14) *it holds that $\phi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$.*

Proof. Recall that $\beta := \frac{p-N}{2(p-1)}$ and

$$\phi = \phi_1\phi_2 = |x - a_1|^{\frac{p-N}{2(p-1)}} |x - a_2|^{\frac{p-N}{2(p-1)}} .$$

By direct computation we formally obtain,

$$\nabla\phi = \beta|x - a_1|^{\frac{p-N}{2(p-1)}-2}|x - a_2|^{\frac{p-N}{2(p-1)}-2} \left[|x - a_2|^2 \, (x - a_1) + |x - a_1|^2 \, (x - a_2)\right]. \tag{3.18}$$

By squaring the relation (3.18), we get

$$\left|\nabla\phi\right|^2 = 4\beta^2|x - a_1|^{2\left(\frac{p-N}{2(p-1)}-1\right)}|x - a_2|^{2\left(\frac{p-N}{2(p-1)}-1\right)}|x - a|^2 .$$

Hence,

$$\left|\nabla\phi\right| = 2\left|\beta\right||x - a_1|^{\frac{p-N}{2(p-1)}-1}|x - a_2|^{\frac{p-N}{2(p-1)}-1}|x - a| .$$

Let $0 < r < \frac{|a_1-a_2|}{4}$ and $R \geq \max\left\{2|a_1|\,, 2|a_2|\right\} + 2r$. Define $B_r^i := B(a_i, r)$ to be the ball centered at a_i and of radius r, $i = 1, 2$, and $B_R := B(a, R)$, where $a = \frac{a_1+a_2}{2}$, to be the ball centered at a and of radius R. By the choice of r and R, we can see that B_R contains both B_r^1 and B_r^2. We prove the L^p-integrability of $\nabla\phi$, for $2 < p < N$, as follows. We split

$$\int_{\mathbb{R}^N} \left|\nabla\phi\right|^p \, dx = \int_{\mathbb{R}^N \setminus B_R} \left|\nabla\phi\right|^p \, dx + \int_{B_R} \left|\nabla\phi\right|^p \, dx \;:=\; I_1 + I_2.$$

First, we notice that, in $\mathbb{R}^N \setminus B_R$,

$$|x - a_i| < |x| + |a_i| < |x| + R < 2|x| , \tag{3.19}$$

$$|x - a_i| > |x| - |a_i| > |x| - \frac{R}{2} = \frac{|x|}{2} + \frac{|x| - R}{2} > \frac{|x|}{2}. \tag{3.20}$$

Similarly, we have that

$$|x| \leq |x - a| \leq 2|x| \quad \text{in } \mathbb{R}^N \setminus B_R. \tag{3.21}$$

Therefore, $|x-a|$ and $|x-a_i|$ behave asymptotically as $|x|$, for $x \in \mathbb{R}^N \setminus B_R$. Next we will write " $\simeq$ " and " $\lesssim$ " instead of usual notations, meaning that the equality or inequality holds up to some constant. By (3.19), (3.20), (3.21) and co-area formula, we get

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^N \setminus B_R} |x|^{\frac{(p-N)p}{2(p-1)}-p} |x|^{\frac{(p-N)p}{2(p-1)}-p} |x|^p \, dx \\
&= \int_{\mathbb{R}^N \setminus B_R} |x|^{\frac{(p-N)p}{p-1}-p} \, dx \\
&= \int_{\mathbb{R}^N \setminus B_R} |x|^{\frac{p(1-N)}{p-1}} \, dx \\
&\simeq \int_R^\infty s^{\frac{p(1-N)}{p-1}} s^{N-1} ds \\
&= \int_R^\infty s^{\frac{1-N}{p-1}} ds,
\end{aligned}$$

which is finite when $\frac{1-N}{p-1}+1<0$. So, $I_1<\infty$ for any $p<N$.
We now estimate I_2 by splitting it in two terms.

$$I_2 = \int_{B_r^1 \cup B_r^2} \left|\nabla\phi\right|^p dx + \int_{B_R \setminus (B_r^1 \cup B_r^2)} \left|\nabla\phi\right|^p dx.$$

The integral on $\overline{B_R \setminus (B_r^1 \cup B_r^2)}$ is finite, since the function under integration is continuous. On the other hand, we have:

$$\begin{aligned}
\int_{B_r^i} \left|\nabla\phi\right|^p dx &\lesssim \int_{B_r^i} |x-a_i|^{\frac{(p-N)p}{2(p-1)}-p} \, dx \\
&\simeq \int_0^r s^{\frac{(p-N)p}{2(p-1)}-p} s^{N-1} ds \\
&= \int_0^r s^{\frac{(p-N)(2-p)}{2(p-1)}-1} ds.
\end{aligned}$$

For $2<p<N$, the integral above is finite, for $i=1,2$. In conclusion, ϕ belongs to $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for any $2<p<N$. □

Taking into account that $V=-\frac{\Delta_p\phi}{\phi^p}$ and integrating by parts in (3.17), we get

$$\int_{\mathbb{R}^N} V\left|\phi\right|^p dx = \int_{\mathbb{R}^N} -\frac{\Delta_p\phi}{\phi^{p-1}}\phi^p dx = \int_{\mathbb{R}^N} \mathtt{div}\Big(\left|\nabla\phi\right|^{p-2}\nabla\phi\Big)\phi dx = \int_{\mathbb{R}^N} \left|\nabla\phi\right|^p dx, \tag{3.22}$$

which concludes the proof of Theorem 2.2.

4 Proof of Theorem 2.3

First we need the following lemma.

Lemma 4.1. *Assume $p \geq 2$. Let ϕ be a positive function in $\mathbb{R}^N$ with $\phi \in C^2(\mathbb{R}^N \setminus \{a_1, a_2\})$ and let $V \in L^1_{loc}(\mathbb{R}^N)$ be a continuous potential on $\mathbb{R}^N \setminus \{a_1, a_2\}$ such that*

$$-\Delta_p \phi(x) - V\phi(x)^{p-1} \geq 0\,, \quad \forall x \in \mathbb{R}^N \setminus \{a_1, a_2\}. \tag{4.1}$$

Then there exists $c_1(p)$ such that

$$c_1(p)\int_{\mathbb{R}^N} \left|\nabla\left(\frac{u}{\phi}\right)\right|^p \phi^p dx \leq \int_{\mathbb{R}^N} |\nabla u|^p\, dx - \int_{\mathbb{R}^N} V|u|^p\, dx\,,\ \ \forall\, u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, a_2\}). \tag{4.2}$$

Moreover, assume we have equality in (4.1). *Then the following reverse inequality holds for any $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, a_2\})$:*

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \int_{\mathbb{R}^N} V|u|^p\, dx \leq \frac{p(p-1)}{2}\int_{\mathbb{R}^N} \left(\phi\left|\nabla\left(\frac{u}{\phi}\right)\right| + \frac{u}{\phi}|\nabla\phi|\right)^{p-2} \phi^2 \left|\nabla\left(\frac{u}{\phi}\right)\right|^2. \tag{4.3}$$

Proof. The proof of inequality (4.2) is a trivial adaptation of Theorem 2.2 from [9]. We focus now on the proof of (4.3)
Using the hypothesis and integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} V|u|^p\, dx &= \int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} \frac{\Delta_p\phi}{\phi^{p-1}}|u|^p\, dx \\ &= \int_{\mathbb{R}^N} \left|\phi\nabla\left(\frac{u}{\phi}\right) + \frac{u}{\phi}\nabla\phi\right|^p - p\left(\frac{u}{\phi}\right)^{p-1}\phi\left|\nabla\phi\right|^{p-2}\nabla\left(\frac{u}{\phi}\right)\cdot\nabla\phi - \left(\frac{u}{\phi}\right)^p \left|\nabla\phi\right|^p dx \end{aligned} \tag{4.4}$$

We employ an inequality from [14]: for $p \geq 2$ it holds

$$\left|x+y\right|^p - p\left|y\right|^{p-1} y\cdot x - \left|y\right|^p \leq \frac{p(p-1)}{2}\Big(|x| + \left|y\right|\Big)^{p-2}|x|^2\,,\ \ \forall\, x,\ y \in \mathbb{R}^N. \tag{4.5}$$

Applying (4.5) for $x = \phi\nabla\left(\frac{u}{\phi}\right)$ and $y = \frac{u}{\phi}\nabla\phi$ in (4.4) we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \int_{\mathbb{R}^N} V|u|^p\, dx \leq \frac{p(p-1)}{2}\int_{\mathbb{R}^N} \left(\phi\left|\nabla\left(\frac{u}{\phi}\right)\right| + \frac{u}{\phi}|\nabla\phi|\right)^{p-2} \phi^2 \left|\nabla\left(\frac{u}{\phi}\right)\right|^2. \tag{4.6}$$

□

We can easily extend the result above to functions u in $W_0^{1,p}(\mathbb{R}^N)$.

4.1 Asymptotic behavior of V_1 and V_2

This section is also useful for the proof of optimality of μ_1 in (2.7). In order to compare the potentials V_1 and V_2 we analyze their behavior at the singular points a_1, a_2, at the degenerate point a and at infinity, respectively. Recall that

$$V_1 = \frac{|a_1 - a_2|^2|x-a|^{p-2}}{|x-a_1|^p|x-a_2|^p}.$$

Fix p such that $2 < p < N$. Then one can easily see that

$$\lim_{x \to a_i} |x - a_i|^p V_1 = 2^{2-p} =: c_1.$$

In the middle point $a = \frac{a_1+a_2}{2}$, V_1 tends to 0, as

$$\lim_{x \to a} |x - a|^{2-p} V_1 = 4^p |a_1 - a_2|^{2(1-p)} =: c_2.$$

At infinity, we have

$$\lim_{|x| \to \infty} |x|^{p+2} V_1 = |a_1 - a_2|^2 =: c_3.$$

In consequence, for $p > 2$ and $i = 1, 2$, we have

$$V_1(x) = \begin{cases} c_1 |x - a_i|^{-p}, & \text{as } x \to a_i \\ c_2 |x - a|^{p-2}, & \text{as } x \to a \\ c_3 |x|^{-(p+2)}, & \text{as } x \to \infty \end{cases}$$

Now we look at V_2:

$$V_2(x) = \frac{|x - a|^{p-4}}{|x - a_1|^p |x - a_2|^p} \left[|x - a_1|^2 |x - a_2|^2 - \big((x - a_1) \cdot (x - a_2) \big)^2 \right].$$

Taking into account the asymptotic behavior of V_1 around the singularities a_i, we further emphasize that V_2 is dominated by V_1 in a neighbourhood of a_i.

Proposition 4.2. *There exists $r_0 > 0$ such that, for any $\delta > 0$, it holds*

$$\mu_2 V_2 - \frac{\delta}{2} V_1 < 0. \tag{4.7}$$

Proof. Let $\delta > 0$. Denote by $M := |a_1 - a_2|$ and $\alpha := \cos(\varphi) \in [-1, 1]$, where $\cos(\varphi) = \frac{(x-a_1)\cdot(x-a_2)}{|x-a_1||x-a_2|}$. Let $r_0 > 0$ aimed to be small and $x \in B(a_i, r_0)$. Then

$$\begin{aligned} & \mu_2 V_2 - \frac{\delta}{2} V_1 < 0 \\ \iff & \mu_2 |x - a|^{p-4} \left[|x - a_1|^2 |x - a_2|^2 - \big((x - a_1) \cdot (x - a_2) \big)^2 \right] - \frac{\delta}{2} |x - a|^{p-2} |a_1 - a_2|^2 < 0 \\ \iff & 2\mu_2 (1 - \alpha^2) |x - a_1|^2 |x - a_2|^2 - \delta |x - a|^2 |a_1 - a_2|^2 < 0 \\ \iff & 8\mu_2 (1 - \alpha^2) r_0^2 (r_0 + M)^2 - \delta M^4 < 0. \end{aligned}$$

When $r_0 \to 0$, the left hand-side tends to $-\delta M^4 < 0$, which proves our statement. ◻

4.2 Proof of Theorem 2.3

Finally, we are ready to prove that it holds

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \mu_1 \int_{\mathbb{R}^N} V_1 |u|^p$$

for any $2 \leq p < N$ and that the constant μ_1 is sharp.
The inequality follows from Theorem 2.2 and Remark 2.4. We will prove here the sharpness of the constant. The case $p = 2$ is proved in [8], so here we prove it for $p > 2$. In order to do this, assume there exists $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx - (\mu_1 + \varepsilon_0) \int_{\mathbb{R}^N} V_1 |u|^p \, dx \geq 0, \quad \forall u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, a_2\}). \tag{4.8}$$

Denote

$$L[u] := \int_{\mathbb{R}^N} |\nabla u|^p \, dx - (\mu_1 + \varepsilon_0) \int_{\mathbb{R}^N} V_1 |u|^p \, dx.$$

We add and subtract $\mu_2 V_2 |u|^p$ in the second integral:

$$\begin{aligned} L[u] &= \int_{\mathbb{R}^N} |\nabla u|^p \, dx - \int_{\mathbb{R}^N} \big(\mu_1 V_1 + \mu_2 V_2\big) |u|^p \, dx + \int_{\mathbb{R}^N} \big(\mu_2 V_2 - \varepsilon_0 V_1\big) |u|^p \, dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^p \, dx - V |u|^p \, dx + \int_{\mathbb{R}^N} \big(\mu_2 V_2 - \varepsilon_0 V_1\big) |u|^p \, dx \end{aligned} \tag{4.9}$$

By Proposition 4.2, for $\delta = \varepsilon_0$, we get that

$$\mu_2 V_2 - \varepsilon_0 V_1 < -\frac{\varepsilon_0}{2} V_1 < 0. \tag{4.10}$$

For any $\varepsilon > 0$ chosen under the assumption that $\varepsilon < \min\{\frac{1}{2}, r_0^2\}$, define the following cut-off function:

$$\theta_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in B_{\varepsilon^2}(a_i), \quad \text{for } i = 1, 2 \\ \frac{\log\left(|x - a_i|/\varepsilon^2\right)}{\log \frac{1}{\varepsilon}}, & \text{if } x \in B_\varepsilon(a_i) \setminus B_{\varepsilon^2}(a_i), \quad \text{for } i = 1, 2 \\ \frac{\log\left(\varepsilon/|x - a_i|^2\right)}{\log \frac{1}{\varepsilon}}, & \text{if } x \in B_{\varepsilon^{1/2}}(a_i) \setminus B_\varepsilon(a_i), \quad \text{for } i = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Consider $u_\varepsilon = \phi \theta_\varepsilon$, where ϕ is defined in (3.2). Due to the fact that $\theta_\varepsilon \in W_0^{1,p}(\mathbb{R}^N)$ we conclude that u_ε is also in $W_0^{1,p}(\mathbb{R}^N)$. Taking $u = u_\varepsilon$ in (4.9) we get

$$\begin{aligned} L[u_\varepsilon] &= \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p - V |u_\varepsilon|^p \, dx + \int_{\mathbb{R}^N} \big(\mu_2 V_2 - \varepsilon_0 V_1\big) |u_\varepsilon|^p \, dx \\ &=: I_\varepsilon + J_\varepsilon. \end{aligned} \tag{4.11}$$

We will use Lemma 4.1 in order to estimate I_ε. By direct computation, the gradient of θ_ε is

$$\nabla\theta_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in B_{\varepsilon^2}(a_i), \quad \text{for } \ i = 1,2 \\ \big(\log\frac{1}{\varepsilon}\big)^{-1}\frac{x-a_i}{|x-a_i|^2}, & \text{if } x \in B_\varepsilon(a_i) \setminus B_{\varepsilon^2}(a_i), \quad \text{for } \ i = 1,2 \\ -2\big(\log\frac{1}{\varepsilon}\big)^{-1}\frac{x-a_i}{|x-a_i|^2}, & \text{if } x \in B_{\varepsilon^{1/2}}(a_i) \setminus B_\varepsilon(a_i), \quad \text{for } \ i = 1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Restricting to the support of θ_ε, we get

$$I_\varepsilon = \sum_{i=1}^{2}\int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} |\nabla u_\varepsilon|^p - V|u_\varepsilon|^p\,dx + \sum_{i=1}^{2}\int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_\varepsilon(a_i)} |\nabla u_\varepsilon|^p - V|u_\varepsilon|^p\,dx =: I_{1,\varepsilon} + I_{2,\varepsilon}.$$

Recall that

$$\phi(x) = |x-a_1|^\beta|x-a_2|^\beta \ , \quad \beta = \frac{p-N}{2(p-1)}.$$

By (4.3) in Lemma 4.1 and using the co-area formula, we get the estimate

$$\begin{aligned} I_{1,\varepsilon} &= \sum_{i=1}^{2}\int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} |\nabla u_\varepsilon|^p - V|u_\varepsilon|^p\,dx \\ &\lesssim \sum_{i=1}^{2}\int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \Big(\phi|\nabla\theta_\varepsilon| + \theta_\varepsilon\big|\nabla\phi\big|\Big)^{p-2}\phi^2|\nabla\theta_\varepsilon|^2\,dx \\ &\lesssim \sum_{i=1}^{2}\int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \Big(\big(\log\frac{1}{\varepsilon}\big)^{-1}|x-a_i|^{\beta-1} + \big(\log\frac{1}{\varepsilon}\big)^{-1}\log\frac{|x-a_i|}{\varepsilon^2}|x-a_i|^{\beta-1}\Big)^{p-2}\times \\ &\times|x-a_i|^{2\beta-2}\big(\log\frac{1}{\varepsilon}\big)^{-2}dx \\ &\simeq \Big(\log\frac{1}{\varepsilon}\Big)^{-p}\sum_{i=1}^{2}\int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \Big(1+\log\frac{|x-a_i|}{\varepsilon^2}\Big)^{p-2}|x-a_i|^{p(\beta-1)}\,dx \\ &\simeq \Big(\log\frac{1}{\varepsilon}\Big)^{-p}\int_{\varepsilon^2}^{\varepsilon}\Big(1+\log\frac{s}{\varepsilon^2}\Big)^{p-2}s^{p(\beta-1)+N-1}ds \\ &\lesssim \Big(\log\frac{1}{\varepsilon}\Big)^{-p}\Big(1+\log\frac{1}{\varepsilon}\Big)^{p-2}\int_{\varepsilon^2}^{\varepsilon} s^{(p-N)\left(\frac{p}{2(p-1)}-1\right)-1}ds \\ &\lesssim \Big(\log\frac{1}{\varepsilon}\Big)^{-2}\varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}. \end{aligned}$$

Similarly,

$$
\begin{aligned}
I_{2,\varepsilon} &= \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_\varepsilon(a_i)} |\nabla u_\varepsilon|^p - V|u_\varepsilon|^p \, dx \\
&\lesssim \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_\varepsilon(a_i)} \Big(\phi|\nabla\theta_\varepsilon| + \theta_\varepsilon\big|\nabla\phi\big|\Big)^{p-2} \phi^2 |\nabla\theta_\varepsilon|^2 \, dx \\
&\lesssim \sum_{i=1}^{2} \int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \Big(\big(\log\frac{1}{\varepsilon}\big)^{-1}|x-a_i|^{\beta-1} + \big(\log\frac{1}{\varepsilon}\big)^{-1}\log\frac{\varepsilon}{|x-a_i|^2}|x-a_i|^{\beta-1}\Big)^{p-2} \times \\
&\quad \times |x-a_i|^{2\beta-2}\big(\log\frac{1}{\varepsilon}\big)^{-2} dx \\
&\simeq \Big(\log\frac{1}{\varepsilon}\Big)^{-p} \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_\varepsilon(a_i)} \Big(1 + \log\frac{\varepsilon}{|x-a_i|^2}\Big)^{p-2} |x-a_i|^{p(\beta-1)} \, dx \\
&\simeq \Big(\log\frac{1}{\varepsilon}\Big)^{-p} 2 \int_\varepsilon^{\varepsilon^{1/2}} \Big(1 + \log\frac{\varepsilon}{s^2}\Big)^{p-2} s^{p(\beta-1)+N-1} ds \\
&\lesssim \Big(\log\frac{1}{\varepsilon}\Big)^{-p} \Big(1 + \log\frac{1}{\varepsilon}\Big)^{p-2} \int_\varepsilon^{\varepsilon^{1/2}} s^{(p-N)\left(\frac{p}{2(p-1)}-1\right)-1} ds \\
&\lesssim \Big(\log\frac{1}{\varepsilon}\Big)^{-2} \varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}.
\end{aligned}
$$

Hence, the estimate for I_ε is

$$
I_\varepsilon = I_{1,\varepsilon} + I_{2,\varepsilon} \leq C_1 \Big(\log\frac{1}{\varepsilon}\Big)^{-2} \varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}. \tag{4.12}
$$

for some positive constant C_1 independent of ε. Now we split J_ε:

$$
J_\varepsilon = \sum_{i=1}^{2} \int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \big(\mu_2 V_2 - \varepsilon_0 V_1\big)|u_\varepsilon|^p \, dx + \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_\varepsilon(a_i)} \big(\mu_2 V_2 - \varepsilon_0 V_1\big)|u_\varepsilon|^p \, dx =: J_{1,\varepsilon} + J_{2,\varepsilon}.
$$

Using (4.10) and co-area formula, we get

$$
\begin{aligned}
J_{1,\varepsilon} &= \sum_{i=1}^{2} \int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \big(\mu_2 V_2 - \varepsilon_0 V_1\big)|u_\varepsilon|^p \, dx \\
&< -\frac{\varepsilon_0}{2} \sum_{i=1}^{2} \int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} V_1 \theta_\varepsilon^p \phi^p \, dx \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \sum_{i=1}^{2} \int_{B_\varepsilon(a_i)\setminus B_{\varepsilon^2}(a_i)} \Big(\log\frac{|x-a_i|}{\varepsilon^2}\Big)^p |x-a_i|^{-p}|x-a_i|^{p\beta} \, dx \\
&\simeq -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} 2\int_{\varepsilon^2}^{\varepsilon} \Big(\log\frac{s}{\varepsilon^2}\Big)^p s^{p(\beta-1)+N-1} ds \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \int_{\varepsilon/2}^{\varepsilon} \Big(\log\frac{s}{\varepsilon^2}\Big)^p s^{p(\beta-1)+N-1} ds \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \Big(\log\frac{1}{2\varepsilon}\Big)^p \int_{\varepsilon/2}^{\varepsilon} s^{p(\beta-1)+N-1} ds \\
&\lesssim -\varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
J_{2,\varepsilon} &= \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_{\varepsilon}(a_i)} \big(\mu_2 V_2 - \varepsilon_0 V_1\big)|u_\varepsilon|^p \, dx \\
&< -\frac{\varepsilon_0}{2} \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_{\varepsilon}(a_i)} V_1 \theta_\varepsilon^p \phi^p \, dx \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \sum_{i=1}^{2} \int_{B_{\varepsilon^{1/2}}(a_i)\setminus B_{\varepsilon}(a_i)} \Big(\log\frac{\varepsilon}{|x-a_i|^2}\Big)^p |x-a_i|^{-p}|x-a_i|^{p\beta} \, dx \\
&\simeq -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} 2\int_{\varepsilon}^{\varepsilon^{1/2}} \Big(\log\frac{\varepsilon}{s^2}\Big)^p s^{p(\beta-1)+N-1} ds \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \int_{\varepsilon}^{\varepsilon^{2/3}} \Big(\log\frac{\varepsilon}{s^2}\Big)^p s^{p(\beta-1)+N-1} ds \\
&\lesssim -\Big(\log\frac{1}{\varepsilon}\Big)^{-p} \Big(\log\frac{1}{\varepsilon^{1/3}}\Big)^p \int_{\varepsilon}^{\varepsilon^{2/3}} s^{p(\beta-1)+N-1} ds \\
&\lesssim -\varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}.
\end{aligned}
$$

Combining the above two estimates, we get that

$$
J_\varepsilon = J_{1,\varepsilon} + J_{2,\varepsilon} < -C_2 \varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}, \tag{4.13}
$$

where C_2 is a positive constant, independent of ε. By (4.11), (4.12) and (4.13) we obtain

$$
\begin{aligned}
L[u_\varepsilon] &= I_\varepsilon + J_\varepsilon \\
&< C_1\Big(\log\frac{1}{\varepsilon}\Big)^{-2}\varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)} - C_2\varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)} \\
&= \varepsilon^{(p-N)\left(\frac{p}{2(p-1)}-1\right)}\left(C_1\Big(\log\frac{1}{\varepsilon}\Big)^{-2} - C_2\right) \xrightarrow{\varepsilon\to 0} 0_-. \qquad (4.14)
\end{aligned}
$$

Clearly, inequality (4.14) provides a contradiction with the assumption (4.8). The proof of Theorem 2.3 is finished.

Appendix: Proof of Proposition 2.2

Recall that, by (3.16),

$$
\begin{aligned}
V = \frac{p-1}{4}\left(\frac{N-p}{p-1}\right)^{p}\frac{|a_1-a_2|^2|x-a|^{p-2}}{|x-a_1|^p|x-a_2|^p} + \frac{p-2}{2}\left(\frac{N-p}{p-1}\right)^{p-1}\times \\
\times\frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\Big[|x-a_1|^2|x-a_2|^2 - \big((x-a_1)\cdot(x-a_2)\big)^2\Big]. \qquad (4.15)
\end{aligned}
$$

It is clear, due to Cauchy-Schwarz inequality, that $V \geq 0$ for $p \in (2, N)$. Also, for $p = 2$, $V = \mu_1 V_1$, which is positive.
Let $p \in (1, 2)$. After some computations, we obtain V in the following form

$$
\begin{aligned}
V = \frac{p-1}{8}\left(\frac{N-p}{p-1}\right)^{p-1}\frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\Bigg[\frac{N-p}{2}\Big(|x-a_1|^4+|x-a_2|^4\Big) \\
+\Big(N+3p-8\Big)|x-a_1|^2|x-a_2|^2 + 2\Big(4-p-N\Big)\Big((x-a_1)\cdot(x-a_2)\Big)^2\Bigg]
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
V &\geq \frac{p-1}{8}\left(\frac{N-p}{p-1}\right)^{p-1}\frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\times \\
&\times\left[\frac{N-p}{2}\Big(|x-a_1|^4+|x-a_2|^4\Big) - (N-p)|x-a_1|^2|x-a_2|^2\right] \\
&= \frac{p-1}{16}\left(\frac{N-p}{p-1}\right)^{p}\frac{|x-a|^{p-4}}{|x-a_1|^p|x-a_2|^p}\Big(|x-a_1|^2-|x-a_2|^2\Big)^2.
\end{aligned}
$$

It is clear that the above quantity is positive for any $1 < p < 2$. The proof is done. ☐

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