Some functional inequalities under lower Bakry-Émery-Ricci curvature bounds with ε -range

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Abstract

For *n*-dimensional weighted Riemannian manifolds, lower *m*-Bakry-Émery-Ricci curvature bounds with ε -range, introduced by Lu-Minguzzi-Ohta [10], integrate constant lower bounds and certain variable lower bounds in terms of weight functions. In this paper, we prove a Cheng type inequality and a local Sobolev inequality under lower *m*-Bakry-Émery-Ricci curvature bounds with ε -range. These generalize those inequalities under constant curvature bounds for $m \in (n, \infty)$ to $m \in (-\infty, 1] \cup \{\infty\}$.

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1 Introduction

The Ricci curvature plays an important role in geometric analysis. For example, lower bounds of Ricci curvature imply comparison theorems such as the Laplacian comparison theorem and Bishop-Gromov volume comparison theorem. This paper is concerned with the Bakry-Émery-Ricci curvature Ric_{ψ}^{m} , which is a generalization of the Ricci curvature for weighted Riemannian manifolds and m is a real parameter called the effective dimension. The condition $\mathrm{Ric}_{\psi}^{m} \geq K$ for $K \in \mathbb{R}$ implies many comparison geometric results similar to those for Riemannian manifolds with Ricci curvature bounded from below by K and dimension bounded from above by m. Especially the case of $m \geq n$ is now classical and well investigated. Recently, there is a growing interest in the m-Bakry-Émery-Ricci curvature in the case of $m \in (-\infty, 1]$. For this range, some Poincaré inequalities [3] (see also [11] for its rigidity), Beckner inequality [2] and the curvature-dimension condition [13] were studied.

It is known that some comparison theorems (such as the Bishop-Gromov volume comparison theorem and the Laplacian comparison theorem) under the constant curvature bound $\mathrm{Ric}_{\psi}^m \geq Kg$ hold only for $m \in [n,\infty)$ and fail for $m \in (-\infty,1] \cup \{\infty\}$. Nonetheless, Wylie-Yeroshkin [20] introduced a variable curvature bound

$$\operatorname{Ric}_{\psi}^{1} \ge K e^{-\frac{4}{n-1}\psi} g$$

associated with the weight function ψ , and established several comparison theorems. They were then generalized to

$$\operatorname{Ric}_{\psi}^{m} \ge K e^{\frac{4}{m-n}\psi} g$$

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with $m \in (-\infty, 1)$ by Kuwae-Li [4]. In [10], Lu-Minguzzi-Ohta gave a further generalization of the form

$$\operatorname{Ric}_{\psi}^{m} \geq K e^{\frac{4(\varepsilon-1)}{n-1}\psi} g$$

for an additional parameter ε in an appropriate range, depending on m, called the ε -range (see also [9] for a proceeding work on singularity theorems in Lorentz-Finsler geometry). This is not only a generalization of [20] and [4], but also a unification of both constant and variable curvature bounds by choosing appropriate ε . We refer to [5, 6, 7] for further investigations on the ε -range.

In this paper, we assume lower bounds of the m-Bakry-Émery-Ricci curvature with ε -range and study analytic applications on non-compact manifolds. The main contributions of this paper are the following:

- We give an upper bound of the L^p_μ -spectrum. In particular, when p=2, this gives an upper bound of the first nonzero eigenvalue of the weighted Laplacian.
- We give an explicit form of a local Sobolev inequality.

An upper bound of the first nonzero eigenvalue of the Laplacian under lower Ricci curvature bounds was first investigated in [1] in 1975 and it is called the Cheng type inequality. Some variants of the Cheng type inequality are known (we refer to [16], for example) under lower bounds of the m-Bakry-Émery-Ricci curvature in the case of $m \in [n, \infty]$. Our Theorem 6 generalizes them. The local Sobolev inequality is an important tool for the DeGiorgi-Nash-Moser theory. Recently in [17], they obtained a Liouville type theorem for the weighted p-Laplacian by using a local Sobolev inequality and Moser's iteration techniques. Our results in Theorem 8 are consistent with the local Sobolev inequality in [17] in the case of constant curvature bounds and the effective dimension $m \in [n, \infty)$.

This paper is organized as follows. In Section 2, we briefly review the Bakry-Émery-Ricci curvature and Cheng type inequalities and local Sobolev inequalities. We show a Cheng type inequality in Section 3 and a local Sobolev inequality in Section 4 under lower bounds of the Bakry-Émery-Ricci curvature with ε -range. In Appendix, we give a variant of Cheng type inequality for deformed metrics under lower bounds of the Bakry-Émery-Ricci curvature with ε -range.

2 Preliminaries

2.1 ε -range

Let (M, g, μ) be an n-dimensional weighted Riemannian manifold. We assume that M is non-compact in this paper. We set $\mu = e^{-\psi}v_g$ where v_g is the Riemannian volume measure and ψ is a C^{∞} function on M. For $m \in (-\infty, 1] \cup [n, +\infty]$, the m-Bakry-Émery-Ricci curvature is defined as follows:

$$\operatorname{Ric}_{\psi}^{m} := \operatorname{Ric}_{g} + \nabla^{2} \psi - \frac{\mathrm{d}\psi \otimes \mathrm{d}\psi}{m-n},$$

where when $m = +\infty$, the last term is interpreted as the limit 0 and when m = n, we only consider a constant function ψ , and set $\operatorname{Ric}_{\psi}^{n} := \operatorname{Ric}_{q}$.

In [10], [9], they introduced the notion of ε -range:

$$\varepsilon = 0 \text{ for } m = 1, \quad |\varepsilon| < \sqrt{\frac{m-1}{m-n}} \text{ for } m \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } m = n.$$
 (1)

In this ε -range, for $K \in \mathbb{R}$, they considered the condition

$$\mathrm{Ric}_{\psi}^m(v) \geq K \mathrm{e}^{\frac{4(\varepsilon-1)}{n-1}\psi(x)} g(v,v), \quad v \in T_x M.$$

We also define the associated constant c as

$$c = \frac{1}{n-1} \left(1 - \varepsilon^2 \frac{m-n}{m-1} \right) > 0 \tag{2}$$

for $m \neq 1$ and $c = (n-1)^{-1}$ for m = 1. We define the comparison function \mathbf{s}_{κ} as

$$\mathbf{s}_{\kappa}(t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t) & \kappa > 0, \\ t & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & \kappa < 0. \end{cases}$$
 (3)

We denote $B(x,r) = \{y \in M \mid d(x,y) < r\}, V(x,r) = \mu(B(x,r)) \text{ and } tB = B(x,tr) \text{ if } B = B(x,r).$

Theorem 1. ([10, Theorem 3.11], Bishop-Gromov volume comparison theorem) Let (M, g, μ) be a complete weighted Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the ε -range (1), $K \in \mathbb{R}$ and $b \geq a > 0$. Assume that

$$\operatorname{Ric}_{\psi}^{m}(v) \geq K e^{\frac{4(\varepsilon-1)}{n-1}\psi(x)} g(v,v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \le e^{-\frac{2(\varepsilon-1)}{n-1}\psi} \le b.$$

Then we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \le \frac{b}{a} \frac{\int_0^{\min\{R/a, \pi/\sqrt{cK}\}} \mathbf{s}_{cK}(\tau)^{1/c} d\tau}{\int_0^{r/b} \mathbf{s}_{cK}(\tau)^{1/c} d\tau}$$

for all $x \in M$ and 0 < r < R, where $R \le b\pi/\sqrt{cK}$ when K > 0 and we set $\pi/\sqrt{cK} := \infty$ for $K \le 0$.

We briefly review the argument in [10] (where they considered, more generally, Finsler manifolds equipped with measures). Given a unit tangent vector $v \in T_xM$, let $\eta:[0,l) \longrightarrow \mathbb{R}$ be the geodesic with $\dot{\eta}(0) = v$. We take an orthonormal basis $\{e_i\}_{i=1}^n$ of T_xM with $e_n = v$ and consider the Jacobi fields

$$E_i(t) := (d \exp_x)_{tv}(te_i), \quad i = 1, 2, \dots, n - 1,$$

along η . Define the $(n-1) \times (n-1)$ matrices $A(t) = (a_{ij}(t))$ by

$$a_{ij}(t) := g(E_i(t), E_j(t)).$$

We define

$$h_0(t) := (\det A(t))^{1/2(n-1)}, \qquad h(t) := e^{-c\psi(\eta(t))} (\det A(t))^{c/2}, \qquad h_1(\tau) := h(\varphi_{\eta}^{-1}(\tau))$$

for $t \in [0, l)$ and $\tau \in [0, \varphi_{\eta}(l))$, where

$$\varphi_{\eta}(t) := \int_0^t e^{\frac{2(\varepsilon - 1)}{n - 1} \psi(\eta(s))} ds.$$

By the definition, we have the following relationship:

$$(e^{-\psi(\eta)}h_0^{n-1})(t) = h(t)^{1/c} = h_1(\varphi_\eta(t))^{1/c}$$

According to the argument in [10, Theorem 3.6], the condition $\operatorname{Ric}_{\psi}^{m}(v) \geq K e^{\frac{4(\varepsilon-1)}{n-1}\psi(x)}g(v,v)$ implies that

$$(e^{-\psi(\eta)}h_0^{n-1})/\mathbf{s}_{cK}(\varphi_\eta)^{1/c}$$
 is non-increasing. (4)

This plays the key role in proving Theorem 1 above.

2.2 Upper bounds of the L^p_μ -spectrum

In this subsection, we explain Cheng type inequalities under lower Bakry-Émery-Ricci curvature bounds by constants. We generalize these results to the ε -range in Section 3. For p>1, the L^p_μ -spectrum is defined by

$$\lambda_{\mu,p}(M) := \inf_{\phi \in C_0^{\infty}(M)} \frac{\int_M |\nabla \phi|^p \mathrm{d}\mu}{\int_M |\phi|^p \mathrm{d}\mu}.$$

When p=2, the L^p_{μ} spectrum is the first nonzero eigenvalue of the weighted Laplacian. Under lower m-Bakry-Émery-Ricci curvature bounds with $m \in [n, \infty)$, we have the following theorems.

Theorem 2. ([16, Theorem 3.2]) Let (M, g, μ) be an n-dimensional weighted complete Riemannian manifold. Assume that $\operatorname{Ric}_{\psi}^m \geq -K$ $(K \geq 0)$. Then the L_{μ}^p -spectrum satisfies

$$\lambda_{\mu,p}(M) \le \left(\frac{\sqrt{(m-1)K}}{p}\right)^p.$$

An additional assumption on the weight function leads to the following Cheng type inequality under a lower ∞ -Bakry-Émery-Ricci curvature bound.

Theorem 3. ([16, Theorem 3.3]) Let (M, g, μ) be an n-dimensional complete weighted Riemannian manifold. We fix a point $q \in M$. Assume that $\mathrm{Ric}_{\psi}^{\infty} \geq -K$ $(K \geq 0)$ and $\frac{\partial \psi}{\partial r} \geq -k$ $(k \geq 0)$ along all minimal geodesic segments from the fixed point $q \in M$, where r is the distance from q. Then the L_{μ}^{p} -spectrum satisfies

$$\lambda_{\mu,p}(M) \le \left(\frac{\sqrt{(n-1)K} + k}{p}\right)^p.$$

These results are generalizations of the original Cheng type inequality in [1].

2.3 Local Sobolev inequality

We have the following local Sobolev inequality under lower bounds of the m-Bakry-Émery-Ricci curvature in the case of $m \in (n, \infty)$ and $n \ge 2$. We generalize the following result in Section 4. We refer to [12] for the case of $m = \infty$.

Theorem 4. ([17, Lemma 3.2]) Let (M, g, μ) be an n-dimensional weighted complete Riemannian manifold. If $\mathrm{Ric}_{\psi}^m \geq -(m-1)K$ for some $K \geq 0$ and $m > n \geq 2$, then there exists a constant C, depending on m, such that for all $B(o, r) \subset M$ we have for $f \in C_0^{\infty}(B(o, r))$,

$$\left(\int_{B(o,r)} |f|^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \le e^{C(1+\sqrt{K}r)} \mu(B(o,r))^{-\frac{2}{m}} r^2 \int_{B(o,r)} (|\nabla f|^2 + r^{-2} f^2) d\mu.$$

We will use the next theorem in Subsection 4.2 to prove a local Sobolev inequality under lower Bakry-Émery-Ricci curvature bounds with ε -range.

Theorem 5. ([14, Theorem 2.2]) Let e^{-tA} be a symmetric submarkovian semigroup acting on the spaces $L^p(M, d\mu)$. Given $\nu > 2$, the following three properties are equivalent.

1.
$$\|e^{-tA}f\|_{\infty} \le C_0 t^{-\nu/2} \|f\|_1$$
 for $0 < t < t_0$.

2.
$$||f||_{2\nu/(\nu-2)}^2 \le C_1 \left(||A^{1/2}f||_2^2 + t_0^{-1}||f||_2^2 \right)$$
.

3.
$$||f||_2^{2+4/\nu} \le C_2 \left(||A^{1/2}f||_2^2 + t_0^{-1}||f||_2^2 \right) ||f||_1^{4/\nu}$$
.

Moreover, 3. implies 1. with $C_0 = (\nu C C_2)^{\nu/2}$ and 1. implies 2. with $C_1 = C C_0^{2/\nu}$, where C is some numerical constant.

3 Upper bound of the L^p_u -spectrum with ε -range

Theorem 6. Let (M, g, μ) be an n-dimensional weighted complete Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the ε -range (1), K > 0 and $b \ge a > 0$. Assume that

$$\operatorname{Ric}_{\psi}^{m}(v) \ge -Ke^{\frac{4(\varepsilon-1)}{n-1}\psi(x)}g(v,v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \le e^{-\frac{2(\varepsilon - 1)}{n - 1}\psi} \le b. \tag{5}$$

Then, for p > 1, we have

$$\lambda_{\mu,p}(M) \le \left(\sqrt{\frac{K}{c}} \frac{1}{pa}\right)^p.$$

Proof. We apply the argument in [16, Theorem 3.2]. For an arbitrary $\delta > 0$, we set

$$\alpha := -\frac{\sqrt{\frac{K}{c}}\frac{1}{a} + \delta}{p}$$

and, for $x \in M$ and $R \ge 2$,

$$\phi(y) := \exp(\alpha r(y)) \varphi(y),$$

where r(y) = d(x, y) and φ is a cut off function on B(x, R) such that $\varphi = 1$ on B(x, R - 1), $\varphi = 0$ on $M \setminus B(x, R)$ and $|\nabla \varphi| \le C_3$, where C_3 is a constant independent of R. For an arbitrary $\zeta > 0$, we have

$$\begin{split} |\nabla \phi|^p &= |\alpha \mathrm{e}^{\alpha r} \varphi \nabla r + \mathrm{e}^{\alpha r} \nabla \varphi|^p \\ &\leq \mathrm{e}^{p \alpha r} (-\alpha \varphi + |\nabla \varphi|)^p \\ &\leq \mathrm{e}^{p \alpha r} \left[(1+\zeta)^{p-1} (-\alpha \varphi)^p + \left(\frac{1+\zeta}{\zeta} \right)^{p-1} |\nabla \varphi|^p \right]. \end{split}$$

By the definition of $\lambda_{\mu,p}(M)$, we find

$$\lambda_{\mu,p}(M) \leq (1+\zeta)^{p-1}(-\alpha)^{p} + \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\int_{M} e^{p\alpha r} |\nabla \varphi|^{p} d\mu}{\int_{M} e^{p\alpha r} \varphi^{p} d\mu}$$

$$= (1+\zeta)^{p-1}(-\alpha)^{p} + \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\int_{B(x,R)\setminus B(x,R-1)} e^{p\alpha r} |\nabla \varphi|^{p} d\mu}{\int_{B(x,R)} e^{p\alpha r} \varphi^{p} d\mu}$$

$$\leq (1+\zeta)^{p-1}(-\alpha)^{p} + C_{3}^{p} \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{e^{p\alpha(R-1)} \mu (B(x,R))}{\int_{B(x,1)} e^{p\alpha r} d\mu}$$

$$\leq (1+\zeta)^{p-1}(-\alpha)^{p} + C_{3}^{p} \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{e^{p\alpha(R-1)} \mu (B(x,R))}{e^{p\alpha} \mu (B(x,1))}.$$
(6)

It follows from Theorem 1 that

$$\mu(B(x,R)) \le \mu(B(x,1)) \frac{b}{a} \frac{\int_0^{R/a} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau}{\int_0^{1/b} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau}.$$
 (7)

To estimate the RHS of (7), we observe

$$(\sqrt{cK})^{1/c} \int_0^{R/a} \mathbf{s}_{-cK}^{1/c}(\tau) d\tau = \int_0^{R/a} \left[\frac{1}{2} \left\{ \exp(\sqrt{cK}\tau) - \exp(-\sqrt{cK}\tau) \right\} \right]^{1/c} d\tau$$

$$\leq \int_0^{R/a} \exp\left(\sqrt{\frac{K}{c}}\tau\right) d\tau$$

$$= \sqrt{\frac{c}{K}} \left\{ \exp\left(\sqrt{\frac{K}{c}}\frac{R}{a}\right) - 1 \right\}.$$

Thus, we have

$$\mu(B(x,R)) \le \mu(B(x,1)) \frac{b}{a} \frac{1}{\int_0^{1/b} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau} \sqrt{\frac{c}{K}} \exp\left(\sqrt{\frac{K}{c}} \frac{R}{a}\right) \frac{1}{(\sqrt{cK})^{1/c}} d\tau$$

This implies

$$\frac{e^{p\alpha(R-1)}\mu\left(B(x,R)\right)}{e^{p\alpha}\mu\left(B(x,1)\right)} \le C_4 \exp\left(p\alpha R + \sqrt{\frac{K}{c}}\frac{R}{a}\right) = C_4 \exp(-\delta R) \to 0$$

as $R \to \infty$, where C_4 is a constant depending on c, a, b, K, δ . Hence, (6) yields

$$\lambda_{\mu,p}(M) \le (1+\zeta)^{p-1} (-\alpha)^p.$$

Since $\zeta > 0$ and $\delta > 0$ are arbitrary, this implies the theorem.

When $m \in [n, \infty)$, $\varepsilon = 1$ and a = b = 1, then $c = \frac{1}{m-1}$ and it holds

$$\lambda_{\mu,p}(M) \le \left(\frac{\sqrt{(m-1)K}}{p}\right)^p,$$

which recovers Theorem 2.

4 Functional inequalities with ε -range

4.1 Local Poincaré inequality

In this subsection, we prove the following Poincaré inequality.

Theorem 7. (Local Poincaré inequality) Let (M, g, μ) be an n-dimensional complete weighted Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the ε -range (1), K > 0 and $b \geq a > 0$. Assume that

$$\operatorname{Ric}_{\psi}^{m}(v) \ge -Ke^{\frac{4(\varepsilon-1)}{n-1}\psi(x)}g(v,v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \le e^{-\frac{2(\varepsilon-1)}{n-1}\psi} \le b. \tag{8}$$

Then we have

$$\forall f \in C^{\infty}(M), \quad \int_{B} |f - f_{B}|^{2} d\mu \leq 2^{n+3} \left(\frac{2b}{a}\right)^{1/c} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a}\right) r^{2} \int_{2B} |\nabla f|^{2} d\mu \tag{9}$$

for all balls $B \subset M$ of radius $0 < r < \infty$, where

$$f_B := \frac{1}{\mu(B)} \int_B f \mathrm{d}\mu.$$

Proof. We apply the argument in [15, Theorem 5.6.6, Lemma 5.6.7]. For any pair of points $(x, y) \in M \times M$, let

$$\gamma_{x,y}: [0,d(x,y)] \to M$$

be a geodesic from x to y parametrized by arclength. We also set

$$l_{x,y}(t) = \gamma_{x,y}(td(x,y))$$

for $t \in [0, 1]$. We have, using Jensen's inequality,

$$\int_{B} |f - f_{B}|^{2} d\mu = \int_{B} \left| \int_{B} (f(x) - f(y)) \frac{d\mu(y)}{\mu(B)} \right|^{2} d\mu(x)
\leq \frac{1}{\mu(B)} \int_{B} \int_{B} |f(l_{x,y}(1)) - f(l_{x,y}(0))|^{2} d\mu(x) d\mu(y)
\leq \frac{1}{\mu(B)} \int_{B} \int_{B} \left\{ \int_{0}^{1} \left| \frac{d(f \circ l_{x,y})}{dt}(t) \right| dt \right\}^{2} d\mu(x) d\mu(y)
\leq \frac{1}{\mu(B)} \int_{B} \int_{B} \int_{0}^{1} \left| \frac{d(f \circ l_{x,y})}{dt}(t) \right|^{2} dt d\mu(x) d\mu(y)
= \frac{2}{\mu(B)} \int_{B} \int_{B} \int_{1/2}^{1} \left| \frac{d(f \circ l_{x,y})}{dt}(t) \right|^{2} dt d\mu(x) d\mu(y).$$

To obtain the last equality we decompose the set

$$\{(x, y, t) : x, y \in B, t \in (0, 1)\}$$

into two pieces,

$$\{(x, y, s) : x, y \in B, t \in (1/2, 1)\}$$

and

$$\{(x, y, s) : x, y \in B, t \in (0, 1/2)\},\$$

then use $l_{x,y}(t) = l_{y,x}(1-t)$. Now, suppose that we can bound the Jacobian $J_{x,t}$ of the map

$$\Phi_{x,t}: y \mapsto l_{x,y}(t)$$

from below by

$$\forall x, y \in B, \forall s \in [1/2, 1], \quad J_{x,t}(y) \ge 1/F(r),$$
 (10)

where r is the radius of the ball B. Then

$$\begin{split} \int_{B} \int_{B}^{1} \left| \frac{\mathrm{d}f(l_{x,y}(t))}{\mathrm{d}t} \right|^{2} \mathrm{d}t \; \mathrm{d}\mu(x) \; \mathrm{d}\mu(y) & \leq F(r) \int_{B} \int_{B}^{1} \int_{1/2}^{1} \left| \frac{\mathrm{d}f(l_{x,y}(t))}{\mathrm{d}t} \right|^{2} J_{x,t}(y) \mathrm{d}t \; \mathrm{d}\mu(x) \; \mathrm{d}\mu(y) \\ & \leq F(r) \int_{B} \int_{B}^{1} \left| \nabla f(l_{x,y}(t)) \right|^{2} d(x,y)^{2} J_{x,t}(y) \mathrm{d}t \; \mathrm{d}\mu(x) \; \mathrm{d}\mu(y) \\ & \leq (2r)^{2} F(r) \int_{0}^{1} \int_{B} \int_{B} \left| \nabla f(l_{x,y}(t)) \right|^{2} J_{x,t}(y) \mathrm{d}\mu(y) \; \mathrm{d}\mu(x) \; \mathrm{d}t \\ & = (2r)^{2} F(r) \int_{0}^{1} \left[\int_{B} \left(\int_{\Phi_{x,t}(B)} \left| \nabla f(z) \right|^{2} \mathrm{d}\mu(z) \right) \mathrm{d}\mu(x) \right] \mathrm{d}t \\ & \leq (2r)^{2} F(r) \mu(B) \int_{2B} \left| \nabla f(z) \right|^{2} \mathrm{d}\mu(z) \right. \end{split}$$

We finally prove (10). Let ξ be the unit tangent vector at x such that $\partial_s \gamma_{x,y}(s)|_{s=0} = \xi$. Let $I(x,s,\xi)$ be the Jacobian of the map $\exp_x : T_x M \to M$ at $s\xi$ with respect to μ . Then

$$d\mu = I(x, s, \xi) ds d\xi,$$

where $d\xi$ is the usual measure on the sphere. Using the notation in Subsection 2.1, we have $I(x, s, \xi) = e^{-\psi(\eta(s))} h_0^{n-1}(s)$. According to (4), we find that

$$s \to \frac{I(x, s, \xi)}{\mathbf{s}_{-cK}(\varphi_{\eta}(s))^{1/c}}$$

is non-increasing. Under the relationship $l_{x,y}(t) = \gamma_{x,y}(s)$, it follows that

$$J_{x,t}(y) = \left(\frac{s}{d(x,y)}\right)^n \frac{I(x,s,\xi)}{I(x,d(x,y),\xi)} \ge \left(\frac{1}{2}\right)^n \frac{\mathbf{s}_{-cK}(\varphi_{\eta}(s))^{1/c}}{\mathbf{s}_{-cK}(\varphi_{\eta}(d(x,y)))^{1/c}}$$

for all $s \in (d(x,y)/2, d(x,y))$. Thus, we have, since $s/b \le \varphi_{\eta}(t) \le s/a$,

$$J_{x,t}(y) \geq \left(\frac{1}{2}\right)^{n} \frac{\mathbf{s}_{-cK}(\varphi_{\eta}(d(x,y)/2))^{1/c}}{\mathbf{s}_{-cK}(\varphi_{\eta}(d(x,y)))^{1/c}}$$

$$\geq \left(\frac{1}{2}\right)^{n} \left(\frac{\varphi_{\eta}(d(x,y)/2)}{\varphi_{\eta}(d(x,y))}\right)^{1/c} \exp\left(-\sqrt{\frac{K}{c}}\varphi_{\eta}(d(x,y))\right)$$

$$\geq \left(\frac{1}{2}\right)^{n} \left(\frac{ad(x,y)/2}{bd(x,y)}\right)^{1/c} \exp\left(-\sqrt{\frac{K}{c}}\frac{d(x,y)}{a}\right)$$

$$\geq \left(\frac{1}{2}\right)^{n} \left(\frac{a}{2b}\right)^{1/c} \exp\left(-\sqrt{\frac{K}{c}}\frac{2r}{a}\right).$$

This proves (10) with $F(r) = \left\{ \left(\frac{1}{2}\right)^n \left(\frac{a}{2b}\right)^{1/c} \exp\left(-\sqrt{\frac{K}{c}} \frac{2r}{a}\right) \right\}^{-1}$ and the theorem follows.

Given that we have the local Poincaré inequality and the volume doubling property (obtained explicitly later in (12)), we can apply [15, Corollary 5.3.5] and we obtain the following inequality.

Corollary 1. Under the same assumptions as in Theorem 7, there exist constants C, P such that

$$\forall f \in C^{\infty}(M), \quad \int_{B} |f - f_{B}|^{2} d\mu \leq P e^{Cr} r^{2} \int_{B} |\nabla f|^{2} d\mu$$

for all balls $B \subset M$ of radius r > 0.

4.2 Local Sobolev inequality

It is shown in [14] that the volume doubling property and Poincaré inequality imply a local Sobolev inequality. We follow this line with Theorems 1 and 7.

Theorem 8. (Local Sobolev inequality) Let (M, g, μ) be an n-dimensional complete weighted Riemannian manifold with $n \geq 3$ and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the ε -range (1), K > 0 and $b \geq a > 0$. Assume that

$$\operatorname{Ric}_{\psi}^{m}(v) \ge -Ke^{\frac{4(\varepsilon-1)}{n-1}\psi(x)}g(v,v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \le e^{-\frac{2(\varepsilon - 1)}{n - 1}\psi} \le b. \tag{11}$$

Then there exist constants D, E depending on c, a, b, n such that for all $B(o, r) \subset M$ we have for $f \in C_0^{\infty}(B(o, r))$,

$$\left(\mu(B(o,r))^{-1} \int_{B(o,r)} |f|^{\frac{2(1+c)}{1-c}} d\mu\right)^{\frac{1-c}{1+c}} \le E \exp\left(D\left(1+\sqrt{\frac{K}{c}}\right) \frac{r}{a}\right) r^2 \mu(B(o,r))^{-1} \int_{B(o,r)} (|\nabla f|^2 + r^{-2} f^2) d\mu.$$

We first prove two lemmas. We set

$$f_s(x) = \int \chi_s(x, z) f(z) d\mu(z),$$

where $V(x,s) = \mu(B(x,s))$ and $\chi_s(x,z) = \frac{1}{V(x,s)} 1_{B(x,s)}(z)$.

Lemma 1. Under the same assumptions as in Theorem 8, there exists a constant C_5 such that for all $y \in M$ and all 0 < s < r, we have

$$||f_s||_2 \le C_5 V^{-\frac{1}{2}} \left(\frac{r}{s}\right)^{\frac{1}{2}\left(1+\frac{1}{c}\right)} ||f||_1,$$

for all $f \in C_0^{\infty}(B)$, where B = B(y,r) and V = V(r) = V(y,r).

Proof. We apply the argument in [14, Lemma 2.3]. We use the notations in Subsection 2.1. For $\tau \geq 0$, 0 < s < r, we set

$$t := \frac{r}{a} \frac{b}{s} \tau.$$

Since $\frac{r}{a}\frac{b}{s} \ge 1$, we have $\tau \le t$. Hence, by direct computations, we obtain

$$\mathbf{s}_{-cK}(t)^{1/c} \le \mathbf{s}_{-cK}(\tau)^{1/c} \left(\frac{t}{\tau}\right)^{1/c} \exp\left(\sqrt{\frac{K}{c}}t\right).$$

Integrating both sides in t from 0 to r/a, we have

$$\int_{0}^{r/a} \mathbf{s}_{-cK}(t)^{1/c} dt \leq \int_{0}^{r/a} \mathbf{s}_{-cK}(\tau)^{1/c} \left(\frac{t}{\tau}\right)^{1/c} \exp\left(\sqrt{\frac{K}{c}}t\right) dt
\leq \left(\frac{r}{a}\frac{b}{s}\right)^{1/c} \exp\left(\sqrt{\frac{K}{c}}\frac{r}{a}\right) \int_{0}^{r/a} \mathbf{s}_{-cK}(\tau)^{1/c} dt
= \left(\frac{r}{a}\frac{b}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}}\frac{r}{a}\right) \int_{0}^{s/b} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau.$$

According to Theorem 1, we find

$$\frac{V(r)}{V(s)} \leq \frac{b}{a} \frac{\int_0^{r/a} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau}{\int_0^{s/b} \mathbf{s}_{-cK}(\tau)^{1/c} d\tau} \\
\leq \left(\frac{b}{a}\right)^{2+\frac{1}{c}} \left(\frac{r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{r}{a}\right).$$

Therefore, we also have the doubling property:

$$V(2r) \le V(r) \left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a}\right). \tag{12}$$

For $x, z \in M$ satisfying d(z, x) < s, we have

$$\begin{array}{lcl} V(z,s) & \leq & V(x,2s) \\ & \leq & V(x,s) \left(\frac{b}{a}\right)^{2+\frac{1}{c}} 2^{\frac{1}{c}+1} \exp\left(\sqrt{\frac{K}{c}} \frac{2s}{a}\right). \end{array}$$

This implies

$$\chi_s(x,z) \le \left(\frac{b}{a}\right)^{2+\frac{1}{c}} 2^{\frac{1}{c}+1} \exp\left(\sqrt{\frac{K}{c}} \frac{2s}{a}\right) \chi_s(z,x).$$

Thus,

$$||f_s||_1 \le \left(\frac{b}{a}\right)^{2+\frac{1}{c}} 2^{\frac{1}{c}+1} \exp\left(\sqrt{\frac{K}{c}} \frac{2s}{a}\right) ||f||_1.$$
 (13)

We moreover assume $B \cap B(x,s) \neq \emptyset$. Since

$$\frac{V(x,2r+s)}{V(x,s)} \le \left(\frac{b}{a}\right)^{2+\frac{1}{c}} \left(\frac{2r+s}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{2r+s}{a}\right)$$

and

$$\frac{V(x,4r)}{V(x,2r+s)} \leq \left(\frac{b}{a}\right)^{2+\frac{1}{c}} \left(\frac{4r}{2r+s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{4r}{a}\right),$$

we have

$$\frac{1}{V(x,s)} \leq \left(\frac{b}{a}\right)^{2+\frac{1}{c}} \left(\frac{2r+s}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{2r+s}{a}\right) \frac{1}{V(x,2r+s)}$$

$$\leq \left(\frac{b}{a}\right)^{2\left(2+\frac{1}{c}\right)} \left(\frac{2r+s}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{2r+s}{a}\right) \left(\frac{4r}{2r+s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{4r}{a}\right) \frac{1}{V(x,4r)}$$

$$\leq \left(\frac{b}{a}\right)^{2\left(2+\frac{1}{c}\right)} \left(\frac{4r}{s}\right)^{1+\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{6r+s}{a}\right) \frac{1}{V(y,r)}.$$

Hence.

$$||f_s||_{\infty} = \left\| \int \chi_s(x, z) f(z) d\mu(z) \right\|_{\infty}$$

$$\leq \left(\frac{b}{a} \right)^{2\left(2 + \frac{1}{c}\right)} \left(\frac{4r}{s} \right)^{1 + \frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{6r + s}{a} \right) \frac{||f||_1}{V(y, r)}.$$

Using (13), we have

$$||f_{s}||_{2} = \left(\int f_{s}^{2} d\mu\right)^{\frac{1}{2}}$$

$$\leq \sqrt{||f_{s}||_{\infty}} \sqrt{||f_{s}||_{1}}$$

$$\leq \left(\frac{b}{a}\right)^{2+\frac{1}{c}} \left(\frac{4r}{s}\right)^{\frac{1}{2}(1+\frac{1}{c})} \exp\left(\sqrt{\frac{K}{c}} \frac{6r+s}{2a}\right) \frac{1}{\sqrt{V(y,r)}} \left(\frac{b}{a}\right)^{\frac{1}{2}(2+\frac{1}{c})} 2^{\frac{1}{2}(\frac{1}{c}+1)} \exp\left(\sqrt{\frac{K}{c}} \frac{s}{a}\right) ||f||_{1}$$

$$= \left(\frac{b}{a}\right)^{3+\frac{3}{2c}} 2^{\frac{1}{2}(1+\frac{1}{c})} \left(\frac{4r}{s}\right)^{\frac{1}{2}(1+\frac{1}{c})} \exp\left(\sqrt{\frac{K}{c}} \frac{6r+3s}{2a}\right) \frac{1}{\sqrt{V(y,r)}} ||f||_{1}.$$

Setting
$$C_5 = \left(\frac{b}{a}\right)^{3+\frac{3}{2c}} 2^{\frac{1}{2}\left(1+\frac{1}{c}\right)} 4^{\frac{1}{2}\left(1+\frac{1}{c}\right)} \exp\left(\sqrt{\frac{K}{c}} \frac{9r}{2a}\right)$$
, we get the desired inequality.

Lemma 2. We fix a constant r > 0. Under the same assumptions as in Theorem 8, there exists C_6 depending only on c, a, b, r, K, n such that

$$||f - f_s||_2 \le C_6 s ||\nabla f||_2, \quad f \in C_0^{\infty}(M)$$

for all 0 < s < r.

Proof. We apply the argument in [14, Lemma 2.4]. Fix a > 0, let $\{B_j : j \in J\}$ be a collection of balls of radius s/2 such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $M = \bigcup_{i \in J} 2B_i$. For $z \in M$, let $J(z) = \{i \in J : z \in 8B_i\}$ and N(z) = #J(z). We first estimate N(z) from above. Let B_z be a ball in $\{B_j : j \in J\}$ such that $z \in 2B_z$. For $i \in J(z)$, we have $B_z \subset 16B_i$. Hence,

$$\mu(B_z) \le \mu(16B_i) \le C_7^4 \mu(B_i),$$

where

$$C_7 := \left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{8r}{a}\right) \ge \left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{8s}{a}\right).$$

Therefore, we have

$$\sum_{i \in J(z)} \mu(B_i) \ge N(z) \frac{\mu(B_z)}{C_7^4}.$$

On the other hand, for $i \in \{j \in J: z \in 8B_j\}$, we have $B_i \subset 16B_z$. Hence,

$$\sum_{i \in J(z)} \mu(B_i) \le \mu(16B_z) \le C_7^4 \mu(B_z).$$

Therefore, we find

$$N(z)\frac{\mu(B_z)}{C_7^4} \le C_7^4 \mu(B_z).$$

Letting $N_0 := C_7^8$, we have $N(z) \le N_0$. We now estimate $||f - f_s||_2$. Note that

$$||f - f_s||_2^2 \le \sum_{i \in J} \left(2 \int_{2B_i} |f(x) - f_{4B_i}|^2 + |f_{4B_i} - f_s(x)|^2 d\mu(x) \right).$$
 (14)

By the Poincaré inequality (9), we have

$$\int_{4B_i} |f(x) - f_{4B_i}|^2 d\mu(x) \le 2^{n+3} \left(\frac{2b}{a}\right)^{\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{4s}{a}\right) (2s)^2 \int_{8B_i} |\nabla f|^2 d\mu \le C_8 s^2 \int_{8B_i} |\nabla f|^2 d\mu, \quad (15)$$

where

$$C_8 = 2^{n+5} \left(\frac{2b}{a}\right)^{\frac{1}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{4r}{a}\right).$$

Since for any $x \in 2B_i = B(x_i, s)$,

$$V(x_i, s) \le V(x, 2s) \le V(x, s) \left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{2s}{a}\right),$$

we have

$$\int_{2B_{i}} |f_{4B_{i}} - f_{s}(x)|^{2} d\mu(x) = \int_{2B_{i}} \left| \int_{B(x,s)} \frac{1}{V(x,s)} \{f_{4B_{i}} - f(z)\} d\mu(z) \right|^{2} d\mu(x)$$

$$\leq \int_{2B_{i}} \int_{B(x,s)} \frac{1}{V(x,s)} |f_{4B_{i}} - f(z)|^{2} d\mu(z) d\mu(x)$$

$$\leq \frac{1}{V(x_{i},s)} \left(\frac{b}{a}\right)^{2+\frac{1}{c}} 2^{\frac{1}{c}+1} \exp\left(\sqrt{\frac{K}{c}} \frac{2s}{a}\right) \int_{2B_{i}} \int_{4B_{i}} |f_{4B_{i}} - f(z)|^{2} d\mu(z) d\mu(x)$$

$$\leq \left(\frac{b}{a}\right)^{2+\frac{1}{c}} 2^{\frac{1}{c}+1} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a}\right) C_{8}s^{2} \int_{8B_{i}} |\nabla f|^{2} d\mu. \tag{16}$$

Using (14), (15), (16), we have

$$||f - f_s||_2^2 \le C_9 s^2 \sum_{i \in J} \int_{8B_i} |\nabla f|^2 d\mu \le C_9 N_0 s^2 ||\nabla f||_2^2,$$

where

$$C_9 = 4\left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a}\right) C_8 \ge 2\left\{\left(\frac{b}{a}\right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a}\right) C_8 + C_8\right\}.$$

Therefore, setting

$$C_6 := \sqrt{N_0 C_9}$$

$$= \left\{ \left(\frac{b}{a} \right)^{2 + \frac{1}{c}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{8r}{a} \right) \right\}^4 \times \sqrt{4 \left(\frac{b}{a} \right)^{2 + \frac{1}{c}}} 2^{\frac{1}{c} + 1} \exp\left(\sqrt{\frac{K}{c}} \frac{2r}{a} \right) \times \sqrt{2^{n+5} \left(\frac{2b}{a} \right)^{\frac{1}{c}}} \exp\left(\sqrt{\frac{K}{c}} \frac{4r}{a} \right)$$

$$= 2^{8 + \frac{5}{c} + \frac{n}{2}} \left(\frac{b}{a} \right)^{9 + \frac{5}{c}} \exp\left(\sqrt{\frac{K}{c}} \frac{35r}{a} \right),$$

we have the desired inequality.

Proof of Theorem 8

We apply the argument in [14, Theorem 2.1]. Fix $x \in M, r > 0$. For $0 < s \le r$ and $f \in C_0^{\infty}(B(x, r))$, we have

$$||f||_2 \le ||f - f_s||_2 + ||f_s||_2$$

It follows from Lemmas 1, 2 that

$$||f||_2 \le C_6 s ||\nabla f||_2 + C_5 V^{-\frac{1}{2}} \left(\frac{r}{s}\right)^{\frac{\nu}{2}} ||f||_1,$$

where $\nu = 1 + \frac{1}{c}$. Hence, we obtain

$$||f||_{2} \le 4C_{6}s \left(||\nabla f||_{2} + \frac{1}{r}||f||_{2} \right) + C_{5}V^{-\frac{1}{2}} \left(\frac{r}{s} \right)^{\frac{\nu}{2}} ||f||_{1}.$$

$$(17)$$

To obtain the minimum of the RHS of (17), we consider its differential with respect to s > 0. At s > 0 which attains the minimum, we have

$$4C_6 \left(\|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right) + C_5 V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \left(-\frac{\nu}{2} \right) s^{-\frac{\nu}{2} - 1} \|f\|_1 = 0.$$

Thus,

$$s^{\frac{\nu}{2}+1} = C_{10} \frac{V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \|f\|_{1}}{\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2}},\tag{18}$$

where $C_{10} = \frac{\nu}{2} \frac{C_5}{4C_6}$. Substituting (18) to the RHS of (17), we obtain

$$\begin{split} \|f\|_{2} & \leq 4C_{6} \left\{ C_{10} \frac{V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \|f\|_{1}}{\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2}} \right\}^{\frac{2}{2+\nu}} \left(\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2} \right) + C_{5} V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \left\{ \left(C_{10} \frac{V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \|f\|_{1}}{\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2}} \right)^{\frac{2}{2+\nu}} \right\}^{\frac{\nu}{2}+\nu} \\ & = 4C_{6} C_{10}^{\frac{2}{2+\nu}} \left(\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2} \right)^{-\frac{2}{2+\nu}+1} \left(V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \|f\|_{1} \right)^{\frac{2}{2+\nu}} \\ & + C_{5} C_{10}^{\frac{-\nu}{2+\nu}} \left(\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2} \right)^{\frac{\nu}{2+\nu}} V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \left(V^{-\frac{1}{2}} r^{\frac{\nu}{2}} \|f\|_{1} \right)^{-\frac{\nu}{2+\nu}} \|f\|_{1} \\ & = \left(\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2} \right)^{\frac{\nu}{2+\nu}} \left\{ 4C_{6} C_{10}^{\frac{2}{2+\nu}} V^{-\frac{1}{2}} \left(\frac{2}{2+\nu} \right) r^{\frac{\nu}{2}} \left(\frac{2}{2+\nu} \right) \|f\|_{1}^{\frac{2}{2+\nu}} \right. \\ & + C_{5} C_{10}^{\frac{-\nu}{2+\nu}} V^{-\frac{1}{2}} \left(1 - \frac{\nu}{2+\nu} \right) \|f\|_{1}^{1-\frac{\nu}{2+\nu}} \right\} \\ & = \left\{ 4C_{6} C_{10}^{\frac{2}{2+\nu}} + C_{5} C_{10}^{\frac{-\nu}{2+\nu}} \right\} \left(\|\nabla f\|_{2} + \frac{1}{r} \|f\|_{2} \right)^{\frac{\nu}{2+\nu}} V^{-\frac{1}{2}} \left(\frac{2}{2+\nu} \right) r^{\frac{\nu}{2}} \left(\frac{2}{2+\nu} \right) \|f\|_{1}^{\frac{2}{2+\nu}} \right. \end{split}$$

Hence

$$||f||_{2}^{2+\frac{4}{\nu}} \le \left\{ 4C_{6}C_{10}^{\frac{2}{2+\nu}} + C_{5}C_{10}^{\frac{-\nu}{2+\nu}} \right\}^{2+\frac{4}{\nu}} V^{-\frac{2}{\nu}} r^{2} \left\{ 2\left(\|\nabla f\|_{2}^{2} + \frac{\|f\|_{2}^{2}}{r^{2}} \right) \right\} ||f||_{1}^{\frac{4}{\nu}}.$$

Recalling the expressions of C_5 and C_6 in the proofs of Lemmas 1, 2, we choose constants E_1 , E_2 depending on c, a, b, n such that

$$C_6 C_{10}^{\frac{2}{2+\nu}} = E_1 \exp\left(\sqrt{\frac{K}{c}} \left(\frac{35r}{a} - \frac{61r}{2a} \frac{2}{2+\nu}\right)\right)$$

and

$$C_5 C_{10}^{\frac{-\nu}{2+\nu}} = E_2 \exp\left(\sqrt{\frac{K}{c}} \left(\frac{9r}{2a} - \frac{61r}{2a} \frac{-\nu}{2 + \nu}\right)\right).$$

Thus, there exist constants D, E_3 such that

$$\left\{4C_6C_{10}^{\frac{2}{2+\nu}} + C_5C_{10}^{\frac{-\nu}{2+\nu}}\right\}^{2+\frac{4}{\nu}} < E_3 \exp\left(D\left(1+\sqrt{\frac{K}{c}}\right)\frac{r}{a}\right).$$

We remark that D, E_3 depend only on c, a, b, n. Since $c \le \frac{1}{n-1}$, we have $\nu = 1 + \frac{1}{c} \ge n > 2$ when $n \ge 3$. Hence, we can use Theorem 5 and Theorem 8 follows.

Remark 1. At the end of the proof of Theorem 8, we use the fact that $1 + \frac{1}{c} > 2$, it is the only reason why we need the assumption $n \ge 3$. In the case of n = 2, we have the local Sobolev inequality when $\varepsilon \ne 0$ under the same curvature bound and (11) in Theorem 8.

Remark 2. One of the possible subjects of further research is the gradient estimate of eigenfunctions of the weighted Laplacian with ε -range, which turned out to be difficult. If Ric_{ψ}^{m} is bounded from below with m > n, then one way to obtain the gradient estimate is to apply the Li-Yau trick as described in [18], [19] and another way is to use the DeGiorgi-Nash-Moser theory [8] as described in [12]. Once we obtained the gradient estimate by the Li-Yau trick, an upper bound of eigenvalues of the weighted Laplacian is obtained as in [18], [19]. However, it seems that the Li-Yau trick and Moser's iteration argument in [17] do not work well in the case where Ric_{ψ}^{m} is bounded from below with $m \leq 1$. The main difficulty stems from the lack of a suitable Bochner formula for analyzing lower Bakry-Émery-Ricci curvature bounds with ε -range. Although [5, Lemma 2.1] obtained the Bochner formula for the distance function with ε -range, a suitable Bochner formula for eigenfunctions of the weighted Laplacian is yet to be known. Finding a suitable Bochner formula for eigenfunctions is our future work.

Appendix: Upper bound of the L^p -spectrum for deformed measures

Although we considered the Riemannian distance d, it is also possible to study comparison theorems associated with a metric deformed by using the weight function (we refer [20], [4], [5], for example). In this appendix, we start from a volume comparison theorem in [5] and prove a variant of Cheng type inequality for the L^p -spectrum.

Let $(M, g, \mu = e^{-\psi}v_g)$ be an *n*-dimensional weighted Riemannian manifold, $m \in (-\infty, 1] \cup [n, +\infty]$ and $\varepsilon \in \mathbb{R}$ in the range (1). We fix a point $q \in M$. We define lower semi continuous functions $s_q : M \to \mathbb{R}$ by

$$s_q(x) := \inf_{\gamma} \int_0^{d(q,x)} e^{-\frac{2(1-\varepsilon)\psi(\gamma(\xi))}{n-1}} d\xi,$$

where the infimum is taken over all unit speed minimal geodesics $\gamma:[0,d(q,x)]\to M$ from q to x. For r>0, we define

$$B_{\psi,a}(r) := \{ x \in M \mid s_a(x) < r \},$$

and also define measures

$$\mu := e^{-\psi} v_g, \quad \nu := e^{-\frac{2(1-\varepsilon)\psi}{n-1}} \mu.$$

We set

$$\mathcal{S}_{-K}(r) := \int_0^r \mathbf{s}_{-K}^{1/c}(s) \, \mathrm{d}s$$

for K > 0. In [5], they obtained the following theorem.

Theorem 9. ([5, Proposition 4.6], Volume comparison) Let (M, g, μ) be an n-dimensional weighted Riemannian manifold. We assume $\operatorname{Ric}_{\psi}^{m} \geq -K \mathrm{e}^{\frac{4(\varepsilon-1)\psi}{n-1}} g$ for K > 0. Then for all r, R > 0 with $r \leq R$ we have

$$\frac{\nu(B_{\psi,q}(R))}{\nu(B_{\psi,q}(r))} \le \frac{\mathcal{S}_{-cK}(R)}{\mathcal{S}_{-cK}(r)}.$$

In the following argument, we start from Theorem 9 instead of Theorem 1 to prove a Cheng type inequality of the L^p -spectrum for the deformed measure ν .

Theorem 10. Let (M, g, μ) be a complete weighted Riemannian manifold. We assume that s_q is smooth and there exists a constant k > 0 such that

$$|\nabla s_q(x)| \le k$$

holds for arbitrary $x \in M$. We also assume

$$\operatorname{Ric}_{\psi}^{m} \geq -K e^{\frac{4(\varepsilon-1)\psi}{n-1}} g$$

for K > 0. Then we have

$$\lambda_{\nu,p}(M) \le \left(\frac{k}{p}\sqrt{\frac{K}{c}}\right)^p. \tag{19}$$

Proof. We apply the argument in Theorem 6.

For $R \geq 2$, let $\eta : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function such that $\eta = 1$ on (-(R-1), R-1), $\eta = 0$ on $\mathbb{R} \setminus (-R, R)$ and $|\eta'| \leq C_3$, where C_3 is a constant independent of R. We set, for an arbitrary $\delta > 0$,

$$\alpha = -\frac{\sqrt{K/c} + \delta}{p}$$

and

$$\phi(y) := \exp(\alpha s_q(y))\varphi(y),$$

where $\varphi(y) := \eta(s_q(y))$. By the assumption of s_q , we have

$$|\nabla \varphi| = |\eta'(s_q)||\nabla s_q| \le kC_3.$$

As in the proof of Theorem 6, we find for an arbitrary $\zeta > 0$,

$$\begin{aligned} |\nabla \phi|^p &= |\alpha e^{\alpha s_q} \varphi \nabla s_q + e^{\alpha s_q} \nabla \varphi|^p \\ &\leq e^{p\alpha s_q} (-k\alpha \varphi + |\nabla \varphi|)^p \\ &\leq e^{p\alpha s_q} \left[(1+\zeta)^{p-1} (-k\alpha \varphi)^p + \left(\frac{1+\zeta}{\zeta}\right)^{p-1} |\nabla \varphi|^p \right]. \end{aligned}$$

By the definition of $\lambda_{\nu,p}(M)$, we obtain

$$\lambda_{\nu,p}(M) \leq (1+\zeta)^{p-1}(-k\alpha)^{p} + \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\int_{M} \exp(p\alpha s_{q})|\nabla\varphi|^{p} d\nu}{\int_{M} \exp(p\alpha s_{q})\varphi^{p} d\nu}$$

$$= (1+\zeta)^{p-1}(-k\alpha)^{p} + \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\int_{B_{\psi,q}(R)\setminus B_{\psi,q}(R-1)} \exp(p\alpha s_{q})|\nabla\varphi|^{p} d\nu}{\int_{B_{\psi,q}(R)} \exp(p\alpha s_{q})\varphi^{p} d\nu}$$

$$\leq (1+\zeta)^{p-1}(-k\alpha)^{p} + (kC_{3})^{p} \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\exp(p\alpha(R-1))\nu (B_{\psi,q}(R))}{\int_{B_{\psi,q}(1)} \exp(p\alpha s_{q}) d\nu}$$

$$\leq (1+\zeta)^{p-1}(-k\alpha)^{p} + (kC_{3})^{p} \left(\frac{1+\zeta}{\zeta}\right)^{p-1} \frac{\exp(p\alpha(R-1))\nu (B_{\psi,q}(R))}{\exp(p\alpha)\nu (B_{\psi,q}(1))}.$$
(20)

From Theorem 9 and

$$(\sqrt{cK})^{1/c} \mathcal{S}_{-cK}(R) \leq \int_0^R \left[\frac{1}{2} \left\{ \exp(\sqrt{cK}s) - \exp(-\sqrt{cK}s) \right\} \right]^{1/c} ds$$

$$\leq \sqrt{\frac{c}{K}} \exp\left(\sqrt{\frac{K}{c}}R\right),$$

we deduce

$$\frac{e^{p\alpha(R-1)}\nu(B_{\psi,q}(R))}{e^{p\alpha}\nu(B_{\psi,q}(1))} \leq \frac{e^{p\alpha R}}{e^{2p\alpha}} \frac{1}{(\sqrt{cK})^{1/c}\mathcal{S}_{-cK}(1)} \sqrt{\frac{c}{K}} \exp\left(\sqrt{\frac{K}{c}}R\right)$$

$$= \frac{1}{e^{2p\alpha}(\sqrt{cK})^{1/c}\mathcal{S}_{-cK}(1)} \sqrt{\frac{c}{K}} \exp\left(p\alpha R + \sqrt{\frac{K}{c}}R\right) \to 0$$

as $R \to \infty$. Letting $R \to \infty$ in (20), we obtain

$$\lambda_{\nu,p}(M) \le (1+\zeta)^{(p-1)} (-k\alpha)^p.$$
 (21)

Since $\zeta > 0$ and $\delta > 0$ are arbitrary, the theorem follows.

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