

**HANKEL DETERMINANT OF TYPE  $H_2(3)$  FOR INVERSE  
FUNCTIONS OF SOME CLASSES OF UNIVALENT FUNCTIONS  
WITH MISSING SECOND COEFFICIENT**

MILUTIN OBRADOVIĆ AND NIKOLA TUNESKI

ABSTRACT. In this paper we determine the upper bounds of  $|H_2(3)|$  for the inverse functions of functions of some classes of univalent functions, where  $H_2(3)(f) = a_3a_5 - a_4^2$  is the Hankel determinant of a special type.

1. INTRODUCTION AND PRELIMINARIES

Let  $\mathcal{A}$  be the class containing functions that are analytic in the unit disk  $\mathbb{D} := \{|z| < 1\}$  and are normalized such that  $f(0) = 0 = f'(0) - 1$ , i.e.,

$$(1) \quad f(z) = z + a_2z^2 + a_3z^3 + \dots$$

By  $\mathcal{S}$  we denote the class of functions from  $\mathcal{A}$  which are univalent in  $\mathbb{D}$ .

Also, we need the classes of functions of bounded turning, of convex functions, of starlike functions, and of functions starlike with respect to symmetric points, subclasses of  $\mathcal{S}$ , defined respectively in the following way

$$\begin{aligned} \mathcal{R} &= [f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}], \\ \mathcal{C} &= \left[ f \in \mathcal{A} : \operatorname{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > 0, z \in \mathbb{D} \right], \\ \mathcal{S}^* &= \left[ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, z \in \mathbb{D} \right], \\ \mathcal{S}_s^* &= \left[ f \in \mathcal{A} : \operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} > 0, z \in \mathbb{D} \right]. \end{aligned}$$

In his paper [4] Zaprawa considered the Hankel determinant of the type

$$H_2(3)(f) = a_3a_5 - a_4^2,$$

defined for the coefficients of the function  $f$  given by (1). The author treated bounds of  $|H_2(3)(f)|$  for the classes  $\mathcal{R}, \mathcal{C}, \mathcal{S}^*$  and gave sharp results in the case  $a_2 = 0$ . He also investigated the general case of these classes. In the same paper it is proved that

$$\max \{|H_2(3)(f)| : f \in \mathcal{S}\} > 1.$$

The object of current paper is to obtain the bounds of the modulus of the Hankel determinant  $H_2(3)(f^{-1})$  of coefficients of the inverse of function from the classes  $\mathcal{R}, \mathcal{C}, \mathcal{S}^*$  and  $\mathcal{S}_s^*$ , defined before, as well as for the class  $\mathcal{S}$ . In all cases we suppose that function  $f$  is missing its second coefficient, i.e.,  $a_2 = 0$ .

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Namely, for every univalent function in  $\mathbb{D}$  exists inverse at least on the disk with radius  $1/4$  (due to the famous Koebe's  $1/4$  theorem). If the inverse has an expansion

$$(2) \quad f^{-1}(w) = w + A_2w^2 + A_3w^3 + \dots,$$

then, by using the identity  $f(f^{-1}(w)) = w$ , from (1) and (2) we receive

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned}$$

Especially, when  $a_2 = 0$ , we have

$$A_2 = 0, \quad A_3 = -a_3, \quad A_4 = -a_4, \quad A_5 = -a_5 + 3a_3^2.$$

So, in this case,

$$(3) \quad H_2(3)(f^{-1}) = A_3A_5 - A_4^2 = a_3a_5 - a_4^2 - 3a_3^3,$$

i.e.,

$$(4) \quad H_2(3)(f^{-1}) = H_2(3)(f) - 3a_3^3.$$

For our further consideration we need the next lemma given by Carlson [1].

**Lemma 1.** *Let*

$$(5) \quad \omega(z) = c_1z + c_2z^2 + \dots$$

*be a Schwartz function, i.e., a function analytic in  $\mathbb{D}$ ,  $\omega(0) = 0$  and  $|\omega(z)| < 1$ . Then*

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \quad \text{and} \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2.$$

## 2. MAIN RESULTS

**Theorem 1.** *Let  $f \in \mathcal{A}$  is given by (1) and let  $a_2 = 0$ . Then*

- (a)  $|H_3(2)(f^{-1})| \leq \frac{28}{45}$  if  $f \in \mathcal{R}$ ;
- (b)  $|H_3(2)(f^{-1})| \leq \frac{2}{45}$  if  $f \in \mathcal{C}$ ;
- (c)  $|H_3(2)(f^{-1})| \leq 2$  if  $f \in \mathcal{S}^*$ ;
- (d)  $|H_3(2)(f^{-1})| \leq 2$  if  $f \in \mathcal{S}_s^*$ .

*All these results are sharp.*

*Proof.*

- (a) Since  $f \in \mathcal{R}$  is equivalent to

$$f'(z) = \frac{1 + \omega(z)}{1 - \omega(z)},$$

for certain Schwartz function  $\omega$ , we receive that

$$(6) \quad f'(z) = 1 + 2\omega(z) + 2\omega^2(z) + \dots$$

Using the notations for  $f$  and  $\omega$  given by (1) and (5), and equating the coefficients in (6), we receive

$$(7) \quad \begin{cases} a_2 = c_1, \\ a_3 = \frac{2}{3}(c_2 + c_1^2), \\ a_4 = \frac{1}{2}(c_3 + 2c_1c_2 + c_1^3), \\ a_5 = \frac{2}{5}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_2^2 + c_1^4). \end{cases}$$

Since  $a_2 = 0$ , by (7) we have  $c_1 = 0$ , and the appropriate coefficients have the next form:

$$(8) \quad a_3 = \frac{2}{3}c_2, \quad a_4 = \frac{1}{2}c_3, \quad a_5 = \frac{2}{5}(c_4 + c_2^2).$$

Now, from (3) and (8), after simple computation, we obtain

$$H_3(2)(f^{-1}) = \frac{4}{15}c_2c_4 - \frac{1}{4}c_3^2 - \frac{28}{45}c_2^3,$$

and further,

$$|H_3(2)(f^{-1})| \leq \frac{4}{15}|c_2||c_4| + \frac{1}{4}|c_3|^2 + \frac{28}{45}|c_2|^3.$$

Applying Lemma 1 (with  $c_1 = 0$ ) we receive

$$|H_3(2)(f^{-1})| \leq \frac{4}{15}|c_2|(1 - |c_2|^2) + \frac{1}{4}(1 - |c_2|^2)^2 + \frac{28}{45}|c_2|^3.$$

and, finally,

$$(9) \quad |H_3(2)(f^{-1})| \leq \frac{1}{4} + \frac{4}{15}|c_2| - \frac{1}{2}|c_2|^2 + \frac{16}{45}|c_2|^3 + \frac{1}{4}|c_2|^4 =: \varphi_1(|c_2|),$$

where  $0 \leq |c_2| \leq 1$ . Since

$$\begin{aligned} \varphi_1'(|c_2|) &= \frac{4}{15} - |c_2| + \frac{16}{15}|c_2|^2 + |c_2|^3 \\ &= \frac{4}{15}(1 - 2|c_2|)^2 + \frac{1}{15}|c_2| + |c_2|^3 > 0, \end{aligned}$$

we have  $\varphi_1(|c_2|) \leq \varphi_1(1) = \frac{28}{45}$ , and from (9),

$$|H_3(2)(f^{-1})| \leq \frac{28}{45} = 0.622\dots$$

This result is best possible as the function  $f_1(z) = \ln \frac{1+z}{1-z} - z$  defined by  $f_1'(z) = \frac{1+z^2}{1-z^2}$ , shows.

- (b) We apply the same method as in the previous case. Namely, from the definition of the class  $\mathcal{C}$  we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

where  $\omega$  is a Schwartz function, and from here

$$(10) \quad (zf'(z))' = [1 + 2(\omega(z) + \omega^2(z) + \dots)] \cdot f'(z).$$

Using the notations (1) and (5), and comparing the coefficients in the relation (10), after some simple calculations, we obtain

$$(11) \quad \begin{cases} a_2 = c_1, \\ a_3 = \frac{1}{3}(c_2 + 3c_1^2), \\ a_4 = \frac{1}{6}(c_3 + 5c_1c_2 + 6c_1^3) \\ a_5 = \frac{1}{30}(3c_4 + 14c_1c_3 + 43c_1^2c_2 + 30c_1^4 + 6c_2^2). \end{cases}$$

If  $a_2 = 0$ , then by (11) we have  $c_1 = 0$ , which implies

$$(12) \quad a_3 = \frac{1}{3}c_2, \quad a_4 = \frac{1}{6}c_3, \quad a_5 = \frac{1}{10}(c_4 + 2c_2^2).$$

Using (3) and (12) we obtain

$$H_3(2)(f^{-1}) = \frac{1}{180}(6c_2c_4 - 5c_3^2 - 8c_2^3).$$

From the last relation we get

$$|H_3(2)(f^{-1})| \leq \frac{1}{180}(6|c_2||c_4| + 5|c_3|^2 + 8|c_2|^3),$$

and further, after applying Lemma1 (with  $c_1 = 0$ ),

$$|H_3(2)(f^{-1})| \leq \frac{1}{180}(6|c_2|(1 - |c_2|^2) + 5(1 - |c_2|^2)^2 + 8|c_2|^3),$$

i.e.,

$$(13) \quad |H_3(2)(f^{-1})| \leq \frac{1}{180}(5 + 6|c_2| - 10|c_2|^2 + 2|c_2|^3 + 5|c_2|^4) =: \varphi_2(|c_2|),$$

where  $0 \leq |c_2| \leq 1$ . Since

$$\varphi_2'(|c_2|) = \frac{1}{90}(3 - 10|c_2| + 3|c_2|^2 + 10|c_2|^3),$$

which, after considering this polynomial in the interval  $[0, 1]$ , gives  $\varphi_1(|c_2|) \leq \varphi_2(1) = \frac{2}{45}$ , and further, from (13),

$$|H_3(2)(f^{-1})| \leq \frac{2}{45} = 0.044\dots$$

The function  $f_2(z) = \operatorname{artanh} z$  satisfying  $1 + \frac{zf_2''(z)}{f_2'(z)} = \frac{1+z^2}{1-z^2}$  shows that the result is the best possible.

- (c) From the definition of the class  $\mathcal{S}^*$  we have that there exists a Schwartz function  $\omega$  such that

$$\frac{zf'(z)}{f(z)} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

and from here

$$(14) \quad zf'(z) = [1 + 2(\omega(z) + \omega^2(z) + \dots)] \cdot f(z).$$

As in the two previous cases ((a) and (b)), by comparing the coefficients in the relation (14), and some simple calculations, we have

$$\begin{cases} a_2 = 2c_1 \\ a_3 = c_2 + 3c_1^2 \\ a_4 = \frac{2}{3}(c_3 + 5c_1c_2 + 6c_1^3) \\ a_5 = \frac{1}{2}\left(c_4 + \frac{14}{3}c_1c_3 + \frac{43}{3}c_1^2c_2 + 10c_1^4 + 2c_2^2\right). \end{cases}$$

For the case  $a_2 = 0$  we have the next

$$(15) \quad a_3 = c_2, \quad a_4 = \frac{2}{3}c_3, \quad a_5 = \frac{1}{2}(c_4 + 2c_2^2).$$

So, from (3) and (15) we obtain

$$H_3(2)(f^{-1}) = \frac{1}{18}(9c_2c_4 - 8c_3^2 - 36c_2^3),$$

and from here

$$|H_3(2)(f^{-1})| \leq \frac{1}{18}(9|c_2||c_4| + 8|c_3|^2 + 36|c_2|^3).$$

Using estimates for  $|c_4|$  and  $|c_3|$  from Lemma 1 (with  $c_1 = 0$ ) from the last relation we receive

$$(16) \quad |H_3(2)(f^{-1})| \leq \frac{1}{18}(8 + 9|c_2| - 16|c_2|^2 + 27|c_2|^3 + 8|c_2|^4) =: \varphi_3(|c_2|),$$

where  $0 \leq |c_2| \leq 1$ . Since

$$\begin{aligned} \varphi_3'(|c_2|) &= \frac{1}{18}(9 - 32|c_2| + 81|c_2|^2 + 32|c_2|^3) \\ &= \frac{1}{18}[9(1 - 3|c_2|)^2 + 22|c_2| + 32|c_2|^3] > 0, \end{aligned}$$

then  $\varphi_3(|c_2|) \leq \varphi_3(1) = 2$ , and from (16),

$$|H_3(2)(f^{-1})| \leq 2.$$

The result is the best possible as the function  $f_3(z) = \frac{z}{1-z^2}$  shows.

- (d) From the definition of the class  $\mathcal{S}_s^*$  we have that there exists a Schwartz function  $\omega$  such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

and from here

$$(17) \quad 2zf'(z) = [1 + 2(\omega(z) + \omega^2(z) + \dots)] \cdot [f(z) - f(-z)].$$

Similarly as in previous cases, by comparing the coefficients in the relation (17), after some simple calculations, we receive

$$(18) \quad \begin{cases} a_2 = c_1 \\ a_3 = c_2 + c_1^2 \\ a_4 = \frac{1}{2}(c_3 + 3c_1c_2 + 2c_1^3) \\ a_5 = \frac{1}{2}(c_4 + 2c_1c_3 + 5c_1^2c_2 + 2c_1^4 + 2c_2^2). \end{cases}$$

For  $a_2 = 0$  ( $c_1 = 0$ ), from (18) we get

$$a_3 = c_2, \quad a_4 = \frac{1}{2}c_3, \quad a_5 = \frac{1}{2}(c_4 + 2c_2^2),$$

and using (3),

$$H_3(2)(f^{-1}) = \frac{1}{4}(2c_2c_4 - c_3^2 - 8c_2^3),$$

and from here

$$|H_3(2)(f^{-1})| \leq \frac{1}{4}(2|c_2||c_4| + |c_3|^2 + 8|c_2|^3).$$

Using the estimates for  $|c_4|$  and  $|c_3|$  from Lemma 1 (with  $c_1 = 0$ ) from the last relation we have

$$(19) \quad |H_3(2)(f^{-1})| \leq \frac{1}{4}(1 + 2|c_2| - 2|c_2|^2 + 6|c_2|^3 + |c_2|^4) =: \varphi_4(|c_2|),$$

where  $0 \leq |c_2| \leq 1$ . Since

$$\begin{aligned} \varphi_4'(|c_2|) &= \frac{1}{2}(1 - 2|c_2| + 9|c_2|^2 + 2|c_2|^3) \\ &= \frac{1}{2}[(1 - |c_2|)^2 + 8|c_2| + 2|c_2|^3] > 0, \end{aligned}$$

then  $\varphi_4$  is an increasing function and  $\varphi_4(|c_2|) \leq \varphi_4(1) = 2$ . So, from (19),

$$|H_3(2)(f^{-1})| \leq 2.$$

This result is the best possible as the function  $f_4$  defined by

$$\frac{2zf_4'(z)}{f_4(z) - f_4(-z)} = \frac{1+z^2}{1-z^2}$$

shows. □

**Remark 1.** From the relation (4) we get the following.

(a) For  $f \in \mathcal{R}$ ,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3\left(\frac{2}{3}|c_2|\right)^3 \leq \frac{8}{9},$$

and the result is the best possible as the function  $f_1$  shows (in this case  $H_3(2)(f_1) = \frac{4}{15}$  and  $H_3(2)(f_1^{-1}) = -\frac{28}{45}$ ).

(b) For  $f \in \mathcal{C}$ ,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3\left(\frac{|c_2|}{3}\right)^3 \leq \frac{1}{9},$$

and the result is the best possible as the function  $f_2$  shows.

(c) For  $f \in \mathcal{S}^*$ ,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3|c_2|^3 \leq 3,$$

and the result is the best possible as the function  $f_3$  shows.

(d) For  $f \in \mathcal{S}_s^*$ ,

$$|H_3(2)(f^{-1}) - |H_3(2)(f)|| = 3|a_3|^3 = 3|c_2|^3 \leq 3,$$

and the result is the best possible for the function  $f_4$ .

For obtaining the corresponding result for the whole class  $\mathcal{S}$  we will use method based on Grunsky coefficients. In the proof we will use mainly the notations and results given in the book of N.A. Lebedev ([3]).

Here are basic definitions and results.

Let  $f \in \mathcal{S}$  and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where  $\omega_{p,q}$  are the Grunsky's coefficients with property  $\omega_{p,q} = \omega_{q,p}$ . For those coefficients we have the next Grunsky's inequality ([2, 3]):

$$(20) \quad \sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_p|^2}{p},$$

where  $x_p$  are arbitrary complex numbers such that last series converges.

Further, it is well-known that if the function  $f$  given by (1) belongs to  $\mathcal{S}$ , then also

$$(21) \quad \tilde{f}_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \dots$$

belongs to the class  $\mathcal{S}$ . Then, for the function  $\tilde{f}_2$  we have the appropriate Grunsky's coefficients of the form  $\omega_{2p-1,2q-1}^{(2)}$  and the inequality (20) has the form:

$$(22) \quad \sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}.$$

Here, and further in the paper we omit the upper index (2) in  $\omega_{2p-1,2q-1}^{(2)}$  if compared with Lebedev's notation.

If in the inequality (22) we put  $x_1 = 1$  and  $x_{2p-1} = 0$  for  $p = 2, 3, \dots$ , then we receive

$$(23) \quad |\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 + 7|\omega_{17}|^2 \leq 1.$$

As it has been shown in [3, p.57], if  $f$  is given by (1) then the coefficients  $a_2$ ,  $a_3$ ,  $a_4$  and  $a_5$  are expressed by Grunsky's coefficients  $\omega_{2p-1,2q-1}$  of the function  $\tilde{f}_2$  given by (21) in the following way:

$$(24) \quad \begin{aligned} a_2 &= 2\omega_{11}, \\ a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\ a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\ a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\ 0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}, \\ 0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4. \end{aligned}$$

We note that in the cited book of Lebedev there exists a typing mistake for the coefficient  $a_5$ . Namely, instead of the term  $5\omega_{13}^2$ , there is  $5\omega_{15}^2$ .

**Theorem 2.** *Let  $f \in \mathcal{S}$  is given by (1) and let  $a_2 = 0$ . Then*

$$|H_3(2)(f^{-1})| \leq \frac{\sqrt{3}}{6\sqrt{7}} + 2\sqrt{3} = 3.57321 \dots$$

*Proof.* In the case when  $a_2 = 0$ , from (24) we have  $\omega_{11} = 0$ , and so

$$(25) \quad a_3 = 2\omega_{13}, \quad a_4 = 2\omega_{33}, \quad a_5 = 2\omega_{35} + 5\omega_{13}^2, \quad \omega_{33} = \omega_{15}, \quad \omega_{35} = \omega_{17} - \omega_{13}^2.$$

Using (3) and (25), we have

$$H_3(2)(f^{-1}) = 4\omega_{13}\omega_{35} - 14\omega_{13}^3 - 4\omega_{33}^2,$$

and after applying the two last relations from (25),

$$H_3(2)(f^{-1}) = 4\omega_{13}\omega_{17} - 18\omega_{13}^3 - 4\omega_{15}^2.$$

From here we have

$$|H_3(2)(f^{-1})| \leq 4|\omega_{13}||\omega_{17}| + 18|\omega_{13}|^3 + 4|\omega_{15}|^2,$$

or finally, using  $|\omega_{17}| \leq \frac{1}{\sqrt{7}}\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}$  (from (23)) we get

$$(26) \quad \begin{aligned} |H_3(2)(f^{-1})| &\leq \frac{1}{\sqrt{7}}|\omega_{13}|\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2} + 18|\omega_{13}|^3 + 4|\omega_{15}|^2 \\ &=: \frac{1}{\sqrt{7}}\psi_1(|\omega_{13}|, |\omega_{15}|) + 2\psi_2(|\omega_{13}|, |\omega_{15}|), \end{aligned}$$

where

$$\psi_1(y, z) = y\sqrt{1 - 3y^2 - 5z^2}, \quad \psi_2(y, z) = 9y^3 + 2z^2,$$

with  $0 \leq y = |\omega_{13}| \leq \frac{1}{\sqrt{3}}$  and  $0 \leq z = |\omega_{15}| \leq \frac{1}{\sqrt{5}}\sqrt{1 - 3y^2}$  (where we used the inequality (23)). It is easy to verify that for these range of  $y$  and  $z$ ,  $\psi_1(y, z) \leq \psi_1(1/\sqrt{6}, 0) = \frac{\sqrt{3}}{6}$  and  $\psi_2(y, z) \leq \psi_2(1/\sqrt{3}, 0) = \sqrt{3}$ , so that from (26)) we have

$$|H_3(2)(f^{-1})| \leq \frac{\sqrt{3}}{6\sqrt{7}} + 2\sqrt{3} = 3.57321 \dots$$

□

**Remark 2.** *From the relation (4) we get for  $f \in \mathcal{S}$ :*

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3|2\omega_{13}|^3 \leq 3\left(2 \cdot \frac{1}{\sqrt{3}}\right)^3 = \frac{8}{\sqrt{3}} = 4.6188 \dots$$

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DEPARTMENT OF MATHEMATICS, FACULTY OF CIVIL ENGINEERING, UNIVERSITY OF BELGRADE,  
BULEVAR KRALJA ALEKSANDRA 73, 11000, BELGRADE, SERBIA.

*Email address:* obrad@grf.bg.ac.rs

DEPARTMENT OF MATHEMATICS AND INFORMATICS, FACULTY OF MECHANICAL ENGINEERING,  
SS. CYRIL AND METHODIUS UNIVERSITY IN SKOPJE, KARPOŠ II B.B., 1000 SKOPJE, REPUBLIC OF  
NORTH MACEDONIA.

*Email address:* nikola.tuneski@mf.ukim.edu.mk