

HOLOMORPHIC EXTENSION IN HOLOMORPHIC FIBER BUNDLES WITH (1,0)-COMPACTIFIABLE FIBER

S. V. FEKLISTOV

ABSTRACT. We use the Leray spectral sequence for the sheaf cohomology groups with compact supports to obtain a vanishing result. The stalks of sheaves $R^\bullet\phi_!\mathcal{O}$ for the structure sheaf \mathcal{O} on the total space of a holomorphic fiber bundle ϕ has canonical topology structures. Using the standard Čech argument we prove a density lemma for QDFS-topology on this stalks. In particular, we obtain a vanishing result for holomorphic fiber bundles with Stein fibers. Using Künnet formulas, properties of an inductive topology (with respect to the pair of spaces) on the stalks of the sheaf $R^1\phi_!\mathcal{O}$ and a cohomological criterion for the Hartogs phenomenon we obtain the main result on the Hartogs phenomenon for the total space of holomorphic fiber bundles with (1,0)-compactifiable fibers.

1. INTRODUCTION

The classical Hartogs extension theorem states that for every domain $W \subset \mathbb{C}^n (n > 1)$ and a compact set $K \subset W$ such that $W \setminus K$ is connected, the restriction homomorphism

$$\mathcal{O}(W) \rightarrow \mathcal{O}(W \setminus K)$$

is an isomorphism.

A natural question arises if this is true for complex analytic spaces.

Definition 1.1. *We say that a noncompact connected complex analytic space X admits the Hartogs phenomenon if for any domain $W \subset X$ and a compact set $K \subset W$ such that $W \setminus K$ is connected, the restriction homomorphism*

$$\mathcal{O}(W) \rightarrow \mathcal{O}(W \setminus K)$$

is an isomorphism.

In this or a similar formulation this phenomenon has been extensively studied in many situations, including Stein manifolds and spaces, $(n - 1)$ -complete normal complex spaces and so on [1, 2, 3, 5, 15, 16, 14, 20, 18, 19, 21, 28].

Our goal is to study the Hartogs phenomenon in holomorphic fiber bundle with noncompact (1,0)-compactifiable fiber. Note that, a noncompact complex manifold F is called (1,0)-compactifiable (for more details, see Section 5) if it admits a compactification F' with the following properties:

- (1) F' is a compact complex manifold;
- (2) $F' \setminus F$ is a proper connected analytic set;
- (3) $H^1(F', \mathcal{O}_{F'}) = 0$.

Let X, Y be a locally compact space and let $\phi: X \rightarrow Y$ be a continuous map. We use the Leray spectral sequence for a group cohomology with compact supports to obtain that $H_c^i(X, \mathcal{F}) = 0$ for all $i < q$ provided $R^i\phi_!\mathcal{F} = 0$ for all $i < q$, where

$\phi_!$ is the direct image with compact supports between sheaves categories $\mathcal{SH}(X)$ and $\mathcal{SH}(Y)$, $R^i\phi_!$ is the i -th derived functor (Corollary 2.1).

Now let $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with noncompact fiber F , and $y \in Y$ be a point. The stalk $(R^i\phi_!\mathcal{O}_X)_y$ has the canonical topological vector space structure which is QDFS-space (see Section 3). Using the standard Čech argument we prove that the canonical homomorphism

$$\mathcal{O}_y \otimes H_c^i(F_y, \mathcal{O}_{F_y}) \rightarrow (R^i\phi_!\mathcal{O}_X)_y$$

has a dense image with respect to the QDFS-topology (Lemma 4.1). In particular, if $(R^i\phi_!\mathcal{O}_X)_y$ is a separated space, then the condition $H_c^i(F, \mathcal{O}_F) = 0$ for all $i < q$ implies that $H_c^i(X, \mathcal{O}_X) = 0$ for all $i < q$ (Corollary 4.3).

If the fiber F is a Stein manifold, then $H_c^i(X, \mathcal{O}_X) = 0$ for all $i < \dim F$ (Corollary 4.4). For $F = \mathbb{C}^n$ and $n > 1$ we obtain the Dworkin's result about vanishing $H_c^1(X, \mathcal{O}_X)$ [7, Corollary 1.4].

Now, assume that the fiber F is $(1,0)$ -compactifiable and $\dim F > 1$. In this case, using the Künnet formulas for algebraic and topological tensor products, the long exact sequence for the pair (F, F') and Lemma 4.1 we obtain that the condition $H_c^1(F, \mathcal{O}_F) = 0$ implies that $R^1\phi_!\mathcal{O}_X = 0$. In particular $H_c^1(X, \mathcal{O}_X) = 0$ provided $H_c^1(F, \mathcal{O}_F) = 0$ (Corollary 5.1).

Using the cohomological criterion for the Hartogs phenomenon (see Section 6) we obtain the following main result (Theorem 6.2).

Theorem 1.1. *Let $\phi: X \rightarrow Y$ be a holomorphic fibre bundle with $(1,0)$ -compactifiable fiber F , $\dim F > 1$. If F admits the Hartogs phenomenon, then X also admits the Hartogs phenomenon.*

For example, let G be a semiabelian Lie group (i.e. G is an extension of an abelian manifold A by an algebraic torus $T \cong (\mathbb{C}^*)^n$). We have a principle T -bundle $G \rightarrow A$. Let F be a toric T -manifold which has only one topological end (about toric varieties see, for instance, [22]), $\dim F > 1$, and F admits the Hartogs phenomenon. Then the total space of an associated fiber bundle $G \times^T F \rightarrow A$ admits the Hartogs phenomenon (see Example 6.1).

2. THE LERAY SPECTRAL SEQUENCE FOR SHEAF COHOMOLOGY GROUPS WITH COMPACT SUPPORTS

In this section we recall the Leray spectral sequence (see [29, 4.1.3]).

2.1. The spectral sequence of composed functors. Now we recall the spectral sequence of composed functors (see [29, Section 4.1.2]).

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories, where \mathcal{A} and \mathcal{B} have sufficiently many injective objects. Let $S: \mathcal{A} \rightarrow \mathcal{B}$, $S': \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors. Assume that S transform the injective objects of \mathcal{A} into S' -acyclic objects of \mathcal{B} .

Then we have the following result (see [29, Theorem 4.9]) which is based on the Cartan-Eilenberg resolution.

Theorem 2.1. *There is a canonical filtration F on the objects $R^i(S' \circ S)(M)$ and a spectral sequence*

$$E_r^{p,q} \implies R^{p+q}(S' \circ S)(M)$$

with $E_2^{p,q} = R^p F'(R^q S(M))$, $E_\infty^{p,q} = Gr_F^p R^{p+q}(S' \circ S)(M)$.

2.2. The Leray spectral sequenses. Recall the classical Leray spectral sequence (see [29, 4.1.3]). It is a special case of spectral sequence of a composed functors. Let X, Y be topological spaces and $\phi: X \rightarrow Y$ be a continuous map. Denote by $\mathcal{SH}(Z)$ the category of sheaves of abelian group on the topological manifold Z . Let \mathcal{AB} be the category of abelian groups.

Consider the direct image functor $\phi_*: \mathcal{SH}(X) \rightarrow \mathcal{SH}(Y)$ and the global sections functor $\Gamma(Y, \bullet): \mathcal{SH}(Y) \rightarrow \mathcal{AB}$. Note that, for any sheaf $\mathcal{F} \in \mathcal{SH}(X)$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Y, \phi_*\mathcal{F})$.

Since each injective sheaf on X is a flabby, each flabby sheaf is Γ -acyclic and $\phi_*\mathcal{F}$ is flabby sheaf on Y for every flabby sheaf \mathcal{F} on X , it follows that for every injective sheaf \mathcal{I} we have that $\phi_*\mathcal{I}$ is a Γ -acyclic sheaf on Y .

We obtain the following theorem ([29, 4.1.3]):

Theorem 2.2. *For every sheaf $\mathcal{F} \in \mathcal{SH}(X)$, there exists a canonical filtration F on $H^q(X, \mathcal{F})$, and a spectral sequence $E_2^{p,q} \implies H^{p+q}(X, \mathcal{F})$ that is canonical starting from E_2 , and satisfies $E_2^{p,q} = H^p(Y, R^q\phi_*\mathcal{F})$, $E_\infty^{p,q} = Gr_F^p H^{p+q}(X, \mathcal{F})$.*

We have to study the group cohomology with compact supports of sheaf on X . In this case we consider the functors Γ_c and $\phi_!$ instead Γ and ϕ_* .

Let X be a locally compact space. Recall that the support of a global section $s \in \Gamma(X, \mathcal{F})$ is the set $\text{supp}(s) = \{x \in X \mid s_x \neq 0\}$.

Definition 2.1. *For a sheaf $\mathcal{F} \in \mathcal{SH}(X)$ put*

$$\Gamma_c(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \text{supp}(s) \text{ is compact}\}.$$

This defines a left exact functor $\Gamma_c(X, -): \mathcal{SH}(X) \rightarrow \mathcal{AB}$. The i 'th derived functor of $\Gamma_c(X, -)$ evaluated on the sheaf \mathcal{F} will be denoted $H_c^i(X, \mathcal{F})$.

Remark 2.1. *Let X be a locally compact space. Note that any soft sheaf $S \in \mathcal{SH}(X)$ is $\Gamma_c(X, -)$ -acyclic [8, Section III, Theorem 2.7].*

Now let $\phi: X \rightarrow Y$ be a continuous map between locally compact spaces. Define the functor $\phi_!$ (see [8, Sections III and VII]).

Definition 2.2. *For a sheaf $\mathcal{F} \in \mathcal{SH}(X)$ and an open subset $V \subset Y$ put*

$$\Gamma(V, \phi_!\mathcal{F}) := \{s \in \Gamma(\phi^{-1}(V), \mathcal{F}) \mid \phi|_{\text{supp}(s)}: \text{supp}(s) \rightarrow Y \text{ is a proper map}\}.$$

We consider $\phi_!\mathcal{F}$ as a subpresheaf of $\phi_\mathcal{F}$.*

Remark 2.2. *Now we list some properties of the presheaf $\phi_!\mathcal{F}$ (see [8, Section VII]).*

- (1) *The presheaf $\phi_!\mathcal{F}$ is a sheaf on Y ; In particular we obtain the left exact functor $\phi_!: \mathcal{SH}(X) \rightarrow \mathcal{SH}(Y)$ and it has the derived functors which will be denoted $R^i\phi_!$.*

- (2) *For $y \in Y$ we have a natural isomorphism*

$$(R^i\phi_!\mathcal{F})_y \cong H_c^i(\phi^{-1}(y), \mathcal{F}|_{\phi^{-1}(y)}).$$

- (3) *A soft sheaf $S \in \mathcal{SH}(X)$ is transformed into a soft sheaf $\phi_!S \in \mathcal{SH}(Y)$. Moreover $R^i\phi_!S = 0$ for $i > 0$.*

The following lemma allows to apply Theorem 2.1.

Lemma 2.1. *Let X, Y be locally compact spaces and $\phi: X \rightarrow Y$ be a continuous map.*

- (1) For any sheaf $\mathcal{F} \in \mathcal{SH}(X)$ we have $\Gamma_c(Y, \phi_! \mathcal{F}) = \Gamma_c(X, \mathcal{F})$;
- (2) If $\mathcal{I} \in \mathcal{SH}(X)$ is an injective sheaf, then $\phi_! \mathcal{I}$ is $\Gamma_c(Y, -)$ -acyclic.

Proof. If $s \in \Gamma_c(X, \mathcal{F})$, then $\phi|_{\text{supp}(s)}: \text{supp}(s) \rightarrow Y$ is a proper map. It follows that $\Gamma_c(Y, \phi_! \mathcal{F}) = \Gamma_c(X, \mathcal{F})$.

Futher, since each injective sheaf \mathcal{I} is soft, it follows that $\phi_! \mathcal{I}$ is soft (by Remark 2.2). But each soft sheaf is $\Gamma_c(Y, -)$ -acyclic. \square

So, we obtain the following result:

Theorem 2.3. *Let X, Y be locally compact spaces and let $\phi: X \rightarrow Y$ be a continuous map. For every sheaf $\mathcal{F} \in \mathcal{SH}(X)$, there exists a canonical filtration F on $H_c^q(X, \mathcal{F})$, and a spectral sequence*

$$E_r^{p,q} \implies H_c^{p+q}(X, \mathcal{F})$$

that is canonical starting from E_2 , and satisfies $E_2^{p,q} = H_c^p(Y, R^q \phi_! \mathcal{F})$, $E_\infty^{p,q} = Gr_F^p H_c^{p+q}(X, \mathcal{F})$.

Proof. We apply Theorem 2.1 to $\mathcal{A} = \mathcal{SH}(X)$, $\mathcal{B} = \mathcal{SH}(Y)$, $\mathcal{C} = \mathcal{AB}$, $F = \phi_!$, $F' = \Gamma_c(Y, -)$ and use Lemma 2.1. \square

In particular we obtain the following corollary

Corollary 2.1. *Let X, Y be locally compact spaces and let $\phi: X \rightarrow Y$ be a continuous map and $\mathcal{F} \in \mathcal{SH}(X)$. If $R^i \phi_! \mathcal{F} = 0$ for all $i < q$, then $H_c^i(X, \mathcal{F}) = 0$ for all $i < q$ and $H_c^q(X, \mathcal{F}) \cong \Gamma_c(Y, R^q \phi_! \mathcal{F})$.*

Proof. Note that, $F^s H^i(X, \mathcal{F}) = 0$ for all $s > i$. Since $E_\infty^{k,l} = 0$ for all $k+l = i < q$, it follows that $H_c^i(X, \mathcal{F}) = 0$ for all $i < q$.

Futher, we have $E_2^{p,i} = 0$ for all p and for all $i < q$. It follows that $E_\infty^{i,q-i} = Gr_F^i H_c^q(X, \mathcal{F}) = 0$ for all $0 < i \leq q$. This means that $F^i H_c^q(X, \mathcal{F}) = F^{i+1} H_c^q(X, \mathcal{F}) = 0$ for all $1 \leq i \leq q$.

Since the differentials $d_r: E_r^{0,q} \rightarrow E_r^{r,q-r+1} = 0$ and $0 = E_r^{-r,q+r-1} \rightarrow E_r^{0,q}$ are zero, it follows that $H_c^q(X, \mathcal{F}) = Gr_F^0 H_c^q(X, \mathcal{F}) = E_\infty^{0,q} \cong E_2^{0,q} = \Gamma_c(Y, R^q \phi_! \mathcal{F})$. \square

2.3. Differential-geometric description of $R^i \phi_! \mathcal{O}_X$. For the structure sheaf \mathcal{O}_X on the total space X of a holomorphic fiber bundle $\phi: X \rightarrow Y$ with a fiber F we have a more explicit formula for the sheaves $R^i \phi_! \mathcal{O}_X$. For the sheaf \mathcal{O}_X there exists the Dolbeault resolution $(\mathcal{A}_X^{0,\bullet}, \bar{\partial})$. Since the sheaves $\mathcal{A}_X^{0,\bullet}$ are c-soft, then the resolution $(\mathcal{A}_X^{0,\bullet}, \bar{\partial})$ is $\phi_!$ -injective. It follows that

$$R^i \phi_! \mathcal{O}_X \cong \frac{\text{Ker}(\bar{\partial}: \phi_! \mathcal{A}_X^{0,i} \rightarrow \phi_! \mathcal{A}_X^{0,i+1})}{\text{Im}(\bar{\partial}: \phi_! \mathcal{A}_X^{0,i-1} \rightarrow \phi_! \mathcal{A}_X^{0,i})}.$$

For any section $\omega \in R^i \phi_! \mathcal{O}_X(U)$ there exists a covering $U = \bigcup_{\alpha \in A} U_\alpha$ and $\omega_\alpha \in \mathcal{A}^{0,i}(U_\alpha \times F)$ such that

- (1) $\bar{\partial} \omega_\alpha = 0$;
- (2) $pr_1: \text{supp}(\omega_\alpha) \rightarrow U_\alpha$ is a proper map;
- (3) For any α, β there exists a covering $U_\alpha \cap U_\beta = \bigcup_i U_{\alpha\beta,i}$ and

$$\eta_{\alpha\beta,i} \in \mathcal{A}^{0,i-1}(U_{\alpha\beta,i} \times F)$$

such that $\text{supp}(s_i) \rightarrow U_{\alpha\beta,i}$ is a proper map and

$$\bar{\partial} \eta_{\alpha\beta,i} = (\omega_\alpha - \omega_\beta)|_{U_{\alpha\beta,i}}.$$

But this description of the sheaves $R^i\phi_!\mathcal{O}_X$ is not used in this paper.

3. CANONICAL TOPOLOGIES ON THE SHEAF COHOMOLOGY GROUPS AND THE TOPOLOGICAL KÜNNET FORMULA

In this section we recall some notions and facts from topological vector spaces (for more details see [13, 27, 24]) and its applications to the cohomology theory of sheaves (see [3]).

3.1. FS and DFS type spaces and properties. A locally convex topological vector space over \mathbb{C} , in addition metrizable and complete is called a Fréchet space. A Fréchet space is called FS space (Fréchet-Schwartz space) if for any neighbourhood of zero U there is a neighbourhood of zero V such that: for any $\varepsilon > 0$ there are $x_1, \dots, x_n \in V$ with the property $V \subset \bigcup_{i=1}^n (x_i + \varepsilon U)$. A locally convex topological vector space is called a DFS space if it is the strong dual of FS space. If E is a locally convex space, then we denote by E' the strong dual space of E . By QFS and QDFS we mean quotients of such spaces.

Let $f: E \rightarrow F$ be a continuous \mathbb{C} -linear map between topological vector spaces over \mathbb{C} . The map f is called a topological homomorphism if the quotient topology on $f(E)$ coincides with the induced topology from F .

Two topological vector spaces E, F are in topological duality if there exists a bilinear and separately continuous map $E \times F \rightarrow \mathbb{C}$ such that the induced maps $E \rightarrow F', F \rightarrow E'$ are topological isomorphisms.

A complex (E^\bullet, d) is called complexes of topological vector spaces if every E^i is a topological vector space over \mathbb{C} and $d^i: E^i \rightarrow E^{i+1}$ is a continuous \mathbb{C} -linear map. A continuous morphism between two TVS complexes is a morphism of complexes (of zero degree) whose components are continuous and \mathbb{C} -linear. If $f: E^\bullet \rightarrow F^\bullet$ is such a morphism, then the linear maps

$$H^n(f): H^n(E^\bullet) \rightarrow H^n(F^\bullet)$$

are continuous.

Now we collect some facts about FS and DFS spaces(for details see [3]).

- FS(DFS) space is reflexive;
- Any closed subspace of FS (DFS) space is an FS (DFS) space;
- Quotient of an FS (DFS) space by a closed subspace is also an FS (DFS) space;
- A surjective continuous \mathbb{C} -linear map between two FS (DFS) spaces is open;
- Any separated locally convex topological vector space, countable inductive limit of Fréchet spaces such that the maps of the inductive system are compact, is DFS;
- Any locally convex topological vector space, countable inductive limit of Fréchet spaces such that the maps of the inductive system are compact and injective, is separated and DFS;
- If $E = \varinjlim E_n$ is an inductive limit (in the category of locally convex topological vector spaces) of FS spaces such that the maps $E_n \rightarrow E_{n+1}$ are compact and in addition E is separated, then any bounded subset of E is the image of a bounded subset of some E_n ;

- Let $f: E \rightarrow F$ be a continuous linear map between FS (DFS) spaces, then f is a topological homomorphism if and only if $f(E)$ is a closed subspace F ;
- If $E \xrightarrow{f} F \xrightarrow{g} G$ is a complex of FS (DFS) spaces, then the cohomology space $\ker g / \operatorname{im} f$ is separated if and only if f is a topological homomorphism;
- Let $E \xrightarrow{f} F \xrightarrow{g} G$ be a complex between locally convex spaces and g is a topological homomorphism, let $G' \xrightarrow{g'} F' \xrightarrow{f'} E'$ be the corresponding topological dual complex. Then there is a natural algebraic isomorphism

$$\operatorname{Ker} f' / \operatorname{Im} g' \cong (\operatorname{Ker} g / \operatorname{Im} f)';$$

- Let $f: E^\bullet \rightarrow F^\bullet$ be a continuous morphism between two FS (DFS) complexes. If f is an algebraic quasi-isomorphism, then it is a topological quasi-isomorphism (induces topological isomorphism to cohomology).
- Let E^\bullet be an acyclic complex of FS or DFS spaces. Then the dual complex E'^\bullet is acyclic.
- Let $f: E^\bullet \rightarrow F^\bullet$ be a topological quasi-isomorphism between FS or DFS complexes. Then the dual morphism f' is a topological quasi-isomorphism.

Now we consider some classical examples.

Example 3.1. Let X be a complex manifold, $\dim X = n$. Let $\mathcal{A}_X^{p,q}$ ($\mathcal{D}_X^{p,q}$) be the sheaf of germs of differential forms of the type (p, q) with coefficients C^∞ -functions (distributions respectively), \mathcal{O}_X be the sheaf of germs of holomorphic functions.

- (1) The space of global section $\Gamma(X, \mathcal{A}_X^{p,q})$ is an FS space (with respect to the topology of the uniform convergence of the coefficients of the form and all their derivatives);
- (2) The space of global section with compact support $\Gamma_c(X, \mathcal{A}_X^{p,q})$ is a DFS space (since $\Gamma_c(X, \mathcal{A}_X^{p,q}) = (\Gamma(X, \mathcal{A}_X^{n-p, n-q}))'$);
- (3) The space $\Gamma(X, \mathcal{O}_X)$ is an FS space (with respect to the topology of the uniform convergence on compacts);
- (4) If U is a relatively compact open subset of X , then the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(U)$ is compact. Let K be a compact subset of X and $\{U_n\}_{n \geq 0}$ a fundamental system of neighbourhoods of K such that $U_{n+1} \subseteq U_n$ for any $n > 0$. Suppose in addition that any connected component of each U_n does intersect K . The restriction maps $\mathcal{O}(U_n) \rightarrow \mathcal{O}(U_{n+1})$ are compact and injective. It follows that the space $\mathcal{O}(K) = \varinjlim \mathcal{O}(U_n)$ (with topology of inductive limit in the category of locally convex topological vector space) is separated, a DFS and LF space. And any bounded subset of $\mathcal{O}(K)$ is the image of a bounded subset of some $\mathcal{O}(U_n)$.
- (5) The spaces $H_c^\bullet(X, \mathcal{O}_X)$ has a natural QDFS topology, since it can be computed by means of the complex $\Gamma_c(X, \mathcal{D}_X^{0,\bullet})$ and $\Gamma_c(X, \mathcal{D}_X^{0,\bullet}) = (\Gamma(X, \mathcal{E}^{n, n-\bullet}))'$.

Let X be a complex space and \mathcal{F} be a coherent analytic sheaf on X . The space of global section $\Gamma(X, \mathcal{F})$ has natural structure of a topological vector space, which is FS if X has a countable topology (a countable basis of open subsets) [11]. Thus \mathcal{F} becomes a Fréchet-Schwartz sheaf, that is a sheaf such that for any open subset U , $\Gamma(U, \mathcal{F})$ has an FS structure and for any two open subsets $V \subset U$, the restriction $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(V, \mathcal{F})$ are continuous.

If $\mathcal{F} \rightarrow \mathcal{G}$ is a morphism of coherent analytic sheaves on X , then the map $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G})$ is continuous. Moreover, if X is a Stein space, then the map $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G})$ is a topological homomorphism.

Remark 3.1. [3, Chapter I, Proposition 2.10 and Lemma 2.11] *Let X be a complex space, $K \subset X$ a Stein compact. Let \mathcal{F}, \mathcal{G} be coherent sheaves on X . Then*

- (1) *the topology of inductive limit (in the category of local convex topological vector spaces) on $\Gamma(K, \mathcal{F})$ is separated, LF and DFS;*
- (2) *the map $\Gamma(K, \mathcal{F}) \rightarrow \Gamma(K, \mathcal{G})$ is a topological homomorphism.*

3.2. QFS-topology on $H^\bullet(X, \mathcal{F})$. Let X be a complex space with countable topology, \mathcal{F} be a coherent sheaf on X . Let \mathfrak{U} be a countable Stein open covering of X . We have a canonical isomorphism $H^\bullet(\mathfrak{U}, \mathcal{F}) \cong H^\bullet(X, \mathcal{F})$. Let $U \in \mathfrak{U}$; then $\Gamma(U, \mathcal{F})$ has the FS topology, then the Čech complex

$$C^\bullet(\mathfrak{U}, \mathcal{F}) = \prod_{s=(s_0, \dots, s_p)} \Gamma(U_{s_0} \cap \dots \cap U_{s_p}, \mathcal{F})$$

becomes an FS complex. So, by passing to cohomology, one gets the natural QFS topology on $H^\bullet(X, \mathcal{F})$ (which is independent on the covering \mathfrak{U}). If $\mathcal{F} \rightarrow \mathcal{G}$ is a morphism of coherent sheaves, then the maps $H^\bullet(X, \mathcal{F}) \rightarrow H^\bullet(X, \mathcal{G})$ are continuous. Moreover, if $H^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{G})$ is surjective, then it is topological.

If we have an exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

then the connection homomorphisms $\delta: H^q(X, \mathcal{F}'') \rightarrow H^{q+1}(X, \mathcal{F}')$ are continuous.

3.3. QDFS-topology on $H_c^\bullet(X, \mathcal{F})$. We now consider topologies on the groups $H_c^\bullet(X, \mathcal{F})$. For any Stein compact K the space $\Gamma(K, \mathcal{F})$ has a natural LF structure given by the equality

$$\Gamma(K, \mathcal{F}) = \varinjlim_{U \supset K} \Gamma(U, \mathcal{F}).$$

This topological structure is DFS.

Let \mathfrak{K} be a locally finite covering of X by Stein compacts (compact subsets of a complex space, admitting a fundamental system of Stein neighbourhoods).

Note that $H^q(K, \mathcal{F}) = 0$ for $q \geq 1$ and for any Stein compact K . Define the complex of finite cochains

$$C_c^p(\mathfrak{K}, \mathcal{F}) := \bigoplus_{s=(s_0, \dots, s_p)} \Gamma(K_{s_0} \cap \dots \cap K_{s_p}, \mathcal{F}).$$

Each $C_c^p(\mathfrak{K}, \mathcal{F})$ has a natural DFS structure and one can easily show that the differentials $C_c^p(\mathfrak{K}, \mathcal{F}) \rightarrow C_c^{p+1}(\mathfrak{K}, \mathcal{F})$ are continuous. Then on the groups $H_c^\bullet(X, \mathcal{F})$ a QDFS topology is obtained (which is independent of the covering) and it is called the natural QDFS topology.

Remark 3.2. [3, Chapter VII, Proposition 4.5] *Let X be a complex manifold with countable basis and \mathcal{F} be a coherent analytic sheaf on X , then the natural topologies on $H^\bullet(X, \mathcal{F})$ and $H_c^\bullet(X, \mathcal{F})$ obtained by means of the Čech cohomology, coincide with the topologies obtained by means of the Dolbeault-Grothendieck resolutions.*

3.4. QDFS-topology on $H_c^\bullet(Z_y, \mathcal{F} |_{Z_y})$. Now, let $X = Y \times F$ be the product of complex manifolds with countable basis, \mathcal{F} be a coherent sheaf on X . Let $Z \subset F$ be a closed complex submanifold, $F_y = F \times \{y\}$, $Z_y = Z \times \{y\}$. Now we define the natural QDFS-topology on $H_c^\bullet(Z_y, \mathcal{F} |_{Z_y})$.

Let \mathfrak{K} be a locally finite covering by Stein compacts of Z_y . Note that by Siu's theorem (see [10, Theorem 3.1.1]) any Stein compact $K \subset Z_y$ has a countable system of Stein neighbourhoods of the form $U \times V \subset X$, where U is a Stein neighbourhood of the point y in Y , and V is a Stein neighbourhood of the set K in F_y . It follows that for $q > 0$:

$$H^q(K, \mathcal{F} |_{Z_y}) = \lim_{U \ni y, V \supset K} H^q(U \times V, \mathcal{F}) = 0.$$

Then the group cohomology $H_c^\bullet(Z_y, \mathcal{F} |_{Z_y})$ is obtained as cohomology of the Čech complex $C_c^\bullet(\mathfrak{K}, \mathcal{F} |_{Z_y})$ where

$$C_c^p(\mathfrak{K}, \mathcal{F} |_{Z_y}) := \bigoplus_{s=(s_0, \dots, s_p)} \Gamma(K_{s_0} \cap \dots \cap K_{s_p}, \mathcal{F} |_{Z_y}).$$

Note that $\Gamma(K_{s_0} \cap \dots \cap K_{s_p}, \mathcal{F} |_{Z_y})$ has a natural LF structure which is separated and DFS (see Remark 3.1). So, $C_c^p(\mathfrak{K}, \mathcal{F} |_{Z_y})$ is also separated and DFS, the group cohomology $H_c^\bullet(F_y, \mathcal{F} |_{Z_y})$ has a QDFS structure.

Remark 3.3. *Obviously, if $\phi: X \rightarrow Y$ is a holomorphic fiber bundle with fiber F and $Z \subset F$ a complex submanifold, we obtain likewise that $H_c^\bullet(Z_y, \mathcal{F} |_{Z_y})$ has the natural QDFS-topology.*

3.5. Inductive topology on $H_c^\bullet(F_y, \mathcal{F} |_{F_y})$ with respect to the pair of spaces. Using the long exact sequence of the pair we can define an inductive topology on $H_c^\bullet(F_y, \mathcal{F} |_{F_y})$.

Let $X' = Y \times F'$ be the product of complex manifolds with countable basis, \mathcal{F} be a coherent analytic sheaf on X' . Assume that F' is a compact complex manifold and let $Z \subset F'$ be a proper closed complex submanifold. We put $F = F' \setminus Z$, $F_y := \{y\} \times F$, $F'_y := \{y\} \times F'$, $Z_y = F'_y \setminus F_y$ and $X = Y \times F$.

For the pair (F_y, F'_y) and the sheaf $\mathcal{F} |_{F'_y}$ we have the following short exact sequence of sheaves (here we denote by $i: Z_y \hookrightarrow F'_y$ the canonical embedding):

$$0 \rightarrow (\mathcal{F} |_{F_y})^{F'_y} \rightarrow \mathcal{F} |_{F'_y} \rightarrow i_*(\mathcal{F} |_{Z_y}) \rightarrow 0$$

which induces the following long exact sequence (recall that $(\mathcal{F} |_{F_y})^{F'_y}$ is the extension by zero of $\mathcal{F} |_{F_y}$):

$$(3.1) \quad \dots \longrightarrow H^p(F'_y, \mathcal{F} |_{F'_y}) \xrightarrow{f} H^p(Z_y, \mathcal{F} |_{Z_y}) \xrightarrow{g} H^{p+1}(F_y, \mathcal{F} |_{F_y}) \longrightarrow \\ \longrightarrow H^{p+1}(F'_y, \mathcal{F} |_{F'_y}) \longrightarrow \dots$$

The spaces $H^\bullet(F'_y, \mathcal{F} |_{F'_y})$, $H^\bullet(Z_y, \mathcal{F} |_{Z_y})$ have canonical QDFS-topologies. Consider the cokernel of f which is the quotient

$$H^p(Z_y, \mathcal{F} |_{Z_y}) / f(H^p(F'_y, \mathcal{F} |_{F'_y}))$$

with quotient topology. So, on the space $H_c^{p+1}(F_y, \mathcal{F} |_{F_y})$ we may consider the inductive topology with respect to the canonical embedding

$$H^p(Z_y, \mathcal{F} |_{Z_y})/f(H^p(F'_y, \mathcal{F} |_{F'_y})) \hookrightarrow H_c^{p+1}(F_y, \mathcal{F} |_{F_y}).$$

This is the finest locally convex topology on $H_c^{p+1}(F_y, \mathcal{F} |_{F_y})$ for which the canonical embedding is a continuous map (in particular, this topology is finer than the canonical QDFS-topology). This topology we call the inductive topology with respect to the pair (F'_y, F_y) .

Now, we may also define this topology using the notion of a mapping cone. Let \mathfrak{K}' be a locally finite covering by Stein compacts of F_y and \mathfrak{K}'' be a locally finite covering by Stein compacts of Z_y . Then $\mathfrak{K} := \mathfrak{K}' \cup \mathfrak{K}''$ is a locally finite covering by Stein compacts of F'_y (by Siu's theorem [10, Theorem 3.1.1], each Stein compact of Z_y is a Stein compact of F'_y). Note that for any $K' \in \mathfrak{K}'$ and for any $K'' \in \mathfrak{K}''$ we have $K' \cap K'' = \emptyset$.

We have the following short exact sequence of Čech complexes

$$0 \rightarrow C_c^\bullet(\mathfrak{K}', \mathcal{F} |_{F_y}) \xrightarrow{h} C_c^\bullet(\mathfrak{K}, \mathcal{F} |_{F'_y}) \xrightarrow{f} C_c^\bullet(\mathfrak{K}'', \mathcal{F} |_{Z_y}) \rightarrow 0$$

which is split and also induces the long exact sequence (3.1).

Let $C^\bullet(h)$ be a cone of the map h . Recall that

$$C^p(h) = C_c^{p+1}(\mathfrak{K}', \mathcal{F} |_{F_y}) \oplus C_c^p(\mathfrak{K}, \mathcal{F} |_{F'_y}),$$

$$\delta_{C(h)} = \begin{pmatrix} -\delta' & 0 \\ h & \delta \end{pmatrix}$$

where δ', δ are Čech differentials in $C_c^\bullet(\mathfrak{K}', \mathcal{F} |_{F_y}), C_c^\bullet(\mathfrak{K}, \mathcal{F} |_{F'_y})$ respectively.

There exist the following maps:

$$\alpha: C_c^\bullet(\mathfrak{K}, \mathcal{F} |_{F'_y}) \rightarrow C^\bullet(h), y \rightarrow (0, y),$$

$$\beta: C^\bullet(h) \rightarrow C_c^{\bullet+1}(\mathfrak{K}', \mathcal{F} |_{F_y}), (x, y) \rightarrow -x.$$

We have an algebraical quasi-isomorphism of complexes:

$$s: C^\bullet(h) \rightarrow C_c^\bullet(\mathfrak{K}'', \mathcal{F} |_{Z_y}), (x, y) \rightarrow f(y).$$

We obtain the following isomorphism of exact sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^p(F'_y, \mathcal{F} |_{F'_y}) & \xrightarrow{f} & H^p(Z_y, \mathcal{F} |_{Z_y}) & \xrightarrow{g} & H_c^{p+1}(F_y, \mathcal{F} |_{F_y}) & \longrightarrow & \cdots \\ & & \searrow \alpha & & \uparrow s=iso & & \nearrow \beta & & \\ \cdots & & & & H^p(C^\bullet(h)) & & & & \cdots \end{array}$$

Note that the s induces a topological isomorphism to cohomology. It follows that the inductive topology on $H_c^{p+1}(F_y, \mathcal{F} |_{F_y})$ with respect to the pair (F'_y, F_y) is the inductive topology with respect to the canonical embedding

$$\beta: H^p(C^\bullet(h))/\alpha(H^p(F'_y, \mathcal{F} |_{F'_y})) \hookrightarrow H_c^{p+1}(F_y, \mathcal{F} |_{F_y}).$$

3.6. The duality theorems. For the complex manifolds there are the duality theorems due to Serre and Malgrange. We need the result for the structure sheaf.

Theorem 3.1. [3, Chapter VII, Theorems 4.1, 4.2] *Let X be a complex manifold with countable basis.*

- (1) *The associated separated spaces of $H_c^p(X, \mathcal{O}_X)$ and $H^{n-p}(X, \Omega_X^n)$ are in topological duality;*
- (2) *The space $H_c^p(X, \mathcal{O}_X)$ is separated if and only if the space $H^{n-p+1}(X, \Omega_X^n)$ is separated.*

3.7. The nuclear spaces and topological Künnet formula. Now we recall the topological tensor product and the Künnet formula. For more details about the topological tensor product see [6, 24, 27], about the Künnet theorem for topological tensor product see [17] or [6, Chapter IX].

Let E, F be locally convex topological vector space. The topological tensor product $E \widehat{\otimes}_\pi F$ (resp. $E \widehat{\otimes}_\varepsilon F$) is the Hausdorff completion $E \otimes F$, equipped with the family of semi-norms $p \widehat{\otimes}_\pi q$ (resp. $p \widehat{\otimes}_\varepsilon q$) associated to fundamental families of semi-norms on E and F which is defined by formulas:

$$p \widehat{\otimes}_\pi q(t) = \inf \left\{ \sum_{1 \leq j \leq N} p(x_j)q(y_j) \mid t = \sum_{1 \leq j \leq N} x_j \otimes y_j, x_j \in E, y_j \in F \right\},$$

$$p \widehat{\otimes}_\varepsilon q(t) = \sup \left\{ |l \otimes m(t)| \mid l \in E', m \in F', \|l\|_p \leq 1, \|m\|_q \leq 1 \right\}.$$

A morphism $f: E \rightarrow F$ of complete locally convex spaces is said to be nuclear if f can be written as $f(x) = \sum \lambda_j \varepsilon_j(x) y_j$, where $\{\lambda_j\}$ is a sequence of scalar with $\sum |\lambda_j| < \infty$, $\varepsilon_j \in E'$ an equicontinuous sequence of linear forms and $y_j \in F$ a bounded sequence.

A locally convex space E is called nuclear if every continuous linear map of E into any Banach space is nuclear.

If E is an FS space and $\{p_j\}$ is an increasing sequence of semi-norms on E defining the topology of E , we have $E = \varprojlim \widehat{E}_{p_j}$, where \widehat{E}_{p_j} is the Hausdorff completion of (E, p_j) and $\widehat{E}_{p_{j+1}} \rightarrow \widehat{E}_{p_j}$ the canonical morphism.

An FS space E is nuclear if and only if the topology of E can be defined by an increasing sequence of semi-norms p_j such that each canonical morphism $\widehat{E}_{p_{j+1}} \rightarrow \widehat{E}_{p_j}$ of Banach spaces is nuclear.

Now we list some facts about nuclear spaces.

- If E or F is nuclear, we obtain an isomorphism $E \widehat{\otimes}_\pi F \cong E \widehat{\otimes}_\varepsilon F$;
- If E, F are nuclear spaces, then $E \widehat{\otimes} F$ is nuclear;
- Every subspace and every separated quotient space of a nuclear space is nuclear;
- The product of an arbitrary family of nuclear spaces is nuclear;
- The locally convex direct sum of a countable family of nuclear spaces is a nuclear space;
- The projective limit of any family of nuclear spaces, and the inductive limit of a countable family of nuclear spaces, are nuclear;
- Let $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ be an exact sequence of topological homomorphisms between Fréchet spaces and let F be a Fréchet space. If E_2 or F is nuclear, there is an exact sequence of topological homomorphisms

$$0 \rightarrow E_1 \widehat{\otimes} F \rightarrow E_2 \widehat{\otimes} F \rightarrow E_3 \widehat{\otimes} F \rightarrow 0;$$

Example 3.2. (1) For discrete spaces I, J there is an isometry

$$l^1(I) \widehat{\otimes}_\pi l^1(J) \cong l^1(I \times J);$$

- (2) If X, Y are compact topological spaces and $C(X), C(Y)$ are their algebras of continuous functions with sup-norms, then $C(X) \widehat{\otimes}_\varepsilon C(Y) \cong C(X \times Y)$;
(3) Let D be a polydisk in \mathbb{C}^n . Then $\mathcal{O}(D)$ is a nuclear FS space.
(4) If \mathcal{F} is a coherent analytic sheaf on a complex analytic variety X , then $\mathcal{F}(X)$ is a nuclear space.
(5) Let \mathcal{F}, \mathcal{G} be coherent analytic sheaves on complex analytic varieties X, Y respectively. Then there is a canonical topological isomorphism

$$\mathcal{F} \boxtimes \mathcal{G}(X \times Y) \cong \mathcal{F}(X) \widehat{\otimes} \mathcal{G}(Y)$$

where $\mathcal{F} \boxtimes \mathcal{G} = p_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} p_2^* \mathcal{G}$ is the analytic external tensor product of \mathcal{F} and \mathcal{G} . In particular,

$$\mathcal{O}_{X \times Y}(X \times Y) \cong \mathcal{O}_X(X) \widehat{\otimes} \mathcal{O}_Y(Y).$$

- (6) Let \mathcal{F} be a coherent analytic sheaf on a complex analytic variety X . If the spaces $H^\bullet(X, \mathcal{F}), H_c^\bullet(X, \mathcal{F})$ are separated, then they are nuclear spaces.
(7) Let $X = Y \times F$ be the product of complex manifolds, \mathcal{F} be a coherent analytic sheaf on X , $Z \subset F$ be a closed complex submanifold, $Z_y = Z \times \{y\}$. If the space $H_c^\bullet(Z_y, \mathcal{F}|_{Z_y})$ is separated, then it is a nuclear space.

We need the following proposition.

Proposition 3.1. [6, Chapter IX, Theorem 5.23], [17, Theorem I] *Let \mathcal{F}, \mathcal{G} be coherent analytic sheaves over complex analytic varieties and suppose that the cohomology spaces $H^\bullet(X, \mathcal{F})$ and $H^\bullet(Y, \mathcal{G})$ are separated spaces. Then there is a topological isomorphism*

$$\bigoplus_{p+q=n} H^p(X, \mathcal{F}) \widehat{\otimes} H^q(Y, \mathcal{G}) \cong H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}).$$

3.8. On separatedness of the space $H_c^\bullet(F_y, \mathcal{O}_X|_{F_y})$. Now let $X = Y \times F$ be the product of complex manifolds with countable basis, Y be a Stein manifold. Let Ω^p be the sheaf of holomorphic p -form.

Lemma 3.1. *There exists an isomorphism $\Omega_X^n \cong \bigoplus_{p+q=n} \Omega_Y^p \boxtimes \Omega_F^q$ as Fréchet-Schwartz sheaves.*

Proof. Let $p_1: X \rightarrow Y$ and $p_2: X \rightarrow F$ be canonical projections. We have the following canonical algebraic isomorphisms:

$$\begin{aligned} \Omega_X^n &= \wedge^n \Omega_X^1 = \wedge^n (p_1^* \Omega_Y^1 \oplus p_2^* \Omega_F^1) \cong \\ &\cong \bigoplus_{p+q=n} \wedge^p (p_1^* \Omega_Y^1) \otimes \wedge^q (p_2^* \Omega_F^1) \cong \bigoplus_{p+q=n} p_1^* \Omega_Y^p \otimes p_2^* \Omega_F^q = \bigoplus_{p+q=n} \Omega_Y^p \boxtimes \Omega_F^q. \end{aligned}$$

Let $d_{X,n} = \dim(\wedge^n(\mathbb{C}^{\dim X}))$, $d_{Y,p} = \dim(\wedge^p(\mathbb{C}^{\dim Y}))$, $d_{F,q} = \dim(\wedge^q(\mathbb{C}^{\dim F}))$. Note that $d_{X,n} = \sum_{p+q=n} d_{Y,p} d_{F,q}$.

Locally, for an open set $U \times V \subset X$ we have the topological isomorphisms:

$$\begin{aligned} \Omega_X^n(U \times V) &\cong (\mathcal{O}_X(U \times V))^{\oplus d_{X,n}} \cong (\mathcal{O}_Y(U) \widehat{\otimes} \mathcal{O}_F(V))^{\oplus d_{X,n}} \cong \\ &\cong \bigoplus_{p+q=n} (\mathcal{O}_Y(U))^{\oplus d_{Y,p}} \widehat{\otimes} (\mathcal{O}_F(V))^{\oplus d_{F,q}} \cong \bigoplus_{p+q=n} \Omega_Y^p(U) \widehat{\otimes} \Omega_F^q(V). \end{aligned}$$

It follows that for any open set $W \subset X$ we have topological isomorphism

$$\Omega_X^n(W) \cong \bigoplus_{p+q=n} (\Omega_Y^p \boxtimes \Omega_F^q)(W).$$

It concludes the proof of Lemma. \square

Proposition 3.2. *Let $y \in Y$ be any point. If for every k, l the spaces $H^k(F, \Omega_F^l)$ are separated, then*

- (1) *for any $i > 0$ the spaces $H_c^i(Y \setminus \{y\} \times F, \mathcal{O}_X)$, $H_c^i(Y \times F, \mathcal{O}_X)$ are separated;*
- (2) *for any $i > 0$ the canonical map $H_c^i(Y \setminus \{y\} \times F, \mathcal{O}_X) \rightarrow H_c^i(Y \times F, \mathcal{O}_X)$ is surjective.*

Proof. Since Y is a Stein space, it follows that $H^i(Y, \Omega_Y^p) = 0$ for all $i > 0$ and for all p , $\Omega^p(Y)$ is separated for all p and $H^j(Y \setminus \{y\}, \Omega_Y^q)$ is separated for all $j \geq 0$ and for all q .

By the Künnet formula (Proposition 3.1) we obtain the following topological isomorphisms:

$$\begin{aligned} \Omega^p(Y) \widehat{\otimes} H^{n-i}(F, \Omega_F^q) &\cong H^{n-i}(Y \times F, \Omega_Y^p \boxtimes \Omega_F^q); \\ \bigoplus_{k+l=n-i} H^k(Y \setminus \{y\}, \Omega_Y^p) \widehat{\otimes} H^l(F, \Omega_F^q) &\cong H^{n-i}(Y \setminus \{y\} \times F, \Omega_Y^p \boxtimes \Omega_F^q). \end{aligned}$$

It follows that for any $i \geq 0$ the spaces

$$H^{n-i}(Y \setminus \{y\} \times F, \Omega_X^n), H^{n-i}(Y \times F, \Omega_X^n)$$

are separated. By the duality theorem 3.1 we have that the spaces

$$H_c^i(Y \setminus \{y\} \times F, \mathcal{O}_X), H_c^i(Y \times F, \mathcal{O}_X)$$

are separated. Moreover, we obtain the topological isomorphisms:

$$H_c^i(Y \setminus \{y\} \times F, \mathcal{O}_X) \cong (H^{n-i}(Y \setminus \{y\} \times F, \Omega_X^n))',$$

$$H_c^i(Y \times F, \mathcal{O}_X) \cong (H^{n-i}(Y \times F, \Omega_X^n))',$$

Since $\Omega^p(Y)$ is a closed subspace of a nuclear space $\Omega^p(Y \setminus \{y\})$, it follows that we have the injective topological homomorphism

$$0 \rightarrow \Omega^p(Y) \widehat{\otimes} H^{n-i}(F, \Omega_F^q) \rightarrow \Omega^p(Y \setminus \{y\}) \widehat{\otimes} H^{n-i}(F, \Omega_F^q).$$

So, the canonical map

$$H^{n-i}(Y \times F, \Omega_X^n) \rightarrow H^{n-i}(Y \setminus \{y\} \times F, \Omega_X^n)$$

is injective topological homomorphism for all i . It follows that the dual map is surjective. This means that for any $i > 0$ the canonical map

$$H_c^i(Y \setminus \{y\} \times F, \mathcal{O}_X) \rightarrow H_c^i(Y \times F, \mathcal{O}_X)$$

is surjective. \square

In particular we have the following

Corollary 3.1. *Let $F_y = \{y\} \times F$. If for every k, l the spaces $H^k(F, \Omega_F^l)$ are separated, then $H_c^i(F_y, \mathcal{O}_X|_{F_y})$ are separated spaces for all i .*

Proof. Since Y is a noncompact manifold, it follows that

$$H_c^0(Y \setminus \{y\} \times F, \mathcal{O}_X) = H_c^0(Y \times F, \mathcal{O}_X) = 0.$$

We have the following long exact sequence of the pair $(X, X \setminus F_y)$ [4, Chapter II, Section 10.3]:

(3.2)

$$\begin{aligned} 0 \longrightarrow H_c^0(F_y, \mathcal{O}_X|_{F_y}) \longrightarrow H_c^1(Y \setminus \{y\} \times F, \mathcal{O}_X) \xrightarrow{f_1} H_c^1(Y \times F, \mathcal{O}_X) \longrightarrow \\ \longrightarrow H_c^1(F_y, \mathcal{O}_X|_{F_y}) \longrightarrow H_c^2(Y \setminus \{y\} \times F, \mathcal{O}_X) \xrightarrow{f_2} H_c^2(Y \times F, \mathcal{O}_X) \longrightarrow \dots \end{aligned}$$

Since by Proposition 3.2 we have that f_i are surjective continuous maps between separated spaces, so $\ker f_i$ are closed subspaces. It follows that $H_c^i(F_y, \mathcal{O}_X|_{F_y})$ are separated spaces. \square

4. DENSITY LEMMA FOR HOLOMORPHIC FIBER BUNDLE AND STEIN FIBERS CASE

Let Y, F be complex manifolds with countable basis. Let $X = Y \times F$, $F_y = \{y\} \times F$, let $Z_y \subset F_y$ be a closed complex submanifold. Let $p_1: X \rightarrow Y$, $p_2: X \rightarrow F$ be canonical projections.

We have the canonical homomorphisms of sheaves

$$p_1^{-1}\mathcal{O}_Y \hookrightarrow \mathcal{O}_X, p_2^{-1}\mathcal{O}_F \hookrightarrow \mathcal{O}_X,$$

and we obtain the canonical injective homomorphism of sheaves

$$p_1^{-1}\mathcal{O}_Y \otimes p_2^{-1}\mathcal{O}_F \hookrightarrow \mathcal{O}_X, f \otimes g \rightarrow fg.$$

For any open set of the form $U \times V \in X$ we obtain that the injective map

$$\mathcal{O}_Y(U) \otimes \mathcal{O}_F(V) \hookrightarrow \mathcal{O}_X(U \times V)$$

has a dense image (Remark 3.2).

Note that we have the canonical isomorphisms of the sheaves

$$(p_1^{-1}\mathcal{O}_Y \otimes p_2^{-1}\mathcal{O}_F)|_{Z_y} \cong \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}$$

where $\mathcal{O}_{Y,y}$ considered as the constant sheaf on the fiber F_y .

For any open subset $W \subset Z_y$ we obtain that the canonical injective map

$$\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}(W) \hookrightarrow \mathcal{O}_X|_{Z_y}(W)$$

has a dense image.

Lemma 4.1. *The canonical homomorphism*

$$\mathcal{O}_{Y,y} \otimes H_c^\bullet(Z_y, \mathcal{O}_{F_y}|_{Z_y}) \rightarrow H_c^\bullet(Z_y, \mathcal{O}_X|_{Z_y})$$

has a dense image with respect to the QDFS-topology on $H_c^\bullet(Z_y, \mathcal{O}_X|_{Z_y})$.

Proof. Let \mathfrak{K} be a locally finite covering of Z_y by Stein compacts. Then the group cohomology $H_c^\bullet(Z_y, \mathcal{O}_X|_{Z_y})$ is obtained as cohomology of the Čech complex $C_c^\bullet(\mathfrak{K}, \mathcal{O}_X|_{Z_y})$ where

$$C_c^p(\mathfrak{K}, \mathcal{O}_X|_{Z_y}) := \bigoplus_{s=(s_0, \dots, s_p)} \Gamma(K_{s_0} \cap \dots \cap K_{s_p}, \mathcal{O}_X|_{Z_y}).$$

Now, we consider the sheaf $\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}$ on Z_y .

By the universal coefficient theorem [4, Theorem 15.3] and by Siu's theorem (see [10, Theorem 3.1.1]) for any Stein compact $K \in \mathfrak{K}$ we have the following:

- for global sections

$$\Gamma(K, \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}) \cong \mathcal{O}_{Y,y} \otimes \Gamma(K, \mathcal{O}_{F_y}|_{Z_y});$$

- for group cohomologies

$$H^q(K, \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}) \cong \mathcal{O}_{Y,y} \otimes H^q(K, \mathcal{O}_{F_y}|_{Z_y}) = 0.$$

For the sheaf $\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}$ on Z_y we have the Čech complex

$$C_c^\bullet(\mathfrak{K}, \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}) = \mathcal{O}_{Y,y} \otimes C_c^\bullet(\mathfrak{K}, \mathcal{O}_{F_y}|_{Z_y})$$

The group $H_c^\bullet(F_y, \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y})$ is obtained as cohomology of the Čech complex $\mathcal{O}_{Y,y} \otimes C_c^\bullet(\mathfrak{K}, \mathcal{O}_{F_y}|_{Z_y})$. Moreover, we have the canonical isomorphism

$$H_c^\bullet(F_y, \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}) = \mathcal{O}_{Y,y} \otimes H_c^\bullet(F_y, \mathcal{O}_{F_y}|_{Z_y}).$$

Note that the canonical injective morphism of complexes

$$\mathcal{O}_{Y,y} \otimes C_c^\bullet(\mathfrak{K}, \mathcal{O}_{F_y}|_{Z_y}) \hookrightarrow C_c^\bullet(\mathfrak{K}, \mathcal{O}_X|_{Z_y})$$

has a dense image.

Now, every cochain $f = (f_{s_0, \dots, s_p})$ in $C_c^p(\mathfrak{K}, \mathcal{O}_X|_{Z_y})$ can be represented as $f = \sum_{I \in \mathbb{Z}_{\geq 0}^m} f_I z^I$, where $z = (z_1, \dots, z_m)$ is a local coordinates with origin at the point $y \in Y$, $f_I = (f_{I, (s_0, \dots, s_p)}) \in C_c^p(\mathfrak{K}, \mathcal{O}_{F_y}|_{Z_y})$ and the power series $\sum_{I \in \mathbb{Z}_{\geq 0}^m} f_{I, (s_0, \dots, s_p)} z^I$

uniformly converges on compact sets in $U \times V$ (here $U \subset Y$ is a small polydisk with center $y \in Y$, V is a Stein neighbourhood of K_{s_0, \dots, s_p} in F_y).

Assume that $\delta(f) = 0$; this means that for any (s_0, \dots, s_{p+1}) we have

$$(4.1) \quad 0 = \delta(f)_{s_0, \dots, s_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \left(\sum_{I \in \mathbb{Z}_{\geq 0}^m} f_{I, (s_0, \dots, \widehat{k}, \dots, s_{p+1})} z^I \right) = \\ = \sum_{I \in \mathbb{Z}_{\geq 0}^m} \left(\sum_{k=0}^{p+1} (-1)^k f_{I, (s_0, \dots, \widehat{k}, \dots, s_{p+1})} \right) z^I.$$

Therefore $\delta(f_I) = 0$. It follows that for every q the canonical injective morphism

$$\mathcal{O}_{Y,y} \otimes Z_c^q(\mathfrak{K}, \mathcal{O}_{F_y}|_{Z_y}) \hookrightarrow Z_c^q(\mathfrak{K}, \mathcal{O}_X|_{Z_y})$$

has a dense image.

Therefore the canonical homomorphism

$$\mathcal{O}_{Y,y} \otimes H_c^\bullet(F_y, \mathcal{O}_{F_y}|_{Z_y}) \rightarrow H_c^\bullet(F_y, \mathcal{O}_X|_{Z_y})$$

has a dense image. \square

Let X, Y, F be complex manifolds with countable basis and $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with connected fiber F . Recall that by Remark 2.2 we have the canonical isomorphism $(R^\bullet\phi_!\mathcal{O}_X)_y \cong H_c^\bullet(F_y, \mathcal{O}_X|_{F_y})$, where $F_y = \phi^{-1}(y)$.

Corollary 4.1. *Let $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with fiber F . Let $F_y = \phi^{-1}(y)$. Then for any $y \in Y$ there is the canonical homomorphism*

$$\mathcal{O}_{Y,y} \otimes H_c^\bullet(F_y, \mathcal{O}_{F_y}) \rightarrow H_c^\bullet(F_y, \mathcal{O}_X|_{F_y})$$

which has a dense image with respect to the QDFS-topology on $H_c^\bullet(F_y, \mathcal{O}_X|_{F_y})$.

Proof. The problem is local. We may assume that $X = Y \times F$ and apply Lemma 4.1 for the case $Z = F$. \square

Corollary 4.2. *Let $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with noncompact fiber F . Then $\phi_!\mathcal{O}_X = 0$. In particular, $H_c^1(X, \mathcal{O}_X) \cong \Gamma_c(Y, R^1\phi_!\mathcal{O}_X)$.*

Proof. Let $F_y = \phi^{-1}(y)$. We have $(\phi_!\mathcal{O}_X)_y \cong \Gamma_c(F_y, \mathcal{O}_X|_{F_y})$. Since $\Gamma_c(F_y, \mathcal{O}_X|_{F_y})$ is separated DFS space and it has the dense subset $\mathcal{O}_y \otimes \Gamma_c(F_y, \mathcal{O}_{F_y}) = 0$, it follows that $\phi_!\mathcal{O}_X = 0$. By Corollary 2.1 we obtain the last statement. \square

So, we obtain the following vanishing result.

Corollary 4.3. *Let $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with noncompact fiber F . Assume that for any $i < q$ sheaves $R^i\phi_!\mathcal{O}_X$ are separated sheaves. If $H_c^i(F, \mathcal{O}_F) = 0$ for any $i < q$, then $H_c^i(X, \mathcal{O}_X) = 0$ for any $i < q$.*

Proof. By Lemma 4.1 we have $R^i\phi_!\mathcal{O}_X = 0$. By Corollary 2.1 we obtain $H^i(X, \mathcal{O}_X) = 0$ for any $i < q$. \square

If the fiber F is a Stein manifold then the spaces $H^k(F, \Omega_F^l)$ are separated. We obtain the following vanishing result.

Corollary 4.4. *Let $\phi: X \rightarrow Y$ be a holomorphic fiber bundle with fiber F which is a Stein manifold. Then $H_c^i(X, \mathcal{O}_X) = 0$ for all $i < \dim F$.*

Proof. We may assume that $\phi: X \rightarrow Y$ is a trivial holomorphic fiber bundle over a Stein manifold Y . This means that $X = Y \times F$ and ϕ is a projection onto the first factor Y . We put $F_y := \{y\} \times F$.

By Corollary 3.1 we have that $H_c^i(F_y, \mathcal{O}_X|_{F_y})$ are separated spaces for any q . Since F is a Stein manifold, $H_c^i(F, \mathcal{O}_F) = 0$ for all $i < \dim F$. Then by Corollary 4.3 we obtain that $H_c^i(X, \mathcal{O}_X) = 0$ for all $i < \dim F$. \square

In particular, for $F = \mathbb{C}^n$ and $n > 1$ we obtain Dwiłewicz's result about vanishing $H_c^1(X, \mathcal{O}_X)$ for the total space X of a complex fiber bundle [7, Corollary 1.4].

Remark 4.1. *Note that, the space $\mathcal{O}_{Y,y}$ is nuclear. Assume that the spaces*

$$H_c^\bullet(Z_y, \mathcal{O}_X|_{Z_y}), H_c^\bullet(Z_y, \mathcal{O}_{F_y}|_{Z_y})$$

are separated spaces.

For any open subset $W \subset Z_y$ the canonical injective map

$$\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}(W) \hookrightarrow \mathcal{O}_X|_{Z_y}(W)$$

is continuous with respect to the projective tensor product topology on

$$\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F_y}|_{Z_y}(W)$$

and the canonical DFS-topology on $\mathcal{O}_X|_{Z_y}(W)$.

It follows that the canonical map

$$\mathcal{O}_{Y,y} \otimes H_c^\bullet(Z_y, \mathcal{O}_{F_y} |_{Z_y}) \rightarrow H_c^\bullet(Z_y, \mathcal{O}_X |_{Z_y})$$

is continuous with respect to the projective tensor product topology and QDFS-topology. Then the canonical continuous map

$$\mathcal{O}_{Y,y} \widehat{\otimes} H_c^\bullet(Z_y, \mathcal{O}_{F_y} |_{Z_y}) \rightarrow H_c^\bullet(Z_y, \mathcal{O}_X |_{Z_y})$$

is surjective.

5. HOLOMORPHIC FIBER BUNDLE WITH $(1, 0)$ -COMPACTIFIABLE FIBERS

We introduce the following definition.

Definition 5.1. *A noncompact complex manifold X is called (b, σ) -compactifiable if it admits a compactification X' with the following properties:*

- (1) X' is a compact complex manifold;
- (2) $X' \setminus X$ is a proper analytic subset and it has b connected components;
- (3) $\dim_{\mathbb{C}} H^1(X', \mathcal{O}_{X'}) = \sigma$.

The numbers (b, σ) for a complex manifold X are independent on the compactification X' . More precisely, we have the following remark.

- Remark 5.1.**
- (1) *If X is a (b, σ) -compactifiable complex manifold, then X has exactly b topological ends (about topological ends see [9], for CW-complexes see [23]);*
 - (2) *The number σ is a birational invariant (see, for example, [25, Corollary 1.4]);*
 - (3) *Let $A(X')$ be the Albanese manifold for a compact complex algebraic manifold X' , then $\dim A(X') = \sigma$.*

Let X, Y, F be complex manifolds with countable basis. Now we have the following important lemma on the stalks of sheaf $R^1\phi_!\mathcal{O}_X$. Recall that

$$(R^1\phi_!\mathcal{O}_X)_y \cong H_c^1(F_y, \mathcal{O}_X |_{F_y}),$$

where $F_y = \phi^{-1}(y)$.

Lemma 5.1. *Let $\phi: X \rightarrow Y$ be a holomorphic fibre bundle with noncompact fiber F and $F_y = \phi^{-1}(y)$. Assume that F is $(1, 0)$ -compactifiable and $\dim F > 1$. Then the inductive topology on $H_c^1(F_y, \mathcal{O}_X |_{F_y})$ with respect to the pair (F'_y, F_y) is separated and the canonical map $\mathcal{O}_{Y,y} \otimes H_c^1(F_y, \mathcal{O}_{F_y}) \rightarrow H_c^1(F_y, \mathcal{O}_X |_{F_y})$ is injective with dense image with respect to this topology.*

Proof. By assumption there exists a compact complex manifold F' such that $Z = F' \setminus F$ is a connected proper analytic subset and $H^1(F', \mathcal{O}_{F'}) = 0$. From the long exact sequence for the pair (F, F') [4, Chapter II, Section 10.3] we obtain the short exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(Z, \mathcal{O}_F |_Z) \longrightarrow H_c^1(F, \mathcal{O}_F) \longrightarrow 0.$$

Now, since the problem is local, we may assume that $\phi: X \rightarrow Y$ is a trivial holomorphic fiber bundle over a Stein manifold Y . This means that $X = Y \times F$ and ϕ is a projection onto the first factor Y . We have $F_y = \{y\} \times F$, $Z_y = F'_y \setminus F_y$ and $X' = Y \times F'$.

For the pair (F_y, F'_y) and the sheaf $\mathcal{O}_{X'}|_{F'_y}$ we have the following long exact sequence (note that $H_c^0(F_y, \mathcal{O}_X|_{F_y}) = 0$, because F_y is a noncompact complex manifold):

$$\begin{aligned} 0 \longrightarrow H^0(F'_y, \mathcal{O}_{X'}|_{F'_y}) \longrightarrow H^0(Z_y, \mathcal{O}_{X'}|_{Z_y}) \longrightarrow H_c^1(F_y, \mathcal{O}_X|_{F_y}) \longrightarrow \\ \longrightarrow H^1(F'_y, \mathcal{O}_{X'}|_{F'_y}) \longrightarrow \dots \end{aligned}$$

Recall that

$$H^i(F'_y, \mathcal{O}_{X'}|_{F'_y}) \cong \varinjlim_{U \supset y} H^i(U \times F', \mathcal{O}_{X'})$$

where the inductive limit may be taken over a cofinal system of Stein neighbourhoods of the point y .

By the Künneth theorem for topological tensor product (Proposition 3.1) we obtain

$$H^0(U \times F', \mathcal{O}_{X'}) \cong H^0(U, \mathcal{O}_Y) \widehat{\otimes} H^0(F', \mathcal{O}_{F'}) = H^0(U, \mathcal{O}_Y),$$

$$H^1(U \times F', \mathcal{O}_{X'}) \cong H^1(U, \mathcal{O}_Y) \widehat{\otimes} H^0(F', \mathcal{O}_{F'}) \oplus H^0(U, \mathcal{O}_Y) \widehat{\otimes} H^1(F', \mathcal{O}_{F'}) = 0.$$

(by assumption, we have $H^1(F', \mathcal{O}_{F'}) = 0$, and for the Stein neighbourhood U of the point y we have $H^1(U, \mathcal{O}_Y) = 0$). Therefore,

$$H^0(F'_y, \mathcal{O}_{X'}|_{F'_y}) \cong \varinjlim_{U \supset y} H^0(U, \mathcal{O}_Y) = \mathcal{O}_{Y,y},$$

$$H^1(F'_y, \mathcal{O}_{X'}|_{F'_y}) = 0.$$

So, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_{Y,y} \longrightarrow H^0(Z_y, \mathcal{O}_{X'}|_{Z_y}) \longrightarrow H_c^1(F_y, \mathcal{O}_X|_{F_y}) \longrightarrow 0.$$

For the algebraic tensor product $p_1^{-1}\mathcal{O}_Y \otimes p_2^{-1}\mathcal{O}_{F'}$ we have the following canonical isomorphisms

$$(p_1^{-1}\mathcal{O}_Y \otimes p_2^{-1}\mathcal{O}_{F'})|_{F'_y} \cong \mathcal{O}_{Y,y} \otimes \mathcal{O}_{F'_y},$$

$$(p_1^{-1}\mathcal{O}_Y \otimes p_2^{-1}\mathcal{O}_{F'})|_{Z_y} \cong \mathcal{O}_{Y,y} \otimes (\mathcal{O}_{F'_y}|_{Z_y}).$$

Apply the universal coefficient theorem [4, Theorem 15.3] to the long exact sequence of the cohomology groups of the sheaf $\mathcal{O}_{Y,y} \otimes \mathcal{O}_{F'_y}$ under the pair (F_y, F'_y) we also have the following short exact sequence.

$$0 \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y,y} \otimes H^0(Z_y, \mathcal{O}_{F'_y}|_{Z_y}) \longrightarrow \mathcal{O}_{Y,y} \otimes H_c^1(F_y, \mathcal{O}_{F_y}) \longrightarrow 0$$

Therefore we obtain the following commutative diagram with exact rows and the maps h_1, h_2 have dense images (by Lemma 4.1).

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{O}_{Y,y} & \longrightarrow & \mathcal{O}_{Y,y} \otimes H^0(Z_y, \mathcal{O}_{F'_y}|_{Z_y}) & \longrightarrow & \mathcal{O}_{Y,y} \otimes H_c^1(F_y, \mathcal{O}_{F_y}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow h_1 & & \downarrow h_2 & & \\ 0 \longrightarrow \mathcal{O}_{Y,y} & \xrightarrow{g_1} & H^0(Z_y, \mathcal{O}_{X'}|_{Z_y}) & \longrightarrow & H_c^1(F_y, \mathcal{O}_X|_{F_y}) & \longrightarrow & 0 \end{array}$$

Note that $\mathcal{O}_{Y,y}$ is a closed subspace in $H^0(Z_y, \mathcal{O}_{X'}|_{Z_y})$. Actually, let $\{f_n\}$ be a sequence of holomorphic function f_n in a neighbourhood $U \times F$ of F'_y which are constant on fibers. If $f = \lim_{n \rightarrow \infty} f_n$ in a small neighbourhood $U' \times V$ of Z_y (here $V \supset Z_y, y \in U' \subset U$), then f is constant on fibers in $U' \times V$ over U' . It defines a function on $U' \times F$ which is constant on fibers.

Note that since h_1 is injective, it follows that h_2 is also injective.

The quotient space $H^0(Z_y, \mathcal{O}_{X'}|_{Z_y})/\mathcal{O}_{Y,y}$ is a separated space which is algebraically isomorphic to $H_c^1(F_y, \mathcal{O}_X|_{F_y})$. The inductive topology on $H_c^1(F_y, \mathcal{O}_X|_{F_y})$ with respect to the pair (F'_y, F_y) is exactly the separated quotient topology. Since h_1 has a dense image, it follows that the space $\mathcal{O}_{Y,y} \otimes H_c^1(F_y, \mathcal{O}_{F_y})$ is dense in the separated quotient topology in $H_c^1(F_y, \mathcal{O}_X|_{F_y})$. \square

Corollary 5.1. *Let $\phi: X \rightarrow Y$ be a holomorphic fibre bundle with $(1, 0)$ -compactifiable fiber F , $\dim F > 1$. If $H_c^1(F, \mathcal{O}_F) = 0$, then $H_c^1(X, \mathcal{O}_X) = 0$.*

Proof. By Lemma 5.1 and Corollary 2.1. \square

Remark 5.2. *Note that, in the case of $\dim F = 1$ we have $H_c^1(F, \mathcal{O}_F) \neq 0$. It follows that $H_c^1(X, \mathcal{O}_X)$ is not necessary trivial. For example, for the tautological line bundle $X = \mathcal{O}(-1)$ over projective space \mathbb{P}^n we have $H_c^1(X, \mathcal{O}_X) = 0$, but for trivial line bundle $X = \mathbb{P}^n \times \mathbb{C}$ we have*

$$H_c^1(X, \mathcal{O}_X) \cong H^0(\mathbb{P}^n \times \{\infty\}, \mathcal{O}_X|_{\mathbb{P}^n \times \{\infty\}})/\mathbb{C} \cong \mathcal{O}_{\mathbb{P}(\mathbb{C}), \infty}/\mathbb{C} \neq 0.$$

6. COHOMOLOGICAL CRITERION FOR THE HARTOGS PHENOMENON

J.-P. Serre proved the Hartogs phenomenon for Stein manifolds by using triviality of the cohomology group with compact supports $H_c^1(X, \mathcal{O}_X)$ where \mathcal{O}_X is the sheaf of holomorphic functions [26].

We consider a class of normal complex analytic varieties such that for this class the finite dimensional of the group $H_c^1(X, \mathcal{O}_X)$ is a necessary and sufficient condition for the Hartogs phenomenon for X .

First for a compact set K of X define $\mu(K)$ to be the union of K with all connected components of $X \setminus K$ that are relatively compact in X (see [28] and [11, Chapter VII, Section D]). We call a complex analytic variety X to be connected at boundary if for every compact K of X the set $X \setminus \mu(K)$ is connected. Define the space of germs of holomorphic functions on boundary

$$\mathcal{O}_X(\partial X) := \varinjlim_S H^0(X \setminus S, \mathcal{O}_X),$$

where the inductive limit is taken over all compact subsets S of X (or over a cofinal part of them) (see [11, Chapter VII, Section D]).

We need the following lemmas.

Lemma 6.1. *Let X be a noncompact normal complex analytic variety which is connected at boundary. Let $K \subset X$ be a compact set such that $X \setminus K$ is connected. If $\dim H_c^1(X, \mathcal{O}_X) < \infty$, then the restriction homomorphism $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X \setminus K)$ is an isomorphism.*

Proof. We consider the following exact sequence of cohomology groups [3]:

(6.1)

$$0 \longrightarrow H_K^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X) \xrightarrow{R_K} H^0(X \setminus K, \mathcal{O}_X) \xrightarrow{F_K} H_K^1(X, \mathcal{O}_X) \longrightarrow \dots$$

Obviously, $H_K^0(X, \mathcal{O}_X) = 0$. So, the restriction homomorphism R_K is injective.

Recall that if $S, T \subset X$ are compact sets and $S \subset T$, then we obtain the canonical homomorphism $\phi_{ST}: H_S^1(X, \mathcal{O}_X) \rightarrow H_T^1(X, \mathcal{O}_X)$. Moreover, we have the canonical isomorphism [3]

$$\varinjlim_S H_S^1(X, \mathcal{O}_X) \cong H_c^1(X, \mathcal{O}_X)$$

where the inductive limit is taken over all compact subsets S of X (or over a cofinal part of them).

So, we obtain the following long exact sequence:

$$(6.2) \quad 0 \longrightarrow \mathcal{O}_X(X) \xrightarrow{r} \mathcal{O}_X(\partial X) \xrightarrow{c} H_c^1(X, \mathcal{O}_X) \longrightarrow \dots$$

Now we apply the Andreotti-Hill technique (see [2, corollary 4.3]). Let $m = \dim H_c^1(X, \mathcal{O}_X)$.

We can assume that $\mathcal{O}_X(\partial X) \neq \mathbb{C}$. Consider an equivalence class $f \in \mathcal{O}_X(\partial X)$ of a non-constant holomorphic function on $X \setminus K$. The elements

$$c(1), c(f), \dots, c(f^m)$$

are linearly dependent. This means that there exists a polynomial $P \in \mathbb{C}[T]$ of degree m such that $c(P(f)) = 0$. It follows that there is a holomorphic function $H \in \mathcal{O}_X(X)$ which is non-constant and such that $r(H) = P(f)$.

Now for the elements

$$c(f), c(r(H)f), \dots, c(r(H)^m f)$$

there exists a polynomial $P_1 \in \mathbb{C}[T]$ such that $c(P_1(r(H)f)) = 0$. It follows that there is a holomorphic function $F \in \mathcal{O}_X(X)$ such that $r(F) = P_1(r(H)f)$. Setting $G = P_1(H)$, we obtain $r(F) = r(G)f$.

Note that since $r(G^m H) = r(G^m P(F/G))$, it follows that $G^m H = G^m P(F/G)$. Hence on $X \setminus \{G = 0\}$ we obtain $H = P(F/G)$. It follows that

$$F/G \in \mathcal{O}_X(X \setminus \{G = 0\})$$

and locally bounded in $X \setminus \{G = 0\}$. Since $G \neq 0$, the Riemann extension theorem implies that $F/G \in \mathcal{O}_X(X)$. Furthermore, $r(F/G) = f$.

Therefore, the induced homomorphism $r: \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(\partial X)$ is an isomorphism.

Let $f \in \mathcal{O}_X(X \setminus K)$. Let $t_K: \mathcal{O}_X(X \setminus K) \rightarrow \mathcal{O}_X(\partial X)$ be a natural homomorphism. There exists a function $\hat{f} \in \mathcal{O}_X(X)$ such that $r(\hat{f}) = t_K(f)$.

$$(6.3) \quad \begin{array}{ccc} \mathcal{O}_X(X) & \xrightarrow{r_K} & \mathcal{O}_X(X \setminus K) \\ & \searrow r_{K'} & \swarrow r_{K/K'} \\ & \mathcal{O}_X(X \setminus K') & \\ & \downarrow t_{K'} & \downarrow t_K \\ & \mathcal{O}_X(\partial X) & \end{array}$$

Then there exists a compact set $K' \subset X$ such that $\widehat{f}|_{X \setminus K'} = f|_{X \setminus K'}$ (see diagramm 6.3). Recall that $X \setminus K$ is a connected set. Using the uniqueness theorem we obtain $\widehat{f}|_{X \setminus K} = f$. The proof of the lemma is complete. \square

In fact, applying the Andreotti-Hill technique to the exact sequence (6.1) as in Lemma 6.1 we obtain the following lemma.

Lemma 6.2. *Let X be a noncompact normal complex analytic variety. Let $K \subset X$ be a compact set such that $X \setminus K$ is connected. If $\dim H_K^1(X, \mathcal{O}_X) < \infty$, then the restriction homomorphism $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X \setminus K)$ is an isomorphism.*

Now we have the following cohomological criterion for the Hartogs phenomenon.

Theorem 6.1. *Let X be a noncompact normal complex analytic variety which is connected at boundary. Assume that X admits an open embedding (not necessary with dense image) $X \hookrightarrow X'$ into a topological space X' and there exists a sheaf of \mathbb{C} -vector spaces \mathcal{F} on X' with $\dim H^1(X', \mathcal{F}) < \infty$ and $\mathcal{F}|_X = \mathcal{O}_X$. Then X admits the Hartogs phenomenon if and only if $\dim H_c^1(X, \mathcal{O}_X) < \infty$.*

Proof. Assume that $\dim H_c^1(X, \mathcal{O}_X) < \infty$. Let $W \subset X$ be a domain and $K \subset W$ be a compact set such that $W \setminus K$ is connected. Note that since $X, W, W \setminus K$ are connected sets, it follows that $X \setminus K$ is connected set.

Further on, we have the following commutative diagram for $K \subset W \subset X \subset X'$ with exact rows.

$$\begin{array}{ccccccc} \mathcal{F}(X') & \xrightarrow{r_1} & \mathcal{F}(X' \setminus K) & \longrightarrow & H_K^1(X', \mathcal{F}) & \longrightarrow & H^1(X', \mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow h_1 & & \\ 0 & \longrightarrow & \mathcal{O}_X(X) & \xrightarrow{r_2} & \mathcal{O}_X(X \setminus K) & \longrightarrow & H_K^1(X, \mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow h_2 & & \\ 0 & \longrightarrow & \mathcal{O}_X(W) & \xrightarrow{r_3} & \mathcal{O}_X(W \setminus K) & \longrightarrow & H_K^1(W, \mathcal{O}_X) \end{array}$$

Since r_2 is an isomorphism and $X \setminus K = (X' \setminus K) \cap X$, then r_1 is surjective. Therefore, $\dim H_K^1(X', \mathcal{O}_{X'}) < \infty$. Using the excision property (see [3]) we obtain h_1 and h_2 are canonical isomorphisms. It follows that $\dim H_K^1(W, \mathcal{O}_X) < \infty$. By Lemma 6.2, r_3 is an isomorphism.

Now we assume that X admits the Hartogs phenomenon. We define

$$E^0(X') := \varinjlim_K H^0(X' \setminus K, \mathcal{F}),$$

and

$$E_c^1(X') := \varinjlim_K H_K^1(X', \mathcal{F})$$

where the inductive limit is taken over all compact subsets K of X (note that not every compact set of X' is a compact set in X). The excision property (see [3]) implies that

$$E_c^1(X') \cong H_c^1(X, \mathcal{O}_X).$$

Since for every compact set $K \subset X$ we have $X \setminus \mu(K)$ is a connected set, then the restriction homomorphism $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X \setminus \mu(K))$ is an isomorphism. It follows that the restriction homomorphism $\mathcal{F}(X') \rightarrow \mathcal{F}(X' \setminus \mu(K))$ is a surjective map.

We have the following exact sequence:

$$\mathcal{F}(X') \xrightarrow{r_1} E^0(X') \longrightarrow E_c^1(X') \longrightarrow H^1(X', \mathcal{F})$$

where r_1 is surjective. It follows that $\dim H_c^1(X, \mathcal{O}_X) \leq \dim H^1(X', \mathcal{F}) < \infty$. The proof of the theorem is complete. \square

In particular, if X as above in Theorem 6.1 and admits the Hartogs phenomenon, then $\dim H_c^1(X, \mathcal{O}_X) \leq \dim H^1(X', \mathcal{F})$ and we obtain the following corollary.

Corollary 6.1. *Let X be a noncompact normal complex analytic variety which is connected at boundary. Assume that X admits an open embedding (not necessary with dense image) $X \hookrightarrow X'$ into a topological space X' and there exists a sheaf of \mathbb{C} -vector spaces \mathcal{F} on X' with $H^1(X', \mathcal{F}) = 0$ and $\mathcal{F}|_X = \mathcal{O}_X$. If X admits the Hartogs phenomenon, then $H_c^1(X, \mathcal{O}_X) = 0$.*

Let us note that for the implication " \Leftarrow " in Theorem 6.1 we could use the one-point compactification and the extension by zero of the sheaf \mathcal{O}_X . Namely, we have the following

Corollary 6.2. *Let X be a noncompact normal complex analytic variety which is connected at boundary. If $\dim H_c^1(X, \mathcal{O}_X) < \infty$, then X admits the Hartogs phenomenon.*

Proof. Let X' be the one-point compactification of X and \mathcal{F} be the extension by zero of the sheaf \mathcal{O}_X . Since $H^1(X', \mathcal{F}) = H_c^1(X', \mathcal{F}) \cong H_c^1(X, \mathcal{O}_X)$, it follows that X admits the Hartogs phenomenon by Theorem 6.1. \square

For example, for $(1, \sigma)$ -compactifiable complex manifolds we obtain the following

Corollary 6.3. *Let X be a $(1, \sigma)$ -compactifiable complex analytic manifold. Then X admits the Hartogs phenomenon if and only if $\dim H_c^1(X, \mathcal{O}_X) < \infty$.*

Remark 6.1. *Let X be a Stein manifold with $\dim X > 1$. Since X is connected at boundary [11, Chapter VII, Section D, Theorem 2] and $H_c^1(X, \mathcal{O}_X) = 0$, it follows that X admits the Hartogs phenomenon.*

Let X, Y, F be complex manifolds with countable basis. We obtain the following main result.

Theorem 6.2. *Let $\phi: X \rightarrow Y$ be a holomorphic fibre bundle with $(1, 0)$ -compactifiable fiber F , $\dim F > 1$. If F admits the Hartogs phenomenon, then X also admits the Hartogs phenomenon.*

Proof. If F admits the Hartogs phenomenon, then $H_c^1(F, \mathcal{O}_F) = 0$ (by Corollary 6.1). By Corollary 5.1 we obtain $H_c^1(X, \mathcal{O}_X) = 0$. So, by Corollary 6.2 we obtain that X admits the Hartogs phenomenon.

Of course, instead Corollary 6.2 we could use the standard $\bar{\partial}$ -technique. \square

Example 6.1. *Let G be a semiabelian Lie group (i.e. G is an extension of an abelian manifold A by an algebraic torus $T \cong (\mathbb{C}^*)^n$). We have a principle T -bundle $G \rightarrow A$. Consider a toric T -manifold F (i.e. T algebraically acts on F with an open dense T -orbit which is isomorphic to T) which has only one topological end, and an associated fiber bundle $G \times^T F \rightarrow A$.*

Any toric T -manifold F admits a T -equivariant compactification F' and moreover $H^1(F', \mathcal{O}_{F'}) = 0$ (see, for instance, [22]). It follows that any toric T -manifold F which has only one topological end is $(1, 0)$ -compactifiable.

It follows that if the toric T -manifold F has only one topological end, $\dim F > 1$ and F admits the Hartogs phenomenon, then $G \times^T F$ admits the Hartogs phenomenon.

For example, $F = \mathcal{O}_{\mathbb{P}^1}(-2)$ has a structure of a toric $(\mathbb{C}^)^2$ -action and it has only one topological end. Note that $\mathcal{O}_{\mathbb{P}^1}(-2)$ is isomorphic to the blow-up $Bl_0(\text{Cone}(Q))$ of the normal 2-dimensional affine cone $\text{Cone}(Q) \subset \mathbb{C}^3$ at point $(0, 0, 0)$, where $Q = \{[z_0 : z_1 : z_2] \in \mathbb{C}\mathbb{P}^2 \mid z_1 z_3 - z_2^2 = 0\}$. It follows that F admits the Hartogs phenomenon, because blow-down map is a proper map and $\text{Cone}(Q)$ admits the Hartogs phenomenon (about Hartogs phenomenon in normal Stein spaces see, for instance, [21]).*

Note that each toric variety F is encoded by a fan Σ_F — a collection of strictly convex cones in a real vector space $\mathbb{R}^{\dim F}$ with the common apex that may intersect only along their common faces (see, for instance, [22]). In the fan language the $(1, 0)$ -compactifiability of F means that the support $|\Sigma_F|$ of Σ_F satisfies $\mathbb{R}^{\dim F} \setminus |\Sigma_F|$ is a connected set. Moreover, a toric manifold F with a fan Σ_F admits the Hartogs phenomenon if and only if the convex hull of $\mathbb{R}^{\dim F} \setminus |\Sigma_F|$ is whole $\mathbb{R}^{\dim F}$ provided $\mathbb{R}^{\dim F} \setminus |\Sigma_F|$ is connected. About the Hartogs phenomenon in toric varieties (and, more general, in spherical varieties) and examples see in the papers [15, 16].

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