LEFT-INVARIANT RICCI COLLINEATIONS ASSOCIATED TO CANONICAL CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

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ABSTRACT. In this paper, we classify Left-invariant Ricci collineations associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups.

1. INTRODUCTION

The concept of symmetry was firstly proposed by ancient Greek and since the earliest days of natural philosophy, symmetry has furnished insight into the laws of physics and the nature of the cosmos. The two outstanding theoretical achievements of the 20th century, relativity and quantum mechanics, involve notions of symmetry in a fundamental way, being been studied in depth because of their interest from both a mathematical and a physical viewpoint.For example, they simplify Einstein equations and provide a classification of the spacetimes according to the structure of the corresponding Lie algebra,which contains Ricci and curvature collineations ,among others(see [3],[6],[7],[8],[9]).

In this article we shall continue to concentrate on Ricci collineations which is, $L_\xi \widetilde{Ric}^i = 0$. In [10],Michael and Pantelis had done two purposes, the first of which was to present a useful method, which reduced the computation of the Ricci collineations and the Matter collineations of a given metric to the computation of Killing vectors ,the second of which was to apply this method and determine all hypersurface orthogonal locally rotationally symmetric spacetime metrics, which admit proper Matter collineations and proper Ricci collineations. In[11], The mathematical concept of left-invariant Ricci collineationsis proposed to denote that Ricci tensors has a value of zero with a left-invariant Ricci collineation ξ, having the mathematical meaning of a differential homomorphism of preserving Ricci tensors. The purpose of [11] is to determine all left-invariant Ricci collineations on three-dimensional Lie groups. Since the connection and Ricci tensor are built from the metric tensor, it must inherit its symmetries where any homothetic vector field is a Ricci collineation. In [12],Wang computed canonical connections and Kobayashi-Nomizu connections and their curvature on three-dimensional Lorentzian Lie groups with a special product structure.

In this paper, our motivation is to classify all left-invariant Ricci collineations associated to acanonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie group. More precisely,the concept of Ricci collineation was extended to canonical connection and Kobayashi-Nomizu connection on three-deimensional Lorentzian Lie group equiped with a product structure. We classify all left-invariant Ricci collineations on seven different three-dimensional Lie groups.

In section 2, we recall the definition of the canonical connection and the Kobayashi-Nomizu connection on three-dimensional Lorentzian Lie group with a product structure and their corresponding symmetric tensors and the definition of Ricci collineation with some remarks. In section 3, we classify all left-invariant Ricci collineations associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional unimodular Lorentzian Lie groups. In section 4, we classify all left-invariant Ricci collineations associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional non-unimodular Lorentzian Lie groups.

2. Preliminaries

2.1 The symmetric tensors associated to cannonical connection and Kobayashi-Nomizu connection

In this section, we recall the definition of the canonical connection and the Kobayashi-Nomizu connection on three-dimensional Lorentzian Lie group with a product structure and their corresponging symmetric tensors.

Let ∇ be the Levi-Civita connection of G_i and R its curvature tensor, taken with the convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Ricci tensor of (G_i, g) is defined by

$$\mathrm{Ric}(X,Y) = -g(R(X,e_1)Y, e_1) - g(R(X,e_2)Y, e_2) + g(R(X,e_3)Y, e_3),$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 timelike and the Ricci operator Ric is given by

$$\mathrm{Ric}(X,Y) = g(\mathrm{Ric}(X), Y).$$

A product structure J on G_i was defined by

$$Je_1 = e_1, \ \ Je_2 = e_2, \ \ Je_3 = -e_3,$$

then $J^2 = \mathrm{id}$ and $g(Je_j, Je_j) = g(e_j, e_j)$.

By [5], the canonical connection and the Kobayashi-Nomizu connection as follows:

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}(\nabla_X J)JY,$$

$$\nabla^1_X Y = \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY} J)X].$$

Then their curvature was given by

$$R^0(X,Y)Z = \nabla^0_X \nabla^0_Y Z - \nabla^0_Y \nabla^0_X Z - \nabla^0_{[X,Y]} Z,$$

$$R^1(X,Y)Z = \nabla^1_X \nabla^1_Y Z - \nabla^1_Y \nabla^1_X Z - \nabla^1_{[X,Y]} Z.$$

The Ricci tensors of (G_i, g) associated to the canonical connection and the Kobayashi-Nomizu connection are defined by

$$\mathrm{Ric}^0(X,Y) = -g(R^0(X,e_1)Y, e_1) - g(R^0(X,e_2)Y, e_2) + g(R^0(X,e_3)Y, e_3),$$

$$\mathrm{Ric}^1(X,Y) = -g(R^1(X,e_1)Y, e_1) - g(R^1(X,e_2)Y, e_2) + g(R^1(X,e_3)Y, e_3).$$

The Ricci operators Ric^0 and Ric^1 is given by

$$\mathrm{Ric}^0(X,Y) = g(\mathrm{Ric}^0(X), Y), \ \ \mathrm{Ric}^1(X,Y) = g(\mathrm{Ric}^1(X), Y).$$

Let

$$\widetilde{\mathrm{Ric}}^0(X,Y) = \frac{\mathrm{Ric}^0(X,Y) + \mathrm{Ric}^0(Y,X)}{2}, \ \ \widetilde{\mathrm{Ric}}^1(X,Y) = \frac{\mathrm{Ric}^1(X,Y) + \mathrm{Ric}^1(Y,X)}{2},$$

and

$$\widetilde{\mathrm{Ric}}^0(X,Y) = g(\widetilde{\mathrm{Ric}}^0(X), Y), \ \ \widetilde{\mathrm{Ric}}^1(X,Y) = g(\widetilde{\mathrm{Ric}}^1(X), Y).$$

All the analysis so far tells us that $\widetilde{\mathrm{Ric}}^0$ *and* $\widetilde{\mathrm{Ric}}^1$ are symmetric tensors.

Considering some different three-dimensional Lorentzian Lie groups, we have specific classification about three-dimensional Lorentzian Lie groups in [2, 4](see Theorem 2.1 and Theorem 2.2 in [1]). We shall denote the connected three-dimensional unimodular Lie groups equipped with a left-invariant Lorentzian metric g by $\{G_i\}_{i=1,\cdots,4}$ and having a corresponding Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,4}$.

2.2 Three-dimensional unimodular Lorentzian Lie groups

Theorem 2.1. *Assume (G, g, J) was a three-dimensional unimodular Lorentzian Lie group,equipped with a left-invariant Lorentzian metric g and a product structure J ,and had a Lie algebra $\boldsymbol{g}$ being one of the following:*

$\boldsymbol{g}_1$:
$$[e_1, e_2] = \alpha e_1 - \beta e_3, \ [e_1, e_3] = -\alpha e_1 - \beta e_2, \ [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \ \alpha \neq 0.$$

$\boldsymbol{g}_2$:
$$[e_1, e_2] = \gamma e_2 - \beta e_3, \ [e_1, e_3] = -\beta e_2 - \gamma e_3, \ [e_2, e_3] = \alpha e_2, \ \gamma \neq 0.$$

$\boldsymbol{g}_3$:
$$[e_1, e_2] = -\gamma e_3, \ [e_1, e_3] = e_1 - \beta e_2, \ [e_2, e_3] = \alpha e_1, \ \alpha \neq 0.$$

$\boldsymbol{g}_4$:
$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \eta = 1 \ or \ \eta = -1, \ [e_1, e_3] = -\beta e_2 + e_3, \ [e_2, e_3] = \alpha e_1, \ .$$

Next we shall denote the connected three-dimensional non-unimodular Lie groups equipped with a left-invariant Lorentzian metric g by $\{G_i\}_{i=5,6,7}$ and having a corresponding Lie algebra $\{\boldsymbol{g}\}_{i=5,6,7}$.

2.3 Three-dimensional non-unimodular Lorentzian Lie groups

Theorem 2.2. *Assume (G, g, J) was a three-dimensional non-unimodular Lorentzian Lie group,equipped with a left-invariant Lorentzian metric g and a product structure J ,and had a Lie algebra $\boldsymbol{g}$ being one of the following:*

$\boldsymbol{g}_5$:
$$[e_1, e_2] = 0, \ [e_1, e_3] = \alpha e_1 + \beta e_2, \ [e_2, e_3] = \gamma e_1 + \delta e_2, \ \alpha + \delta \neq 0 \ , \alpha\gamma + \beta\delta = 0.$$

$\boldsymbol{g}_6$:
$$[e_1, e_2] = \alpha e_2 + \beta e_3, \ [e_1, e_3] = \gamma e_2 + \delta e_3, \ [e_2, e_3] = 0, \ \alpha + \delta \neq 0 \ , \alpha\gamma - \beta\delta = 0.$$

$\boldsymbol{g}_7$:
$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \ [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \ [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \ \alpha + \delta \neq 0 \ , \alpha\gamma = 0.$$

Definition 2.3. *A unimodular Lorentzian Lie group of $\{G_i\}_{i=1,\cdots,4}$ admits left-Ricci collineations if and only if it satisfies:*

$$(2.1) \qquad\qquad\qquad\qquad \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^i = 0,$$

for $i = 1, 2$,where ξ is a element of three-dimensional unimodular Lie group and $\xi = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$,and $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis with e_3 timelike. Let

$$(2.2) \qquad\qquad \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^0 (X, Y) = \xi(\widetilde{\mathrm{Ric}}^0(X, Y) - \widetilde{\mathrm{Ric}}^1([\xi, X], Y) - \widetilde{\mathrm{Ric}}^0(X, [\xi, Y]),$$

and

$$(2.3) \qquad\qquad \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^1 (X, Y) = \xi(\widetilde{\mathrm{Ric}}^1(X, Y) - \widetilde{\mathrm{Ric}}^1([\xi, X], Y) - \widetilde{\mathrm{Ric}}^1(X, [\xi, Y]),$$

According to this definition,we have

$$\mathrm{L}_{e_1} \widetilde{\mathrm{Ric}}^i (e_1, e_1) = e_1(\widetilde{\mathrm{Ric}}^i(e_1, e_1) - \widetilde{\mathrm{Ric}}^i([e_1, e_1], e_1) - \widetilde{\mathrm{Ric}}^i(e_1, [e_1, e_1]) = 0,$$

$$\mathrm{L}_{e_2} \widetilde{\mathrm{Ric}}^i (e_2, e_2) = e_2(\widetilde{\mathrm{Ric}}^i(e_2, e_2) - \widetilde{\mathrm{Ric}}^i([e_2, e_2], e_2) - \widetilde{\mathrm{Ric}}^i(e_2, [e_2, e_2]) = 0,$$

$$\mathrm{L}_{e_3} \widetilde{\mathrm{Ric}}^i (e_3, e_3) = e_3(\widetilde{\mathrm{Ric}}^i(e_3, e_3) - \widetilde{\mathrm{Ric}}^i([e_3, e_3], e_3) - \widetilde{\mathrm{Ric}}^i(e_3, [e_3, e_3]) = 0,$$

in the following discussion,we will not mention the calculation of these six items $(i = 1, 2)$.owing to $\widetilde{\mathrm{Ric}}^0$ and $\widetilde{\mathrm{Ric}}^1$ are symmetric tensors,we have

$$\mathrm{L}_\xi \widetilde{\mathrm{Ric}}^i (e_1, e_2) = \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^i (e_2, e_1), \ \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^i (e_1, e_3) = \mathrm{L}_\xi \widetilde{\mathrm{Ric}}^i (e_3, e_1),$$

Remark 2.4. $\mathbb{V}_{\mathbb{RC}}$ *is spanned by Ricci collineations.*

3. Left-invariant Ricci collineations associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional unimodular Lorentzian Lie groups

3.1 Left-invariant Ricci collineation of G_1

In [1], we have for G_1, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_1 satisfies

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \ [e_1, e_3] = -\alpha e_1 - \beta e_2, \ [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \ \alpha \neq 0.$$

By (2.25) in [12], we have

Lemma 3.1. *Ricci symmetric tensors of (G_1, g, J) associated to canonical connection are given by*

(3.1)
$$\widetilde{\text{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\left(\alpha^2 + \frac{\beta^2}{2}\right) & 0 & -\frac{\alpha\beta}{4} \\ 0 & -\left(\alpha^2 + \frac{\beta^2}{2}\right) & -\frac{\alpha^2}{2} \\ \frac{\alpha\beta}{4} & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (2.2) and (3.1), we have

Lemma 3.2. *For G_1, the following equalities hold*

(3.2)
$$\mathrm{L}_{e_1}\widetilde{\text{Ric}}^0(e_1, e_2) = \alpha(\alpha^2 + \frac{3}{4}\beta^2), \ \mathrm{L}_{e_1}\widetilde{\text{Ric}}^0(e_1, e_3) = -\alpha(\alpha^2 + \frac{1}{2}\beta^2), \ \mathrm{L}_{e_1}\widetilde{\text{Ric}}^0(e_2, e_2) = \alpha^2\beta,$$

$$\mathrm{L}_{e_1}\widetilde{\text{Ric}}^0(e_2, e_3) = -\beta(\frac{5}{4}\alpha^2 + \frac{1}{2}\beta^2), \ \mathrm{L}_{e_1}\widetilde{\text{Ric}}^0(e_3, e_3) = \frac{3}{2}\alpha^2\beta, \ \mathrm{L}_{e_2}\widetilde{\text{Ric}}^0(e_1, e_1) = -2\alpha(\alpha^2 + \frac{3}{4}\beta^2)$$

$$\mathrm{L}_{e_2}\widetilde{\text{Ric}}^0(e_1, e_2) = -\frac{1}{2}\alpha^2\beta, \ \mathrm{L}_{e_2}\widetilde{\text{Ric}}^0(e_1, e_3) = \beta(\alpha^2 + \frac{1}{2}\beta^2), \mathrm{L}_{e_2}\widetilde{\text{Ric}}^0(e_2, e_3) = \frac{1}{2}\alpha(\alpha^2 + \beta^2),$$

$$\mathrm{L}_{e_2}\widetilde{\text{Ric}}^0(e_3, e_3) = -\alpha(\alpha^2 + \frac{1}{2}\beta^2), \mathrm{L}_{e_3}\widetilde{\text{Ric}}^0(e_1, e_1) = 2\alpha(\alpha^2 + \frac{1}{2}\beta^2), \ \mathrm{L}_{e_3}\widetilde{\text{Ric}}^0(e_1, e_2) = \frac{1}{4}\alpha^2\beta$$

$$\mathrm{L}_{e_3}\widetilde{\text{Ric}}^0(e_1, e_3) = -\frac{3}{4}\alpha^2\beta, \mathrm{L}_{e_3}\widetilde{\text{Ric}}^0(e_2, e_2) = -\alpha(\alpha^2 + \beta^2), \mathrm{L}_{e_3}\widetilde{\text{Ric}}^0(e_2, e_3) = \frac{1}{2}\alpha(\alpha^2 + \frac{1}{2}\beta^2).$$

By (3.2) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_1, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi\widetilde{\text{Ric}}^0_{ij} = 0, for \ i = 1, 2, 3 \ and \ j = 1, 2, 3$, so

(3.3)
$$\begin{cases} -\alpha(\alpha^2 + \frac{3}{4}\beta^2)\lambda_2 + \alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_3 = 0, \\[2mm] \alpha(\alpha^2 + \frac{3}{4}\beta^2)\lambda_1 - \frac{1}{2}\alpha^2\beta\lambda_2 + \frac{1}{4}\alpha^2\beta\lambda_3 = 0, \\[2mm] -\alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 + \beta(\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 - \frac{3}{4}\alpha^2\beta\lambda_3 = 0, \\[2mm] \alpha^2\beta\lambda_1 - \alpha(\alpha^2 + \beta^2)\lambda_3 = 0, \\[2mm] -\beta(\frac{5}{4}\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 + \frac{1}{2}\alpha(\alpha^2 + \beta^2)\lambda_2 + \frac{1}{2}\alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_3 = 0, \\[2mm] \frac{3}{2}\alpha^2\beta\lambda_1 - \alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 = 0. \end{cases}$$

By solving (3.3), we get

Theorem 3.3. *the unimodular Lorentzian Lie group (G_1, g, J) does not admit left-invariant Ricci collineations associated to the canonical connection ∇^0 .*

Proof 3.4. *We analyze each one of these factors by separate. Because $\alpha \neq 0$, then we have*

(3.4)
$$
\begin{cases}
-(\alpha^2 + \frac{3}{4}\beta^2)\lambda_2 + (\alpha^2 + \frac{1}{2}\beta^2)\lambda_3 = 0, \\[2mm]
(\alpha^2 + \frac{3}{4}\beta^2)\lambda_1 - \frac{1}{2}\alpha\beta\lambda_2 + \frac{1}{4}\alpha\beta\lambda_2 = 0, \\[2mm]
-\alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 + \beta(\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 - \frac{3}{4}\alpha^2\beta\lambda_3 = 0, \\[2mm]
\alpha\beta\lambda_1 - (\alpha^2 + \beta^2)\lambda_3 = 0, \\[2mm]
-\beta(\frac{5}{4}\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 + \frac{1}{2}\alpha(\alpha^2 + \beta^2)\lambda_2 - \frac{1}{2}\alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_3 = 0, \\[2mm]
\frac{3}{2}\alpha\beta\lambda_1 - (\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 = 0.
\end{cases}
$$

Now if $\beta = 0$, we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$. If $\beta \neq 0$, naturelly we have

$$
\lambda_3 = \frac{\alpha\beta}{\alpha^2 + \beta^2}\lambda_1, \lambda_2 = \frac{3\alpha\beta}{2\alpha^2 + \beta^2}\lambda_1,
$$

substitute these into Eq.(3.4) to get

$$
[(\alpha^2 + \frac{1}{2}\beta^2)(2\alpha^2 + \beta^2) - (\alpha^2 + \frac{3}{4}\beta^2)(\alpha^2 + \beta^2)]\lambda_1 = 0,
$$

$$
[(4\alpha^2 + 3\beta^2)(2\alpha^2 + \beta^2)(\alpha^2 + \beta^2) - \alpha^2\beta^2(4\alpha^2 + 5\beta^2)]\,\lambda_1 = 0
$$

Solving these, we have $\lambda_1 = \lambda_2 = \lambda_3 = 0, \alpha\beta \neq 0$.
In summary, the unimodular Lorentzian Lie group (G_1, g, J) does not admit left-invariant Ricci collineations associated to the canonical connection ∇^0.

By (2.23) in [12], we have

Lemma 3.5. *Ricci symmetric tensors of (G_1, g, J) associated to Kobayasha-Nomizu connection are given by*

(3.5)
$$
\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta^2) & \alpha\beta & \frac{\alpha\beta}{2} \\ \alpha\beta & -(\alpha^2 + \beta^2) & -\frac{\alpha^2}{2} \\ -\frac{\alpha\beta}{2} & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
$$

By (2.3) and (3.5), we have

Lemma 3.6. *For G_1, the following equalities hold*

(3.6)
$$
\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \alpha(\alpha^2 + \frac{1}{2}\beta^2), \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = -\alpha^3, \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = -\alpha^2\beta,
$$

$$
\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \beta(\frac{1}{2}\alpha^2 - \beta^2), \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = -2\alpha(\alpha^2 + \frac{1}{2}\beta^2)
$$

$$
\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \frac{1}{2}\alpha^2\beta, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \beta^3, \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \frac{1}{2}\alpha^3,
$$

$$
\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = \alpha(\beta^2 - \alpha^2), \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 2\alpha^3, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = -\frac{1}{2}\alpha^2\beta
$$

$$
\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = -\alpha^3, \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \frac{1}{2}\alpha(\alpha^2 - \beta^2).
$$

By (3.6) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_1, g, J) admits left-invariant Ricci collineations associated to the Kobayasha-Nomizu connection ∇^1, then $L_\xi \widetilde{\text{Ric}}_{ij}^{1} = 0, for\ i = 1, 2, 3\ and\ j = 1, 2, 3$, so

(3.7)
$$\begin{cases} -2\alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 + 2\alpha^3\lambda_3 = 0, \\ \alpha(\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 + \frac{1}{2}\alpha^2\beta\lambda_2 - \frac{1}{2}\alpha^2\beta\lambda_3 = 0, \\ -\alpha^3\lambda_1 + \beta^3\lambda_2 = 0, \\ \alpha^2\beta\lambda_1 + \alpha^3\lambda_3 = 0, \\ \beta(\frac{1}{2}\alpha^2 - \beta^2)\lambda_1 + \frac{1}{2}\alpha^3\lambda_2 + \frac{1}{2}\alpha(\alpha^2 - \beta^2)\lambda_3 = 0, \\ \alpha(\beta^2 - \alpha^2)\lambda_2 = 0. \end{cases}$$

By solving (3.7), we get

Theorem 3.7. *the unimodular Lorentzian Lie group (G_1, g, J) does not admit left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 .*

Proof 3.8. *We analyze each one of these factors by separate. Because $\alpha \neq 0$, then we have*

(3.8)
$$\begin{cases} -(\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 + \alpha^2\lambda_3 = 0, \\ (\alpha^2 + \frac{1}{2}\beta^2)\lambda_1 - \frac{1}{2}\alpha\beta\lambda_2 - \frac{1}{2}\alpha\beta\lambda_2 = 0, \\ -\alpha^3\lambda_1 + \beta^3\lambda_2 = 0, \\ \beta\lambda_1 + \alpha\lambda_3 = 0, \\ \beta(\frac{1}{2}\alpha^2 - \beta^2)\lambda_1 + \frac{1}{2}\alpha^3\lambda_2 + \frac{1}{2}\alpha(\alpha^2 - \beta^2)\lambda_3 = 0, \\ (\beta^2 - \alpha^2)\lambda_2 = 0. \end{cases}$$

Now if $\beta^2 - \alpha^2 = 0$, we have $\alpha = \beta$ or $\alpha + \beta = 0$, .If $\alpha = \beta$,naturelly we have $\lambda_1 = \lambda_2 = -\lambda_3$,substitute these into Eq.(3.8), we get

$$-(2\alpha^2 + \frac{1}{2}\beta^2)\lambda_2 = 0,$$

Solving these, we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
If $\beta^2 - \alpha^2 \neq 0$,, then we have $\lambda_2 = 0$. Similarly we put this into Eq.(3.8) ,we have $\lambda_2 = \lambda_3 = 0$.
In summary,the unimodular Lorentzian Lie group (G_1, g, J) does not admit left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 .

3.2 Left-invariant Ricci collineation of G_2

In [1], we have for G_2, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_2 satisfies

$$[e_1, e_2] = \gamma e_2 - \beta e_3,\ [e_1, e_3] = -\beta e_2 - \gamma e_3,\ [e_2, e_3] = \alpha e_1,\ \gamma \neq 0.$$

By (2.44) in [12], we have

Lemma 3.9. *Ricci symmetric tensors of (G_2, g, J) associated to canonical connection are given by*

$$(3.9) \qquad \widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\left(\gamma^2 + \frac{\alpha\beta}{2}\right) & 0 & 0 \\ 0 & -\left(\gamma^2 + \frac{\alpha\beta}{2}\right) & \frac{\alpha\gamma}{4} - \frac{\beta\gamma}{2} \\ 0 & \frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (2.2) and (3.9), we have

Lemma 3.10. *For G_2, the following equalities hold*

$$(3.10) \qquad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = \gamma(2\gamma^2 + \beta^2 + \frac{1}{2}\alpha\beta),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = -\beta(\gamma^2 + \frac{1}{2}\alpha\beta) \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = \beta(\beta\gamma - \frac{1}{2}\alpha\gamma), \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = -\gamma(\gamma^2 + \frac{1}{2}\beta^2 + \frac{1}{4}\alpha\beta), \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\gamma^2(\beta + \frac{3}{2}\alpha) + \frac{1}{2}\alpha^2\beta \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = \frac{1}{2}\gamma^2(\beta - \frac{3}{2}\alpha) + \frac{1}{2}\alpha\beta(\beta - \alpha)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\beta\gamma(\frac{1}{2}\alpha - \beta), \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0.$$

By (3.10) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_2, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^0_{ij} = 0, for \ i = 1, 2, 3 \ and \ j = 1, 2, 3$, so

$$(3.11) \qquad \begin{cases} \gamma(\gamma^2 + \frac{1}{2}\beta^2 + \frac{1}{4}\alpha\beta)\lambda_2 - \left[\frac{1}{2}\gamma^2(\beta - \frac{3}{2}\alpha) + \frac{1}{2}\alpha\beta(\beta - \alpha)\right]\lambda_3 = 0, \\[2mm] \left[\frac{1}{2}\gamma^2(\beta + \frac{3}{2}\alpha) + \frac{1}{2}\alpha^2\beta\right]\lambda_2 + \frac{1}{2}\beta\gamma(\frac{1}{2}\alpha - \beta)\lambda_3 = 0, \\[2mm] \gamma(2\gamma^2 + \beta^2 + \frac{1}{2}\alpha\beta)\lambda_1 = 0, \\[2mm] \beta(\gamma^2 + \frac{1}{2}\alpha\beta)\lambda_1 = 0, \\[2mm] \beta(\beta\gamma - \frac{1}{2}\alpha\gamma)\lambda_1 = 0. \end{cases}$$

By solving (3.11), we get

Theorem 3.11. *the unimodular Lorentzian Lie group (G_2, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if*
$(1)\alpha = \beta = 0, \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

$(2)\alpha \neq 0, \beta \neq 0, \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{C}{D}e_3 \rangle.$

Proof 3.12. *We analyze each one of these factors by separate. Because $\gamma \neq 0$, then we assume that*

$$(3.12) \qquad \begin{cases} A = \gamma(2\gamma^2 + \beta^2 + \frac{1}{2}\alpha\beta), \\[2mm] B = \gamma^2(\frac{3}{2}\alpha - \beta) + \alpha\beta(\alpha - \beta), \\[2mm] C = \gamma^2(\beta + \frac{3}{2}\alpha) + \alpha^2\beta, \\[2mm] D = \beta\gamma(\frac{1}{2}\alpha - \beta). \end{cases}$$

We get

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 4\alpha^4\beta^2 - 4\alpha^3\beta^3 + 12\alpha^3\beta\gamma^2 - 7\alpha^2\beta^2\gamma^2 + 9\alpha^2\beta^4 - 4\alpha\beta^3\gamma^2 + 4\alpha\beta\gamma^4 + 4\beta^4\gamma^2 + 4\beta^2\gamma^4$$

If $\alpha = 0$, we have $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \beta^2\gamma^2(\beta^2 + \gamma^2).$

Note that if beta $\neq 0$,$\alpha = 0$, we have $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0$,*then* $\lambda_1 = \lambda_2 = \lambda_3 = 0.$

Note that if beta $= 0$, then $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0$, *and we have* $\lambda_1 = \lambda_2 = 0.$

If $\alpha \neq 0, \beta = 0$, we have

(3.13)
$$\begin{cases} \gamma^3\lambda_2 + \frac{3}{4}\alpha\gamma^2\lambda_3 = 0, \\ \alpha\gamma^2\lambda_2 = 0, \\ \gamma^3\lambda_1 = 0. \end{cases}$$

Then we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Notice that when $\alpha \neq 0, \beta \neq 0$, we have $\lambda_1 = 0$, whenever $\alpha = 2\beta$ or $\alpha \neq 2\beta$. If $\alpha \neq 0, \beta \neq 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0$,then $\lambda_2 = \lambda_3 = 0$.

And if $\alpha \neq 0, \beta \neq 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0$,then $\lambda_3 = -\frac{C}{D}\lambda_2$.

In summary,the unimodular Lorentzian Lie group (G_2, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if
(1)$\alpha = \beta = 0, \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$,
(2)$\alpha \neq 0, \beta \neq 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{C}{D}e_3 \rangle$.

By (2.54) in [12], we have

Lemma 3.13. *Ricci symmetric tensors of (G_2, g, J) associated to Kobayasha-Nomizu connection are given by*

(3.14)
$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\beta^2 + \gamma^2) & 0 & 0 \\ 0 & -(\gamma^2 + \alpha\beta) & \frac{\alpha\gamma}{2} \\ 0 & -\frac{\alpha\gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (2.3) and (3.14),we have

Lemma 3.14. *For G_2, the following equalities hold*

(3.15)
$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 2\gamma(\gamma^2 + \frac{1}{2}\alpha\beta),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = -\beta(\gamma^2 + \alpha\beta) \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = -\alpha\beta\gamma, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = -\gamma(\gamma^2 + \frac{1}{2}\alpha\beta), \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \alpha(\beta^2 + \frac{1}{2}\gamma^2) \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \gamma^2(\beta - \frac{1}{2}\alpha)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \frac{1}{2}\alpha\beta\gamma, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0.$$

By (3.15) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_2, g, J) is left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1, then $L_\xi \widetilde{\mathrm{Ric}}_{ij}^1 = 0, i = 1, 2, 3$ *and* $j = 1, 2, 3$, so

(3.16)
$$\begin{cases} \gamma(\gamma^2 + \frac{1}{2}\alpha\beta)\lambda_2 - \gamma^2(\beta - \frac{1}{2}\alpha)\lambda_3 = 0, \\[2mm] \alpha(\beta^2 + \frac{1}{2}\gamma^2)\lambda_2 + \frac{1}{2}\alpha\beta\gamma\lambda_3 = 0, \\[2mm] \gamma(\gamma^2 + \frac{1}{2}\alpha\beta)\lambda_1 = 0, \\[2mm] \beta(\gamma^2 + \alpha\beta)\lambda_1 = 0, \\[2mm] \alpha\beta\gamma\lambda_1 = 0. \end{cases}$$

By solving (3.16), we get

Theorem 3.15. *the unimodular Lorentzian Lie group (G_2, g, J) is left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 if and only if*
$(1) \alpha = \beta = \lambda_2 = \lambda_1 = 0, \mathbb{V}_{\mathbb{RC}} = < e_3 >,$

$(2) \alpha = \lambda_1 = 0, \beta \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\gamma}{\beta} e_3 \rangle,$

$(3) \lambda_1 = 0, \alpha \neq 0, \beta \neq 0, \lambda_3 = -\frac{2\beta^2 + \gamma^2}{\beta\gamma}\lambda_2, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{2\beta^2 + \gamma^2}{\beta\gamma} e_3 \rangle.$

Proof 3.16. *We analyze each one of these factors by separate. Because $\gamma \neq 0$, then we have*

(3.17)
$$\begin{cases} (\gamma^2 + \frac{1}{2}\alpha\beta)\lambda_2 - \gamma(\beta - \frac{1}{2}\alpha)\lambda_3 = 0, \\[2mm] \alpha(\beta^2 + \frac{1}{2}\gamma^2)\lambda_2 + \frac{1}{2}\alpha\beta\gamma\lambda_3 = 0, \\[2mm] (\gamma^2 + \frac{1}{2}\alpha\beta)\lambda_1 = 0, \\[2mm] \beta(\gamma^2 + \alpha\beta)\lambda_1 = 0, \\[2mm] \beta(\frac{1}{2}\alpha^2 - \beta^2)\lambda_1 + \frac{1}{2}\alpha^3\lambda_2 + \frac{1}{2}\alpha(\alpha^2 - \beta^2)\lambda_3 = 0, \\[2mm] \alpha\beta\lambda_1 = 0. \end{cases}$$

Notice that compare the third equation with the third fifth equation, λ_1 must be equal to 0.
If $\alpha = 0$, we get $\gamma\lambda_2 - \beta\lambda_3 = 0$. At the same time, if $\beta = 0$, then $\lambda_2 = 0$. And if $\beta \neq 0$, then $\lambda_3 = \frac{\gamma}{\beta}\lambda_2$.
If $\alpha \neq 0$, we get

(3.18)
$$\begin{cases} (2\gamma^2 + \alpha\beta)\lambda_2 + \gamma(\alpha - 2\beta)\lambda_3 = 0, \\[2mm] (2\beta^2 + \gamma^2)\lambda_2 + \beta\gamma\lambda_3 = 0. \end{cases}$$

Now if $\beta = 0$, then simply we have $\lambda_2 = \lambda_3 = 0$.
we assume that

(3.19)
$$\begin{cases} A = 2\gamma^2 + \alpha\beta, \\[2mm] B = \gamma(\alpha - 2\beta), \\[2mm] C = 2\beta^2 + \gamma^2, \\[2mm] D = \beta\gamma. \end{cases}$$

We get

$$\left| \begin{array}{cc} A & B \\ C & D \end{array} \right| = (\alpha - 4\beta)(\beta^2 + \gamma^2)$$

Now if $(\alpha - 4\beta)(\beta^2 + \gamma^2) \neq 0$, equal to $\alpha \neq 4\beta$, we have $\lambda_2 = \lambda_3 = 0$. If $(\alpha - 4\beta)(\beta^2 + \gamma^2) = 0$, equal to $\alpha = 4\beta$, we have $\lambda_3 = -\frac{2\beta^2 + \gamma^2}{\beta\gamma}\lambda_2$.
In summary, the unimodular Lorentzian Lie group (G_2, g, J) is left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 if and only if
$(1)\alpha = \beta = \lambda_2 = \lambda_1 = 0, \mathbb{V}_{\mathbb{RC}} = <e_3>$,

$(2)\alpha = \lambda_1 = 0, \beta \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\gamma}{\beta}e_3\rangle$,

$(3)\lambda_1 = 0, \alpha \neq 0, \beta \neq 0, \lambda_3 = -\frac{2\beta^2 + \gamma^2}{\beta\gamma}\lambda_2, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{2\beta^2 + \gamma^2}{\beta\gamma}e_3\rangle$.

3.3 Left-invariant Ricci collineation of G_3

By [1], we have for G_3, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_3 satisfies

$$[e_1, e_2] = \gamma e_3, \ [e_1, e_3] = -\beta e_2, \ [e_2, e_3] = \alpha e_1, \ .$$

By (2.64) in [12], we have

Lemma 3.17. *Ricci symmetric tensors of (G_3, g, J) associated to canonical connection are given by*

(3.20)
$$\widetilde{\mathrm{Ric}}^0 \left(\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right) = \left(\begin{array}{ccc} -\gamma a_3 & 0 & 0 \\ 0 & -\gamma a_3 & 0 \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right).$$

where

$$a_1 = \frac{1}{2}(\alpha - \beta - \gamma), a_2 = \frac{1}{2}(\alpha - \beta + \gamma), a_3 = \frac{1}{2}(\alpha + \beta - \gamma)$$

By (2.2) and (3.20), we have

Lemma 3.18. *For G_2, the following equalities hold*

(3.21)
$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 0,$$
$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = -\beta\gamma a_3 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0$$
$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \alpha\gamma a_3 \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0,$$
$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = (\beta - \alpha)\gamma a_3$$
$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0.$$

By (3.21) and definition 2.3, we have

   If the unimodular Lorentzian Lie group (G_3, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $L_\xi \widetilde{\mathrm{Ric}}^0_{ij} = 0, i = 1, 2, 3$ *and* $j = 1, 2, 3$, so

(3.22)
$$\begin{cases} (\beta - \alpha)\gamma a_3 \lambda_3 = 0, \\\\ \alpha\gamma a_3 \lambda_2 = 0, \\\\ \beta\gamma a_3 \lambda_1 = 0, \end{cases}$$

By solving (3.22), we get

Theorem 3.19. *the unimodular Lorentzian Lie group (G_3, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if*
(1)$\gamma = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(2)$\gamma \neq 0, \alpha = \beta = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(3)$\alpha = 0, \gamma \neq 0, \beta \neq 0, \beta = \gamma, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(4)$\alpha = 0, \gamma \neq 0, \beta \neq 0, \beta \neq \gamma, \mathbb{V}_{\mathbb{RC}} = \langle e_2 \rangle$,

(5)$\gamma \neq 0, \alpha \neq 0, \gamma = \alpha + \beta, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(6)$\gamma \neq 0, \alpha \neq 0, \gamma \neq \alpha + \beta, \alpha = \beta, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$.

Proof 3.20. *We analyze each one of these factors by separate. Because $a_3 = \frac{1}{2}(\alpha + \beta - \gamma)$, then we have*

(3.23)
$$\begin{cases} \gamma(\beta - \alpha)(\alpha + \beta - \gamma)\lambda_3 = 0, \\\\ \alpha\gamma\lambda_2(\alpha + \beta - \gamma) = 0, \\\\ \beta\gamma\lambda_1(\alpha + \beta - \gamma) = 0, \end{cases}$$

If $\gamma = 0$, we get $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$.
If $\gamma \neq 0$, we get

(3.24)
$$\begin{cases} (\beta - \alpha)\lambda_3(\alpha + \beta - \gamma) = 0, \\\\ \alpha\lambda_2(\alpha + \beta - \gamma) = 0, \\\\ \beta\lambda_1(\alpha + \beta - \gamma) = 0. \end{cases}$$

Now if $\alpha = 0, \beta = 0$,then simply we have $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$.
If $\alpha = 0, \beta \neq 0$, we have

(3.25)
$$\begin{cases} (\beta - \gamma)\lambda_3 = 0, \\\\ (\beta - \gamma)\lambda_1 = 0. \end{cases}$$

then if $\beta - \gamma = 0$, simply having $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$ and if $\beta - \gamma \neq 0$, simply having $\lambda_2 = \lambda_3 = 0$.
Now if $\alpha \neq 0$,, we have

(3.26)
$$\begin{cases} (\beta - \alpha)(\alpha + \beta - \gamma)\lambda_3 = 0, \\\\ (\alpha + \beta - \gamma)\lambda_2 = 0, \\\\ \beta(\alpha + \beta - \gamma)\lambda_1 = 0. \end{cases}$$

In this case, if $\gamma = \alpha + \beta$, we have $\lambda_1 \in R$, $\lambda_2 \in R$, $\lambda_3 \in R$ and if $\gamma \neq \alpha + \beta$, we have $\lambda_2 = 0$. Furthermore if $\beta \neq 0, \alpha \neq \beta$, we get $\lambda_1 = \lambda_3 = 0$, and if $\beta \neq 0, \alpha = \beta$, we get $\lambda_1 = 0$ where if $\beta = 0$, then we have $\lambda_3 = 0$.

In summary,the unimodular Lorentzian Lie group (G_3, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if

(1)$\gamma = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(2)$\gamma \neq 0, \alpha = \beta = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(3)$\alpha = 0, \gamma \neq 0, \beta \neq 0, \beta = \gamma, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(4)$\alpha = 0, \gamma \neq 0, \beta \neq 0, \beta \neq \gamma, \mathbb{V}_{\mathbb{RC}} = \langle e_2 \rangle$,

(5)$\gamma \neq 0, \alpha \neq 0, \gamma = \alpha + \beta, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(6)$\gamma \neq 0, \alpha \neq 0, \gamma \neq \alpha + \beta, \alpha = \beta, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$.

By (2.69) in [12], we have

Lemma 3.21. *Ricci symmetric tensors of (G_3, g, J) associated to Kobayashi-Nomizu connection are given by*

(3.27)
$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \gamma(a_1 - a_3) & 0 & 0 \\ 0 & -\gamma(a_2 + a_3) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (2.3) and (3.27), we have

Lemma 3.22. *For G_3, the following equalities hold*

(3.28)
$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 0,$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = -\beta\gamma(a_2 + a_3) \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \alpha\gamma(a_3 - a_1) \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \beta\gamma(a_2 + a_3) + \alpha\gamma(a_1 - a_3)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0.$$

By (3.28) and definition 2.3, we get

If the unimodular Lorentzian Lie group (G_3, g, J) admits left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^1_{ij} = 0, i = 1, 2, 3$ *and* $j = 1, 2, 3$, so

(3.29)
$$\begin{cases} [\beta\gamma(a_2 + a_3) + \alpha\gamma(a_1 - a_3)]\lambda_3 = 0, \\ \\ \alpha\gamma(a_3 - a_1)\lambda_2 = 0, \\ \\ \beta\gamma(a_2 + a_3)\lambda_1 = 0. \end{cases}$$

By solving (3.29), we get

Theorem 3.23. *the unimodular Lorentzian Lie group (G_3, g, J) is left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 if and only if (easy to prove)*

(1)$\alpha\beta\gamma = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(2)$\alpha\beta\gamma \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$.

3.4 Left-invariant Ricci collineation of G_4

By [1], we have for G_4, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_4 satisfies

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \eta = 1 \ or \ \eta = -1, \ [e_1, e_3] = -\beta e_2 + e_3, \ [e_2, e_3] = \alpha e_1 \ .$$

By (2.81) in [12], we have

Lemma 3.24. *Ricci symmetric tensors of* (G_4, g, J) *associated to canonical connection are given by*

$$(3.30) \qquad \widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} [(2\eta - \beta)b_3 - 1] & 0 & 0 \\ 0 & [(2\eta - \beta)b_3 - 1] & \frac{\beta - b_3}{2} \\ 0 & \frac{b_3 - \beta}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

where

$$b_1 = \frac{\alpha}{2} + \eta - \beta, b_2 = \frac{\alpha}{2} - \eta, b_3 = \frac{\alpha}{2} + \eta$$

By (3.30) and (2.3)

Lemma 3.25. *For* G_4, *the following equalities hold*

$$(3.31) \quad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = (2\eta - \beta)(b_3 + \beta) - 2,$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = \beta[(2\eta - \beta)b_3 - 1] \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = \beta(b_3 - \beta), \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = (b_3 + \beta)(\frac{1}{2}\beta - \eta) + 1, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \alpha[(\beta - 2\eta)b_3 + 1] + \frac{1}{2}(\beta - b_3) \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = (\alpha - \beta)[(2\eta - \beta)b_3 - 1] + \frac{1}{2}(b_3 - \beta) \ ,$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\beta(\beta - b_3), \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0.$$

By (3.31) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_4, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^0_{ij} = 0, for \ i = 1, 2, 3 \ and \ j = 1, 2, 3$, so

$$(3.32) \quad \begin{cases} \left[(b_3 + \beta)(\frac{1}{2}\beta - \eta) + 1\right]\lambda_2 + \{(\alpha - \beta)\left[(2\eta - \beta)b_3 - 1\right] + \frac{1}{2}(b_3 - \beta)\}\lambda_3 = 0, \\[2mm] \{\alpha[(\beta - 2\eta)b_3 + 1] + \frac{1}{2}(\beta - b_3)\}\lambda_2 + \frac{1}{2}\beta(\beta - b_3)\lambda_3 = 0, \\[2mm] \left[(2\eta - \beta)(b_3 + \beta) - 2\right]\lambda_1 = 0, \\[2mm] \beta\left[(2\eta - \beta)b_3 - 1\right]\lambda_1 = 0, \\[2mm] \beta(b_3 - \beta)\lambda_1 = 0. \end{cases}$$

By solving (3.32), we get

Theorem 3.26. *the unimodular Lorentzian Lie group* (G_4, g, J) *admits left-invariant Ricci collineations associated to the canonical connection* ∇^0 *if and only if*
$(1)\beta = 0, \eta = 1, \alpha = 0, \lambda_2 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

$(2)\beta = 0, \eta = 1, (\alpha + 2)(\alpha + \frac{1}{4}) = 0, \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

$(3)\beta = 0, \eta = -1, \alpha = 0, \lambda_2 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

$(4)\beta = 0, \eta = -1, (\alpha - 2)(\alpha - \frac{1}{4}) = 0, \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

$(5)\beta \neq 0, \eta = 1, \alpha = 0, \beta = 1, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle,$

$(6)\beta \neq 0, \eta = 1, \alpha = 2, \beta = 2, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

$(7)\beta \neq 0, \eta = 1, \alpha + 2 \neq 2\beta, \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{C}{D}e_3 \rangle,$

$(8)\beta \neq 0, \eta = -1, \alpha = 0, \beta = -1, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle,$

$(9)\beta \neq 0, \eta = -1, \alpha = -2, \beta = -2, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

$(10)\beta \neq 0, \eta = -1, \alpha - 2 \neq 2\beta, \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{C}{D}e_3 \rangle.$

Proof 3.27. *We analyze each one of these factors by separate. Because $b_3 = \frac{1}{2}\alpha + \beta$, if $\beta = 0$, then we have*

$$(3.33) \qquad \begin{cases} (2\eta b_3 - 2)\lambda_1 = 0, \\ (-b_3\eta + 1)\lambda_2 + (2\alpha\eta b_3 + \frac{1}{2}b_3)\lambda_3 = 0, \\ (-2\alpha\eta b_3 + \alpha - \frac{1}{2}b_3)\lambda_2 = 0. \end{cases}$$

Now if $\eta = 1$, we have

$$(3.34) \qquad \begin{cases} \alpha\lambda_1 = 0, \\ -\frac{\alpha}{2}\lambda_2 + (\alpha^2 + \frac{9}{4}\alpha + \frac{1}{2})\lambda_3 = 0, \\ (\alpha^2 + \frac{5}{4}\alpha + \frac{1}{2})\lambda_2 = 0. \end{cases}$$

Furthermore if $\alpha = 0$, then we get $\lambda_2 = \lambda_3 = 0$. If $\alpha \neq 0$ and $(\alpha + 2)(\alpha + \frac{1}{4}) \neq 0$, we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$. If $\alpha \neq 0$ and $(\alpha + 2)(\alpha + \frac{1}{4}) = 0$, we get $\lambda_1 = \lambda_2 = 0$.
Although we assume $\eta = -1$, we still have

$$(3.35) \qquad \begin{cases} \alpha\lambda_1 = 0, \\ \frac{\alpha}{2}\lambda_2 + (-\alpha^2 + \frac{9}{4}\alpha - \frac{1}{2})\lambda_3 = 0, \\ (\alpha^2 - \frac{5}{4}\alpha + \frac{1}{2})\lambda_2 = 0. \end{cases}$$

Furthermore if $\alpha = 0$, then we get $\lambda_2 = \lambda_3 = 0$. If $\alpha \neq 0$ and $(\alpha - 2)(\alpha - \frac{1}{4}) \neq 0$, we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$. If $\alpha \neq 0$ and $(\alpha - 2)(\alpha - \frac{1}{4}) = 0$, we get $\lambda_1 = \lambda_2 = 0$.

If $\beta \neq 0$, then we have

(3.36)
$$
\begin{cases}
\left[(b_3 + \beta)(\tfrac{1}{2}\beta - \eta) + 1\right]\lambda_2 + \{(\alpha - \beta)\left[(2\eta - \beta)b_3 - 1\right] + \tfrac{1}{2}(b_3 - \beta)\}\lambda_3 = 0, \\[2mm]
\{\alpha[(\beta - 2\eta)b_3 + 1] + \tfrac{1}{2}(\beta - b_3)\}\lambda_2 + \tfrac{1}{2}\beta(\beta - b_3)\lambda_3 = 0, \\[2mm]
\left[(2\eta - \beta)(b_3 + \beta) - 2\right]\lambda_1 = 0, \\[2mm]
\left[(2\eta - \beta)b_3 - 1\right]\lambda_1 = 0, \\[2mm]
(b_3 - \beta)\lambda_1 = 0.
\end{cases}
$$

Now if $\eta = 1$, we have

(3.37)
$$
\begin{cases}
\left[(\tfrac{\alpha}{2} + 1 + \beta)(\tfrac{1}{2}\beta - 1) + 1\right]\lambda_2 + \{(\alpha - \beta)\left[(2 - \beta)(\tfrac{\alpha}{2} + 1) - 1\right] + \tfrac{1}{2}(\tfrac{\alpha}{2} + 1 - \beta)\}\lambda_3 = 0, \\[2mm]
\{\alpha[(\beta - 2)(\tfrac{\alpha}{2} + 1) + 1] + \tfrac{1}{2}(\beta - \tfrac{\alpha}{2} - 1)\}\lambda_2 + \tfrac{1}{2}\beta(\beta - \tfrac{\alpha}{2} - 1)\lambda_3 = 0, \\[2mm]
\left[(2 - \beta)(\tfrac{\alpha}{2} + 1 + \beta) - 2\right]\lambda_1 = 0, \\[2mm]
\left[(2 - \beta)(\tfrac{\alpha}{2} + 1) - 1\right]\lambda_1 = 0, \\[2mm]
(\tfrac{\alpha}{2} + 1 - \beta)\lambda_1 = 0.
\end{cases}
$$

We notice that if $\alpha + 2 = 2\beta$, Eq.(3.37) would reduce to

(3.38)
$$
\begin{cases}
(\beta - 1)^2\lambda_2 + (2 - \beta)(\beta - 1)^2\lambda_3 = 0, \\[2mm]
(\beta - 1)^3\lambda_2 = 0, \\[2mm]
(\beta - 1)\lambda_1 = 0.
\end{cases}
$$

So when $\beta = 1$, then $\lambda_1 = 0, \lambda_2 \in R, \lambda_3 \in R$. When $\beta \neq 1, \beta \neq 2$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. When $\beta \neq 1, \beta = 2$, then $\lambda_1 = \lambda_2 = 0, \lambda_3 \in R$.
But if $\alpha + 2 \neq 2\beta$, then we have $\lambda_1 = 0$.
We assume that

(3.39)
$$
\begin{cases}
A = \left[(\tfrac{\alpha}{2} + 1 + \beta)(\tfrac{1}{2}\beta - 1) + 1\right], \\[2mm]
B = \{(\alpha - \beta)\left[(2 - \beta)(\tfrac{\alpha}{2} + 1) - 1\right] + \tfrac{1}{2}(\tfrac{\alpha}{2} + 1 - \beta)\}, \\[2mm]
C = \{\alpha[(\beta - 2)(\tfrac{\alpha}{2} + 1) + 1] + \tfrac{1}{2}(\beta - \tfrac{\alpha}{2} - 1)\}, \\[2mm]
D = \tfrac{1}{2}\beta(\beta - \tfrac{\alpha}{2} - 1).
\end{cases}
$$

We get

(3.40)
$$
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 4\alpha^4\beta^2 - 16\alpha^4\beta + 16\alpha^4 - 4\alpha^3\beta^3 + 32\alpha^3\beta^2 - 68\alpha^3\beta + 40\alpha^3 - 16\alpha^2\beta^3 + 73\alpha^2\beta^2 -
$$
$$
98\alpha^2\beta + 41\alpha^2 - 20\alpha\beta^3 + 60\alpha\beta^2 - 60\alpha\beta + 20\alpha + 4\beta^4 - 16\beta^3 + 24\beta^2 - 16\beta + 4
$$

Next if

(3.41)
$$
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0,
$$

we get $\lambda_3 = -\frac{C}{D}\lambda_2$.

We continue to assume that $\eta = 1$, we have

$$
(3.42) \quad
\begin{cases}
\left[(\frac{\alpha}{2}+1-\beta)(\frac{1}{2}\beta+1)+1\right]\lambda_2 + \{(\alpha-\beta)\left[(-2-\beta)(\frac{\alpha}{2}-1)-1\right]+\frac{1}{2}(\frac{\alpha}{2}-1-\beta)\}\lambda_3 = 0, \\[2mm]
\{\alpha[(\beta+2)(\frac{\alpha}{2}+1)-1]+\frac{1}{2}(\beta-\frac{\alpha}{2}+1)\}\lambda_2 + \frac{1}{2}\beta(\beta-\frac{\alpha}{2}+1)\lambda_3 = 0, \\[2mm]
\left[(-2-\beta)(\frac{\alpha}{2}-1+\beta)-2\right]\lambda_1 = 0, \\[2mm]
\left[(-2-\beta)(\frac{\alpha}{2}-1)-1\right]\lambda_1 = 0, \\[2mm]
(\frac{\alpha}{2}-1-\beta)\lambda_1 = 0.
\end{cases}
$$

We notice that if $\alpha = 2 + 2\beta$, Eq.(3.42) would reduce to

$$
(3.43) \quad
\begin{cases}
(\beta+1)^2\lambda_2 - (2+\beta)(\beta-1)^2\lambda_3 = 0, \\[2mm]
(\beta+1)^3\lambda_2 = 0, \\[2mm]
(\beta+1)\lambda_1 = 0.
\end{cases}
$$

So when $\beta = -1$, then $\lambda_1 = 0, \lambda_2 \in R, \lambda_3 \in R$. When $\beta \neq -1, \beta \neq -2$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. When $\beta \neq -1, \beta = -2$, then $\lambda_1 = \lambda_2 = 0, \lambda_3 \in R$.

But if $\alpha + 2 \neq 2\beta$, then we have $\lambda_1 = 0$.

We assume that

$$
(3.44) \quad
\begin{cases}
A = \left[(\frac{\alpha}{2}-1+\beta)(\frac{1}{2}\beta+1)+1\right], \\[2mm]
B = \{(\alpha-\beta)\left[(-2-\beta)(\frac{\alpha}{2}-1)-1\right]+\frac{1}{2}(\frac{\alpha}{2}-1-\beta)\}, \\[2mm]
C = \{\alpha[(\beta+2)(\frac{\alpha}{2}-1)+1]+\frac{1}{2}(\beta-\frac{\alpha}{2}+1)\}, \\[2mm]
D = \frac{1}{2}\beta(\beta-\frac{\alpha}{2}+1).
\end{cases}
$$

We get

$$
(3.45) \quad
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 4\alpha^4\beta^2 + 16\alpha^4\beta + 16\alpha^4 - 4\alpha^3\beta^3 - 32\alpha^3\beta^2 - 68\alpha^3\beta - 40\alpha^3 + 16\alpha^2\beta^3 + 73\alpha^2\beta^2 +
$$
$$
98\alpha^2\beta + 41\alpha^2 - 20\alpha\beta^3 - 60\alpha\beta^2 - 60\alpha\beta - 20\alpha + 4\beta^4 + 16\beta^3 + 24\beta^2 + 16\beta + 4
$$

Next if

$$
(3.46) \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0,
$$

we get $\lambda_3 = -\frac{C}{D}\lambda_2$.

By (2.89) in [12], we have

Lemma 3.28. *Ricci symmetric tensors of (G_4, g, J) associated to Kobayashi-Nomizu connection are given by*

$$
(3.47) \quad \widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -[1+(\beta-2\eta)(b_3-b_1)] & 0 & 0 \\ 0 & -[1+(\beta-2\eta)(b_2+b_3)] & \frac{b_1+\beta-\alpha-b_3}{2} \\ 0 & \frac{\alpha+b_3-b_1-\beta}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
$$

By (3.47) and (2.3), we have

Lemma 3.29. *For G_4, the following equalities hold*

(3.48) $\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 0,\ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0\ ,\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = -2 - \alpha(\beta - 2\eta),$

$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = -\beta[1 + \alpha(\beta - 2\eta)]\ ,\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = \alpha\beta,\ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0$

$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 1 + \frac{1}{2}\alpha(\beta - 2\eta),\ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \alpha[\frac{1}{2} + \beta(\beta - 2\eta)]\ ,\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0,$

$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0\ ,\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0,\ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \beta - \frac{\alpha}{2}$

$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = -\frac{1}{2}\alpha\beta,\ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 0\ ,\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0.$

By (3.48) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_4, g, J) admits left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^1_{ij} = 0, i = 1, 2, 3$ *and* $j = 1, 2, 3$, so

(3.49)
$$\begin{cases} \left[1 + \frac{1}{2}\alpha(\beta - 2\eta)\right]\lambda_2 + (\beta - \frac{\alpha}{2})\lambda_3 = 0, \\ \alpha\left[\frac{1}{2} + \beta(\beta - 2\eta)\right]\lambda_2 - \frac{1}{2}\alpha\beta\lambda_3 = 0, \\ \left[-2 - \alpha(\beta - 2\eta)\right]\lambda_1 = 0, \\ \beta\left[1 + \alpha(\beta - 2\eta)\right]\lambda_1 = 0, \\ \alpha\beta\lambda_1 = 0. \end{cases}$$

By (3.49), we get

Theorem 3.30. *the unimodular Lorentzian Lie group (G_4, g, J) admits left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1 if and only if*
(1)$\alpha = \beta = \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle,$

(2)$\alpha = \lambda_1 = 0, \beta \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{1}{\beta}e_3 \rangle,$

(3)$\alpha \neq 0, \beta = 0, \alpha\eta = 1, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

(4)$\alpha \neq 0, \beta \neq 0, \eta = 1, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\alpha^2 - 8\alpha + 8}{2\alpha}e_3 \rangle,$

(5)$\alpha \neq 0, \beta \neq 0, \eta = 1, \alpha = 2, \beta = 1, \lambda_2 = -\lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - e_3 \rangle,$

(6)$\alpha \neq 0, \beta \neq 0, \eta = -1, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\alpha^2 + 8\alpha + 8}{2\alpha}e_3 \rangle,$

(7)$\alpha \neq 0, \beta \neq 0, \eta = -1, \alpha = -2, \beta = -1, \lambda_2 = \lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + e_3 \rangle,$

Proof 3.31. *We analyze each one of these factors by separate and because $b_1 = \frac{\alpha}{2} + \eta - \beta, b_3 = \frac{\alpha}{2} + \eta$, if $\alpha = 0$, then Eq.(3.49) would reduce to*

(3.50)
$$\begin{cases} \lambda_2 + \beta\lambda_3 = 0, \\ \lambda_1 = 0, \end{cases}$$

Furthermore if $\beta = 0$, then we have $\lambda_1 = \lambda_2 = 0$, and if $\beta \neq 0$, then we have $\lambda_3 = -\frac{1}{\beta}\lambda_2$.
But if $\alpha \neq 0, \beta = 0$, we get

(3.51)
$$\begin{cases} (-2 + 2\alpha\eta)\lambda_1 = 0, \\\\ (1 - \alpha\eta)\lambda_2 - \frac{1}{2}\alpha\lambda_3 = 0, \\\\ \lambda_2 = 0. \end{cases}$$

as long as $\alpha\eta = 1$, then $\lambda_1 \in R, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle$, otherwise $\mathbb{V}_{\mathbb{RC}}$ is empty set.
Considering $\alpha \neq 0, \beta \neq 0$, then naturely $\lambda_1 = 0$, and when $\eta = 1$, we have

(3.52)
$$\begin{cases} \left[1 + \frac{1}{2}\alpha(\beta - 2)\right]\lambda_2 + (\beta - \frac{\alpha}{2})\lambda_3 = 0, \\\\ \left[\frac{1}{2} + \beta(\beta - 2)\right]\lambda_2 - \frac{1}{2}\beta\lambda_3 = 0, \end{cases}$$

We assume that

(3.53)
$$\begin{cases} A = \left[1 + \frac{1}{2}\alpha(\beta - 2)\right], \\\\ B = \beta - \frac{\alpha}{2}, \\\\ C = \left[\frac{1}{2} + \beta(\beta - 2)\right], \\\\ D = -\frac{1}{2}\beta. \end{cases}$$

We get

(3.54)
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = (\frac{1}{4}\alpha - \beta)(\beta - 1)^2$$

If $\beta = 1$, we have $(1 - \frac{1}{2}\alpha)(\lambda_2 + \lambda_3) = 0$, so $\lambda_2 + \lambda_3$ equals to 0 with $\alpha = 2$.
If $\alpha = 4\beta$, we have $(1 + 2\beta^2 - 4\beta) - \beta\lambda_3 = 0$, so $\lambda_3 = \frac{1+2\beta^2-4\beta}{\beta}\lambda_2 = \frac{\alpha^2-8\alpha+8}{2\alpha}\lambda_2$.
Now if $\eta = -1$, we have

(3.55)
$$\begin{cases} \left[1 + \frac{1}{2}\alpha(\beta + 2)\right]\lambda_2 + (\beta - \frac{\alpha}{2})\lambda_3 = 0, \\\\ \left[\frac{1}{2} + \beta(\beta + 2)\right]\lambda_2 - \frac{1}{2}\beta\lambda_3 = 0, \end{cases}$$

We assume that

(3.56)
$$\begin{cases} A = \left[1 + \frac{1}{2}\alpha(\beta + 2)\right], \\\\ B = \beta - \frac{\alpha}{2}, \\\\ C = \left[\frac{1}{2} + \beta(\beta + 2)\right], \\\\ D = -\frac{1}{2}\beta. \end{cases}$$

We get

(3.57)
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = (\frac{1}{4}\alpha - \beta)(\beta + 1)^2$$

If $\beta = -1$, we have$(1 + \frac{1}{2}\alpha)(\lambda_2 - \lambda_3) = 0$, so $\lambda_2 + \lambda_3$ equals to 0 with $\alpha = -2$.
If $\alpha = 4\beta$, we have$(1 + 2\beta^2 + 4\beta) - \beta\lambda_3 = 0$, so $\lambda_3 = \frac{1+2\beta^2+4\beta}{\beta}\lambda_2 = \frac{\alpha^2+8\alpha+8}{2\alpha}\lambda_3$.

In summary,the unimodular Lorentzian Lie group (G_4, g, J) admits left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1 if and only if
(1)$\alpha = \beta = \lambda_1 = \lambda_2 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$,

$(2) \alpha = \lambda_1 = 0, \beta \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{1}{\beta} e_3 \rangle,$

$(3) \alpha \neq 0, \beta = 0, \alpha \eta = 1, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

$(4) \alpha \neq 0, \beta \neq 0, \eta = 1, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\alpha^2 - 8\alpha + 8}{2\alpha} e_3 \rangle,$

$(5) \alpha \neq 0, \beta \neq 0, \eta = 1, \alpha = 2, \beta = 1, \lambda_2 = -\lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - e_3 \rangle,$

$(6) \alpha \neq 0, \beta \neq 0, \eta = -1, \alpha = 4\beta, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\alpha^2 + 8\alpha + 8}{2\alpha} e_3 \rangle,$

$(7) \alpha \neq 0, \beta \neq 0, \eta = -1, \alpha = -2, \beta = -1, \lambda_2 = \lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + e_3 \rangle,$

4. Left-invariant Ricci collineations associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional non-unimodular Lorentzian Lie groups

4.1 Left-invariant Ricci collineation of G_5

By [1], we have for G_5, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_5 satisfies

$$[e_1, e_2] = 0, \ [e_1, e_3] = \alpha e_1 + \beta e_2, \ [e_2, e_3] = \gamma e_1 + \delta e_2, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0, \ .$$

By (3.5) in [12], we have

Lemma 4.1. *Ricci symmetric tensors of* (G_5, g, J) *associated to canonical connection are given by*

$$(4.1) \qquad \widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Lemma 4.2. *For G_5, the following equalities hold* $\mathrm{L}_\xi \widetilde{\mathrm{Ric}}^0 (i, j) = 0, for \ i = 1, 2, 3 \ and \ j = 1, 2, 3.$

If the unimodular Lorentzian Lie group (G_5, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi \widetilde{\mathrm{Ric}}^0_{ij} = 0, i = 1, 2, 3 \ and \ j = 1, 2, 3$, so there are no constraint conditions for G_5.

Naturelly, we get

Theorem 4.3. *the unimodular Lorentzian Lie group (G_5, g, J) admits definitely left-invariant Ricci collineations associated to the canonical connection ∇^0 .*

By [12], we have

Lemma 4.4. *Ricci symmetric tensors of* (G_5, g, J) *associated to Kobayashi-Nomizu connection are given by*

$$(4.2) \qquad \widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Lemma 4.5. *For G_5, the following equalities hold* $\mathrm{L}_\xi \widetilde{\mathrm{Ric}}^1(i,j) = 0, i = 1,2,3$ *and* $j = 1,2,3$.

If the unimodular Lorentzian Lie group (G_5, g, J) admits left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1, then $\mathrm{L}_\xi \widetilde{\mathrm{Ric}}_{ij} = 0, i = 1,2,3$ *and* $j = 1,2,3$, so there are no constraint conditions for G_5. Naturelly, we get

Theorem 4.6. *the unimodular Lorentzian Lie group (G_5, g, J) admits definitely left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1 .*

4.2 Left-invariant Ricci collineation of G_6

By [1], we have for G_6, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_6 satisfies

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \ [e_1, e_3] = \gamma e_2 + \delta e_3, \ [e_2, e_3] = 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0 \ .$$

By (3.18) in [12], we have

Lemma 4.7. *Ricci symmetric tensors of (G_6, g, J) associated to canonical connection are given by*

$$(4.3) \qquad \widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 & 0 & 0 \\ 0 & \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 & \frac{1}{2}[\gamma\alpha - \frac{1}{2}\delta(\beta - \gamma)] \\ 0 & \frac{1}{2}[-\gamma\alpha + \frac{1}{2}\delta(\beta - \gamma)] & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (4.3) and (2.2)

Lemma 4.8. *For G_6, the following equalities hold*

$$(4.4) \qquad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 2\alpha^3 + \alpha\beta\gamma - \beta(\beta - \gamma)(\alpha + \frac{1}{2}\delta),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = \frac{1}{2}\alpha\gamma(3\alpha + \delta) + \frac{1}{2}(\gamma - \beta)(\alpha\delta + \delta^2 + \beta\gamma), \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = \gamma\left[\alpha\gamma + \frac{1}{2}\delta(\gamma - \beta)\right],$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0 \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = -\alpha^3 - \frac{1}{2}\alpha\beta\gamma + \frac{1}{2}\beta(\beta - \gamma)(\alpha + \frac{1}{2}\delta),$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\alpha\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right] \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = -\alpha^2\gamma - \frac{1}{2}\alpha\gamma\delta + \frac{1}{2}(\beta - \gamma)(\beta\gamma + \frac{1}{2}\delta^2)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\gamma\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right], \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = 0.$$

By (4.4) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_6, g, J) is left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi \widetilde{\mathrm{Ric}}^0_{ij} = 0, i = 1,2,3$ *and* $j = 1,2,3$, so

$$\begin{cases} \left[-\alpha^3 - \frac{1}{2}\alpha\beta\gamma + \frac{1}{2}\beta(\beta - \gamma)(\alpha + \frac{1}{2}\delta)\right]\lambda_2 + \left[-\alpha^2\gamma - \frac{1}{2}\alpha\gamma\delta + \frac{1}{2}(\beta - \gamma)(\beta\gamma + \frac{1}{2}\delta^2)\right]\lambda_3 = 0, \\[2mm] \frac{1}{2}\alpha\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right]\lambda_2 + \frac{1}{2}\gamma\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right]\lambda_3 = 0, \\[2mm] \left[2\alpha^3 + \alpha\beta\gamma - \beta(\beta - \gamma)(\alpha + \frac{1}{2}\delta)\right]\lambda_1 = 0, \\[2mm] \left[\frac{1}{2}\alpha\gamma(3\alpha + \delta) + \frac{1}{2}(\gamma - \beta)(\alpha\delta + \delta^2 + \beta\gamma)\right]\lambda_1 = 0, \\[2mm] \gamma\left[\alpha\gamma + \frac{1}{2}\delta(\gamma - \beta)\right]\lambda_1 = 0. \end{cases}$$

(4.5)

By solving (4.5), we get

Theorem 4.9. *the unimodular Lorentzian Lie group (G_6, g, J) is left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if*

$(1)\gamma = 0, \alpha(2\alpha^2 - \beta^2) = 0, \mathbb{V}_{RC} = \langle e_1, e_2, e_3 \rangle,$

$(2)\gamma = 0, \alpha(2\alpha^2 - \beta^2) \neq 0,, \mathbb{V}_{RC} = \langle e_3 \rangle,$

$(3)\gamma \neq 0, \alpha = \beta = 0, \delta \neq 0, \lambda_1 = \lambda_3 = 0, \mathbb{V}_{RC} = \langle e_2 \rangle,$

$(4)\gamma \neq 0, \alpha \neq 0, \alpha + \beta = 0, \gamma + \delta = 0, \delta \neq 0, \lambda_1 = 0, \lambda_3 = \frac{\alpha}{\delta}\lambda_2, \mathbb{V}_{RC} = \langle e_2 + \frac{\alpha}{\delta}e_3 \rangle,$

$(5)\gamma \neq 0, \alpha \neq 0, \alpha = \beta, \gamma = \delta, \delta \neq 0, \lambda_1 = 0, \lambda_3 = -\frac{\alpha}{\delta}\lambda_2, \mathbb{V}_{RC} = \langle e_2 - \frac{\alpha}{\delta}e_3 \rangle.$

Proof 4.10. *We analyze each one of these factors by separate ,if $\gamma \neq 0, \alpha = 0$, we have*

$$\begin{cases} \frac{1}{2}\beta\delta(\beta - \gamma)\lambda_2 + (\beta - \gamma)(\beta\gamma + \frac{1}{2}\gamma^2)\lambda_3 = 0, \delta(\beta - \gamma)\lambda_3 = 0, \\[2mm] \beta\delta(\beta - \gamma)\lambda_1 = 0, (\beta - \gamma)(\beta\gamma + \gamma^2)\lambda_1 = 0, \\[2mm] \delta(\beta - \gamma)\lambda_1 = 0, \alpha\gamma - \beta\delta = 0, \alpha + \delta \neq 0. \end{cases}$$

(4.6)

Notice that $\delta \neq 0$ for $\alpha + \delta \neq 0$ and furthermore $\beta = 0$ for $\alpha\gamma - \beta\delta = 0$,so Eq.(4.6) would reduce to

$$\begin{cases} (\beta\gamma + \frac{1}{2}\gamma^2)\lambda_3 = 0, \\[2mm] \delta\gamma\lambda_3 = 0, \\[2mm] \beta\gamma\lambda_1 = 0. \end{cases}$$

(4.7)

Naturelly we get $\lambda_1 = \lambda_3 = 0, \lambda_2 \in R$.
If $\gamma \neq 0, \alpha \neq 0$, we have $\beta \neq 0, \delta \neq 0, \left[\alpha\gamma + \frac{1}{2}\delta(\gamma - \beta)\right] \neq 0$, otherwise $\alpha\gamma - \beta\delta \neq 0$.
Naturelly we have $\lambda_1 = 0$ and we assume that

$$\begin{cases} A = -\alpha^3 - \frac{1}{2}\alpha\beta\gamma + \frac{1}{2}\beta(\beta - \gamma)(\alpha + \frac{1}{2}\delta), \\[2mm] B = -\alpha^2\gamma - \frac{1}{2}\alpha\gamma\delta + \frac{1}{2}(\beta - \gamma)(\beta\gamma + \frac{1}{2}\delta^2), \\[2mm] C = \frac{1}{2}\alpha\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right], \\[2mm] D = \frac{1}{2}\gamma\left[-\alpha\gamma + \frac{1}{2}\delta(\beta - \gamma)\right]. \end{cases}$$

(4.8)

We get

(4.9)
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{1}{4}\gamma(\alpha + \delta)(\alpha\delta - \beta\gamma) = 0,$$

solving

$$(4.10) \qquad \begin{cases} \alpha\gamma - \beta\delta = 0, \\ \alpha\delta - \beta\gamma = 0. \end{cases}$$

we get $\alpha + \beta = 0, \gamma + \delta = 0, \lambda_3 = \frac{\alpha}{\delta}\lambda_2$ *or* $\alpha = \beta, \gamma = \delta, \lambda_3 = -\frac{\alpha}{\delta}\lambda_2$.
If $\gamma = 0$, *we naturely have* $\beta\delta = 0$, *then we get*

$$(4.11) \qquad \begin{cases} (2\alpha^3 - \alpha\beta^2)\lambda_1 = 0, \\ (2\alpha^3 - \alpha\beta^2)\lambda_2 = 0. \end{cases}$$

When $\alpha(2\alpha^2 - \beta^2) = 0$, *we get* $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$ *and when* $\alpha(2\alpha^2 - \beta^2) \neq 0$, *we get* $\lambda_1 = \lambda_2 = 0, \lambda_3 \in R$.
In summary,the unimodular Lorentzian Lie group (G_6, g, J) *is left-invariant Ricci collineations associated to the canonical connection* ∇^0 *if and only if*

$(1) \gamma = 0, \alpha(2\alpha^2 - \beta^2) = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

$(2) \gamma = 0, \alpha(2\alpha^2 - \beta^2) \neq 0,, \mathbb{V}_{\mathbb{RC}} = \langle e_3 \rangle$,

$(3) \gamma \neq 0, \alpha = \beta = 0, \delta \neq 0, \lambda_1 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_2 \rangle$,

$(4) \gamma \neq 0, \alpha \neq 0, \alpha + \beta = 0, \gamma + \delta = 0, \delta \neq 0, \lambda_1 = 0, \lambda_3 = \frac{\alpha}{\delta}\lambda_2, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + \frac{\alpha}{\delta}e_3 \rangle$,

$(5) \gamma \neq 0, \alpha \neq 0, \alpha = \beta, \gamma = \delta, \delta \neq 0, \lambda_1 = 0, \lambda_3 = -\frac{\alpha}{\delta}\lambda_2, \mathbb{V}_{\mathbb{RC}} = \langle e_2 - \frac{\alpha}{\delta}e_3 \rangle$.

By (3.23) in [12], we have

Lemma 4.11. *Ricci symmetric tensors of* (G_6, g, J) *associated to Kobayashi-Nomizu connection are given by*

$$(4.12) \qquad \widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta\gamma) & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By(4.12) and (2.3), we have

Lemma 4.12. *For* G_6, *the following equalities hold*

$$(4.13) \qquad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = 0, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0 \ , \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 2\alpha^3,$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \alpha^2\gamma, \ \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = -\alpha^3, \ \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0 \ , \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0,$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = -\alpha^2\gamma,$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = 0, \ \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = 0 \ , \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = 0.$$

By (4.13) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_6, g, J) admits left-invariant Ricci collineations associated to Kobayashi-Nomizu connection ∇^1, then $L_\xi \widetilde{\mathrm{Ric}}_{ij}^1 = 0, for\ i = 1, 2, 3\ and\ j = 1, 2, 3$, so

(4.14)
$$\begin{cases} \alpha^3 \lambda_2 + \alpha^2 \gamma \lambda_3 = 0, \\ 2\alpha^3 \lambda_1 = 0, \\ \alpha^2 \gamma \lambda_1 = 0. \end{cases}$$

By solving (4.14), we get

Theorem 4.13. *the unimodular Lorentzian Lie group (G_6, g, J) admits left-invariant Ricci collineations associated to Kobayshi-Nomizu connection ∇^1 if and only if*
(1)$\alpha = \beta = 0, \delta \neq 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle$,

(2)$\alpha \neq 0, \lambda_2 = -\frac{\gamma}{\alpha}\lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle -\frac{\gamma}{\alpha}e_2 + e_3 \rangle$.

Proof 4.14. *We analyze each one of these factors by separate. If $\alpha = 0$, we have $\delta \neq 0, \beta = 0$ for $\alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0$, and naturely $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$.*
And if $\alpha \neq 0$, we get

(4.15)
$$\begin{cases} \alpha\lambda_2 + \gamma\lambda_3 = 0, \\ \lambda_1 = 0. \end{cases}$$

So $\lambda_2 = -\frac{\gamma}{\alpha}\lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle -\frac{\gamma}{\alpha}e_2 + e_3 \rangle$.

4.3 Left-invariant Ricci collineation of G_7

By [1], we have for G_7, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_7 satisfies

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3,\ [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3,\ [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,\ \alpha + \delta \neq 0,\ \alpha\gamma = 0.$$

By (3.34) in [12], we have

Lemma 4.15. *Ricci symmetric tensors of (G_7, g, J) associated to canonical connection are given by*

(4.16)
$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \frac{\beta\gamma}{2}) & 0 & \frac{1}{2}(\gamma\alpha + \frac{\delta\gamma}{2}) \\ 0 & -(\alpha^2 + \frac{\beta\gamma}{2}) & -\frac{1}{2}(\alpha^2 + \frac{\beta\gamma}{2}) \\ -\frac{1}{2}(\gamma\alpha + \frac{\delta\gamma}{2}) & \frac{1}{2}(\alpha^2 + \frac{\beta\gamma}{2}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

By (4.16) and (2.2),we have

Lemma 4.16. *For G_7, the following equalities hold*

$$(4.17) \quad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = -\alpha^3 - \frac{1}{4}\beta\delta\gamma, \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \alpha^3 + \frac{1}{4}\beta\delta\gamma$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = -\beta(\alpha^2 + \frac{1}{2}\beta\gamma), \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = \beta(\alpha^2 + \frac{1}{2}\beta\gamma),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = -\beta(\alpha^2 + \frac{1}{2}\beta\gamma), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = 2\alpha^3 + \frac{1}{2}\beta\delta\gamma$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = \frac{1}{2}\beta(\alpha^2 + \frac{1}{2}\beta\gamma), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{4}\gamma(\delta^2 - \beta^2) + \frac{1}{2}\beta(\gamma^2 - \alpha^2)$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^0(e_3, e_3) = \delta(\frac{1}{2}\gamma^2 - \alpha^2 - \frac{1}{2}\beta\gamma)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_1) = -2\alpha^3 - \frac{1}{2}\beta\delta\gamma, \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_2) = -(\gamma + \frac{1}{2}\beta)(\alpha^2 + \frac{1}{2}\beta\gamma) - \frac{1}{4}\delta^2\gamma$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_1, e_3) = \frac{1}{2}\beta(\alpha^2 + \frac{1}{2}\beta\gamma), \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_2) = -\delta(\alpha^2 + \frac{1}{2}\beta\gamma)$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^0(e_2, e_3) = -\frac{1}{4}\delta\gamma^2 + \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma).$$

By (4.17) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_7, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^0_{ij} = 0, i = 1, 2, 3$ and $j = 1, 2, 3$, so

$$(4.18) \quad \begin{cases} (2\alpha^3 + \frac{1}{2}\beta\delta\gamma)(\lambda_2 - \lambda_3) = 0, \\[2mm] (-\alpha^3 - \frac{1}{4}\beta\delta\gamma)\lambda_1 + \frac{1}{2}\beta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_2 + \left[-(\gamma + \frac{1}{2}\beta)(\alpha^2 + \frac{1}{2}\beta\gamma) - \frac{1}{4}\delta^2\gamma\right]\lambda_3 = 0, \\[2mm] (\alpha^3 + \frac{1}{4}\beta\delta\gamma)\lambda_1 + \left[\frac{1}{4}\gamma(\delta^2 - \beta^2) + \frac{1}{2}\beta(\gamma^2 - \alpha^2)\right]\lambda_2 + \frac{1}{2}\beta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_3 = 0, \\[2mm] \beta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_1 + \delta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_3 = 0, \\[2mm] \beta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_1 + \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_2 + \left[-\frac{1}{4}\delta\gamma^2 + \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma)\right]\lambda_3 = 0, \\[2mm] -\beta(\alpha^2 + \frac{1}{2}\beta\gamma)\lambda_1 + \delta(\frac{1}{2}\gamma^2 - \alpha^2 - \frac{1}{2}\beta\gamma)\lambda_2 = 0. \end{cases}$$

By solving (4.18), we get

Theorem 4.17. *the unimodular Lorentzian Lie group (G_7, g, J) admits left-invariant Ricci collineations associated to the canonical connection ∇^0 if and only if*
(1)$\alpha = 0, \delta \neq 0, \gamma \neq 0, \beta = 0, \lambda_2 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

(2)$\alpha = 0, \delta \neq 0, \gamma = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1, e_2, e_3 \rangle,$

(3)$\alpha \neq 0, \gamma = \delta = 0, \lambda_1 = 0, \lambda_2 = \lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + e_3 \rangle.$

Proof 4.18. *We analyze each one of these factors by separate. If $\alpha = 0$, we have $\delta \neq 0$, for $\alpha + \delta \neq 0, \alpha\gamma = 0$, and*

(4.19)
$$
\begin{cases}
\beta\gamma(\lambda_2 - \lambda_3) = 0, \\[2mm]
\beta\delta\gamma\lambda_1 - \beta^2\gamma\lambda_2 + \left[\beta\gamma(2\gamma + \beta) + \delta^2\gamma\right]\lambda_3 = 0, \\[2mm]
\beta\delta\gamma\lambda_1 + \left[\gamma(\delta^2 - \beta^2) + 2\beta\gamma^2\right]\lambda_2 + \beta^2\gamma\lambda_3 = 0, \\[2mm]
\beta^2\gamma\lambda_1 + \beta\gamma\delta\lambda_3 = 0, \\[2mm]
2\beta^2\gamma\lambda_1 + \beta\gamma\delta\lambda_2 + (\beta\gamma\delta - \delta\gamma^2)\lambda_3 = 0, \\[2mm]
\beta^2\gamma\lambda_1 + (\beta\gamma\delta - \delta\gamma^2)\lambda_2 = 0.
\end{cases}
$$

Obviously $\lambda_1 \in R, \lambda_2 \in R, \lambda_3 \in R$ with $\gamma = 0$, and when $\beta \neq 0$, we have $\lambda_2 = \lambda_3$. Then we have

(4.20)
$$
\begin{cases}
\beta\delta\lambda_1 + (\delta^2 + 2\beta\gamma)\lambda_2 = 0, \\[2mm]
\beta^2\lambda_1 + \beta\delta\lambda_2 = 0, \\[2mm]
\beta^2\lambda_1 + (\beta\delta - \delta\gamma)\lambda_2 = 0.
\end{cases}
$$

So we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
when $\gamma \neq 0$, Eq.(4.19)would easily reduce to

(4.21)
$$
\begin{cases}
\beta(\lambda_2 - \lambda_3) = 0, \\[2mm]
\beta\delta\lambda_1 - \beta^2\lambda_2 + \left[\beta(2\gamma + \beta) + \delta^2\right]\lambda_3 = 0, \\[2mm]
\beta\delta\lambda_1 + (\delta^2 - \beta^2 + 2\beta\gamma)\lambda_2 + \beta^2\lambda_3 = 0, \\[2mm]
\beta^2\lambda_1 + \beta\delta\lambda_3 = 0, \\[2mm]
2\beta^2\lambda_1 + \beta\delta\lambda_2 + (\beta\delta - \delta\gamma)\lambda_3 = 0, \\[2mm]
\beta^2\lambda_1 + (\beta\delta - \delta\gamma)\lambda_2 = 0.
\end{cases}
$$

simplify this equation, we still get $\lambda_2 = \lambda_3 = 0, \lambda_1 \in R$ with $\beta = 0$.
If $\alpha \neq 0$, then we have $\gamma = 0$. And we get

(4.22)
$$
\begin{cases}
\alpha^3(\lambda_2 - \lambda_3) = 0, \\[2mm]
\alpha^3\lambda_1 + \frac{1}{2}\alpha^2\beta(\lambda_3 - \lambda_2) = 0, \\[2mm]
\beta\lambda_1 + \delta\lambda_2 = 0.
\end{cases}
$$

Because $\alpha \neq 0$, so we get $\lambda_2 = \lambda_3$. Put $\lambda_2 = \lambda_3$ into $\alpha^3\lambda_1 + \frac{1}{2}(\lambda_3 - \lambda_2) = 0$, we have $\lambda_1 = 0$. So another equation would reduce to $\delta\lambda_2 = 0$, and then $\delta = 0$.

By (3.42) in [12], we have

Lemma 4.19. *Ricci symmetric tensors of (G_7, g, J) associated to Kobayashi-Nomizu connection are given by*

(4.23)
$$
\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\alpha^2 & \frac{1}{2}(\beta\delta - \alpha\beta) & -\beta(\alpha + \delta) \\ \frac{1}{2}(\beta\delta - \alpha\beta) & -(\alpha^2 + \beta^2 + \beta\gamma) & -\frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) \\ \beta(\alpha + \delta) & \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
$$

By (4.23) and (2.3),we have

Lemma 4.20. *For G_7, the following equalities hold*

$$(4.24) \qquad \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = -\alpha^3 + \frac{1}{2}\beta^2(3\delta + \alpha), \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \alpha^3 - \frac{1}{2}\beta^2(3\delta + \alpha),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = \beta(2\alpha\delta - 3\alpha^2 - 2\beta^2 - \beta\gamma + 2\delta^2), \mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \beta(\frac{5}{2}\alpha^2 + \frac{1}{2}\alpha\delta + \beta^2 + \beta\gamma),$$

$$\mathrm{L}_{e_1}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = -\beta(2\alpha^2 + 3\alpha\delta + \beta\gamma + 2\delta^2), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = 2\alpha^2 - \beta^2(3\delta + \alpha),$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \frac{1}{2}\beta(3\alpha^2 + 2\beta^2 + \beta\gamma - 2\alpha\delta - 2\delta^2), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = -\beta(\alpha^2 + \frac{5}{2}\delta^2 + 2\alpha\delta + \frac{1}{2}\beta\gamma),$$

$$\mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{1}{2}\alpha\delta), \mathrm{L}_{e_2}\widetilde{\mathrm{Ric}}^1(e_3, e_3) = -\delta(3\beta\gamma + \alpha\delta + 2\delta^2),$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_1) = -2\alpha^3 + \beta^2(3\delta + \alpha), \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_2) = \frac{1}{2}\beta(5\delta^2 - 3\alpha^2 - 2\beta^2 - \beta\gamma + 3\alpha\delta),$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_1, e_3) = \frac{1}{2}\beta(2\alpha^2 + 3\alpha\delta + \beta\gamma + 2\delta^2), \mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_2) = \delta(2\delta^2 + \alpha\delta - 2\alpha^2 - 2\beta^2),$$

$$\mathrm{L}_{e_3}\widetilde{\mathrm{Ric}}^1(e_2, e_3) = \frac{1}{2}\delta(3\beta\gamma + \alpha\delta + 2\delta^2).$$

By (4.24) and definition 2.3, we have

If the unimodular Lorentzian Lie group (G_7, g, J) is left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1, then $\mathrm{L}_\xi\widetilde{\mathrm{Ric}}^1_{ij} = 0, i = 1, 2, 3$ *and* $j = 1, 2, 3$, so

$$(4.25) \quad \begin{cases} \left[2\alpha^2 - \beta^2(3\delta + \alpha)\right]\lambda_2 + \left[-2\alpha^3 + \beta^2(3\delta + \alpha)\right]\lambda_3 = 0, \\[2mm] \left[-\alpha^3 + \frac{1}{2}\beta^2(3\delta + \alpha)\right]\lambda_1 + \frac{1}{2}\beta(3\alpha^2 + 2\beta^2 + \beta\gamma - 2\alpha\delta - 2\delta^2)\lambda_2 + \frac{1}{2}\beta(5\delta^2 - 3\alpha^2 - 2\beta^2 - \beta\gamma + 3\alpha\delta)\lambda_3 = 0, \\[2mm] \left[\alpha^3 - \frac{1}{2}\beta^2(3\delta + \alpha)\right]\lambda_1 - \beta(\alpha^2 + \frac{5}{2}\delta^2 + 2\alpha\delta + \frac{1}{2}\beta\gamma)\lambda_2 + \frac{1}{2}\beta(2\alpha^2 + 3\alpha\delta + \beta\gamma + 2\delta^2)\lambda_3 = 0, \\[2mm] \beta(2\alpha\delta - 3\alpha^2 - 2\beta^2 - \beta\gamma + 2\delta^2)\lambda_1 + \delta(2\delta^2 + \alpha\delta - 2\alpha^2 - 2\beta^2)\lambda_3 = 0, \\[2mm] \beta(\frac{5}{2}\alpha^2 + \frac{1}{2}\alpha\delta + \beta^2 + \beta\gamma)\lambda_1 + \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{1}{2}\alpha\gamma)\lambda_2 + \frac{1}{2}\delta(3\beta\gamma + \alpha\delta + 2\delta^2)\lambda_3 = 0, \\[2mm] \beta(2\alpha^2 + 3\alpha\delta + \beta\gamma + 2\delta^2)\lambda_1 + \delta(3\beta\gamma + \alpha\delta + 2\delta^2)\lambda_2 = 0. \end{cases}$$

By solving (4.25), we get

Theorem 4.21. *the unimodular Lorentzian Lie group (G_7, g, J) is left-invariant Ricci collineations associated to the Kobayashi-Nomizu connection ∇^1 if and only if*
$(1)\alpha = 0, \delta \neq 0, \beta \neq 0, \gamma = 0, \lambda_2 = \lambda_3 = -\frac{\beta}{\delta}\lambda_1, \mathbb{V}_{\mathbb{RC}} = \langle e_1 - \frac{\beta}{\delta}e_2 - \frac{\beta}{\delta}e_3 \rangle,$

$(2)\alpha = 0, \delta \neq 0, \beta = 0, \lambda_2 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle,$

$(3)\alpha \neq 0, \gamma = 0, \delta = 0, \lambda_1 = 0, \lambda_2 = \lambda_3, \mathbb{V}_{\mathbb{RC}} = \langle e_2 + e_3 \rangle.$

Proof 4.22. *We analyze each one of these factors by separate,because $\alpha\gamma = 0$, and if $\alpha = 0$, we have $\delta \neq 0$ for $\alpha + \delta \neq 0$, and Eq.(4.25) would reduce to*

(4.26)
$$\begin{cases} \beta^2(\lambda_2 - \lambda_3) = 0, \\[2mm] \frac{3}{2}\beta^2\delta\lambda_1 + \frac{1}{2}\beta(2\beta^2\beta\gamma - 2\delta^2)\lambda_2 + \beta(\frac{5}{2}\delta^2 - \beta^2 - \frac{1}{2}\beta\gamma)\lambda_3 = 0, \\[2mm] \frac{3}{2}\beta^2\delta\lambda_1 + \beta(\frac{5}{2}\delta^2 + \frac{1}{2}\beta\gamma)\lambda_2 - \beta(\frac{1}{2}\beta\gamma + \delta^2)\lambda_3 = 0, \\[2mm] \beta(-2\beta^2 - \beta\gamma + 2\delta^2)\lambda_1 + \delta(2\delta^2 - 2\beta^2)\lambda_3 = 0, \\[2mm] \beta(\beta^2 + \beta\gamma)\lambda_1 + \delta(\beta^2 - \delta^2)\lambda_2 + \delta(\frac{1}{2}\beta\gamma + \delta^2)\lambda_3 = 0, \\[2mm] \beta(\beta\gamma + 2\delta^2)\lambda_1 + \delta(3\beta\gamma + 2\delta^2)\lambda_2 = 0. \end{cases}$$

If $\beta \neq 0$, we have $\lambda_2 = \lambda_3$. Put $\lambda_2 = \lambda_3$ into Eq.(4.26), we get

(4.27)
$$\begin{cases} \beta\lambda_1 + \delta\lambda_3 = 0, \\[2mm] \gamma\lambda_1 = 0. \end{cases}$$

So $\gamma = 0, -\frac{\beta}{\delta}\lambda_1 = \lambda_2 = \lambda_3$, otherwise $\gamma \neq 0, \lambda_1 = \lambda_2 = \lambda_3 = 0$.
If $\beta = 0$, then we get

(4.28)
$$\begin{cases} \delta^3\lambda_3 = 0, \\[2mm] \delta^3\lambda_2 = 0, \\[2mm] \delta^3(\lambda_3 - \lambda_2) = 0. \end{cases}$$

So we have $\lambda_2 = \lambda_3 = 0, \mathbb{V}_{\mathbb{RC}} = \langle e_1 \rangle$.
If $\alpha \neq 0, \gamma = 0$, we get

(4.29)
$$\begin{cases} \left[2\alpha^2 - \beta^2(3\delta + \alpha)\right](\lambda_2 - \lambda_3) = 0, \\[2mm] \left[-2\alpha^2 + \beta^2(3\delta + \alpha)\right]\lambda_1 + \beta(3\alpha^2 + 2\beta^2 - 2\alpha\delta - \delta^2)\lambda_2 + \beta(5\delta^2 - 3\delta^2 - 2\beta^2 + 3\alpha\delta)\lambda_3 = 0, \\[2mm] \left[-2\alpha^2 + \beta^2(3\delta + \alpha)\right]\lambda_1 + \beta(2\alpha^2 + 4\alpha\delta + 5\delta^2)\lambda_2 - \beta(2\alpha^2 + 3\alpha\delta + 2\delta^2)\lambda_3 = 0, \\[2mm] \beta(\frac{5}{2}\alpha^2 + \frac{1}{2}\alpha\delta + \beta^2)\lambda_1 + \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{1}{2}\alpha\delta)\lambda_2 + \delta(\frac{1}{2}\alpha\delta + \delta^2)\lambda_3 = 0, \\[2mm] \beta(2\alpha^2 + 3\alpha\delta + 2\delta^2)\lambda_1 + \delta(\alpha\delta + 2\delta^2)\lambda_2 = 0. \end{cases}$$

If $2\alpha^2 - \beta^2(3\delta + \alpha) \neq 0$, then we have $\lambda_2 = \lambda_3$. Put $\lambda_2 = \lambda_3$ into other equations, we get

(4.30)
$$\begin{cases} \left[-2\alpha^2 + \beta^2(3\delta + \alpha)\right]\lambda_1 + \beta(3\delta^2 + 3\alpha\delta)\lambda_3 = 0, \\[2mm] \beta(\frac{5}{2}\alpha^2 + \frac{1}{2}\alpha\delta + \beta^2)\lambda_1 + \delta(\alpha^2 + \beta^2)\lambda_3 = 0, \\[2mm] \beta(2\alpha^2 + 3\alpha\delta + 2\delta^2)\lambda_1 + \delta(\alpha\delta + 2\delta^2)\lambda_3 = 0. \end{cases}$$

When $\delta = 0$, so $2\alpha^2 - \beta^2 \neq 0$. Put this into other equations, Eq.(4.30)would reduce to $\lambda_1 = 0$, so $\mathbb{V}_{\mathbb{RC}} = \langle e_2 + e_3 \rangle$.
When $\delta \neq 0$, we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
If $2\alpha^2 - \beta^2(3\delta + \alpha) = 0$, $\beta \neq 0$, otherwise $\alpha = 0$ in error.
Furthermore if $\delta = 0$, we get $2\alpha^2 - \beta^2 = 0, \lambda_0$. Put these into Eq.(4.30) , we get $\lambda_2 = \lambda_3$.
If $\delta \neq 0$, then we have $\delta = \frac{\alpha(2\alpha^2 - \beta^2)}{3\beta^2}$. We assume that

(4.31)
$$\begin{cases} a_{11} = 0, a_{12} = -(\alpha^2 + \frac{5}{2}\delta^2 + 2\alpha\delta), a_{13} = \alpha^2 + \frac{3}{2}\alpha\delta + \delta^2 \\[2mm] a_{21} = \frac{5}{2}\alpha^2 + \frac{1}{2}\alpha\delta + \beta^2, a_{22} = \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{1}{2}\alpha\delta), a_{23} = \delta(\frac{1}{2}\alpha\delta + \delta^2) \\[2mm] a31 = \beta(2\alpha^2 + 3\alpha\delta + 2\delta^2), a_{32} = \delta(\alpha\delta + 2\delta^2), a_{33} = 0. \end{cases}$$

We get

$$(4.32) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$$

So we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

5. Acknowledgements

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