## INVARIANT RICCI COLLINEATIONS ASSOCIATED TO THE BOTT CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

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Abstract. In this paper, we determine all left-invariant Ricci collineations associated to the Bott connection with three distributions on three-dimensional Lorentzian Lie groups .

#### 1. INTRODUCTION

"Symmetry" can be regarded as a one-parameter group of diffeomorphism of spacetime, and it preserves mathematical and physical quantity. In 1969, based on the different geometric objects' invariance properties, Katzin defined them as those vector fields X, such that leave the various relevant geometric quantities and classified Ricci collineations, curvature collineations in [\[1\]](#page-28-0). Collineations are symmetry properties of space-times, so Ricci collineations i.e. Ricci symmetry, which is defined by  $(L_{\xi}Ric) = 0$ . Because Ricci Collineations' close connection with the energy-momentum tensor, inspired more mathematicians' research interesting. After that Carot in [\[2\]](#page-28-1) described the general proper Ricci collineations in type B warped space-times for spherically symmetric metrics in 1997. Next year Qadir and Ziad discussed the Ricci collineations of spherically symmetric spacetimes in [\[3\]](#page-28-2). Ricci Collineations have been discussed and determined in more different spacetimes and for various other ricci tensor in  $[4, 5, 6, 7]$  $[4, 5, 6, 7]$  $[4, 5, 6, 7]$  $[4, 5, 6, 7]$ .

In [\[8,](#page-29-4) [9,](#page-29-5) [10\]](#page-29-6), the Bott connection has been introduced. And three-dimensional Lorentzian Lie groups had been divided into  $\{G_i\}_{i=1,\dots,7}$  in [\[15,](#page-29-7) [17\]](#page-29-8), by this classification, Yong Wang defined a product structure on three-dimensional Lorentzian Lie groups and compute canonical connections and Kobayashi-Nomizu connections and their curvature with this product structure in [\[11\]](#page-29-9). He also classified algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure. Simially, in [\[12\]](#page-29-10) Tong Wu computed the Bott connections and their Ricci tensor and classified affine solitons associated to the Bott connections. In this paper, we determine all leftinvariant Ricci collineations associated to three different Bott connections which is defined by three different distributions in [\[12\]](#page-29-10) on  $\{G_i\}_{i=1,\dots,7}$  ( $\{G_i\}_{i=1,\dots,4}$  is three-dimensional unimodular Lorentzian Lie groups.  $\{G_i\}_{i=5,\cdots,7}$  is three-dimensional non-unimodular Lorentzian Lie groups).

In section 2, we recall the definition of the Bott connection  $\nabla^{B_1}$  with the first distribution, and then give left-invariant Ricci collineations associated to the Bott connection  $\nabla^{B_1}$  on threedimensional Lorentzian unimodular and non-unimodular Lie groups. In section 3, we recall the definition of the Bott connection  $\nabla^{B_2}$  with the second distribution, and then give left-invariant

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Ricci collineations associated to the Bott connection  $\nabla^{B_2}$  on three-dimensional Lorentzian unimodular and non-unimodular Lie groups. In section 4, we recall the definition of the Bott connection  $\nabla^{B_3}$  with the first distribution, and then give left-invariant Ricci collineations associated to the Bott connection  $\nabla^{B_3}$  on three-dimensional Lorentzian unimodular and non-unimodular Lie groups.

## 2. Invariant Ricci collineations associated to the Bott connection on three-dimensional Lorentzian Unimodular Lie groups with the first **DISTRIBUTION**

Calvaruso and Cordero classified three-dimensional Lorentzian Lie groups in [\[15,](#page-29-7) [17\]](#page-29-8)(see Theorem 2.1 and Theorem 2.2 in [\[14\]](#page-29-11)). Throughout this paper, we shall denote the connected by  ${G_i}_{i=1,\dots,7}$ , simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra  $\{\mathfrak{g}\}_{i=1,\cdots,7}$ .

Then we recall the definition of the Bott connection  $\nabla^{B_1}$ . Let M be a smooth manifold, and let  $TM = span\{e_1, e_2, e_3\}$ , then took the frst distribution:  $F_1 = span\{e_1, e_2\}$  and  $F_1^{\perp} = span\{e_3\}$ , where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike. The Bott connection  $\nabla^{B_1}$  is defined as follows: (see  $[8]$ ,  $[9]$ ,  $[10]$ )

(2.1) 
$$
\nabla_X^{B_1} Y = \begin{cases} \pi_{F_1}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_1) \\ \pi_{F_1}([X, Y]), & X \in \Gamma^\infty(F_1^\perp), Y \in \Gamma^\infty(F_1) \\ \pi_{F_1^\perp}([X, Y]), & X \in \Gamma^\infty(F_1), Y \in \Gamma^\infty(F_1^\perp) \\ \pi_{F_1^\perp}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_1^\perp) \end{cases}
$$

where  $\pi_{F_1}$  and  $\pi_F^{\perp}$  $\downarrow_{F_1}$  are respectively the projection on  $F_1$  and  $F_1^{\perp}$ ,  $\nabla^L$  is the Levi-Civita connection of  $G_i$ .

We define the curvature tensor of the Bott connection  $\nabla^{B_1}$ 

(2.2) 
$$
R^{B_1}(X,Y)Z = \nabla_X^{B_1} \nabla_Y^{B_1} Z - \nabla_Y^{B_1} \nabla_X^{B_1} Z - \nabla_{[X,Y]}^{B_1} Z.
$$

The Ricci tensor of  $(G_i, g)$  associated to the Bott connection  $\nabla^{B_1}$  is defined by

$$
(2.3) \qquad \rho^{B_1}(X,Y) = -g(R^{B_1}(X,e_1)Y,e_1) - g(R^{B_1}(X,e_2)Y,e_2) + g(R^{B_1}(X,e_3)Y,e_3).
$$

Let

(2.4) 
$$
Ric^{B_1}(X,Y) = \frac{\rho^{B_1}(X,Y) + \rho^{B_1}(Y,X)}{2}.
$$

We define:

$$
(2.5) \qquad (L_V Ric^{B_1})(X,Y) := V[Ric^{B_1}(X,Y)] - Ric^{B_1}([V,X],Y) - Ric^{B_1}(X,[V,Y])
$$

for vector  $X, Y, V$ .

**Theorem 2.1.**  $(G_i, g)$  *admits left-invariant Ricci collineations associated to the Bott connection*  $\nabla^{B_1}$  *if and only if it satisfies* 

(2.6) 
$$
(L_V Ric^{B_1})(X,Y) = 0,
$$

*where*  $V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$  *is a left-invariant vector field and*  $\lambda_1, \lambda_2, \lambda_3$  *are real numbers.* 

# 2.1 Invariant Ricci collineations of  $G_1$  associated to the Bott connection  $\nabla^{B_1}$

By  $[14]$ , we have the following Lie algebra of  $G_1$  satisfies

$$
(2.7) \qquad [e_1, e_2] = \alpha e_1 - \beta e_3, \; [e_1, e_3] = -\alpha e_1 - \beta e_2, \; [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \; \alpha \neq 0.
$$

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.2.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_1, g)$  associated to the Bott connection  $\nabla^{B_1}$  is deter*mined by*

(2.8) 
$$
Ric^{B_1}(e_1, e_1) = -(\alpha^2 + \beta^2), \ Ric^{B_1}(e_1, e_2) = \alpha\beta, \ Ric^{B_1}(e_1, e_3) = -\frac{\alpha\beta}{2},
$$

$$
Ric^{B_1}(e_2, e_2) = -(\alpha^2 + \beta^2), \ Ric^{B_1}(e_2, e_3) = \frac{\alpha^2}{2}, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By (2.5) and Lemma 2.2, we have

### Lemma 2.3.

(2.9) 
$$
(L_V Ric^{B_1})(e_1, e_1) = -\alpha (2\alpha^2 + \beta^2)\lambda_2 + 2\alpha^3 \lambda_3,
$$

$$
(L_V Ric^{B_1})(e_1, e_2) = \alpha (\alpha^2 + \frac{\beta^2}{2})\lambda_1 + \frac{\alpha^2 \beta}{2} \lambda_2 - \frac{\alpha^2 \beta}{2} \lambda_3,
$$

$$
(L_V Ric^{B_1})(e_1, e_3) = -\alpha^3 \lambda_1 + \beta^3 \lambda_2,
$$

$$
(L_V Ric^{B_1})(e_2, e_2) = -\alpha^2 \beta \lambda_1 - \alpha^3 \lambda_3,
$$

$$
(L_V Ric^{B_1})(e_2, e_3) = \beta (\frac{\alpha^2}{2} - \beta^2)\lambda_1 + \frac{\alpha^3}{2} \lambda_2 + \frac{\alpha}{2} (\alpha^2 - \beta^2)\lambda_3,
$$

$$
(L_V Ric^{B_1})(e_3, e_3) = \alpha (\beta^2 - \alpha^2)\lambda_2.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.3 and Theorem 2.1, we have the following equations:

(2.10)  
\n
$$
\begin{cases}\n-\alpha(2\alpha^2 + \beta^2)\lambda_2 + 2\alpha^3\lambda_3 = 0 \\
\alpha(\alpha^2 + \frac{\beta^2}{2})\lambda_1 + \frac{\alpha^2\beta}{2}\lambda_2 - \frac{\alpha^2\beta}{2}\lambda_3 = 0 \\
-\alpha^3\lambda_1 + \beta^3\lambda_2 = 0 \\
-\alpha^2\beta\lambda_1 - \alpha^3\lambda_3 = 0 \\
\beta(\frac{\alpha^2}{2} - \beta^2)\lambda_1 + \frac{\alpha^3}{2}\lambda_2 + \frac{\alpha}{2}(\alpha^2 - \beta^2)\lambda_3 = 0 \\
\alpha(\beta^2 - \alpha^2)\lambda_2 = 0\n\end{cases}
$$

By solving (2.10) , we get

**Theorem 2.4.**  $(G_1, g, V)$  does not admit left-invariant Ricci collineations associated to the Bott *connection*  $\nabla^{B_1}$ .

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**Proof.** We know that  $\alpha \neq 0$ . By the sixth equation, we get  $(\beta^2 - \alpha^2)\lambda_2 = 0$ , so *case* 1) *If*  $\alpha = \beta$ *, by* (2.10)*,* 

(2.11) 
$$
\begin{cases}\n-3\lambda_2 + 2\lambda_3 = 0 \\
3\lambda_1 + \lambda_2 - \lambda_3 = 0 \\
-\lambda_1 + \lambda_2 = 0 \\
\lambda_1 + \lambda_3 = 0\n\end{cases}
$$

*then we get*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2) *If*  $\alpha = -\beta$ *, by* (2.10)*,* 

(2.12) 
$$
\begin{cases}\n-3\lambda_2 + 2\lambda_3 = 0 \\
3\lambda_1 - \lambda_2 + \lambda_3 = 0 \\
\lambda_1 + \lambda_2 = 0 \\
\lambda_1 - \lambda_3 = 0\n\end{cases}
$$

*then we get*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 3) If  $\lambda_2 = 0$ , by the third equation, we get  $\alpha^3 \lambda_1 = 0$ , i.e.  $\lambda_1 = 0$ , then by the fourth *equation, we get*  $\alpha^3 \lambda_3 = 0$ *, i.e.*  $\lambda_3 = 0$ *, so*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. So we have no non-trivial solution.*

# 2.2 Invariant Ricci collineations of  $G_2$  associated to the Bott connection  $\nabla^{B_1}$

By  $[14]$ , we have the following Lie algebra of  $G_2$  satisfies

(2.13) 
$$
[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1, \quad \gamma \neq 0.
$$

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.5.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_2, g)$  associated to the Bott connection  $\nabla^{B_1}$  is deter*mined by*

(2.14) 
$$
Ric^{B_1}(e_1, e_1) = -(\beta^2 + \gamma^2), \ Ric^{B_1}(e_1, e_2) = 0, \ Ric^{B_1}(e_1, e_3) = 0,
$$

$$
Ric^{B_1}(e_2, e_2) = -(\gamma^2 + \alpha\beta), \ Ric^{B_1}(e_2, e_3) = -\frac{\alpha\gamma}{2}, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By (2.5) and Lemma 2.5, we have

### Lemma 2.6.

(2.15)  
\n
$$
(L_V Ric^{B_1})(e_1, e_1) = 0,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_2) = -\gamma(\gamma^2 + \frac{\alpha \beta}{2})\lambda_2 + \gamma^2(\beta - \frac{\alpha}{2})\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_3) = \alpha(\beta^2 + \frac{\gamma^2}{2})\lambda_2 + \frac{\alpha \beta \gamma}{2}\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_2) = \gamma(2\gamma^2 + \alpha \beta)\lambda_1,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_3) = -\beta(\gamma^2 + \alpha \beta)\lambda_1,
$$
\n
$$
(L_V Ric^{B_1})(e_3, e_3) = -\alpha \beta \gamma \lambda_1.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.4 and Theorem 2.1, we have the following equations:

(2.16)  
\n
$$
\begin{cases}\n-\gamma(\gamma^2 + \frac{\alpha\beta}{2})\lambda_2 + \gamma^2(\beta - \frac{\alpha}{2})\lambda_3 = 0 \\
\alpha(\beta^2 + \frac{\gamma^2}{2})\lambda_2 + \frac{\alpha\beta\gamma}{2}\lambda_3 = 0 \\
\gamma(2\gamma^2 + \alpha\beta)\lambda_1 = 0 \\
-\beta(\gamma^2 + \alpha\beta)\lambda_1 = 0 \\
-\alpha\beta\gamma\lambda_1 = 0\n\end{cases}
$$

By solving (2.16) , we get

**Theorem 2.7.**  $(G_2, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:* 

- (1)  $\alpha = 0, \beta = 0, \gamma \neq 0,$ (2)  $\alpha = 0, \beta \neq 0, \gamma \neq 0,$ (3)  $\alpha \neq 0, \beta \neq 0, \gamma \neq 0, 2\beta^3 - 2\alpha\beta^2 - \frac{\alpha\gamma^2}{2}$  $\frac{1}{2} = 0.$ *Moreover, in these cases, we have* (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 \rangle$ .
- (2)  $\mathscr{V}_{\mathscr{R}C} = \langle \frac{\beta}{\gamma} \rangle$  $\frac{p}{\gamma}e_2 + e_3 >.$ (3)  $\mathscr{V}_{\mathscr{R}C} = \langle -\frac{\beta \gamma}{2 \beta^2} \rangle$  $\frac{P_1}{2\beta^2 + \gamma^2}e_2 + e_3 >.$

where  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_2, g, V)$ *.* 

**Proof.** We know that  $\gamma \neq 0$ , by the fifth equation, we have  $\alpha \beta \lambda_1 = 0$ . *case* 1) *If*  $\lambda_1 \neq 0$ *, then*  $\alpha\beta = 0$ *. Note that by the third equation, we have* 

$$
\gamma(2\gamma^2 + \alpha\beta)\lambda_1 = 2\gamma^3\lambda_1 = 0,
$$

*i.e.*  $\gamma = 0$ *. This is an contracdiction.* 

*case* 2) If  $\lambda_1 = 0$ , then the third, foyrth, fifth equation trivially holds. *case* 2-1) *If*  $\alpha = 0$ *, then by the first equation, we have*  $-\gamma^3 \lambda_2 + \gamma^2 \beta \lambda_3 = 0$ *, i.e.* 

$$
-\gamma \lambda_2 + \beta \lambda_3 = 0.
$$

*case* 2-1-1) *If*  $\beta = 0$ *, then*  $\lambda_2 = 0$ *. We get* (1)*. case* 2-1-2) *If*  $\beta \neq 0$ *, then*  $-\gamma \lambda_2 + \beta \lambda_3 = 0$ *, i.e.*  $\lambda_2 = \frac{\beta}{\gamma}$  $\frac{\beta}{\gamma}\lambda_3$ *.* We get (2). *case* 2-2) *If*  $\alpha \neq 0$ *, by* (2.16)*, we have* 

(2.17) 
$$
\begin{cases} (\gamma^2 + \alpha \beta)\lambda_2 + \gamma(\beta - \frac{\alpha}{2})\lambda_3 = 0\\ (\beta^2 + \frac{\gamma^2}{2})\lambda_2 + \frac{\beta \gamma}{2}\lambda_3 = 0 \end{cases}
$$

*case* 2-2-1) *If*  $\beta = 0$ *, by* (2.17)*, we have* 

(2.18) 
$$
\begin{cases} \gamma^2 \lambda_2 + \frac{\alpha \gamma}{2} \lambda_3 = 0 \\ \frac{\alpha \gamma^2}{2} \lambda_2 = 0 \end{cases}
$$

*then we have*  $\lambda_2 = 0, \lambda_3 = 0$ , *i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ . *case* 2-2-2) *If*  $\beta \neq 0$ *, solve* (2.16) *we have* 

$$
2\beta^3 - 2\alpha\beta^2 - \frac{\alpha\gamma^2}{2} = 0,
$$

*then*  $\lambda_2 = -\frac{\beta \gamma}{2 \beta^2}$  $\frac{\beta}{2\beta^2+\gamma^2}\lambda_3$ *.* We get (3) *case* 2-2) *If*  $\alpha = 0$ *, by* (2.16)*, we have* 

$$
-\gamma \lambda_2 + \beta \lambda_3 = 0.
$$

# 2.3 Invariant Ricci collineations of  $G_3$  associated to the Bott connection  $\nabla^{B_1}$

By [\[14\]](#page-29-11), we have the following Lie algebra of  $G_3$  associated to the Bott connection  $\nabla^{B_1}$  satisfies

(2.19) 
$$
[e_1, e_2] = -\gamma e_3, [e_1, e_3] = -\beta e_2, [e_2, e_3] = \alpha e_1.
$$

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.8.**  $(12)$  *The Ricci tensor of*  $(G_2, g)$  *is determined by* 

(2.20) 
$$
Ric^{B_1}(e_1, e_1) = -\beta \gamma, \ Ric^{B_1}(e_1, e_2) = 0, \ Ric^{B_1}(e_1, e_3) = 0,
$$

$$
Ric^{B_1}(e_2, e_2) = -\alpha \gamma, \ Ric^{B_1}(e_2, e_3) = 0, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By (2.5) and Lemma 2.8, we have

### Lemma 2.9.

(2.21) 
$$
(L_V Ric^{B_1})(e_1, e_1) = 0, (L_V Ric^{B_1})(e_1, e_2) = 0,
$$

$$
(L_V Ric^{B_1})(e_1, e_3) = \alpha \beta \gamma \lambda_2, (L_V Ric^{B_1})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_1})(e_2, e_3) = -\alpha \beta \gamma \lambda_1, (L_V Ric^{B_1})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.7 and Theorem 2.1, we have the following equations:

(2.22) 
$$
\begin{cases} \alpha \beta \gamma \lambda_2 = 0 \\ -\alpha \beta \gamma \lambda_1 = 0 \end{cases}
$$

By solving (2.22) , we get

Theorem 2.10.  $(G_3, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:* (1)  $\alpha\beta\gamma=0$ ,  $(2)$   $\alpha \neq 0, \beta \neq 0, \gamma \neq 0.$ *Moreover, in these cases, we have* (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle$ (2)  $\mathcal{V}_{\mathcal{R}C} = \langle e_3 \rangle$ . *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_3, g, V)$ *.* 

**Proof.** *case* 1) If  $\alpha\beta\gamma = 0$ , (2.22) *trivially holds.* We get (1). *case* 2) *If*  $\alpha\beta\gamma \neq 0$ , *i.e.*  $\alpha \neq 0$ ,  $\beta \neq 0$ ,  $\gamma \neq 0$ , then  $\lambda_1 = \lambda_2 = 0$ . We get (2).

# 2.4 Invariant Ricci collineations of  $G_4$  associated to the Bott connection  $\nabla^{B_1}$

By  $[14]$ , we have the following Lie algebra of  $G_4$  satisfies

(2.23)  $[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \eta = 1 \text{ or } -1, [e_1, e_3] = -\beta e_2 + e_3, [e_2, e_3] = \alpha e_1.$ 

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.11.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_4, g)$  associated to the Bott connection  $\nabla^{B_1}$  is de*termined by*

(2.24) 
$$
Ric^{B_1}(e_1, e_1) = -(\beta - \eta)^2, \ Ric^{B_1}(e_1, e_2) = 0, \ Ric^{B_1}(e_1, e_3) = 0,
$$

$$
Ric^{B_1}(e_2, e_2) = 2\alpha\eta - \alpha\beta - 1, \ Ric^{B_1}(e_2, e_3) = \frac{\alpha}{2}, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By  $(2.5)$  and Lemma $(2.11)$ , we have

### Lemma 2.12.

(2.25)  
\n
$$
(L_V Ric^{B_1})(e_1, e_1) = 0,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_2) = (-\alpha \eta + \frac{\alpha \beta}{2} + 1)\lambda_2 + (\beta + \frac{\alpha}{2} - \alpha \eta^2)\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_3) = \alpha [(\beta - \eta)^2 - \frac{1}{2}]\lambda_2 - \frac{\alpha \beta}{2}\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_2) = (2\alpha \eta - \alpha \beta - 2)\lambda_1,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_3) = \beta (2\alpha \eta - \alpha \beta - 1)\lambda_1,
$$
\n
$$
(L_V Ric^{B_1})(e_3, e_3) = \alpha \beta \lambda_1.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.10 and Theorem 2.1, we have the following equations:

(2.26)  

$$
\begin{cases}\n(-\alpha \eta + \frac{\alpha \beta}{2} + 1)\lambda_2 + (\beta + \frac{\alpha}{2} - \alpha \eta^2)\lambda_3 = 0 \\
\alpha[(\beta - \eta)^2 - \frac{1}{2}]\lambda_2 - \frac{\alpha \beta}{2}\lambda_3 = 0 \\
(2\alpha \eta - \alpha \beta - 2)\lambda_1 = 0 \\
\beta(2\alpha \eta - \alpha \beta - 1)\lambda_1 = 0 \\
\alpha \beta \lambda_1 = 0\n\end{cases}
$$

By solving (2.26) , we get

**Theorem 2.13.**  $(G_4, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:* 

(1)  $\alpha \neq 0, \beta = 0, \eta = 1 \text{ or } -1, \alpha \eta = 1,$ 

(2)  $\alpha = 0, \beta = 0, \eta = 1 \text{ or } -1,$ 

(3)  $\alpha = 0, \beta \neq 0, \eta = 1 \text{ or } -1,$ 

(4) 
$$
\alpha \neq 0, \beta \neq 0, \eta = 1 \text{ or } -1, \alpha - 4\beta = 0,
$$

(5)  $\alpha \neq 0, \beta \neq 0, \eta = 1 \text{ or } -1, \beta = \eta.$ 

*Moreover, in these cases, we have*

(1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle,$ (2)  $\mathscr{V}_{\mathscr{R}C} = \langle e_3 \rangle$ , (3)  $\mathscr{V}_{\mathscr{R}C} = \langle -\beta e_2 + e_3 \rangle,$ (4)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 + (2\beta + \frac{1}{\beta})\rangle$  $\frac{1}{\beta} - 4\eta$ )e<sub>3</sub> >, (5)  $\mathscr{V}_{\mathscr{R}C} = \langle -\eta e_2 + e_3 \rangle$ .

*where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_4, g, V)$ *.* 

**Proof.** We know that  $\eta = 1$  or  $-1$ .

*case* 1) *If*  $\lambda_1 \neq 0$  *by the sixth equation,*  $\alpha \beta = 0$ *.* 

*case* 1-1) *If*  $\alpha = 0$ *, by the third equation we have*  $-2\lambda_1 = 0$ *, i.e.*  $\lambda_1 = 0$ *. This is a contradiction. case* 1-2) If  $\alpha \neq 0$ , then  $\beta = 0$ , by the third equation we have  $2\alpha\eta - 2 = 0$ , i.e.  $\alpha\eta = 1$ . By (2.26)*, we have*

(2.27) 
$$
\begin{cases} -\frac{\alpha}{2}\lambda_3 = 0\\ \frac{\alpha}{2}\lambda_2 = 0 \end{cases}
$$

*i.e.*  $\lambda_2 = \lambda_3 = 0$ *. We get* (1)*. case* 2) If  $\lambda_1 = 0$ , by (2.26),

(2.28) 
$$
\begin{cases} (-\alpha \eta + \frac{\alpha \beta}{2} + 1)\lambda_2 + (\beta - \frac{\alpha}{2})\lambda_3 = 0\\ \alpha[(\beta - \eta)^2 - \frac{1}{2}]\lambda_2 - \frac{\alpha \beta}{2}\lambda_3 = 0 \end{cases}
$$

*case* 2-1) *If*  $\alpha = 0$ *, then*  $\lambda_2 + \beta \lambda_3 = 0$ *. case* 2-1-1) *If*  $\beta = 0$ *, then*  $\lambda_2 = 0$ *. We get* (2)*. case* 2-1-2) *If*  $\beta \neq 0$ *, then*  $\lambda_2 = -\beta \lambda_3$ *. We get* (3)*.* 

*case* 2-2) *If*  $\alpha \neq 0$ *, we get* 

(2.29) 
$$
\begin{cases} (-\alpha \eta + \frac{\alpha \beta}{2} + 1)\lambda_2 + (\beta - \frac{\alpha}{2})\lambda_3 = 0\\ [(\beta - \eta)^2 - \frac{1}{2}]\lambda_2 - \frac{\beta}{2}\lambda_3 = 0 \end{cases}
$$

*case* 2-2-1) *If*  $\beta = 0$ *, we have* 

(2.30) 
$$
\begin{cases} (-\alpha \eta + 1)\lambda_2 - \frac{\alpha}{2}\lambda_3 = 0\\ \frac{1}{2}\lambda_2 = 0 \end{cases}
$$

*then*  $\lambda_2 = 0, \lambda_3 = 0$ *, i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2-2-2) *If*  $\beta \neq 0$ , we have  $[(\beta - \eta)^2 - \frac{1}{2}]$  $\frac{1}{2}$ ]( $\beta - \frac{\alpha}{2}$  $\frac{\alpha}{2}$ ) +  $\frac{\beta}{2}$ (- $\alpha \eta + \frac{\alpha \beta}{2}$  $(\frac{2}{2}+1)]\lambda_2 = 0, i.e.$  $(4\beta - \alpha)(\beta - \eta)^2 \lambda_2 = 0.$ 

If 
$$
\lambda_2 = 0
$$
, then  $\lambda_3 = 0$ , i.e.  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ .  
\nIf  $\alpha = 4\beta$ , then  $\lambda_3 = (2\beta + \frac{1}{\beta} - 4\eta)\lambda_2$ . We get (4).  
\nIf  $\beta = \eta$ , then  $\lambda_2 = -\beta\lambda_3$ . We get (5).

# 2.5 Invariant Ricci collineations of  $G_5$  associated to the Bott connection  $\nabla^{B_1}$

By  $[14]$ , we have the following Lie algebra of  $G_5$  satisfies

$$
(2.31) \qquad [e_1, e_2] = 0, \; [e_1, e_3] = \alpha e_1 + \beta e_2, \; [e_2, e_3] = \gamma e_1 + \delta e_2, \; \alpha + \delta \neq 0, \; \alpha \gamma + \beta \delta = 0.
$$

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.14.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_5, g)$  associated to the Bott connection  $\nabla^{B_1}$  is de*termined by*

(2.32) 
$$
Ric^{B_1}(e_i, e_j) = 0.
$$

*for any pairs (i,j).*

By  $(2.5)$  and Lemma $(2.14)$ , we have

### Lemma 2.15.

(2.33) 
$$
(L_V Ric^{B_1})(e_i, e_j) = 0.
$$

Then we have

**Theorem 2.16.** Any left-invariant vector field on  $(G_5, g, V)$  is a left-invariant Ricci collineations *associated to the Bott connection*  $\nabla^{B_1}$ .

### 2.6 Invariant Ricci collineations of  $G_6$  associated to the Bott connection  $\nabla^{B_1}$

By [\[14\]](#page-29-11), we have the following Lie algebra of  $G_6$  satisfies

(2.34)  $[e_1, e_2] = \alpha e_2 + \beta e_3$ ,  $[e_1, e_3] = \gamma e_2 + \delta e_3$ ,  $[e_2, e_3] = 0$ ,  $\alpha + \delta \neq 0$   $\alpha \gamma - \beta \delta = 0$ . where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.17.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_6, g)$  associated to the Bott connection  $\nabla^{B_1}$  is de*termined by*

(2.35) 
$$
Ric^{B_1}(e_1, e_1) = -(\alpha^2 + \beta\gamma), \ Ric^{B_1}(e_1, e_2) = 0, \ Ric^{B_1}(e_1, e_3) = 0,
$$

$$
Ric^{B_1}(e_2, e_2) = -\alpha^2, \ Ric^{B_1}(e_2, e_3) = 0, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By (2.5) and Lemma 2.17, we have

#### Lemma 2.18.

(2.36) 
$$
(L_V Ric^{B_1})(e_1, e_1) = 0, (L_V Ric^{B_1})(e_1, e_2) = -\alpha^3 \lambda_2 - \alpha^2 \gamma \lambda_3,
$$

$$
(L_V Ric^{B_1})(e_1, e_3) = 0 (L_V Ric^{B_1})(e_2, e_2) = 2\alpha^3 \lambda_1,
$$

$$
(L_V Ric^{B_1})(e_2, e_3) = \alpha^2 \gamma \lambda_1, (L_V Ric^{B_1})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.17 and Theorem 2.1, we have the following equations:

(2.37) 
$$
\begin{cases} -\alpha^3 \lambda_2 - \alpha^2 \gamma \lambda_3 = 0 \\ 2\alpha^3 \lambda_1 = 0 \\ \alpha^2 \gamma \lambda_1 = 0 \end{cases}
$$

By solving (2.37) , we get

**Theorem 2.19.**  $(G_6, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:* 

- (1)  $\alpha = 0, \beta = 0, \delta \neq 0,$ (2)  $\alpha \neq 0, \gamma = 0, \alpha + \delta \neq 0, \beta \delta = 0,$ (3)  $\alpha \neq 0, \gamma \neq 0, \alpha + \delta \neq 0, \alpha \gamma - \beta \delta = 0.$ *Moreover, in these cases, we have*
- 
- (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle$ (2)  $\mathscr{V}_{\mathscr{R}C} = \langle e_3 \rangle,$
- (3)  $\mathscr{V}_{\mathscr{R}C} = \langle -\frac{\gamma}{2} \rangle$  $\frac{1}{\alpha}e_2 + e_3 >$ .

where  $\mathscr{V}_{\mathscr{R}C}$  is the vector space of left-invariant Ricci collineations on  $(G_6, g, V)$ .

**Proof.** We know that  $\alpha + \delta \neq 0$ ,  $\alpha\gamma - \beta\delta = 0$ . *case* 1) If  $\alpha = 0$ , then  $\delta \neq 0$ ,  $\beta = 0$ . (2.37) *trivially holds. case* 2) *If*  $\alpha \neq 0$ *, by the second equation we have*  $\lambda_1 = 0$ *. By* (2.37)*, we have* 

$$
\alpha \lambda_2 + \gamma \lambda_3 = 0.
$$

*case* 2-1) *If*  $\gamma = 0$ *, then*  $\lambda_2 = 0$ *,*  $\beta \delta = 0$ *. We get* (2)*. case* 2-2) *If*  $\gamma \neq 0$ , then  $\lambda_2 = -\frac{\gamma}{\gamma}$  $\frac{1}{\alpha}\lambda_3$ *.* We get (3).

# 2.7 Invariant Ricci collineations of  $G_7$  associated to the Bott connection  $\nabla^{B_1}$

By  $[14]$ , we have the following Lie algebra of  $G_7$  satisfies (2.38)  $[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3$ ,  $[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3$ ,  $[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3$ ,  $\alpha + \delta \neq 0$ ,  $\alpha \gamma = 0$ .

where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$  timelike.

**Lemma 2.20.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_7, g)$  associated to the Bott connection  $\nabla^{B_1}$  is de*termined by*

(2.39) 
$$
Ric^{B_1}(e_1, e_1) = -\alpha^2, \ Ric^{B_1}(e_1, e_2) = \frac{\beta(\delta - \alpha)}{2},
$$

$$
Ric^{B_1}(e_1, e_3) = \beta(\alpha + \delta), Ric^{B_1}(e_2, e_2) = -(\alpha^2 + \beta^2 + \beta\gamma),
$$

$$
Ric^{B_1}(e_2, e_3) = \delta^2 + \frac{\beta\gamma + \alpha\delta}{2}, \ Ric^{B_3}(e_3, e_3) = 0.
$$

By (2.5) and Lemma 2.20, we have

### Lemma 2.21.

$$
(2.40)
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_1) = (2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_2 - (2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_2) = (-\alpha^3 + \frac{3\beta^2 \delta}{2} - \frac{\alpha \beta^2}{2})\lambda_1 + \beta(\frac{3\alpha^2}{2} + \beta^2 - \delta^2 - \alpha \delta + \frac{\beta \gamma}{2})\lambda_2
$$
\n
$$
+ \beta(-\frac{3\alpha^2}{2} - \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha \delta}{2} - \frac{\beta \gamma}{2})\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_1, e_3) = (\alpha^3 - \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha \delta + \frac{\beta \gamma}{2})\lambda_2
$$
\n
$$
+ \beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_2) = \beta(-3\alpha^2 - 2\beta^2 + 2\delta^2 + 2\alpha \delta - \beta \gamma)\lambda_1 + \delta(-2\alpha^2 - 2\beta^2 + 2\delta^2 + \alpha \delta)\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_2, e_3) = \beta(\frac{5\alpha^2}{2} + \beta^2 + \frac{\alpha \delta}{2} + \beta \gamma)\lambda_1 + \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{\alpha \delta}{2})\lambda_2,
$$
\n
$$
+ \delta(\delta^2 + \frac{\alpha \delta}{2} + \frac{3\beta \gamma}{2})\lambda_3,
$$
\n
$$
(L_V Ric^{B_1})(e_3, e_3) = -\beta(2\alpha^2 + 2\delta^2 + 3\alpha \delta + \beta \gamma)\lambda_1 - \delta(-2\delta^2 + \alpha \delta + 3\beta \delta)\lambda_3.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 2.19 and Theorem 2.1, we have the following equations:

$$
\begin{cases}\n(2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_2 - (2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_3 = 0 \\
(-\alpha^3 + \frac{3\beta^2 \delta}{2} - \frac{\alpha \beta^2}{2})\lambda_1 + \beta(\frac{3\alpha^2}{2} + \beta^2 - \delta^2 - \alpha \delta + \frac{\beta \gamma}{2})\lambda_2 \\
+ \beta(-\frac{3\alpha^2}{2} - \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha \delta}{2} - \frac{\beta \gamma}{2})\lambda_3 = 0 \\
(\alpha^3 - \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha \delta + \frac{\beta \gamma}{2})\lambda_2 \\
+ \beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_3 = 0 \\
\beta(-3\alpha^2 - 2\beta^2 + 2\delta^2 + 2\alpha \delta - \beta \gamma)\lambda_1 + \delta(-2\alpha^2 - 2\beta^2 + 2\delta^2 + \alpha \delta)\lambda_3 = 0 \\
\beta(\frac{5\alpha^2}{2} + \beta^2 + \frac{\alpha \delta}{2} + \beta \gamma)\lambda_1 + \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{\alpha \delta}{2})\lambda_2 \\
+ \delta(\delta^2 + \frac{\alpha \delta}{2} + \frac{3\beta \gamma}{2})\lambda_3 = 0 \\
-\beta(2\alpha^2 + 2\delta^2 + 3\alpha \delta + \beta \gamma)\lambda_1 - \delta(2\delta^2 + \alpha \delta + 3\beta \gamma)\lambda_3 = 0\n\end{cases}
$$

By solving (2.41) , we get

Theorem 2.22.  $(G_7, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:*  $(1) \alpha = 0, \beta = 0, \delta \neq 0,$ 

 $(2)\alpha = 0, \delta \neq 0, \beta \neq 0, \gamma = 0,$  $(3)\alpha \neq 0, \gamma = 0, \delta = 0.$ *Moreover, in these cases, we have*  $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle,$  $(2)\mathscr{V}_{\mathscr{R}C}=<-\frac{\delta}{\beta}$  $\frac{6}{\beta}e_1 + e_2 + e_3 >$  $(3)\mathscr{V}_{\mathscr{R}C} = \langle e_2 + e_3 \rangle.$ 

*where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_7, g, V)$ *.* 

**Proof.** We know that  $\alpha + \delta \neq 0, \alpha \gamma = 0$ . *case* 1) *If*  $\alpha = 0$ *, then*  $\delta \neq 0$ *. By the first equation,*  $\beta^2 \delta(\lambda_2 - \lambda_3) = 0$ *. case* 1-1) *If*  $\beta = 0$ *, by* (2.41)

(2.42) 
$$
\begin{cases} \delta^3 \lambda_3 = 0 \\ -\delta^3 \lambda_2 + \delta^3 \lambda_3 = 0 \end{cases}
$$

*then*  $\lambda_2 = \lambda_3 = 0$ *. We get* (1)*.* 

*case* 1-2) *If*  $\beta \neq 0$ , then  $\lambda_2 = \lambda_3$ . By the second equation, we have  $3\beta^2 \delta \lambda_1 + 3\beta \delta^2 \lambda_2 = 0$ , *i.e.*  $\beta \lambda_1 + \delta \lambda_2 = 0$ . *Then by the sixth equation, we have*  $2\beta \gamma \delta \lambda_2 = 0$ *, i.e.*  $\gamma \lambda_2 = 0$ . *case* 1-2-1) *If*  $\gamma \neq 0$ *, then*  $\lambda_2 = \lambda_3 = 0$ ,  $\lambda_1 = 0$ *.* 

*case* 1-2-2) *If*  $\gamma = 0$ *, then*  $\beta \lambda_1 + \delta \lambda_2 = 0$ *, i.e.*  $\lambda_1 = -\frac{\delta}{\beta}$  $\frac{\delta}{\beta}\lambda_2=-\frac{\delta}{\beta}$  $\frac{\sigma}{\beta} \lambda_3$ . (2.41) *trivially holds.* We *get* (2)*. case* 2) *If*  $\alpha \neq 0$ , *then*  $\gamma = 0$ ,  $\alpha + \delta \neq 0$ *.* 

case 2-1) If 
$$
2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2 \neq 0
$$
, then  $\lambda_2 = \lambda_3$ . By (2.41)  
\n
$$
\begin{cases}\n(-\alpha^3 + \frac{3\beta^2 \delta}{2} - \frac{\alpha \beta^2}{2})\lambda_1 + \beta(\frac{3\delta^2}{2} + \frac{\alpha \delta}{2})\lambda_2 = 0 \\
\beta(-3\alpha^2 - 2\beta^2 + 2\delta^2 + 2\alpha \delta)\lambda_1 + \delta(-2\alpha^2 - 2\beta^2 + 2\delta^2 + \alpha \delta)\lambda_2 = 0 \\
\beta(\frac{5\alpha^2}{2} + \beta^2 + \frac{\alpha \delta}{2})\lambda_1 + \delta(\alpha^2 + \beta^2)\lambda_2 = 0 \\
-\beta(2\alpha^2 + 2\delta^2 + 3\alpha \delta)\lambda_1 - \delta(2\delta^2 + \alpha \delta)\lambda_3 = 0\n\end{cases}
$$

*case* 2-1-1) *If*  $\delta = 0$ *, then*  $\lambda_1 = 0$ *.* (2.43) *trivially holds. We get* (3)*. case* 2-1-2) *If*  $\delta \neq 0$ *, then*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2-2) *If*  $2\alpha^3 - 3\beta^2\delta - \alpha\beta^2 = 0$ , *by* (2.41)

$$
(2.44) \quad\n\begin{cases}\n\beta(\frac{3\alpha^2}{2} + \beta^2 - \delta^2 - \alpha\delta)\lambda_2 + \beta(-\frac{3\alpha^2}{2} - \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha\delta}{2})\lambda_3 = 0 \\
-\beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha\delta + \frac{\beta\gamma}{2})\lambda_2 + \beta(\alpha^2 + \delta^2 + \frac{3\alpha\delta}{2} + \frac{\beta\gamma}{2})\lambda_3 = 0 \\
\beta(-3\alpha^2 - 2\beta^2 + 2\delta^2 + 2\alpha\delta)\lambda_1 + \delta(-2\alpha^2 - 2\beta^2 + 2\delta^2 + \alpha\delta)\lambda_3 = 0 \\
\beta(\frac{5\alpha^2}{2} + \beta^2 + \frac{\alpha\delta}{2})\lambda_1 + \delta(\alpha^2 + \beta^2 - \delta^2 - \frac{\alpha\delta}{2})\lambda_2 + \delta(\delta^2 + \frac{\alpha\delta}{2})\lambda_3 = 0 \\
-\beta(2\alpha^2 + 2\delta^2 + 3\alpha\delta)\lambda_1 - \delta(2\delta^2 + \alpha\delta)\lambda_3 = 0\n\end{cases}
$$

*If*  $\beta = 0$ *, then*  $\alpha = 0$ *, this is a contradiction, then*  $\beta \neq 0$ *. case* 2-1-1) *If*  $\delta = 0$ *, then*  $2\alpha^2 - \beta^2 = 0$ ,  $\lambda_1 = 0$ ,  $\lambda_2 = \lambda_3$ *. This case is in* (3)*. case* 2-1-2) *If*  $\delta \neq 0$ *, then*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *.* 

## 3. Invariant Ricci collineations associated to the Bott connection on three-dimensional Lorentzian Unimodular Lie groups with the second **DISTRIBUTION**

Let M be a smooth manifold, and let  $TM = span\{e_1, e_2, e_3\}$ , then took the frst distribution:  $F_2 = span\{e_1, e_3\}$  and  $F_2^{\perp} = span\{e_2\}$ , where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$ timelike. The Bott connection  $\nabla^{B_2}$  is defined as follows: (see [\[8\]](#page-29-4), [\[9\]](#page-29-5), [\[10\]](#page-29-6))

(3.1) 
$$
\nabla_X^{B_2} Y = \begin{cases} \pi_{F_2}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_2) \\ \pi_{F_2}([X, Y]), & X \in \Gamma^\infty(F_2^\perp), Y \in \Gamma^\infty(F_2) \\ \pi_{F_2^\perp}([X, Y]), & X \in \Gamma^\infty(F_2), Y \in \Gamma^\infty(F_2^\perp) \\ \pi_{F_2^\perp}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_2^\perp) \end{cases}
$$

where  $\pi_{F_2}$  and  $\pi_F^{\perp}$  $\frac{1}{F_2}$  are respectively the projection on  $F_2$  and  $F_2^{\perp}$ ,  $\nabla^L$  is the Levi-Civita connection of  $G_i$ .

# 3.1 Invariant Ricci collineations of  $G_1$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.1.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_1, g)$  associated to the Bott connection  $\nabla^{B_2}$  is deter*mined by*

(3.2) 
$$
Ric^{B_2}(e_1, e_1) = \alpha^2 - \beta^2, \ Ric^{B_2}(e_1, e_2) = \frac{\alpha \beta}{2}, \ Ric^{B_2}(e_1, e_3) = -\alpha \beta,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = \frac{\alpha^2}{2}, \ Ric^{B_2}(e_3, e_3) = \beta^2 - \alpha^2.
$$

By  $(2.5)$  and Lemma 3.1, we have

### Lemma 3.2.

(3.3)  
\n
$$
(L_V Ric^{B_2})(e_1, e_1) = 2\alpha^3 \lambda_1 + \alpha(\beta^2 - 2\alpha^2) \lambda_2,
$$
\n
$$
(L_V Ric^{B_2})(e_1, e_2) = -\alpha^3 \lambda_1 - \beta^3 \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_1, e_3) = \alpha(\alpha^2 - \frac{\beta^2}{2})\lambda_1 - \frac{\alpha^2 \beta}{2} \lambda_2 + \frac{\alpha^2 \beta}{2} \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_2, e_2) = \alpha(\beta^2 + \alpha^2) \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_2, e_3) = \beta(\frac{\alpha^2}{2} + \beta^2) \lambda_1 - \frac{\alpha}{2}(\alpha^2 + \beta^2) \lambda_2 - \frac{\alpha^3}{2} \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_3, e_3) = -\alpha^2 \beta \lambda_1 + \alpha^3 \lambda_2.
$$

Then, if a left-invariant vector field  $V$  is a Ricci collineation associated to the Bott connection  $\nabla^{B_2}$ , by Lemma 3.2 and Theorem 2.1, we have the following equations:

(3.4)  
\n
$$
\begin{cases}\n2\alpha^3\lambda_1 + \alpha(\beta^2 - 2\alpha^2)\lambda_2 = 0 \\
-\alpha^3\lambda_1 - \beta^3\lambda_3 = 0 \\
\alpha(\alpha^2 - \frac{\beta^2}{2})\lambda_1 - \frac{\alpha^2\beta}{2}\lambda_2 + \frac{\alpha^2\beta}{2}\lambda_3 = 0 \\
\alpha(\beta^2 + \alpha^2)\lambda_3 = 0 \\
\beta(\frac{\alpha^2}{2} + \beta^2)\lambda_1 - \frac{\alpha}{2}(\alpha^2 + \beta^2)\lambda_2 - \frac{\alpha^3}{2}\lambda_3 = 0 \\
-\alpha^2\beta\lambda_1 + \alpha^3\lambda_2 = 0\n\end{cases}
$$

By solving (3.4) , we get

**Theorem 3.3.**  $(G_1, g, V)$  does not admit left-invariant Ricci collineations associated to the Bott *connection*  $\nabla^{B_2}$ .

**Proof.** We know that  $\alpha \neq 0$ . By the fourth equation, and  $\alpha^2 + \beta^2 = 0$ , we have  $\lambda_3 = 0$ . Then *by the second equation, we have*  $\alpha^3 \lambda_1 = 0$ , *i.e.*  $\lambda_1 = 0$ . Finally by the fifth equation, we have  $\alpha(\alpha^2 + \beta^2)\lambda_2 = 0$ , *i.e.*  $\lambda_2 = 0$ . So we have no non-trivial solution.

3.2 Invariant Ricci collineations of  $G_2$  associated to the Bott connection  $\nabla^{B_2}$ 

**Lemma 3.4.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_2, g)$  associated to the Bott connection  $\nabla^{B_2}$  is deter*mined by*

(3.5) 
$$
Ric^{B_2}(e_1, e_1) = -(\beta^2 + \gamma^2), \ Ric^{B_2}(e_1, e_2) = 0, \ Ric^{B_2}(e_1, e_3) = 0,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = -\frac{\alpha\gamma}{2}, \ Ric^{B_2}(e_3, e_3) = \gamma^2 + \alpha\beta.
$$

By (2.5) and Lemma 3.4, we have

Lemma 3.5.

(3.6)  
\n
$$
(L_V Ric^{B_2})(e_1, e_1) = 0,
$$
\n
$$
(L_V Ric^{B_2})(e_1, e_2) = \frac{\alpha \beta \gamma}{2} \lambda_2 - \alpha (\beta^2 + \frac{\gamma^2}{2}) \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_1, e_3) = \gamma^2 (\frac{\alpha}{2} - \beta) \lambda_2 - \gamma (\frac{\alpha \beta}{2} + \gamma^2) \lambda_3,
$$
\n
$$
(L_V Ric^{B_2})(e_2, e_2) = -\alpha \beta \gamma \lambda_1,
$$
\n
$$
(L_V Ric^{B_2})(e_2, e_3) = \beta (\alpha \beta + \gamma^2) \lambda_1,
$$
\n
$$
(L_V Ric^{B_2})(e_3, e_3) = \gamma (\alpha \beta + 2\gamma^2) \lambda_1.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_2}$ , by Lemma 3.5 and Theorem 2.1, we have the following equations:

(3.7)  
\n
$$
\begin{cases}\n\frac{\alpha\beta\gamma}{2}\lambda_2 - \alpha(\beta^2 + \frac{\gamma^2}{2})\lambda_3 = 0 \\
\gamma^2(\frac{\alpha}{2} - \beta)\lambda_2 - \gamma(\frac{\alpha\beta}{2} + \gamma^2)\lambda_3 = 0 \\
-\alpha\beta\gamma\lambda_1 = 0 \\
\beta(\alpha\beta + \gamma^2)\lambda_1 = 0 \\
\gamma(\alpha\beta + 2\gamma^2)\lambda_1 = 0\n\end{cases}
$$

By solving (3.7) , we get

**Theorem 3.6.**  $(G_2, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_2}$  *if and only if one of the following holds:* 

(1)  $\alpha = 0, \beta = 0, \gamma \neq 0,$  $(2) \alpha = 0, \beta \neq 0, \gamma \neq 0,$ (3)  $\alpha \neq 0, \beta \neq 0, \alpha - 4\beta = 0.$ *Moreover, in these cases, we have* (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 \rangle$ , (2)  $\mathscr{V}_{\mathscr{R}C} = \langle -\frac{\gamma}{\beta} \rangle$  $\frac{1}{\beta}e_2 + e_3 >$ (3)  $\mathscr{V}_{\mathscr{R}C} = \langle \frac{2\beta}{\gamma} \rangle$  $rac{2\beta}{\gamma}+\frac{\gamma}{\beta}$  $(\frac{1}{\beta})e_2 + e_3 >.$ *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_2, g, V)$ *.* 

**Proof.** We know that  $\gamma \neq 0$ . *case* 1) *If*  $\lambda_1 \neq 0$ , by the third equation, then  $\alpha\beta = 0$ . Note that by the fifth equation, we have

$$
\gamma(\alpha\beta + 2\gamma^2)\lambda_1 = 2\gamma^3\lambda_1 = 0,
$$

*i.e.*  $\gamma = 0$ *. This is a contradiction. case* 2) If  $\lambda_1 = 0$ , then the third, fourth, fifth equations trivially holds. *case* 2-1) *If*  $\alpha = 0$ *, then by the second equation, we have*  $\gamma^2 \beta \lambda_2 + \gamma^3 \lambda_3 = 0$ *, i.e.* 

$$
\beta \lambda_2 + \gamma \lambda_3 = 0.
$$

*case* 2-1-1) *If*  $\beta = 0$ *, then*  $\lambda_3 = 0$ *. We get* (1)*. case* 2-1-2) *If*  $\beta \neq 0$ *, then*  $\beta \lambda_2 + \gamma \lambda_3 = 0$ *, i.e.*  $\lambda_2 = -\frac{\gamma}{\beta}$  $\frac{1}{\beta}\lambda_3$ *.* We get (2). *case* 2-2) *If*  $\alpha \neq 0$ *, by* (3.7)*, we have* 

(3.8) 
$$
\begin{cases} \frac{\beta \gamma}{2} \lambda_2 - (\beta^2 + \frac{\gamma^2}{2}) \lambda_3 = 0\\ \gamma(\frac{\alpha}{2} - \beta) \lambda_2 - (\frac{\alpha \beta}{2} + \gamma^2) \lambda_3 = 0 \end{cases}
$$

*case* 2-2-1) *If*  $\beta = 0$ *, by* (3.7)*, we have* 

(3.9) 
$$
\begin{cases} -\frac{\gamma^2}{2}\lambda_3 = 0\\ \frac{\alpha \gamma^2}{2}\lambda_2 - \gamma^2 \lambda_3 = 0 \end{cases}
$$

*then we have*  $\lambda_3 = 0, \lambda_2 = 0$ , *i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2-2-2) *If*  $\beta \neq 0$ *, solve* (3.8) *we have* 

$$
(\frac{\alpha}{2} - 2\beta)(\beta^2 + \gamma^2)\lambda_3 = 0,
$$

If 
$$
\lambda_3 = 0
$$
, then  $\lambda_2 = 0$ , i.e.  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ .  
If  $\alpha - 4\beta = 0$ , then  $\lambda_2 = \left(\frac{2\beta}{\gamma} + \frac{\gamma}{\beta}\right)\lambda_3$ . We get (3).

# 3.3 Invariant Ricci collineations of  $G_3$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.7.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_3, g)$  associated to the Bott connection  $\nabla^{B_2}$  is deter*mined by*

(3.10) 
$$
Ric^{B_2}(e_1, e_1) = -\beta \gamma, \ Ric^{B_2}(e_1, e_2) = 0, \ Ric^{B_2}(e_1, e_3) = 0,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = 0, \ Ric^{B_2}(e_3, e_3) = \alpha \beta.
$$

By (2.5) and Lemma 3.7, we have

### Lemma 3.8.

(3.11) 
$$
(L_V Ric^{B_2})(e_1, e_1) = 0, (L_V Ric^{B_2})(e_1, e_2) = -\alpha \beta \gamma \lambda_3,
$$

$$
(L_V Ric^{B_2})(e_1, e_3) = 0, (L_V Ric^{B_2})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_2})(e_2, e_3) = \alpha \beta \gamma \lambda_1, (L_V Ric^{B_2})(e_3, e_3) = 0.
$$

Ricci collineations and the state of the

Then, if a left-invariant vector field  $V$  is a Ricci collineation associated to the Bott connection  $\nabla^{B_2}$ , by Lemma 3.8 and Theorem 2.1, we have the following equations:

(3.12) 
$$
\begin{cases} -\alpha \beta \gamma \lambda_3 = 0 \\ \alpha \beta \gamma \lambda_1 = 0 \end{cases}
$$

By solving (3.12) , we get

Theorem 3.9.  $(G_3, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_2}$  *if and only if one of the following holds:* (1)  $\alpha\beta\gamma=0$ ,  $(2)$   $\alpha \neq 0, \beta \neq 0, \gamma \neq 0.$ *Moreover, in these cases, we have* (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle$ (2)  $\mathcal{V}_{\mathcal{R}C} = \langle e_2 \rangle$ . *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_3, g, V)$ *.* 

**Proof.** *case* 1) *If*  $\alpha\beta\gamma = 0$ *, the above equations trivially holds. We get* (1)*. case* 2) *If*  $\alpha\beta\gamma \neq 0$ , *i.e.*  $\alpha \neq 0$ ,  $\beta \neq 0$ ,  $\gamma \neq 0$ , then  $\lambda_1 = \lambda_3 = 0$ . We get (2)

# 3.4 Invariant Ricci collineations of  $G_4$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.10.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_4, g)$  associated to the Bott connection  $\nabla^{B_2}$  is de*termined by*

(3.13) 
$$
Ric^{B_2}(e_1, e_1) = -(\beta - \eta)^2, \ Ric^{B_2}(e_1, e_2) = 0, \ Ric^{B_2}(e_1, e_3) = 0,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = \frac{\alpha}{2}, \ Ric^{B_2}(e_1, e_3) = \alpha\beta + 1.
$$

By  $(2.5)$  and Lemma 3.10, we have

### Lemma 3.11.

*.*

(3.14) 
$$
(L_V Ric^{B_2})(e_1, e_1) = 0,
$$

$$
(L_V Ric^{B_2})(e_1, e_2) = \frac{\alpha}{2}(2\eta - \beta)\lambda_2 + \alpha[\frac{1}{2} - (\beta - \eta)^2]\lambda_3,
$$

$$
(L_V Ric^{B_2})(e_1, e_3) = (-\frac{\alpha}{2} + 2\eta - \beta + \alpha\eta^2)\lambda_2 + (\frac{\alpha\beta}{2} + 1)\lambda_3,
$$

$$
(L_V Ric^{B_2})(e_2, e_2) = -\alpha(2\eta - \beta)\lambda_1,
$$

$$
(L_V Ric^{B_2})(e_2, e_3) = -(2\eta - \beta)(\alpha\beta + 1)\lambda_1,
$$

$$
(L_V Ric^{B_2})(e_3, e_3) = -(\alpha\beta + 2)\lambda_1.
$$

(3.15)  

$$
\begin{cases}\n\frac{\alpha}{2}(2\eta - \beta)\lambda_2 + \alpha[\frac{1}{2} - (\beta - \eta)^2]\lambda_3 = 0 \\
(-\frac{\alpha}{2} + 2\eta - \beta + \alpha\eta^2)\lambda_2 + (\frac{\alpha\beta}{2} + 1)\lambda_3 = 0 \\
-\alpha(2\eta - \beta)\lambda_1 = 0 \\
-(2\eta - \beta)(\alpha\beta + 1)\lambda_1 = 0 \\
-(\alpha\beta + 2)\lambda_1 = 0\n\end{cases}
$$

By solving (3.15) , we get

Theorem 3.12.  $(G_4, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_2}$  *if and only if one of the following holds:* 

(1)  $\alpha\beta + 2 = 0, 2\eta - \beta = 0, \eta = 1$  or  $-1$ , (2)  $\alpha = 0, 2\eta - \beta = 0, \eta = 1 \text{ or } -1,$ (3)  $\alpha = 0, 2\eta - \beta \neq 0, \eta = 1 \text{ or } -1,$ (4)  $\alpha \neq 0, 2\eta - \beta \neq 0, \eta = 1 \text{ or } -1, \alpha = -4(2\eta - \beta),$ (5)  $\alpha \neq 0, \beta = \eta, \eta = 1 \text{ or } -1.$ 

*Moreover, in these cases, we have*

- (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle$ ,
- (2)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 \rangle$ ,
- (3)  $\mathscr{V}_{\mathscr{R}C} = < -\frac{1}{2m}$  $\frac{1}{2\eta - \beta}e_2 + e_3 >$

(4) 
$$
\mathscr{V}_{\mathscr{R}C} = \langle -(\frac{\alpha}{2} + 4\eta + \frac{4}{\alpha})e_2 + e_3 \rangle
$$
,  
(5)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 - \eta e_3 \rangle$ .

*where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_4, g, V)$ *.* 

### **Proof.** We know that  $\eta = 1$  or  $-1$ .

*case* 1) *If*  $\lambda_1 \neq 0$ *, by the sixth equation,*  $\alpha\beta + 2 = 0$ *. Then by the fourth equation,*  $(2\eta - \beta)\lambda_1 = 0$ *, i.e.*  $2\eta - \beta = 0$ *. By* (3.15)*, we have* 

(3.16) 
$$
\begin{cases} -\frac{\alpha}{2}\lambda_3 = 0\\ \frac{\alpha}{2}\lambda_2 - \lambda_3 = 0 \end{cases}
$$

*because*  $\eta = 1$  (or − 1)*, we have*  $\alpha = -1$  (or1)  $\neq 0$ *, i.e.*  $\lambda_2 = \lambda_3 = 0$ *. We get* (1)*. case* 2) If  $\lambda_1 = 0$ , then the third, fourth, fifth equations trivially holds. *case* 2-1) *If*  $\alpha = 0$ *, then by the second equation, we have*  $(2\eta - \beta)\lambda_2 + \lambda_3 = 0$ *. case* 2-1-1) *If*  $2\eta - \beta = 0$ *, then*  $\lambda_3 = 0$ *. We get* (2)*. case* 2-1-2) *If*  $2\eta - \beta \neq 0$ , then  $\lambda_2 = -\frac{1}{2\eta}$  $\frac{1}{2\eta-\beta}\lambda_3$ *.* We get (3). *case* 2-2) *If*  $\alpha \neq 0$ *, we get* 

(3.17) 
$$
\begin{cases} (2\eta - \beta)\lambda_2 + [1 - 2(\beta - \eta)^2]\lambda_3 = 0\\ (\frac{\alpha}{2} + 2\eta - \beta)\lambda_2 + (\frac{\alpha\beta}{2} + 1)\lambda_3 = 0 \end{cases}
$$

*case* 2-2-1) *If*  $2\eta - \beta = 0$ *, then*  $\lambda_3 = 0, \lambda_2 = 0$ *, i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2-2-2) *If*  $2\eta - \beta \neq 0$ , we have  $[(\frac{\alpha}{2} + 2\eta - \beta)[1 - 2(\beta - \eta)^2] - (\frac{\alpha\beta}{2})$  $(\frac{2\pi}{2}+1)(2\eta-\beta)]\lambda_2=0, i.e.$  $\left[\frac{\alpha}{2}\right]$  $\frac{\alpha}{2} + 2(2\eta - \beta)[(\beta - \eta)^2 \lambda_2 = 0.$ *If*  $\lambda_2 = 0$ *, then*  $\lambda_3 = 0$ *, i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *.* 

*If*  $\alpha = -4(2\eta - \beta)$ *, then*  $\lambda_2 = -(\frac{\alpha}{2})$  $\frac{\alpha}{2} + \frac{4}{\alpha}$  $\frac{1}{\alpha} + 4\eta \lambda_3$ *. We get* (4)*. If*  $\beta = \eta$ *, then*  $\lambda_3 = -\eta \lambda_2$ *. We get* (5)*.* 

# 3.5 Invariant Ricci collineations of  $G_5$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.13.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_5, g)$  associated to the Bott connection  $\nabla^{B_2}$  is de*termined by*

(3.18) 
$$
Ric^{B_2}(e_1, e_1) = \alpha^2, \ Ric^{B_2}(e_1, e_2) = 0, \ Ric^{B_2}(e_1, e_3) = 0,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = 0, \ Ric^{B_2}(e_3, e_3) = -(\alpha^2 + \beta\gamma).
$$

By (2.5) and Lemma 3.13, we have

#### Lemma 3.14.

(3.19) 
$$
(L_V Ric^{B_2})(e_1, e_1) = -2\alpha^3 \lambda_3, \ (L_V Ric^{B_2})(e_1, e_2) = \alpha^2 \gamma \lambda_3, (L_V Ric^{B_2})(e_1, e_3) = -\alpha^3 \lambda_1 - \alpha^2 \gamma \lambda_2 \ (L_V Ric^{B_2})(e_2, e_2) = 0, (L_V Ric^{B_2})(e_2, e_3) = 0, \ (L_V Ric^{B_2})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_2}$ , by Lemma 3.14 and Theorem 2.1, we have the following equations:

(3.20)  

$$
\begin{cases}\n-2\alpha^3 \lambda_3 = 0 \\
\alpha^2 \gamma \lambda_3 = 0 \\
-\alpha^3 \lambda_1 - \alpha^2 \gamma \lambda_2 = 0\n\end{cases}
$$

By solving (3.20) , we get

Theorem 3.15.  $(G_5, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_2}$  *if and only if one of the following holds:* 

(1)  $\alpha = 0, \beta = 0, \delta \neq 0,$ 

- (2)  $\alpha \neq 0, \gamma = 0, \alpha + \delta \neq 0, \beta \delta = 0,$
- (3)  $\alpha \neq 0, \gamma \neq 0, \alpha + \delta \neq 0, \alpha \gamma + \beta \delta = 0.$ *Moreover, in these cases, we have*
- (1)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle$
- (2)  $\mathscr{V}_{\mathscr{R}C} = \langle e_2 \rangle$ ,
- (3)  $\mathscr{V}_{\mathscr{R}C} = \langle -\frac{\gamma}{2} \rangle$  $\frac{1}{\alpha}e_2 + e_3 >$ .

where  $\mathscr{V}_{\mathscr{R}C}$  is the vector space of left-invariant Ricci collineations on  $(G_5, g, V)$ .

**Proof.** We know that  $\alpha + \delta \neq 0$ ,  $\alpha \gamma + \beta \delta = 0$ . *case* 1) *If*  $\alpha = 0$ *, then*  $\delta \neq 0, \beta = 0$ *.* (3.20) *trivially holds. We get* (1)*. case* 2) *If*  $\alpha \neq 0$ *, by the first equation we have*  $\lambda_3 = 0$ *. By* (3.20)*, we have* 

$$
\alpha \lambda_1 + \gamma \lambda_2 = 0.
$$

*case* 2-1) *If*  $\gamma = 0$ *, then*  $\lambda_1 = 0$ *,*  $\beta \delta = 0$ *. We get* (2)*. case* 2-2) *If*  $\gamma \neq 0$ , then  $\lambda_1 = -\frac{\gamma}{\gamma}$  $\frac{1}{\alpha}\lambda_2$ *.* We get (3).

# 3.6 Invariant Ricci collineations of  $G_6$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.16.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_6, g)$  associated to the Bott connection  $\nabla^{B_2}$  is de*termined by*

(3.21) 
$$
Ric^{B_2}(e_1, e_1) = -(\delta^2 + \beta\gamma), \quad Ric^{B_2}(e_1, e_2) = 0, \quad Ric^{B_2}(e_1, e_3) = 0,
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \quad Ric^{B_2}(e_2, e_3) = 0, \quad Ric^{B_2}(e_3, e_3) = \delta^2.
$$

By  $(2.5)$  and Lemma 3.16, we have

#### Lemma 3.17.

(3.22) 
$$
(L_V Ric^{B_2})(e_1, e_1) = 0, (L_V Ric^{B_2})(e_1, e_2) = 0,
$$

$$
(L_V Ric^{B_2})(e_1, e_3) = \beta \delta^2 \lambda_2 + \delta^3 \lambda_3 (L_V Ric^{B_2})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_2})(e_2, e_3) = -\beta \delta^2 \lambda_1, (L_V Ric^{B_2})(e_3, e_3) = -2\delta^3 \lambda_1.
$$

Then, if a left-invariant vector field  $V$  is a Ricci collineation associated to the Bott connection  $\nabla^{B_2}$ , by Lemma 3.17 and Theorem 2.1, we have the following equations:

(3.23) 
$$
\begin{cases} \beta \delta^2 \lambda_2 + \delta^3 \lambda_3 = 0 \\ -\beta \delta^2 \lambda_1 = 0 \\ -2\delta^3 \lambda_1 = 0 \end{cases}
$$

By solving (3.23) , we get

**Theorem 3.18.**  $(G_6, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:*  $(1)\delta = 0, \gamma = 0, \alpha \neq 0,$  $(2)\delta \neq 0, \beta = 0, \alpha + \delta \neq 0, \alpha \gamma = 0,$  $(3)\delta \neq 0, \beta \neq 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0.$ *Moreover, in these cases, we have*  $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle,$  $(2)\mathcal{V}_{\mathcal{R}C} = \langle e_2 \rangle,$  $(3)\mathscr{V}_{\mathscr{R}C} = \langle e_2 - \frac{\beta}{5} \rangle$  $\frac{\epsilon}{\delta}e_3$  >. where  $\mathscr{V}_{\mathscr{R}C}$  is the vector space of left-invariant Ricci collineations on  $(G_6, g, V)$ .

**Proof.** We know that  $\alpha + \delta \neq 0$ ,  $\alpha\gamma - \beta\delta = 0$ . *case* 1) *If*  $\delta = 0$ *, then*  $\alpha \neq 0, \gamma = 0$ *.* (3.23) *trivially holds. We get* (1)*. case* 2) *If*  $\delta \neq 0$ *, by the third equation we have*  $\lambda_1 = 0$ *. By* (3.23)*, we have* 

$$
\beta \lambda_2 + \delta \lambda_3 = 0.
$$

*case* 2-1) *If*  $\beta = 0$ *, then*  $\lambda_3 = 0$ *,*  $\alpha \gamma = 0$ *. We get* (2)*. case* 2-2) *If*  $\beta \neq 0$ *, then*  $\lambda_3 = -\frac{\beta}{s}$  $\frac{\beta}{\delta} \lambda_2$ *.* We get (3).

# 3.7 Invariant Ricci collineations of  $G_7$  associated to the Bott connection  $\nabla^{B_2}$

**Lemma 3.19.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_7, g)$  associated to the Bott connection  $\nabla^{B_2}$  is de*termined by*

(3.24) 
$$
Ric^{B_2}(e_1, e_1) = \alpha^2, \ Ric^{B_2}(e_1, e_2) = \beta(\alpha + \delta), \ Ric^{B_2}(e_1, e_3) = \frac{\beta(\delta - \alpha)}{2},
$$

$$
Ric^{B_2}(e_2, e_2) = 0, \ Ric^{B_2}(e_2, e_3) = \delta^2 + \frac{\beta\gamma + \alpha\delta}{2}, \ Ric^{B_2}(e_3, e_3) = -\alpha^2 + \beta^2 - \beta\gamma.
$$

By  $(2.5)$  and Lemma 3.19, we have

### Lemma 3.20.

$$
(3.25) \ (L_V Ric^{B_2})(e_1, e_1) = (-2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_2 + (2\alpha^3 + 3\beta^2 \delta + \alpha \beta^2)\lambda_3,\n(L_V Ric^{B_2})(e_1, e_2) = (\alpha^3 + \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_2\n+ \beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha \delta + \frac{\beta \gamma}{2})\lambda_3,\n(L_V Ric^{B_2})(e_1, e_3) = (-\alpha^3 + \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(-\frac{3\alpha^2}{2} + \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha \delta}{2} - \frac{\beta \gamma}{2})\lambda_2\n+ \frac{\beta}{2}(-3\alpha^2 + 2\beta^2 + 2\delta^2 + 2\alpha \delta - \beta \gamma)\lambda_3,\n(L_V Ric^{B_2})(e_2, e_2) = 2\beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_1 + \delta(2\delta^2 + \alpha \delta + 3\beta \gamma)\lambda_3,\n(L_V Ric^{B_2})(e_2, e_3) = \beta(-\frac{5\alpha^2}{2} + \beta^2 - \frac{\alpha \delta}{2} - \beta \gamma)\lambda_1 - \delta(\delta^2 + \frac{\alpha \delta}{2} + \frac{3\beta \gamma}{2})\lambda_2,\n+ \delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha \delta}{2})\lambda_3,\n(L_V Ric^{B_2})(e_3, e_3) = -2\beta(-\frac{3\alpha^2}{2} + \beta^2 + \delta^2 + \alpha \delta - \frac{\beta \gamma}{2})\lambda_1 - 2\delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha \delta}{2})\lambda_3.
$$

$$
\begin{cases}\n(-2\alpha^3 - 3\beta^2 \delta - \alpha \beta^2)\lambda_2 + (2\alpha^3 + 3\beta^2 \delta + \alpha \beta^2)\lambda_3 = 0 \\
(\alpha^3 + \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_2 \\
+ \beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha \delta + \frac{\beta \gamma}{2})\lambda_3 = 0 \\
(-\alpha^3 + \frac{3\beta^2 \delta}{2} + \frac{\alpha \beta^2}{2})\lambda_1 - \beta(-\frac{3\alpha^2}{2} + \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha \delta}{2} - \frac{\beta \gamma}{2})\lambda_2 \\
+ \frac{\beta}{2}(-3\alpha^2 + 2\beta^2 + 2\delta^2 + 2\alpha \delta - \beta \gamma)\lambda_3 = 0 \\
2\beta(\alpha^2 + \delta^2 + \frac{3\alpha \delta}{2} + \frac{\beta \gamma}{2})\lambda_1 + \delta(2\delta^2 + \alpha \delta + 3\beta \gamma)\lambda_3 = 0 \\
\beta(-\frac{5\alpha^2}{2} + \beta^2 - \frac{\alpha \delta}{2} - \beta \gamma)\lambda_1 - \delta(\delta^2 + \frac{\alpha \delta}{2} + \frac{3\beta \gamma}{2})\lambda_2 \\
+ \delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha \delta}{2}) = 0 \\
-2\beta(-\frac{3\alpha^2}{2} + \beta^2 + \delta^2 + \alpha \delta - \frac{\beta \gamma}{2})\lambda_1 - 2\delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha \delta}{2})\lambda_3 = 0\n\end{cases}
$$

By solving (3.26) , we get

Theorem 3.21.  $(G_7, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_2}$  *if and only if one of the following holds:* 

 $(1) \alpha = 0, \beta = 0, \delta \neq 0,$  $(2) \alpha \neq 0, \gamma = 0, \alpha + \delta \neq 0, \beta \neq 0,$  $(3)\alpha \neq 0, \gamma \neq 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0.$ *Moreover, in these cases, we have*  $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle,$  $(2)\mathscr{V}_{\mathscr{R}C} = \langle \frac{\delta}{\rho}e_1 + e_2 + e_3 \rangle,$ 

(3) 
$$
\mathcal{V}_{\mathcal{R}C} = \langle e_2 + e_3 \rangle
$$
.  
\n(3)  $\mathcal{V}_{\mathcal{R}C} = \langle e_2 + e_3 \rangle$ .  
\nwhere  $\mathcal{V}_{\mathcal{R}C}$  is the vector space of left-invariant Ricci collineations on  $(G_7, g, V)$ .

**Proof.** We know that  $\alpha + \delta \neq 0, \alpha \gamma = 0$ . *case* 1) *If*  $\alpha = 0$ *, then*  $\delta \neq 0$ *. By the first equation,*  $\beta^2 \delta(\lambda_2 - \lambda_3) = 0$ *. case* 1-1) *If*  $\beta = 0$ *, by* (3.26)

(3.27) 
$$
\begin{cases} \delta^3 \lambda_3 = 0 \\ -\delta^3 \lambda_2 + \delta^3 \lambda_3 = 0 \end{cases}
$$

*then*  $\lambda_2 = \lambda_3 = 0$ *. We get* (1)*. case* 1-2) If  $\beta \neq 0$ , then  $\lambda_2 = \lambda_3$ . By the second equation, we have  $3\beta^2 \delta \lambda_1 - 3\beta \delta^2 \lambda_2 = 0$ , i.e.

$$
\beta \lambda_1 - \delta \lambda_2 = 0.
$$

*Then by the fifth equation, we have*  $2\gamma \delta \lambda_2 = 0$ *, i.e.*  $\gamma \lambda_2 = 0$ *. case* 1-2-1) *If*  $\gamma \neq 0$ *, then*  $\lambda_2 = \lambda_3 = 0$ ,  $\lambda_1 = 0$ *. case* 1-2-2) *If*  $\gamma = 0$ *, then*  $\beta \lambda_1 - \delta \lambda_2 = 0$ *i.e.* $\lambda_1 = \frac{\delta}{\epsilon}$  $\frac{\delta}{\beta}\lambda_2=\frac{\delta}{\beta}$  $\frac{\delta}{\beta} \lambda_3$ . (3.26) *trivially holds.* We get (2).

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case 2) If 
$$
\alpha \neq 0
$$
, then  $\gamma = 0$ ,  $\alpha + \delta \neq 0$ .  
\ncase 2-1) If  $-2\alpha^3 - 3\beta^2\delta - \alpha\beta^2 \neq 0$ , then  $\lambda_2 = \lambda_3$ . By (3.26)  
\n
$$
\begin{cases}\n(\alpha^3 + \frac{3\beta^2\delta}{2} - \frac{\alpha\beta^2}{2})\lambda_1 + \beta(\frac{3\delta^2}{2} + \frac{\alpha\delta}{2})\lambda_2 = 0 \\
\beta(\alpha^2 + \delta^2 + \frac{3\alpha\delta}{2})\lambda_1 + \delta(\delta^2 + \frac{\alpha\delta}{2})\lambda_2 = 0 \\
\beta(-\frac{5\alpha^2}{2} + \beta^2 - \frac{\alpha\delta}{2})\lambda_1 + \delta(-\alpha^2 + \beta^2)\lambda_2 = 0 \\
\beta(-\frac{3\alpha^2}{2} + \beta^2 + \delta^2 + \alpha\delta)\lambda_1 + \delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha\delta}{2})\lambda_2 = 0\n\end{cases}
$$

*case* 2-1-1) *If*  $\delta = 0$ *, then*  $\lambda_1 = 0$ *.* (3.26) *trivially holds. We get* (3)*. case* 2-1-2) *If*  $\delta \neq 0$ *, then*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *. case* 2-2) *If*  $-2\alpha^3 - 3\beta^2\delta - \alpha\beta^2 = 0$ , *by* (3.25)

$$
(3.29)
$$
\n
$$
\begin{cases}\n-\beta(\alpha^2 + \delta^2 + \frac{3\alpha\delta}{2})\lambda_2 + \beta(\alpha^2 + \frac{5\delta^2}{2} + 2\alpha\delta)\lambda_3 = 0 \\
-2\alpha^3\lambda_1 - \beta(-\frac{3\alpha^2}{2} + \beta^2 + \frac{5\delta^2}{2} + \frac{3\alpha\delta}{2})\lambda_2 \\
+ \frac{\beta}{2}(-3\alpha^2 + 2\beta^2 + 2\delta^2 + 2\alpha\delta)\lambda_3 = 0 \\
2\beta(\alpha^2 + \delta^2 + \frac{3\alpha\delta}{2})\lambda_1 + \delta(2\delta^2 + \alpha\delta)\lambda_3 = 0 \\
\beta(-\frac{5\alpha^2}{2} + \beta^2 - \frac{\alpha\delta}{2})\lambda_1 - \delta(\delta^2 + \frac{\alpha\delta}{2})\lambda_2 + \delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha\delta}{2}) = 0 \\
-2\beta(-\frac{3\alpha^2}{2} + \beta^2 + \delta^2 + \alpha\delta)\lambda_1 - 2\delta(-\alpha^2 + \beta^2 + \delta^2 + \frac{\alpha\delta}{2})\lambda_3 = 0\n\end{cases}
$$

*If*  $\beta = 0$ *, then*  $\alpha = 0$ *, this is a contradiction, then*  $\beta \neq 0$ *. case* 2-2-1) *If*  $\delta = 0$ *, then*  $2\alpha^2 - \beta^2 = 0$ ,  $\lambda_1 = 0$ ,  $\lambda_2 = \lambda_3$ *. This case is in* (3)*. case* 2-2-2) *If*  $\delta \neq 0$ *, then*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *.* 

## 4. Invariant Ricci collineations associated to the Bott connection on three-dimensional Lorentzian Unimodular Lie groups with the third **DISTRIBUTION**

Let M be a smooth manifold, and let  $TM = span\{e_1, e_2, e_3\}$ , then took the third distribution:  $F_3 = span\{e_2, e_3\}$  and  $F_3^{\perp} = span\{e_1\}$ , where  $e_1, e_2, e_3$  is a pseudo-orthonormal basis, with  $e_3$ timelike. The Bott connection  $\nabla^{B_3}$  is defined as follows: (see [\[8\]](#page-29-4), [\[9\]](#page-29-5), [\[10\]](#page-29-6))

(4.1) 
$$
\nabla_X^{B_3} Y = \begin{cases} \pi_{F_3}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_3) \\ \pi_{F_3}([X, Y]), & X \in \Gamma^\infty(F_3^\perp), Y \in \Gamma^\infty(F_3) \\ \pi_{F_3^\perp}([X, Y]), & X \in \Gamma^\infty(F_3), Y \in \Gamma^\infty(F_3^\perp) \\ \pi_{F_3^\perp}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(F_3^\perp) \end{cases}
$$

where  $\pi_{F_3}$  and  $\pi_{F_3}^{\perp}$  $\frac{1}{F_3}$  are respectively the projection on  $F_3$  and  $F_3^{\perp}$ ,  $\nabla^L$  is the Levi-Civita connection of  $G_i$ .

## 4.1 Invariant Ricci collineations of  $G_1$  associated to the Bott connection  $\nabla^{B_3}$

**Lemma 4.1.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_1, g)$  associated to the Bott connection  $\nabla^{B_3}$  is deter*mined by*

(4.2) 
$$
Ric^{B_3}(e_1, e_1) = 0, \ Ric^{B_3}(e_1, e_2) = \frac{\alpha \beta}{2}, \ Ric^{B_3}(e_1, e_3) = -\frac{\alpha \beta}{2},
$$

$$
Ric^{B_3}(e_2, e_2) = -\beta^2, \ Ric^{B_3}(e_2, e_3) = 0, \ Ric^{B_3}(e_3, e_3) = \beta^2.
$$

By (2.5) and Lemma 4.1, we have

### Lemma 4.2.

(4.3)  
\n
$$
(L_V Ric^{B_3})(e_1, e_1) = \alpha \beta^2 \lambda_2 - \alpha \beta^2 \lambda_3,
$$
\n
$$
(L_V Ric^{B_3})(e_1, e_2) = -\frac{\alpha \beta^2}{2} \lambda_1 + \frac{\alpha^2 \beta}{2} \lambda_2 + \beta (\beta^2 - \frac{\alpha^2}{2}) \lambda_3,
$$
\n
$$
(L_V Ric^{B_3})(e_1, e_3) = \frac{\alpha \beta^2}{2} \lambda_1 - \beta (\beta^2 + \frac{\alpha^2}{2}) \lambda_2 + \frac{\alpha^2 \beta}{2} \lambda_3,
$$
\n
$$
(L_V Ric^{B_3})(e_2, e_2) = -\alpha^2 \beta \lambda_1 - \alpha \beta^2 \lambda_3,
$$
\n
$$
(L_V Ric^{B_3})(e_2, e_3) = \alpha^2 \beta \lambda_1 + \frac{\alpha \beta^2}{2} \lambda_2 + \frac{\alpha \beta^2}{2} \lambda_3,
$$
\n
$$
(L_V Ric^{B_3})(e_3, e_3) = -\alpha^2 \beta \lambda_1 - \alpha \beta^2 \lambda_2.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 4.2 and Theorem 2.1, we have the following equations:

(4.4)  
\n
$$
\begin{cases}\n\alpha \beta^2 \lambda_2 - \alpha \beta^2 \lambda_3 = 0 \\
-\frac{\alpha \beta^2}{2} \lambda_1 + \frac{\alpha^2 \beta}{2} \lambda_2 + \beta (\beta^2 - \frac{\alpha^2}{2}) \lambda_3 = 0 \\
\frac{\alpha \beta^2}{2} \lambda_1 - \beta (\beta^2 + \frac{\alpha^2}{2}) \lambda_2 + \frac{\alpha^2 \beta}{2} \lambda_3 = 0 \\
-\alpha^2 \beta \lambda_1 - \alpha \beta^2 \lambda_3 = 0 \\
\alpha^2 \beta \lambda_1 + \frac{\alpha \beta^2}{2} \lambda_2 + \frac{\alpha \beta^2}{2} \lambda_3 = 0 \\
-\alpha^2 \beta \lambda_1 - \alpha \beta^2 \lambda_2 = 0\n\end{cases}
$$

By solving (4.4) , we get

**Theorem 4.3.**  $(G_1, g, V)$  *admits left-invariant Ricci collineations associated to the Bott connection*  $\nabla^{B_3}$  *if and only if*  $\alpha \neq 0, \beta = 0$ *. Moreover, in this case, we have*  $\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle$ , *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_1, g, V)$ *.* 

**Proof.** We know that  $\alpha \neq 0$ . By the first equation, we get  $\beta^2(\lambda_2 - \lambda_3) = 0$ , so *case* 1) *If*  $\beta = 0$ , (4.4) *trivially holds. We get solution. case* 2) *If*  $\beta \neq 0$ *, then*  $\lambda_2 = \lambda_3$ *. By* (4.4)*,* 

(4.5) 
$$
\begin{cases} -\frac{\alpha\beta^2}{2}\lambda_1 + \beta^3\lambda_2 = 0\\ -\alpha^2\beta\lambda_1 - \alpha\beta^2\lambda_2 = 0 \end{cases}
$$

*i.e.*

(4.6) 
$$
\begin{cases} \alpha \lambda_1 - 2\beta \lambda_2 = 0 \\ \alpha \lambda_1 + \beta \lambda_2 = 0 \end{cases}
$$

*then we get*  $\lambda_2 = 0$ *, i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *.* 

# 4.2 Invariant Ricci collineations of  $G_2$  associated to the Bott connection  $\nabla^{B_3}$

**Lemma 4.4.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_2, g)$  associated to the Bott connection  $\nabla^{B_3}$  is deter*mined by*

(4.7) 
$$
Ric^{B_3}(e_1, e_1) = 0, \ Ric^{B_3}(e_1, e_2) = 0, \ Ric^{B_3}(e_1, e_3) = 0,
$$

$$
Ric^{B_3}(e_2, e_2) = -\alpha\beta, \ Ric^{B_3}(e_2, e_3) = -\alpha\gamma, \ Ric^{B_3}(e_3, e_3) = \alpha\beta.
$$

By (2.5) and Lemma 4.4, we have

### Lemma 4.5.

(4.8) 
$$
(L_V Ric^{B_3})(e_1, e_1) = 0, (L_V Ric^{B_3})(e_1, e_2) = \alpha(\beta^2 + \gamma^2)\lambda_3,
$$

$$
(L_V Ric^{B_3})(e_1, e_3) = -\alpha(\beta^2 + \gamma^2)\lambda_2, (L_V Ric^{B_3})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_3})(e_2, e_3) = 0, (L_V Ric^{B_3})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field  $V$  is a Ricci collineation associated to the Bott connection  $\nabla^{B_3}$ , by Lemma 4.5 and Theorem 2.1, we have the following equations:

(4.9) 
$$
\begin{cases} \alpha(\beta^2 + \gamma^2)\lambda_3 = 0\\ -\alpha(\beta^2 + \gamma^2)\lambda_2 = 0 \end{cases}
$$

By solving (4.9) , we get

**Theorem 4.6.**  $(G_2, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_3}$  *if and only if one of the following holds:* 

 $(1) \alpha = 0, \gamma \neq 0,$ 

 $(2)\alpha \neq 0, \gamma \neq 0.$ 

*Moreover, in these cases, we have*

 $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle,$ 

 $(2)\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle.$ 

*where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_2, g, V)$ *.* 

**Proof.** We know that  $\gamma \neq 0$ , then  $\beta^2 + \gamma^2 \neq 0$ . By (4.9), we have

(4.10) 
$$
\begin{cases} \alpha \lambda_3 = 0 \\ -\alpha \lambda_2 = 0 \end{cases}
$$

*case* 1) *If*  $\alpha = 0$ , (4.9) *trivially holds. We get* (1). *case* 2) *If*  $\alpha \neq 0$ *, then*  $\lambda_2 = \lambda_3 = 0$ *. We get* (2)*.* 

### 4.3 Invariant Ricci collineations of  $G_3$  associated to the Bott connection  $\nabla^{B_3}$

**Lemma 4.7.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_3, g)$  associated to the Bott connection  $\nabla^{B_3}$  is deter*mined by*

(4.11) 
$$
Ric^{B_3}(e_1, e_1) = 0, \ Ric^{B_3}(e_1, e_2) = 0, \ Ric^{B_3}(e_1, e_3) = 0,
$$

$$
Ric^{B_3}(e_2, e_2) = 0, \ Ric^{B_3}(e_2, e_3) = 0, \ Ric^{B_3}(e_3, e_3) = \alpha \beta.
$$

By (2.5) and Lemma 4.7, we have

### Lemma 4.8.

*.*

(4.12) 
$$
(L_V Ric^{B_3})(e_1, e_1) = 0, (L_V Ric^{B_3})(e_1, e_2) = 0,
$$

$$
(L_V Ric^{B_3})(e_1, e_3) = -\alpha \beta \gamma \lambda_2, (L_V Ric^{B_3})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_3})(e_2, e_3) = \alpha \beta \gamma \lambda_1, (L_V Ric^{B_3})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_3}$ , by Lemma 4.8 and Theorem 2.1, we have the following equations:

(4.13) 
$$
\begin{cases} -\alpha \beta \gamma \lambda_2 = 0 \\ \alpha \beta \gamma \lambda_1 = 0 \end{cases}
$$

By solving (4.13) , we get

**Theorem 4.9.**  $(G_3, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_3}$  *if and only if one of the following holds:*  $(1)\alpha\beta\gamma=0$ ,  $(2) \alpha \neq 0, \beta \neq 0, \gamma \neq 0.$ *Moreover, in these cases, we have*  $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle,$  $(2)\mathscr{V}_{\mathscr{R}C} = \langle e_3 \rangle.$ *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_3, g, V)$ *.* **Proof.** *case* 1) *If*  $\alpha\beta\gamma = 0$ , (4.13) *trivially holds. We get* (1)*. case* 2) *If*  $\alpha\beta\gamma \neq 0$ , *i.e.*  $\alpha \neq 0$ ,  $\beta \neq 0$ ,  $\gamma \neq 0$ , then  $\lambda_1 = \lambda_2 = 0$ . We get (2)

# 4.4 Invariant Ricci collineations of  $G_4$  associated to the Bott connection  $\nabla^{B_3}$

**Lemma 4.10.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_4, g)$  associated to the Bott connection  $\nabla^{B_3}$  is de*termined by*

(4.14) 
$$
Ric^{B_3}(e_1, e_1) = 0, \ Ric^{B_3}(e_1, e_2) = 0, \ Ric^{B_3}(e_1, e_3) = 0,
$$

$$
Ric^{B_3}(e_2, e_2) = \alpha(2\eta - \beta), \ Ric^{B_3}(e_2, e_3) = \alpha, \ Ric^{B_3}(e_3, e_3) = \alpha\beta.
$$

By  $(2.5)$  and Lemma 4.10, we have

### Lemma 4.11.

(4.15) 
$$
(L_V Ric^{B_3})(e_1, e_1) = 0, (L_V Ric^{B_3})(e_1, e_2) = -\alpha[\beta(2\eta - \beta) - 1]\lambda_3,
$$

$$
(L_V Ric^{B_3})(e_1, e_3) = \alpha[\beta(2\eta - \beta) - 1]\lambda_2, (L_V Ric^{B_3})(e_2, e_2) = 0,
$$

$$
(L_V Ric^{B_3})(e_2, e_3) = 0, (L_V Ric^{B_3})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field V is a Ricci collineation associated to the Bott connection  $\nabla^{B_1}$ , by Lemma 4.11 and Theorem 2.1, we have the following equations:

(4.16) 
$$
\begin{cases} -\alpha[\beta(2\eta - \beta) - 1]\lambda_3 = 0\\ \alpha[\beta(2\eta - \beta) - 1]\lambda_2 = 0 \end{cases}
$$

By solving (4.16) , we get

Theorem 4.12.  $(G_4, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_1}$  *if and only if one of the following holds:* 

- $(1)\alpha = 0, \eta = 1 \text{ or } -1,$  $(2)\alpha \neq 0, \beta = \eta, \eta = 1 \text{ or } -1,$  $(3)\alpha \neq 0, \beta \neq \eta, \eta = 1 \text{ or } -1.$ *Moreover, in these cases, we have*  $(1)\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle,$
- $(2)\mathscr{V}_{\mathscr{R}C} = \langle e_1, e_2, e_3 \rangle,$  $(3)\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle.$ *where*  $\mathscr{V}_{\mathscr{R}C}$  *is the vector space of left-invariant Ricci collineations on*  $(G_4, g, V)$ *.*

**Proof.** We know that  $\eta = 1$  or  $-1$ . *case* 1) If  $\alpha = 0$ , (4.16) *trivially holds.* We get (1). *case* 2) *If*  $\alpha \neq 0$ *, then by* (4.16)*, we have* 

(4.17) 
$$
\begin{cases} [\beta(2\eta - \beta) - 1]\lambda_3 = 0\\ [\beta(2\eta - \beta) - 1]\lambda_2 = 0 \end{cases}
$$

*case* 2-1) *If*  $\beta(2\eta - \beta) - 1 = 0$ *, then case* 2-1-1) *If*  $\eta = 1$ *, we have* 

$$
\beta(2\eta - \beta) - 1 = \beta(2 - \beta) - 1 = 2\beta - \beta^{2} - 1 = -(\beta - 1)^{2} = 0.
$$

*Then*  $\beta = 1$ *, this case is in* (2)*. case* 2-1-2) *If*  $\eta = -1$ *, we have* 

$$
\beta(2\eta - \beta) - 1 = \beta(-2 - \beta) - 1 = -2\beta - \beta^2 - 1 = -(\beta + 1)^2 = 0.
$$

*Then*  $\beta = -1$ *, this case is in* (2)*. case* 2-2) *If*  $\beta(2\eta - \beta) - 1 \neq 0$ , *i.e.*  $\beta \neq \eta$ , we have  $\lambda_2 = \lambda_3 = 0$ . We get (3).

4.5 Invariant Ricci collineations of  $G_5$  associated to the Bott connection  $\nabla^{B_3}$ 

**Lemma 4.13.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_5, g)$  associated to the Bott connection  $\nabla^{B_3}$  is de*termined by*

(4.18) 
$$
Ric^{B_3}(e_1, e_1) = 0, \ Ric^{B_3}(e_1, e_2) = 0, \ Ric^{B_3}(e_1, e_3) = 0,
$$

$$
Ric^{B_3}(e_2, e_2) = \delta^2, \ Ric^{B_3}(e_2, e_3) = 0, \ Ric^{B_3}(e_3, e_3) = -(\delta^2 + \beta\gamma).
$$

By  $(2.5)$  and Lemma 4.13, we have

### Lemma 4.14.

(4.19) 
$$
(L_V Ric^{B_3})(e_1, e_1) = 0, (L_V Ric^{B_3})(e_1, e_2) = \beta \delta^2 \lambda_3, (L_V Ric^{B_3})(e_1, e_3) = 0, (L_V Ric^{B_3})(e_2, e_2) = 2\delta^3 \lambda_3, (L_V Ric^{B_3})(e_2, e_3) = -\beta \delta^2 \lambda_1 - \delta^3 \lambda_2, (L_V Ric^{B_3})(e_3, e_3) = 0.
$$

Then, if a left-invariant vector field  $V$  is a Ricci collineation associated to the Bott connection  $\nabla^{B_3}$ , by Lemma 4.13 and Theorem 2.1, we have the following equations:

(4.20)  

$$
\begin{cases} \beta \delta^2 \lambda_3 = 0 \\ 2 \delta^3 \lambda_3 = 0 \\ -\beta \delta^2 \lambda_1 - \delta^3 \lambda_2 = 0 \end{cases}
$$

By solving (4.20) , we get

Theorem 4.15.  $(G_5, g, V)$  *admits left-invariant Ricci collineations associated to the Bott con-* $\eta$  *nection*  $\nabla^{B_3}$  *if and only if one of the following holds:* 

- (1)  $\delta = 0, \alpha \neq 0, \gamma = 0$ (2)  $\delta \neq 0, \beta = 0, \alpha + \delta \neq 0, \alpha \gamma = 0,$ (3)  $\delta \neq 0, \beta \neq 0, \alpha + \delta \neq 0, \alpha \gamma + \beta \delta = 0.$ *Moreover, in these cases, we have*
- (1)  $\mathcal{V}_{\mathcal{R}C} = \langle e_1, e_2, e_3 \rangle$ (2)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1 \rangle$ ,
- (3)  $\mathscr{V}_{\mathscr{R}C} = \langle e_1 \frac{\beta}{s} \rangle$  $\frac{\epsilon}{\delta}e_2$  >.

where  $\mathscr{V}_{\mathscr{R}C}$  is the vector space of left-invariant Ricci collineations on  $(G_5, g, V)$ .

**Proof.** We know that  $\alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0$ . *case* 1) *If*  $\delta = 0$ *, then*  $\alpha \neq 0, \gamma = 0$ *.* (4.20) *trivially holds. We get* (1)*. case* 2) *If*  $\delta \neq 0$ *, by the second equation we have*  $\lambda_3 = 0$ *. By* (4.20)*, we have* 

$$
\beta \lambda_1 + \delta \lambda_2 = 0.
$$

*case* 2-1) *If*  $\beta = 0$ *, then*  $\lambda_2 = 0$ *,*  $\alpha \gamma = 0$ *. We get* (2)*. case* 2-2) *If*  $\beta \neq 0$ *, then*  $\lambda_2 = -\frac{\beta}{5}$  $\frac{\beta}{\delta} \lambda_1$ *.* We get (3).

4.6 Invariant Ricci collineations of  $G_6$  associated to the Bott connection  $\nabla^{B_3}$ 

**Lemma 4.16.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(66, g)$  associated to the Bott connection  $\nabla^{B_3}$  is deter*mined by*

(4.21) 
$$
Ric^{B_1}(e_i, e_j) = 0.
$$

*for any pairs (i,j).*

By  $(2.5)$  and Lemma 4.16, we have

### Lemma 4.17.

(4.22) 
$$
(L_V Ric^{B_1})(e_i, e_j) = 0.
$$

Then we have

**Theorem 4.18.** Any left-invariant vector field on  $(G_6, g, V)$  is a left-invariant Ricci collineations *associated to the Bott connection*  $\nabla^{B_3}$ .

## 4.7 Invariant Ricci collineations of  $G_7$  associated to the Bott connection  $\nabla^{B_3}$

**Lemma 4.19.** ([\[12\]](#page-29-10)) The Ricci tensor of  $(G_7, g)$  associated to the Bott connection  $\nabla^{B_3}$  is de*termined by*

(4.23) 
$$
Ric^{B_1}(e_1, e_1) = 0, \ Ric^{B_1}(e_1, e_2) = 0, \ Ric^{B_1}(e_1, e_3) = 0,
$$

$$
Ric^{B_1}(e_2, e_2) = -\beta\gamma, \ Ric^{B_1}(e_2, e_3) = \beta\gamma, \ Ric^{B_3}(e_3, e_3) = -\beta\gamma.
$$

By (2.5) and Lemma 4.19, we have

#### Lemma 4.20.

(4.24) 
$$
(L_V Ric^{B_1})(e_i, e_j) = 0.
$$

Then we have

Theorem 4.21. *Any left-invariant vector field on* (G7, g, V ) *is a left-invariant Ricci collineations associated to the Bott connection*  $\nabla^{B_3}$ .

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