ON A FAMILY OF 2-AUTOMATIC SEQUENCES DERIVED FROM ULTIMATELY PERIODIC SEQUENCES AND GENERATING ALGEBRAIC CONTINUED FRACTIONS IN $\mathbb{F}_2((1/t))$

(Suites 2-automatiques à colone vertébrale ultimement périodique)

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Warning: This note is not intended to be officially published. The matter exposed here grew from numerous exchanges during the past two months with Yining Hu (at a very and too large distance !). My aim is to report in a first draft, and in a very private and personal way, on a curious mathematical structure.

We consider a large family $\mathcal{F}$ of infinite sequences over a finite alphabet $\mathcal{A} = \{a_1, a_2, ..., a_k\}$. This family includes a celebrated example of a 2-automatic sequence on the set $\{a, b\}$, called Period-doubling sequence, which has been studied in a previous article [1].

Each sequence $\mathbf{s}$ in $\mathcal{F}$ is built in the following way, from another sequence $\boldsymbol{\varepsilon} = (\varepsilon_i)_{i\geq 0}$ over $\{a_1, a_2, ..., a_k\}$, this last one being ultimately periodic.

Starting from the empty word W_0, we consider the sequence of words $(W_n)_{n\geq 0}$ such that for $n \geq 0$ we have $W_{n+1} = W_n, \varepsilon_n, W_n$ (note the coma is for concatenation and it will be omitted when it is suitable). Hence, we have $W_1 = \varepsilon_0$, $W_2 = \varepsilon_0, \varepsilon_1, \varepsilon_0$ etc...Observe by construction that W_{n+1} starts by W_n and therefore we may consider W_∞ the inductive limit of these words (i.e. the word begining by W_n for all $n \geq 0$). Note that, for all $n \geq 1$, W_n is a palindrome, centered in ε_n, of length $2^n - 1$.

This W_∞ represents the sequence $\mathbf{s}$, which we may denote by $\mathbf{s}(\boldsymbol{\varepsilon})$. Hence, we have :

$$\mathbf{s}(\boldsymbol{\varepsilon}) = \varepsilon_0\varepsilon_1\varepsilon_0\varepsilon_2\varepsilon_0\varepsilon_1\varepsilon_0\varepsilon_3\varepsilon_0\varepsilon_1\varepsilon_0\varepsilon_2\varepsilon_0....$$

In this family $\mathcal{F}$, each sequence $\mathbf{s} = (s_i)_{i\geq 0}$, over $\{a_1, a_2, ..., a_k\}$, generates an infinite continued fraction α in $\mathbb{F}_2((1/t))$, denoted $CF(\mathbf{s})$, by replacing the letters a_i by non-constant polynomials in $\mathbb{F}_2[t]$ (this choice is arbitrary, hence there is a $CF(\mathbf{s})$ for each choice but it is considered as unique in the sequel). For basic information on continued fractions, particularly in power series fields, the reader is refered to [2]. Hence we will write :

$$\alpha = CF(\mathbf{s}) = [s_0, s_1, ..., s_n, ...]$$

where the s_i are the partial quotients in $\mathbb{F}_2[t]$. We recall that the sequence of convergents to α is denoted $(x_n/y_n)_{n\geq 1}$. For $n \geq 1$, we have $x_n/y_n = [s_0, ..., s_{n-1}] = s_0 + 1/(s_1 + 1/....)$, hence $x_1 = s_0$, $y_1 = 1$ and $x_2 = s_0 s_1 + 1$, $y_2 = s_1$ etc...

To be more precise about periodic sequences, we introduce the following definition.

Definition. *Let $l \geq 0$ and $d \geq 1$ be two integers. An ultimately periodic sequence $\boldsymbol{\varepsilon}$ is called of type (l, d) if :*
1) $l = 0$ and $\boldsymbol{\varepsilon}$ is purely periodic, with period of length d, this being denoted by $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_{d-1})^\infty$.
2) $l > 0$ and $\boldsymbol{\varepsilon}$ is ultimately periodic, with a prefix of length l and a period of length d, this being denoted by $\boldsymbol{\varepsilon} = \varepsilon_0, \varepsilon_1, ..., \varepsilon_{l-1}, (\varepsilon_l, \varepsilon_{l+1}, ..., \varepsilon_{l+d-1})^\infty$.
Here all the ε_i's are in $\mathcal{A}$ (assuming that $k \geq l + d$ to allow different values to the terms of the sequence $\boldsymbol{\varepsilon}$).

Let us illustrate the construction of $\mathbf{s}(\boldsymbol{\varepsilon})$ in two basic cases :
1) $\boldsymbol{\varepsilon} = a, b, (c)^\infty$ then

$$\mathbf{s}(\boldsymbol{\varepsilon}) = a, b, a, c, a, b, a, c, ... = (abac)^\infty$$

Note that here $\boldsymbol{\varepsilon}$ is ultimately constant (of type $(3, 1)$) and $\mathbf{s}(\boldsymbol{\varepsilon})$ is periodic. Consequently a basic property on continued fractions shows that $\alpha = CF(\mathbf{s}(\boldsymbol{\varepsilon}))$ is quadratic over $\mathbb{F}_2(t)$.
2) $\boldsymbol{\varepsilon} = (a, b)^\infty$ then

$$\mathbf{s}(\boldsymbol{\varepsilon}) = abaaabababaaaba....$$

Here $\boldsymbol{\varepsilon}$ is of type $(0, 2)$ and $\mathbf{s}(\boldsymbol{\varepsilon})$ is the celebrated sequence mentionned above and called Period-doubling. It has been proved that $\alpha = CF(\mathbf{s}(\boldsymbol{\varepsilon}))$ satisfies an algebraic equation of degree 4 with coefficients in $\mathbb{F}_2[t]$ (see [1]).

As it happens in these two simple cases, we are going to prove in the following theorem that all sequences $\mathbf{s}(\boldsymbol{\varepsilon})$ in $\mathcal{F}$ generate a continued fraction α which is algebraic over $\mathbb{F}_2(t)$. During the proof, the algebraic equation satisfied by α will appear explicitely.

Let $\boldsymbol{\varepsilon}$ be a sequence of type (l, d) then we denote by $\mathbb{F}_2(\boldsymbol{\varepsilon})$ the subfield of $\mathbb{F}_2(t)$ generated by the vector $(\varepsilon_0, ..., \varepsilon_{l+d-1})$ whose $l+d$ coordinates belong to $\mathbb{F}_2[t]$.

Theorem. *Let $l \geq 0$ and $d \geq 1$ be integers. Let $\boldsymbol{\varepsilon}$ be an ultimately periodic sequence of type (l, d). Let $\mathbf{s}(\boldsymbol{\varepsilon})$ the sequence in $\mathbb{F}_2[t]$ and $\alpha =$*

$CF(\mathbf{s}(\boldsymbol{\varepsilon}))$ the continued fraction in $\mathbb{F}_2((1/t))$, both be defined as above. Then there is a polynomial P in $\mathbb{F}_2(\boldsymbol{\varepsilon})[x]$ such that $\deg_x(P) = 2^d$ and $P(\alpha) = 0$. To be more precise, setting $\beta = 1/\alpha$, there are $d+1$ elements in $\mathbb{F}_2(\boldsymbol{\varepsilon})$, A and B_k for $0 \leq k \leq d-1$, such that

$$\beta^{2^d} = A + \sum_{0\leq k\leq d-1} B_k \beta^{2^k}.$$

Note that the case $d = 1$ is trivial as we saw in case 1) above. Indeed, in that case $\boldsymbol{\varepsilon}$ is ultimately constant. Hence $\mathbf{s}(\boldsymbol{\varepsilon}) = W, \varepsilon_l, W, \varepsilon_l, ... = (W, \varepsilon_l)^\infty$ where W is a finite (or empty) word and therefore α is quadratic. In the sequel, we may assume that $d \geq 2$.

The proof of the Theorem lies on the existence of a particular subsequence of convergents to α. These particular convergents are linked to the structure of the word $\mathbf{s}(\boldsymbol{\varepsilon})$. Indeed, it is natural to consider the truncation of the continued fraction α containing the partial quotients from s_0 up to s_{2^n-2} for $n \geq 1$, thus corresponding to the finite word W_n, of length $2^n - 1$, mentionned above. Hence, for $n \geq 1$, we set $u_n/v_n = [s_0, ..., s_{2^n-2}]$. We have

$$(u_1, v_1) = (s_0, 1) \quad \text{and} \quad (u_2, v_2) = (s_0 s_1 s_2 + s_0 + s_2, s_1 s_2 + 1).$$

The first step of the proof was introduced in our previous work concerning the particular case of the Period-doubling sequence. In [1,p. 4 Lemma 3.2.], using basic properties on continuants, we could prove that the pair (u_n, v_n) satisfies a simple recurrence relation.

Indeed, for $n \geq 1$, we have

$$(R) \qquad u_{n+1} = \varepsilon_n u_n^2 \quad \text{and} \quad v_{n+1} = \varepsilon_n u_n v_n + 1,$$

with $(u_1, v_1) = (\varepsilon_0, 1)$. From (R) we get immediately

$$v_{n+1}/u_{n+1} = v_n/u_n + 1/u_{n+1}$$

and therefore we obtain

$$v_n/u_n = \sum_{1\leq i\leq n} 1/u_i \quad \text{for} \quad n \geq 1.$$

Let us consider $\beta = 1/\alpha$. Then $\beta = lim_n(v_n/u_n)$ and consequently we have

$$\beta = \sum_{n\geq 1} 1/u_n. \qquad eq(0)$$

From $eq(0)$, we will show that β is algebraic in the following way. By successive elevation to the power 2, for $0 \leq i \leq d$, we can define inductively $d+1$ sequences, $(\varepsilon(n,i))_{n\geq 0}$, as follows:

$$\varepsilon(n,0)=1 \quad \text{and} \quad \varepsilon(n,i+1)=\varepsilon(n,i)^2\varepsilon_{n+i} \quad \text{for} \quad 0\leq i\leq d-1.$$

Note that, for $n \geq 0$, we have $\varepsilon(n,1)=\varepsilon_n$. Then we observe that, by elevating $eq(0)$ to the power 2, we get

$$\beta^2=\sum_{n\geq 1}1/u_n^2=\sum_{n\geq 1}\varepsilon_n/u_{n+1}=\sum_{n\geq 1}\varepsilon(n,1)/u_{n+1}. \qquad eq(1)$$

Moreover, by successive elevation to the power 2, starting from $eq(0)$ and introducing the sequences $(\varepsilon(n,i))_{n\geq 0}$, we also get, for $0 \leq i \leq d$,

$$\beta^{2^i}=\sum_{n\geq 1}\varepsilon(n,i)/u_{n+i}. \qquad eq(i)$$

Remark. *For all $1 \leq i \leq d$ the sequence $(\varepsilon(n,i))_{n\geq 0}$ is ultimately periodic of type (l,d).*

Proof by induction. This is true for $i=1$. If $(\varepsilon(n,i))_{n\geq 0}$ is a periodic sequence of type (l,d), then we have, for $n \geq l$, $\varepsilon(n+d,i)=\varepsilon(n,i)$ and consequently $\varepsilon(n+d,i+1)=\varepsilon(n+d,i)^2\varepsilon_{n+d+i}=\varepsilon(n,i)^2\varepsilon_{n+i}=\varepsilon(n,i+1)$.

Now we introduce a partition of the set of positive integers into $d+1$ subsets: first the finite set $F=\{k \mid 1\leq k\leq l+d-1\}$ and the d subsets

$$E_j=\{md+l+j \mid m\geq 1\} \quad \text{for} \quad 0\leq j\leq d-1.$$

$$\mathbb{N}^*=\bigcup_{j=0}^{d-1}E_j\bigcup F.$$

Linked to this partition, we introduce d elements, β_j for $0 \leq j \leq d-1$, in $\mathbb{F}_2((1/t))$, defined by

$$\beta_j=\sum_{n\in E_j}1/u_n. \qquad (B)$$

Combining $eq(0)$ and (B), and defining $z_0 \in \mathbb{F}_2(\boldsymbol{\varepsilon})$ by $\sum_{k\in F}1/u_k$, we can write,

$$\beta=z_0+\sum_{0\leq j\leq d-1}\sum_{n\in E_j}1/u_n=z_0+\sum_{0\leq j\leq d-1}\beta_j. \qquad Eq(0)$$

Using the above remark, concerning the periodicity of the sequences $(\varepsilon(n,i))_{n\geq 0}$, for $0 \leq j \leq d-1$ and for $1 \leq i \leq d$, we can write ,

$$\varepsilon(d+l+j-i,i)\beta_j = \sum_{m\geq 1} \varepsilon(md+l+j-i,i)/u_{md+l+j}$$

$$\varepsilon(d+l+j-i,i)\beta_j = \sum_{n+i\in E_j} \varepsilon(n,i)/u_{n+i}.$$

Consequently, we observe that there exists $z_i \in \mathbb{F}_2(\boldsymbol{\varepsilon})$, a finite sum of the first terms in the series appearing in $eq(i)$, such that, for $1 \leq i \leq d$, $eq(i)$ becomes the following equality

$$\beta^{2^i} = z_i + \sum_{0\leq j\leq d-1} \varepsilon(d+l+j-i,i)\beta_j. \qquad Eq(i)$$

(Note that these quantities z_i have a different form depending on the triplet (l,d,i). See the three examples below.)

Now, let us introduce the following square matrix of order d :

$$M(d) = (m_{i,j})_{0\leq i,j\leq d-1} \quad \text{where} \quad m_{i,j} = \varepsilon(d+l+j-i,i).$$

Introducing two column vectors B and C, we observe that the d equations $Eq(i)$, for $0 \leq i \leq d-1$, can be summed up introducing the following linear system (S) : $\quad M(d).B = C \quad$, where

$$B = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \beta + z_0 \\ \beta^2 + z_1 \\ \vdots \\ \beta^{2^{d-1}} + z_{d-1} \end{bmatrix}.$$

We introduce the determinant, $\Delta(d)$, of the matrix $M(d)$ and also the determinant $\Delta(j,d)$ obtained from $\Delta(d)$ by replacing the column vector of rank j by the column vector C. Hence, applying Cramer's rule for solving the linear system (S), we get

$$\beta_j = \Delta(j,d)/\Delta(d) \quad \text{for} \quad 0 \leq j \leq d-1.$$

Finally, reporting these values for β_j in $Eq(d)$, we obtain

$$\beta^{2^d} = z_d + (\sum_{0\leq j\leq d-1} \varepsilon(l+j,d)\Delta(j,d))/\Delta(d). \quad Eq(*)$$

We observe that $\Delta(d)$ belongs to $\mathbb{F}_2(\boldsymbol{\varepsilon})$. While, $\Delta(j,d)$ belongs to $\mathbb{F}_2(\boldsymbol{\varepsilon})[\beta]$. Indeed, developping the determinant $\Delta(j,d)$ along the column of rank j, we get $\Delta(j,d) = c_j + \sum_{0\le k\le d-1} b_{k,j}\beta^{2^k}$. Consequently $Eq(*)$ can be witten as expected (with coefficients in $\mathbb{F}_2(\boldsymbol{\varepsilon})$) :

$$\beta^{2^d} = A + \sum_{0\le k\le d-1} B_k \beta^{2^k}. \qquad Eq(**)$$

So the proof of the theorem is complete.

We present, here below, three examples. In order to avoid unnecessary complications with the subscripts, we use (a,b,c) for the letters $(\varepsilon_0,\varepsilon_1,\varepsilon_2)$.

Example 1: Type (0,2). Period-Doubling sequence . Let us consider the case 2, mentioned above, where

$$\boldsymbol{\varepsilon} = (a,b)^\infty \quad \text{and} \quad \beta = 1/CF(\mathbf{s}(\boldsymbol{\varepsilon})).$$

We have $(\varepsilon_0,\varepsilon_1,\varepsilon_2) = (a,b,a)$ and $(z_0,z_1,z_2) = (1/a,0,1)$.

$$\Delta(2) = \begin{vmatrix} 1 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{vmatrix} = a+b,$$

$$\Delta(0,2) = \begin{vmatrix} \beta+z_0 & 1 \\ \beta^2+z_1 & \varepsilon_2 \end{vmatrix} \quad \text{and} \quad \Delta(1,2) = \begin{vmatrix} 1 & \beta+z_0 \\ \varepsilon_1 & \beta^2+z_1 \end{vmatrix}.$$

Hence we get

$$\Delta(0,2) = \beta^2 + a\beta + 1 \quad \text{and} \quad \Delta(1,2) = \beta^2 + b\beta + b/a.$$

Since $\varepsilon(0,2) = \varepsilon_0^2\varepsilon_1 = a^2b$ and $\varepsilon(1,2) = \varepsilon_1^2\varepsilon_2 = b^2a$, $Eq(*)$ becomes

$$(a+b)\beta^4 = a+b+(\beta^2+a\beta+1)a^2b+(\beta^2+b\beta+b/a)b^2a.$$

From this, we get (as expected, see [1, p 2, Th 1.1]) :

$$\beta^4 = 1+b(a+b)+ab(a+b)\beta+ab\beta^2. \qquad Eq(**)$$

Example 2: Type (1,2). Here we have :

$$\boldsymbol{\varepsilon} = a,(b,c)^\infty \quad \text{and} \quad \beta = 1/CF(\mathbf{s}(\boldsymbol{\varepsilon})).$$

We have $(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3) = (a,b,c,b)$ and $(z_0,z_1,z_2) = (1/a+1/ba^2, 1/a^2, 0)$.

$$\Delta(2) = \begin{vmatrix} 1 & 1 \\ \varepsilon_2 & \varepsilon_3 \end{vmatrix} = b+c$$

$$\Delta(0,2) = \begin{vmatrix} \beta + z_0 & 1 \\ \beta^2 + z_1 & \varepsilon_3 \end{vmatrix} \quad \text{and} \quad \Delta(1,2) = \begin{vmatrix} 1 & \beta + z_0 \\ \varepsilon_2 & \beta^2 + z_1 \end{vmatrix}.$$

Hence we get

$$\Delta(0,2) = \beta^2 + b\beta + b/a \quad \text{and} \quad \Delta(1,2) = \beta^2 + c\beta + 1/a^2 + c/a + c/ba^2.$$

We have $\varepsilon(1,2) = \varepsilon_1^2\varepsilon_2$ and $\varepsilon(2,2) = \varepsilon_2^2\varepsilon_3$. Consequently, $Eq(*)$ becomes

$$(b+c)\beta^4 = (\beta^2 + b\beta + b/a)b^2c + (\beta^2 + c\beta + 1/a^2 + c/a + c/ba^2)c^2b.$$

From this, we get :

$$\beta^4 = bc(b+c)/a + c^2/a^2 + bc(b+c)\beta + bc\beta^2. \qquad Eq(**)$$

(Note that changing c into a, we have $\boldsymbol{\varepsilon} = a, (b,c)^\infty = a, (b,a)^\infty = (a,b)^\infty$ and we regain the previous example and the same algebraic equation for β as above.)

Example 3: Type (0,3). Here we have :

$$\boldsymbol{\varepsilon} = (a,b,c)^\infty \quad \text{and} \quad \beta = 1/CF(\mathbf{s}(\boldsymbol{\varepsilon})).$$

We have $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (a,b,c,a,b)$ and

$$(z_0, z_1, z_2, z_3) = (1/a + 1/ba^2, 1/a^2, 0, 1).$$

$$\Delta(3) = \begin{vmatrix} 1 & 1 & 1 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_1^2\varepsilon_2 & \varepsilon_2^2\varepsilon_3 & \varepsilon_3^2\varepsilon_4 \end{vmatrix} = ba^2(a+c) + ac^2(b+c) + cb^2(a+b)$$

$$\Delta(0,3) = \begin{vmatrix} \beta + z_0 & 1 & 1 \\ \beta^2 + z_1 & a & b \\ \beta^4 + z_2 & c^2a & a^2b \end{vmatrix} = (a+b)\beta^4 + (a^2b + ac^2)\beta^2 + ab(a^2+c^2)\beta + \delta_0$$

$$\Delta(1,3) = \begin{vmatrix} 1 & \beta + z_0 & 1 \\ c & \beta^2 + z_1 & b \\ b^2c & \beta^4 + z_2 & a^2b \end{vmatrix} = (a+b)\beta^4 + (a^2b + ac^2)\beta^2 + ab(a^2+c^2)\beta + \delta_1$$

$$\Delta(2,3) = \begin{vmatrix} 1 & 1 & \beta + z_0 \\ c & a & \beta^2 + z_1 \\ b^2c & c^2a & \beta^4 + z_2 \end{vmatrix} = (a+c)\beta^4 + (ac^2 + b^2c)\beta^2 + ac(c^2+b^2)\beta + \delta_2$$

together with

$$\delta_0 = ab(a^2+c^2)z_0 + (a^2b+c^2a)z_1 + (a+b)z_2$$

$$\delta_1 = cb(a^2+b^2) + (a^2b+b^2c)z_1 + (a+b)z_2$$

$$\delta_2 = ac(c^2+b^2) + (c^2a+b^2c)z_1 + (b+c)z_2.$$

Here, $Eq(*)$ becomes

$$\beta^8 = z_3 + (\sum_{0\le j\le 2} \varepsilon(j,3)\Delta(j,3))/\Delta(3)$$

and we also have

$$\varepsilon(0,3) = a^4b^2c, \qquad \varepsilon(1,3) = b^4c^2a \quad \text{and} \quad \varepsilon(2,3) = c^4a^2b.$$

Finally combining these values and the four values for the determinants given above, from $Eq(*)$, we get the desired outcome :

$$\beta^8 = A + \sum_{0\le k\le 2} B_k\beta^{2^k}. \qquad Eq(**)$$

At last, remarkably enough, we can check that the four coefficients in this last equation do not only belong to $\mathbb{F}_2(\boldsymbol{\varepsilon})$ but are indeed elements in $F_2[t]$ and we have

$$A = a^3b^2c + a^2b^2c^2 + ab^3c^2 + b^4c^2 + ab^2c^3 + abc^4 + a^2bc + ab^2c + abc^2 + c^4 + 1$$

$$B_0 = a^4b2c + a^3b^2c^2 + a^2b^3c^2 + ab^4c^2 + a^2b^2c^3 + a^2bc^4$$

$$B_1 = a^3b^2c + a^2b^2c^2 + ab^3c^2 + a^2bc^3$$

and

$$B_2 = a^2bc + ab^2c + abc^2.$$

An important and last point need to be discussed. Indeed, the reader will probably ask the following question : are the sequences, belonging to the family $\mathcal{F}$, 2-automatic as it is indicated in the title of this note ?
There are different ways to characterize automatic sequences. A direct way is to consider the letters of the infinite world as elements in a finite field $\mathbb{F}_q$ of characteristic p. If a power series γ in $\mathbb{F}_q((1/t))$ is algebraic over $\mathbb{F}_q(t)$, then the sequence of its coefficients is p-automatic (Christol's theorem).

Concerning the sequences $\mathbf{s}(\boldsymbol{\varepsilon})$ described above, in the general case the automaticity will result from a conjecture. First we assume that the

$l+d$ elements defining the sequence are in a finite field $\mathbb{F}_q$ of characteristic 2 with $q = 2^s \geq l+d$, and consequently we may consider the power series $\gamma \in \mathbb{F}_q((1/t))$ associated to this sequence. Beginning by the trivial case $d = 1$, we have observed that $\mathbf{s}(\boldsymbol{\varepsilon})$ is utimately periodic and therefore p-automatic for all p. Note that the power series γ, associated to it, is rational and consequently it satisfies a polynomial of degree 1 over $\mathbb{F}_q(t)$. We make the following conjecture :

Conjecture. *Let $l \geq 0$ and $d \geq 2$ be integers. Let $\boldsymbol{\varepsilon}$ be an ultimately periodic sequence of type (l,d). Let $\mathbf{s}(\boldsymbol{\varepsilon}) = (s_n)_{n\geq 0}$ be the sequence defined above. Then there exists a finite field $\mathbb{F}_q$ of characteristic 2, containing $l+d$ elements identified with the terms of this sequence so that we may consider $\gamma = \sum_{n\geq 0} s_n t^{-n}$ in $\mathbb{F}_q((1/t))$ and there is a polynomial P in $\mathbb{F}_q(t)[x]$ such that $\deg_x(P) = 2^{d-1}$ and $P(\gamma) = 0$.*

In the simpler case $(l,d) = (0,2)$ and $\boldsymbol{\varepsilon} = (a,b)^\infty$, already considered several times above, the resulting sequence $\mathbf{s}(\boldsymbol{\varepsilon})$ (called Period-doubling) is well known to be 2-automatic. More precisely, the above conjecture is true. Indeed, if we identify the pair (a,b) with the pair $(0,1)$ in $\mathbb{F}_2$ then $\gamma = \sum_{n\geq 0} s_n t^{-n} \in \mathbb{F}_2((1/t))$ satisfies $\gamma^2 + t\gamma = t^2/(t^2+1)$.

References

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[2] A. Lasjaunias. Continued fractions. arXiv:1711.11276, 30 Nov 2017.

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